A MONADIC LOGIC FOR CAPACITY QUANTIFIERS

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ABSTRACT. Hoover and Keisler's Probability Logic is a variant of first order logic that replaces the usual quantifiers $\forall$ and $\exists$ with probability quantifiers $(Px \geq r)$. The sentence $(Px \geq r)\varphi(x)$ is interpreted as "the set $\{x : \varphi(x)\}$ has probability at least r." We generalize the Hoover-Keisler logic to capacities and introduce a new logic for capacity quantifiers. A weak model existence theorem and a weak completeness theorem are proved. We conjecture that the strong forms of those theorems can be obtained using nonstandard analysis.

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0. Introduction

Lindström characterized first order logic $L_{\omega\omega}$ using the countable compactness property and the downward Lowenheim-Skolem-Tarski theorem. That is, $L_{\omega\omega}$ is the most powerful logic so that if a countable set of sentences Φ has no model, then some finite subset of Φ has no model, and if a sentence has a model, then it has a countable model. What this means is that, if we try to strengthen $L_{\omega\omega}$ with respect to its expressive power, then we will run into the problem of losing compactness or downward Lowenheim-Skolem-Tarski properties. This gave rise to the the study of *abstract logics*, also known as *abstract model theory* owing to its model theoretic flavor.

In abstract model theory, different types of quantifiers have been studied. For instance, the logic $L_{\omega\omega}(Q_0)$ has formulas of the form $Q_0 x\varphi(x)$, where $\mathfrak{A} = \langle A, \ldots\rangle$ is a model of $Q_0 x\varphi(x)$ if and only if there are infinitely many $a \in A$ so that $\mathfrak{A}$ thinks $\varphi(a)$ is true. The logic $L_{\omega\omega}(Q_1)$ has formulas of the same kind, except that $\mathfrak{A}$ is a model of $Q_1 x\varphi(x)$ if and only if there are uncountably many, that is at least $\aleph_1$-many, elements $a \in A$ so that $\mathfrak{A}$ thinks $\varphi(a)$ is true. In general, the logic $L_{\omega\omega}(Q_\alpha)$ is the logic with its satisfaction relation defined as $\mathfrak{A} \vDash_\alpha Q_\alpha x\varphi(x)$ if and only if there are at least $\aleph_\alpha$-many elements $a \in A$ so that $\mathfrak{A} \vDash_\alpha \varphi(a)$. There is no completeness theorem for $L_{\omega\omega}(Q_0)$, whereas a completeness theorem for $L_{\omega\omega}(Q_1)$ has been proved by Keisler. Moreover, $L_{\omega\omega}(Q_1)$ is countably compact while $L_{\omega\omega}(Q_0)$ is not. One may refer the first chapter of [BFF17], or chapter Thirteen of [BS06] for a comprehensive treatment of generalized quantifiers.

The study of infinitary logics was later rekindled as the logic $L_{\omega\omega}(Q_0)$ did not turn out to be that fountain of "good" properties that the logicians sought. The infinitary logic $L_{\kappa,\lambda}$ has disjunctions and conjunctions of size less than κ, and less than λ-many quantifiers in its formulas, where κ and λ are infinite cardinals. Thus, $L_{\omega\omega}$ is the usual first order logic, and $L_{\omega_1\omega}$ is similar to first order logic with the exception of allowing countably many disjunctions and conjunctions in its formulas. However, this distinction makes $L_{\omega_1\omega}$ vastly different from $L_{\omega\omega}$ from a model theoretic point of view. The compactness theorem fails for $L_{\omega_1\omega}$. Still, when passing from $L_{\omega\omega}$ to $L_{\omega_1\omega}$, a great deal of expressibility power is achieved. We provide some examples of those in section 2 of this paper.

Later, logicians were interested in logics answering the following questions. Given a field of study X in mathematics, how does one define a language $\mathcal{L}_X$ capable of expressing the interesting properties and theorems about X? Having a satisfaction relation $\vDash_X$ defined between structures for $\mathcal{L}_X$ and its sentences, what model theoretic theorems, for instance the compactness theorem, Löwenheim-Skolem-Tarski theorems, and interpolation theorems, can be proved inside the logic L_X corresponding to the language $\mathcal{L}_X$ and the satisfaction relation $\vDash_X$, and so on. In this vein, Keisler and Hoover developed the logic $L_{\mathbb{A}P}$ suitable for probability theory. They introduced the notion of *probability quantifiers* in it. Instead of the usual quantifiers $\forall$ and $\exists$, $L_{\mathbb{A}P}$ has quantifiers of the form $(Px \geq r)\varphi(x)$, which are interpreted as "the set $\{x : \varphi(x)\}$ has probability at least r", and so on. One may refer [Kei85] and [Hoo78a], or Hoover's doctoral dissertation [Hoo78b] for a detailed treatment of $L_{\mathbb{A}P}$. The logic $L_{\mathbb{A}P}$ was improved to allow integral signs in its formulas, and Raškovíc and Tanović developed a probability logic which allows the usual first order quantifiers in its formulas, and they generalized the logic $L_{\mathbb{A}P}$ to $L_{\mathbb{A}M}$ where the probability measures are replaced by σ-finite measures [RT90], [RaÅ85].

In this paper, we introduce the monadic logic $L_{\mathbb{A}\mathscr{I}}$. Instead of the probability quantifiers $(Px \geq r)$, we have inculcated quantifiers of the form $(\mathscr{I}x \geq r)$, called *capacity quantifiers*, to our language. The formula $(\mathscr{I}x \geq r)\varphi(x)$ is interpreted as "the set $\{x : \varphi(x)\}$ has capacity at least r." Capacities bear a strong resemblance to measures, especially probability measures, except that capacities need not be additive. Moreover, while a measure gives the volume of a set, a capacity is a description of the *size* of a set. Given a set F and a paving $\mathcal{F}$ on F, that is, a family of subsets of F containing the empty set, an $\mathcal{F}$-capacity is an extended real-valued monotonically non-decreasing function I on the power set $\mathcal{P}(F)$ of F satisfying continuity from below on $\mathcal{P}(F)$ and continuity from above on $\mathcal{F}$. There are different applications of capacities across different fields of mathematics. Similar to the Lebesgue integral in the context of measures, there are versions of integrals in the context of capacities, namely the *Choquet integral* and the *Sugeno integral*. In this paper, we are mainly interested in *strongly subadditive* capacities, also known as *2-alternating* capacities. Such capacities are characterized by the inequality $I(A \cup B) + I(A \cap B) \leq I(A) + I(B)$ on their domains. These notions have various applications across different areas of mathematics such as probability theory, computability theory, and decision theory.

Our goal is to lay down a set of axioms that correspond to the defining properties of a capacity in the context of capacity quantifiers and prove a model existence theorem for $L_{\mathbb{A}\mathscr{I}}$. But for the time being, we only prove a weak form of the model existence theorem. We conjecture that a strong form can be proved using the ideas found in [Ros90a] about nonstandard capacities. Moreover, a proof theory for $L_{\mathbb{A}\mathscr{I}}$ is left for future work. We speculate that a Completeness Theorem will hold for $L_{\mathbb{A}\mathscr{I}}$ analogous to the one in $L_{\mathbb{A}P}$.

It should be noted that our logic is monadic. As in $L_{\mathbb{A}P}$, we do not have a natural way to extend a capacity to higher dimensions. However, one could follow Keisler and Hoover, and obtain an analogue of graded probability structures for the models of $L_{\mathbb{A}\mathscr{I}}$.

As one might have already guessed, the subscript $\mathbb{A}$ in both $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\mathscr{I}}$ has a special purpose. It represents an *admissible set* $\mathbb{A}$ on which our formulas are built set theoretically, specifically within the set theory called *Kripke-Platek Set Theory*. In the absence of $\mathbb{A}$, we will have the infinitary logic $L_{\omega_1\mathscr{I}}$. Choosing $\mathbb{A}$ carefully allows us to have a nice set of formulas to which we can apply many important theorems of model theory such as *Barwise Compactness* and *Barwise Completeness* theorems, and other important results from computability theory which we will not look at in this paper. Indeed, $L_{\mathbb{A}\mathscr{I}} = \mathbb{A} \cap L_{\omega_1\mathscr{I}}$ is called the admissible fragment of $L_{\omega_1\mathscr{I}}$ determined by $\mathbb{A}$. Thus, when $\mathbb{A}$ is countable admissible, we will have a countable admissible set for our set of formulas of $L_{\mathbb{A}\mathscr{I}}$. The above equality might be confusing at first glance. However, we promise that it will start making sense once we describe how formulas are built set theoretically.

Notice that when writing a capacity quantifier $(\mathscr{I}x \geq r)$, we refer to a number r. But then, if we work in $L_{\mathbb{A}\mathscr{I}}$, one might wonder where those numbers r come from. The answer is the following. We can start building $\mathbb{A}$ above a set of *urelements*, which is just a fancy name for some objects that we do not assume to be sets by themselves. For instance, we can treat rational numbers, and of course real numbers, as urelements and build $\mathbb{A}$ on the set of rational numbers $\mathbb{Q}$, or on the set of real numbers $\mathbb{R}$. But for countability requirements, we will make ourselves content with choosing $\mathbb{Q}$ as our set of urelements. Thus we can treat the reals as the Dedekind cuts of sets of rationals. For a detailed description of these ideas one may refer the first three chapters of [Bar17].

The logic $L_{\mathbb{A}\mathscr{I}}$ somewhat generalizes the logic $L_{\mathbb{A}P}$. *Somewhat*, because as noted earlier, we do not have a natural way to extend a given capacity to higher dimensions. Thus, we are still unaware which theorems of $L_{\mathbb{A}P}$ can be proved in $L_{\mathbb{A}\mathscr{I}}$. However, we can at once get rid of the nuisance of the necessity of the sets of interest, mainly the definable ones, being measurable as a capacity applies to all the subsets of a given set with no discrimination. Moreover, we can expect to enrich our language $\mathcal{L}_{\mathbb{A}\mathscr{I}}$ to include relation symbols for random closed sets, akin to the relations for random variables in [Kei85], and perhaps do a model theoretic analysis in the context of a second order logic for capacity quantifiers.

This paper is divided into four sections and an appendix. The first section is a discussion of capacities and how they arise in the theory of analytic sets. In the second section, we give a brief introduction to infinitary logic $L_{\omega_1\omega}$ as we shall borrow some model theoretic techniques from it, especially a modified version of the notion called a *consistency property*. The third section is intended to serve as a summary of the model existence theorem for $L_{\mathbb{A}P}$. Finally, in the fourth section, we introduce the notion of capacity quantifiers and the logic $L_{\mathbb{A}\mathscr{I}}$. We end it with some conjectures. An appendix is included, the first of which is a brief discussion of the set theoretic formulation of formulas, specifically within the theory KP, admissible sets and Barwise Compactness Theorem, and the second of which is a summary of the nonstandard preliminaries required for the paper.

1. Capacities

We will introduce *capacities* in this section. In Section 3 we describe Hoover and Keisler's *Probability Logic*, a variant of first order logic which replaces the usual quantifiers $\forall$ and $\exists$ with *probability quantifiers*; this is the model we use for the logic for this paper. This is a type of *infinitary logic*, and in Section 2 we will give a quick introduction to that. Infinitary logics are usually badly behaved; one way to make them more tractable is to restrict to *admissible sets*, and we give an introduction to those in Appendix A. Both the Hoover-Keisler logic and the one we introduce here use nonstandard analysis as a tool, and we give a brief introduction to that in Appendix B. Our main definitions and results are in Section 4. The references for this section will be [DM78], [MM05], and [Ros90a].

1.1. Basics of Capacity Theory

Definition 1. *A paving on a set F is a family $\mathcal{F}$ of subsets of E containing the empty set. The pair $\langle F, \mathcal{F}\rangle$ is called a paved set. A paving is a* regular paving *if it is closed under finite unions and finite intersections.*

Example 1. The power set $\mathcal{P}(F)$ and $\{\varnothing\}$ are regular pavings on F.

Example 2. Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{T}$ is a regular paving on X.

Example 3. A σ-algebra $\mathcal{F}$ on a set F is a regular paving which is also closed under complements, countable unions, and countable intersections.

Definition 2. *Let F be a set, and suppose $\mathcal{F}$ is a family of subsets of F. Then the regular paving generated by $\mathcal{F}$ is the smallest regular paving on F containing $\mathcal{F}$.*

Definition 3. *Let $\mathcal{F}$ be a family of subsets of a set F. A set $X \in \mathcal{F}$ is*

(1) $\mathcal{F}_\sigma$ if $X = \bigcup_{n\in\mathbb{N}} F_n$ with $F_n \in \mathcal{F}$ for each $n \in \mathbb{N}$;

(2) $\mathcal{F}_\delta$ if $X = \bigcap_{n\in\mathbb{N}} F_n$ with $F_n \in \mathcal{F}$ for each $n \in \mathbb{N}$;

(3) $\mathcal{F}_{\sigma\delta}$ if $X = \bigcap_{n\in\mathbb{N}} X_n$ with X_n an $\mathcal{F}_\sigma$ for each $n \in \mathbb{N}$.

Definition 4. *Let $\mathcal{F}$ be a regular paving on a set F. An $\mathcal{F}$-*capacity *on F is an extended real-valued set function I on $\mathcal{P}(F)$ with the following properties.*

(1) I is monotone, that is, if $A \subseteq B \subseteq F$ then $I(A) \le I(B)$.

(2) If $\langle A_n \rangle$ is an increasing sequence of subsets of F, then $I\big(\bigcup_{n=1}^\infty A_n\big) = \sup_n I(A_n)$.

(3) If $\langle A_n \rangle$ is a decreasing sequence of elements of $\mathcal{F}$, then $I\big(\bigcap_{n=1}^\infty A_n\big) = \inf_n I(A_n)$.

A subset A of F is called capacitable *if*

$$I(A) = \sup\{I(B) : B \in \mathcal{F}_\delta \text{ and } B \subseteq A\}.$$

Example 4. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Recall that the outer measure generated by $\mathbb{P}$ is given by

$$\mathbb{P}^*(A) = \inf\{\sum_{n=1}^\infty \mathbb{P}(A_n) : A_n \in \mathcal{A} \text{ and } A \subseteq \bigcup_{n=1}^\infty E_n\}$$

for each $A \subseteq \Omega$. Then $\mathbb{P}^*$ is an $\mathcal{A}$-capacity on Ω.

Example 5. Consider the real line $\mathbb{R}$ with its usual topology. Let $\mathcal{K}$ be the set of all compact subsets of $\mathbb{R}$. Suppose I is a $\mathcal{K}$-capacity on $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Define J on $\mathcal{P}(\mathbb{R})$ by $J(A) = I\big(f(A)\big)$. Then J is a $\mathcal{K}$-capacity on $\mathbb{R}$.

There are different kinds of capacities characterized by their behavior on the subsets of their domains. We shall define some of those different capacities below. For us, the most important kind of capacities are the strongly subadditive capacities, and we will see why in Section 4.

Definition 5. *Let $\langle F, \mathcal{F}\rangle$ be a paved set, and let I be an $\mathcal{F}$-capacity on F. Then*

(1) I is subadditive *if and only if $I(A \cup B) \leq I(A) + I(B)$ for all $A, B \in \mathcal{F}$.*

(2) I is additive *if and only if $I(A \cup B) + I(A \cap B) = I(A) + I(B)$ for all $A, B \in \mathcal{F}$.*

(3) I is maxitive *if and only if $I(A \cup B) = \max\{I(A), I(B)\}$ for all $A, B \in \mathcal{F}$.*

(4) I is strongly subadditive *if and only if $I(A \cup B) + I(A \cap B) \leq I(A) + I(B)$ for all $A, B \in \mathcal{F}$.*

In this paper, our attention is restricted to capacities taking values in the interval $[0, 1]$, that is, all the capacities considered here will be *normalized* capacities.

Definition 6. *Let $\mathcal{E} = \{K_i\}_{i \in I}$ be a family of sets. Then $\mathcal{E}$ has the finite intersection property if for each finite $I_0 \subseteq I$ we have $\bigcap_{i \in I_0} K_i \neq \varnothing$.*

Remark 1. The above definition is equivalent to saying that there exists an ultrafilter $\mathcal{U}$ so that $K_i \in \mathcal{U}$ for each $i \in I$. See [CK90] Chapter 4.

Definition 7. *Let $\langle E, \mathcal{E}\rangle$ be a paved set. Then $\mathcal{E}$ is compact if and only if every family of elements of $\mathcal{E}$ with the finite intersection property has a non-empty intersection.*

Example 6. The paving $\mathcal{K}(E)$ consisting of the compact subsets of a Hausdorff topological space E is a compact paving. If $\mathcal{E}$ is a compact paving, then so is $\mathcal{E} \cup \{E\}$.

We are now ready to define $\mathcal{F}$-analytic sets corresponding to a paved set $\langle F, \mathcal{F}\rangle$. This is paragraph 7 of [DM78] on page $41 - III$.

Definition 8. *Let $\langle F, \mathcal{F}\rangle$ be a paved set. A subset $A \subseteq F$ is $\mathcal{F}$-*analytic *if there is an auxiliary compact metrizable space E and a subset $B \subseteq E \times F$ in $\big(\mathcal{K}(E) \times \mathcal{F}\big)_{\sigma\delta}$ so that A is the projection of B onto F. The paving on F consisting of all $\mathcal{F}$-analytic sets is denoted by $(\mathcal{F})$.*

Theorem 1 (Choquet's Theorem)**.** *Let I be an $\mathcal{F}$-capacity. Every $\mathcal{F}$-analytic set is capacitable relative to I.*

Dellacherie and Meyer [DM78] prove Choquet's theorem using two lemmas. First lemma asserts that $\mathcal{F}_{\sigma\delta}$ sets are capacitable. The second lemma states that the composition of the projection map with a capacity is a capacity in some sense.

Lemma 1. *Every element of $\mathcal{F}_{\sigma\delta}$ is capacitable relative to I.*

Lemma 2. *Let E be a compact metric space with its compact paving $\mathcal{E}$. Let I be an $\mathcal{F}$-capacity. Define J on $E \times F$ by $J(H) = I(\pi(H))$ for all $H \subseteq E \times H$. Then J is an $\mathcal{H}$-capacity on $E \times F$, where $\mathcal{H}$ is the paving obtained by closing $(\mathcal{E} \times \mathcal{F})_{\sigma\delta}$ under finite unions and finite intersections.*

Now we want to associate a so-called *outer capacity* to every strongly subadditive increasing set function. In doing so, one obtains a genuine Choquet capacity. Thus, we will use Choquet's theorem to show that analytic sets are measurable. For a proof of the following theorem, see [DM78].

Theorem 2. *Suppose $\mathcal{F}$ is a regular paving on a set F. Let I be a positive, increasing, strongly subadditive set function on $\mathcal{F}$ satisfying the following condition.*

For every increasing sequence $\langle A_n \rangle$ of sets in $\mathcal{F}$ with $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}, I(A) = \sup_n I(A_n)$.

We have the following definitions.

(A) For every $A \in \mathcal{F}_\sigma$, define $I_L(A) = \sup\{I(B) : B \in \mathcal{F} \text{ and } B \subseteq A\}$. Thus, I is extended to $\mathcal{F}_\sigma$. The set function I_L is called an inner capacity*.*

(B) For every $C \subseteq F$, define $I_L(C) = \inf\{I_L(A) : A \in \mathcal{F}_\sigma \text{ and } C \subseteq A\}$. Thus, I is extended to 2^F.

The function I_L has the following properties.

(1) For every increasing sequence $\langle X_n \rangle$ of subsets of F, $I_L(\bigcup_{n=1}^{\infty} X_n) = \sup_n I_L(X_n)$.

(2) If $\langle X_n \rangle$ and $\langle Y_n \rangle$ are two sequences of subsets of F with $X_n \subseteq Y_n$ for each n, then

$$I_L\Big(\bigcup_{n=1}^{\infty} Y_n\Big) + \sum_{n=1}^{\infty} I_L(X_n) \le I_L\Big(\bigcup_{n=1}^{\infty} X_n\Big) + \sum_{n=1}^{\infty} I_L(Y_n).$$

(3) I_L *is an* $\mathcal{F}$*-capacity if and only if* $I_L\left(\bigcap_{n=1}^{\infty} A_n\right) = \inf_n I(A_n)$ *for every decreasing sequence* $\langle A_n \rangle$ *of elements of* $\mathcal{F}$*.*

Example 7 (Analytic sets are measurable). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Let $\mathcal{F} \subseteq \mathcal{A}$ be so that $\mathcal{F}$ is closed under finite unions and finite intersections. Put $I = \mathbb{P} \restriction_{\mathcal{F}}$. Then we have the following.

(a) For each $A \in \mathcal{F}_\sigma$,

$$\begin{aligned} I_L(A) &= \sup\{I(B) : B \in \mathcal{F} \text{ and } B \subseteq A\} \\ &= \sup\{\mathbb{P} \restriction_{\mathcal{F}} (B) : B \in \mathcal{F} \text{ and } B \subseteq A\} \\ &= \mathbb{P}(A) \ . \end{aligned}$$

(b) For each $A \in \mathcal{F}_\delta$,

$$\begin{aligned} I_L(A) &= \inf\{I_L(B) : B \in \mathcal{F}_\sigma \text{ and } B \supseteq A\} \\ &= \inf\{\mathbb{P}(B) : B \in \mathcal{F}_\sigma \text{ and } B \supseteq A\} \text{ (by (a), } \mathbb{P} \text{ and as } I_L \text{ agree on } \mathcal{F}_\sigma) \\ &= \mathbb{P}(A) \ . \end{aligned}$$

Thus, I_L and $\mathbb{P}$ agree on both $\mathcal{F}_\sigma$ and $\mathcal{F}_\delta$. Clearly the conditions *(1)* and *(3)* of the above theorem are satisfied by I_L. Hence I_L is an $\mathcal{F}$-capacity. Now, let $A \subseteq \Omega$ be $\mathcal{F}$-analytic. By Choquet's theorem,

$$I_L(A) = \sup\{I_L(B) : B \in \mathcal{F}_\delta \text{ and } B \subseteq A\} = \sup\{\mathbb{P}(B) : B \in \mathcal{F}_\delta \text{ and } B \subseteq A\}$$

and by the above considerations,

$$I_L(A) = \inf\{\mathbb{P}(B) : B \in \mathcal{F}_\sigma \text{ and } B \supseteq A\}.$$

Thus we have

$$\sup\{\mathbb{P}(B) : B \in \mathcal{F}_\delta \text{ and } B \subseteq A\} = \inf\{\mathbb{P}(B) : B \in \mathcal{F}_\sigma \text{ and } B \supseteq A\}.$$

Therefore, we can find $B' \in \mathcal{F}_{\delta\sigma}$ and $C' \in \mathcal{F}_{\sigma\delta}$ such that $B' \subseteq A \subseteq C'$ with $\mathbb{P}(C') \leq I_L(A) \leq \mathbb{P}(B')$ using an $\left(\frac{1}{n}\right)$-argument. Using the fact that $\mathbb{P}$ is a measure, we then get $\mathbb{P}(B') = \mathbb{P}(C')$. Hence $A \in \mathcal{A}$, that is, A is measurable.

1.2. Suslin Schemes

In this section, we review the notion of Suslin schemes and show that every $\mathcal{F}$-analytic set is the kernel of an $\mathcal{F}$-Suslin scheme. First off, let us fix some notation. Let B be a nonempty set. By $B^{<\mathbb{N}}$ we will denote the set of finite sequences of elements of B including the empty sequence. For $s, t \in B^{<\mathbb{N}}$, $s \prec t$ denotes that s is an initial segment of t, that is $s(0) = t(0), \ldots, s(|s|-1) = t(|s|-1)$, where $s = \langle s(0), \ldots, s(|s|-1)\rangle$ for some $|s| \in \mathbb{N}^+$ called the length of s. We will denote the set of all infinite sequences of elements of B by $B^{\mathbb{N}}$. For $\sigma \in B^{\mathbb{N}}$ and $n \in \mathbb{N}$, $\sigma \upharpoonright n$ is the sequence $\langle \sigma(0), \ldots, \sigma(n-1)\rangle$, where we stipulate that $\sigma \upharpoonright n$ is the empty sequence if $n = 0$.

Definition 9. *Let B be a nonempty set. A system of sets on a set F is a family $\{B_s : s \in B^{<\mathbb{N}}\}$ of subsets of F.*

Definition 10. *Let $\langle F, \mathcal{F}\rangle$ be a paved set. An $\mathcal{F}$-*Suslin scheme *is a mapping*

$$\mathscr{S} : B^{<\mathbb{N}} \to \mathcal{F}; s \mapsto B_s.$$

The kernel *of $\mathscr{S}$ is the set*

$$\mathscr{S}_B\left(\{B_s : s \in B^{<\mathbb{N}}\}\right) = \bigcup_{\sigma \in B^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} B_{\sigma \upharpoonright n}.$$

The following is a weak form the Theorem 76 of [DM78]. Indeed, the paving $\mathcal{E}$ needs only be semi-compact.

Theorem 3. *Let $\langle E, \mathcal{E}\rangle$ and $\langle F, \mathcal{F}\rangle$ be two paved sets. Suppose $\mathcal{E}$ is compact. Let B be an element of $(\mathcal{E} \times \mathcal{F})_{\sigma\delta}$ and let A be the projection of B onto F. Then A is the kernel of an $\mathcal{F}$-Suslin scheme.*

We end this section with the following theorem, which says that every kernel of an $\mathcal{F}$-Suslin scheme is the projection of an element of $(\mathcal{E} \times \mathcal{F})_{\sigma\delta}$, where $\mathcal{E}$ is the paving consisting of all the compact subsets of the compact space $\overline{\mathbb{N}}^{\mathbb{N}}$ and $\overline{\mathbb{N}}$ is the one-point compactification of $\mathbb{N}$, where we equip $\mathbb{N}$ with the discrete topology (see [Eng89]). This is Theorem 77 of [DM78].

Theorem 4. *Let E and $\mathcal{E}$ be as above. Further let $\langle F, \mathcal{F} \rangle$ be a paved set and let A be the kernel of an $\mathcal{F}$-Suslin scheme (A_s). For $s \in \mathbb{N}^{<\mathbb{N}}$, put $I_s = \{\sigma \in \mathbb{N}^{\mathbb{N}} : s < \sigma\}$. Then A is the projection of $B \subseteq \mathbb{N}^{\mathbb{N}} \times F$ onto F, where $B = \bigcap_n \bigcup_{|s|=n} (I_s \times A_s)$. Moreover, B is an $(\mathcal{E} \times \mathcal{F})_{\sigma\delta}$ if we consider $\mathbb{N}^{\mathbb{N}} \times F$ as a subset of $E \times F$.*

1.3. Nonstandard Capacity Theory

The reader may look at Appendix B for the nonstandard preliminaries. To further understand the nonstandard ideas in capacity theory one may look at [Ros90a], [Ros90b], and [HR93].

Definition 11. *Suppose $\mathcal{F}$ is a regular paving on a set F. A normalized strongly subadditive monotone set function on $\mathcal{F}$ is called an $\mathcal{F}$-*precapacity*.*

Definition 12. *Let F be a set with a regular paving $\mathcal{F}$. Let $T : \mathcal{F} \to [0,1]$ be a precapacity on $\mathcal{F}$. Define $L(T) : \mathcal{P}(F) \to [0,1]$ by*

$$L(T)(E) = \inf_{\substack{E \subseteq D \\ D \in \mathcal{F}_\sigma}} \sup_{\substack{X \subseteq D \\ X \in \mathcal{F}}} T(X).$$

Let F be an internal set and let $\mathcal{F}$ be an internal (standardly) regular paving on F. And let $T : \mathcal{F} \to {}^*[0,1]$ be an internal precapacity. Let ${}^\circ T(A)$ be the standard part of $T(A)$ for each $A \in \mathcal{F}$. Thus we get a mapping ${}^\circ T : \mathcal{F} \to [0,1]$. It is easy to see that ${}^\circ T$ is an $\mathcal{F}$-precapacity on F. We have the following theorem.

Theorem 5 ([Ros90a])**.** *The mapping $L({}^\circ T)$ induced by T is a strongly subadditive $\mathcal{F}$-capacity on F.*

2. Infinitary Logic: A Primer

We give a quick introduction to the infinitary logic $L_{\omega_1\omega}$. We will look at the Completeness Theorem and a notion called *consistency properties*, which is modified appropriately for Hoover and Keisler's probability logic and the logic we introduce in Section 4. Main references for this section are [Kei71],[Mar16] and [Bar17].

2.1. An Introduction to the logic $L_{\omega_1\omega}$

We will start with a brief description of infinitary languages which will be a prerequisite for the material to follow. Let $\mathcal{L}$ be a first order language with countably many relation, function, and constant symbols, and the logical symbols $\neg, \wedge, \vee, \forall, \exists$, and $=$. Suppose $\mathcal{L}$ has ω_1 variables $v_0, v_1, \ldots$. Then the language $\mathcal{L}_{\omega_1\omega}$ has the same symbols as $\mathcal{L}$, and the connectives $\wedge$ and $\vee$ can be applied to *countable* sets of formulas from $\mathcal{L}$. The class of formulas of $\mathcal{L}_{\omega_1\omega}$ is the least class X such that:

(i) Each atomic formula of $\mathcal{L}$ is in X.
(ii) If $\varphi \in X$ and $\alpha < \omega_1$, then $\neg\varphi \in X, \forall v_\alpha \varphi \in X$, and $\exists v_\alpha \varphi \in X$.
(iii) If Φ is a finite or countable, non-empty, subset of X, then $\bigwedge \Phi \in X$ and $\bigvee \Phi \in X$.

Remark 2. Formulas can be built up set theoretically. That is, a variable v_α is identified with the ordered pair $\langle v, \alpha \rangle$, and all the other logical symbols are identified with natural numbers. Atomic formulas are finite sequences of symbols. $\neg\varphi$ is $\langle \neg, \varphi \rangle$, $\bigwedge \varphi$ is $\langle \bigwedge, \varphi \rangle$, and $\forall v_\alpha \varphi$ is $\langle \forall, v_\alpha, \varphi \rangle$. Consequently, every formula of $\mathcal{L}_{\omega_1\omega}$ is a set.

We can generalize the above notion of infinitary languages to any regular cardinal κ and get $\mathcal{L}_{\kappa\omega}$. We stipulate that κ be regular for then if φ is a sentence in $\mathcal{L}_{\kappa\omega}$, then φ always has fewer than κ many subformulas. But this fails if κ is singular [Mar16] [KK75]. With $\kappa = \omega$ we get the usual logic with just finitely many disjunctions or conjunctions in its formulas. The language $\mathcal{L}_{\omega_1\omega}$ has uncountably many formulas. The expressive power of $\mathcal{L}_{\omega_1\omega}$ is far greater than that of $\mathcal{L}_{\omega\omega}$ as manifested by the following examples. For more examples see [Kei71].

Example 8.

The class of all finite models can be characterized by the $\mathcal{L}_{\omega_1\omega}$-sentence

$$\bigvee_{n<\omega} \exists x_1 \dots x_n \forall y (y = x_1 \vee \dots \vee y = x_n).$$

Example 9.

Let $\mathcal{L} = \{+, \cdot, S, 0\}$ be the language of Peano arithmetic, where S is the successor function as usual. The $\mathcal{L}_{\omega_1\omega}$–sentence

$$(\text{Peano's Axioms}) \wedge \forall x (x = 0 \vee x = S(0) \vee x = S(S(0)) \vee \cdots)$$

characterizes the class of all models isomorphic to the standard model of arithmetic.

The satisfaction relation is defined in the usual manner (see [CK90]) with the following modification. Let Φ be a countable set of formulas of $\mathcal{L}_{\omega_1\omega}$ and let $\mathfrak{A}$ be a structure for $\mathcal{L}$. Then

$$\mathfrak{A} \models \bigvee_{\varphi\in\Phi} \varphi \text{ if and only if } \mathfrak{A} \models \varphi \text{ for some } \varphi \in \Phi,$$

and

$$\mathfrak{A} \models \bigwedge_{\varphi\in\Phi} \varphi \text{ if and only if } \mathfrak{A} \models \varphi \text{ for all } \varphi \in \Phi.$$

We will reserve calligraphic letters for languages, and an uppercase letter for the logic corresponding to a given language. Albeit having a greater expressive power than $L_{\omega\omega}$, the logic $L_{\omega_1\omega}$ has some of its drawbacks. For instance, the Compactness Theorem and the Upward Löwenheim-Skolem-Tarski Theorem fail for $L_{\omega_1\omega}$.

Example 10. This is the example on page 10 of [Kei71]. For each $n \leq \omega$, let c_n be a constant for $\mathcal{L}$, and let $\Sigma = \{(\forall x)\bigvee_{n<\omega}(x = c_n), (c_\omega \neq c_0), (c_\omega \neq c_1), \dots\}$. Then Σ is finitely satisfiable but Σ does not have a model. To see why the Upward Löwenheim-Skolem-Tarski theorem does not hold for $L_{\omega_1\omega}$, observe that the sentence $(\forall x)\bigvee_{n<\omega}(x = c_n)$ has a countable model while it does not have any uncountable models.

One property of $L_{\omega_1\omega}$ that stands out is the following. If $\mathfrak{A}$ is a countable structure for $\mathcal{L}$, then there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that if $\mathfrak{B}$ is another countable structure for $\mathcal{L}$, then $\mathfrak{B} \vDash \varphi$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Thus, all the countable models of φ are exactly the isomorphic copies of $\mathfrak{A}$. This is called the Scott's Isomorphism Theorem and φ is called a Scott sentence.

The following definition illustrates how the negation symbol is moved inside a formula. Note that we need this "definition" as we are now dealing with possibly infinitely long formulas. However, one ought to believe that this is just a formality.

Definition 13. *For each formula φ of $\mathcal{L}_{\omega_1\omega}$, the formal negation $\varphi\neg$ of φ is defined in the following manner. Here, Φ is an arbitrary countable set of $\mathcal{L}_{\omega_1\omega}$-formulas and x is an arbitrary variable of $\mathcal{L}_{\omega_1\omega}$.*

(1) $\varphi\neg$ is $\neg\varphi$ for atomic φ.

(2) $(\neg\varphi)\neg$ is the same as φ.

(3) $(\bigwedge_{\varphi\in\Phi}\varphi)\neg$ is $\bigvee_{\varphi\in\Phi}\neg\varphi$, and $(\bigvee_{\varphi\in\Phi}\varphi)\neg$ is $\bigwedge_{\varphi\in\Phi}\neg\varphi$.

(4) $(\exists x\varphi)\neg$ is $\forall x\neg\varphi$, and $(\forall x\varphi)\neg$ is $\exists x\neg\varphi$.

Now we will list the axioms for the logic $L_{\omega_1\omega}$ and its rules of inference. We aim to sketch a proof of Karp's completeness theorem in the next section. Let x, y be arbitrary variables of $\mathcal{L}_{\omega_1\omega}$. Further, let φ, ψ be arbitrary $\mathcal{L}_{\omega_1\omega}$-formulas, and let Φ be an arbitrary countable set of $\mathcal{L}_{\omega_1\omega}$-formulas.

Definition 14. *The axioms for $L_{\omega_1\omega}$ are the following.*

(1) Every instance of a tautology of finitary logic.

(2) $(\neg\varphi) \leftrightarrow (\varphi\neg)$.

(3) $\bigwedge\Phi \to \varphi$ for all $\varphi \in \Phi$.

(4) $\forall x\varphi(x\ldots) \to \varphi(t\ldots)$, where $\varphi(x\ldots)$ is a formula, t is a term which is free for x in $\varphi(x\ldots)$, and $\varphi(t\ldots)$ is obtained by replacing each free occurrence of x by t.

(5) $x = x$.

(6) $x = y \to y = x$.

(7) $(\varphi(x\ldots) \wedge t = x) \to \varphi(t\ldots)$, *where* $\varphi(x\ldots)$ *and* $\varphi(t\ldots)$ *are as in (4).*

Definition 15. *The rules of inference for* $L_{\omega_1\omega}$ *are the following.*

(1) From $\psi, \psi \to \varphi$, *infer* φ.

(2) From $\psi \to \varphi(x,\ldots)$, *infer* $\psi \to \forall\varphi(x,\ldots)$, *where* x *does not occur free in* ψ.

(3) From for all $\varphi \in \Phi$, $\psi \to \varphi$, *infer* $\psi \to \bigwedge\Phi$.

2.2. Model Existence Theorem and Completeness Theorem

In this section, we will briefly describe a model building procedure for the logic $L_{\omega_1\omega}$. We start by reviewing the notion of consistency properties. Then we will sketch a proof of the *Model Existence Theorem*, which is used in the proof of Karp's completeness theorem. We will not prove the completeness theorem but only mention it. Our exposition closely follows [Kei71]. The first step is to augment the language by adding a countable set of new constant symbols. We will then pick any set from the consistency property and show that it has a model. The idea is more or less similar to the Henkin construction in first order logic.

Definition 16. *Let* C *be a countable set of constant symbols not in* $\mathcal{L}$. *Then* $\mathcal{M}$ *is the augmented language* $\mathcal{L} \cup C$. *The infinitary logic corresponding to* $\mathcal{M}$ *is denoted by* $M_{\omega_1\omega}$.

Definition 17. *A basic term is either a constant symbol or a term of the form* $F(c_1,\ldots,c_n)$, *where* $c_1,\ldots,c_n \in C$ *and* F *is a function symbol of* $\mathcal{L}$.

Definition 18. *Let* S *be a set of countable sets of* $M_{\omega_1\omega}$*-sentences. Then we call* S *a consistency property if and only if for each* $s \in S$ *all of the following hold.*

C1. (*Consistency Rule*) *Either* $\varphi \notin s$ *or* $(\neg\varphi) \notin s$.

C2. ($\neg$*-Rule*) $(\neg\varphi) \in s$ *implies* $s \cup \{(\varphi\neg)\} \in S$.

C3. ($\bigwedge$*-Rule*) $(\bigwedge\Phi) \in s$ *implies* $s \cup \{\varphi\} \in S$ *for all* $\varphi \in \Phi$.

C4. ($\forall$*-Rule*) $(\forall x\varphi(x)) \in s$ *implies* $s \cup \{\varphi(c)\} \in s$ *for all* $c \in C$.

C5. ($\bigvee$*-Rule*) $(\bigvee\Phi) \in s$ *implies* $s \cup \{\varphi\} \in S$ *for some* $\varphi \in \Phi$.

C6. ($\exists$*-Rule*) $(\exists x\varphi(x)) \in s$ *implies* $s \cup \{\varphi(c)\} \in S$ *for some* $c \in C$.

C7. (*Equality Rules*) *Let t be a basic term, and let $c, d \in C$.*

(α) $(c = d) \in s$ *implies* $s \cup \{d = c\} \in S$.

(β) $c = t, \varphi(t) \in s$ *imply* $s \cup \{\varphi(c)\} \in S$.

(γ) $s \cup \{e = t\} \in S$ *for some* $e \in C$.

Lemma 3. *Let S be a consistency property with $s \in S$, and let $c, d, e \in C$. Then each of the following holds.*

(i) *There exists* $s' \in S$ *such that* $s \subseteq s'$ *and* $(c = c) \in s'$.

(ii) $(c = d), (d = e) \in s$ *implies* $s \cup \{c = e\} \in S$.

(iii) $\varphi, \varphi \to \psi \in s$ *implies there exists* $s' \in S$ *such that* $s \subseteq s'$ *and* $\psi \in s'$.

Proof. This is Lemma 2.4 of [Bar17]. ⊣

We will now sketch a proof of the model existence theorem, which is due to Makkai. This is Theorem 2 of [Kei71].

Theorem 6 (Model Existence Theorem)**.** *If S is a consistency property and $s_0 \in S$, then s_0 has a countable model.*

Proof. First, note that each subset of an element of S is again an element of S, that is, if $s' \subseteq s \in S$, then $s' \in S$. Second, let us form a set Y as follows. We declare that Y is the least set of sentences such that each of the following properties is satisfied by Y.

(i) $s_0 \subseteq Y$.

(ii) Y is closed under subformulas.

(iii) If t is a term, $c \in C$, and $\varphi(t) \in Y$, then $\varphi(c) \in Y$.

(iv) If $(\neg\varphi) \in Y$, then $(\varphi\neg) \in Y$.

(v) If $c, d \in C$, then $(c = d) \in Y$.

Now, notice by the minimality of Y, and since s_0 is countable, breaking the formulas in s_0 into subformulas, if necessary at all, keeps the set Y countable. Therefore, we may enumerate the set X of all sentences in Y as $X = \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$. Let $T = \{t_0, t_1, t_2, \ldots\}$ be the set of all basic terms of $\mathcal{L}$. Observe that we can have T enumerated as the set of constant symbols C is countable and $\mathcal{L}$ has only countably many function symbols.

Next, our goal is to construct a sequence $s_0 \subseteq s_1 \subseteq s_2 \subseteq \cdots$ of elements of S as follows. We will do this recursively. We already have s_0. Suppose we also have $s_n \in S$. Using the definition of a consistency property, s_{n+1} may be chosen so that

(a) $s_n \subseteq s_{n+1} \in S$

(b) $s_n \cup \{\varphi_n\} \in S$ implies $\varphi_n \in S_{n+1}$:

- If φ_n is $\bigvee \Phi$, then $\theta \in s_{n+1}$ for some $\theta \in \Phi$.
- If φ_n is $\exists x \varphi(x)$, then $\varphi(c) \in s_{n+1}$ for some $c \in C$.

(c) For some $c \in C$, $(c = t_n) \in s_{n+1}$.

Let us unravel the first few steps of this construction. s_0 is given. In step (b), if $s_0 \cup \{\varphi_0\} \in S$, then toss φ_0 into s_1. Note that we may have $s_1 = s_0$ in this step as φ_0 may be a proper subformula of a formula of s_0. But eventually, we will have a new superset s_n.

Finally, a model $\mathfrak{A}$ of s_0 is defined in the following manner. Put $s_\omega = \bigcup_{n<\omega} s_n$. Define a relation $\sim$ on C by declaring $c \sim d$ if and only if $(c = d) \in s_\omega$. Using lemma 3, we can show that $\sim$ is an equivalence relation on C. Let $\mathfrak{A}$ have the universe $A = \{[c] : c \in C\}$, where $[c]$ is the equivalence class of c. We want to make sure that different representatives of an equivalence class do not work differently for our purpose. To that end, suppose $\varphi(c_1, \ldots, c_n) \in s_\omega$ and $c_i \sim d_i$, where $c_i, d_i \in C$ for $1 \le i \le n$. By $C7$-(β) of definition 18, since $(c_i = d_i) \in s_\omega$, and by construction of s_ω ($(c_i = d_i) \in s_{n_i}$ for some n_i, thus choose n large enough so that every $(c_i = d_i)$ falls into a single s_n), we see that $\varphi(d_1, \ldots, d_n) \in s_\omega$. Thus, interpret the relation, function, and constant symbols of $\mathcal{M}$ in the following manner.

- If t is a basic term and $c \in C$, then let $\mathfrak{A} \models (c = t)$ if and only if $(c = t) \in s_\omega$.
- If R is an n-ary relation symbol and $c_1, \ldots, c_n \in C$, then let $\mathfrak{A} \models R(c_1, \ldots, c_n)$ if and only if $R(c_1, \ldots, c_n) \in s_\omega$.

These conditions determine the model $\mathfrak{A}$, and by the properties of a consistency property and an argument by induction on the complexity of formulas, it can be shown that $\mathfrak{A} \models s_0$. Countability of $\mathfrak{A}$ follows by construction. $\dashv$

Corollary 1 (Extended Model Existence Theorem). *Suppose S is a consistency property and let Γ be a countable set of sentences in $M_{\omega_1\omega}$. Further, suppose for all $s \in S$ and $\varphi \in \Gamma$, $s \cup \{\varphi\} \in S$. Then for all $s \in S$, $s \cup \Gamma$ has a model.*

Definition 19. *φ is a theorem of $L_{\omega_1\omega}$ if and only if $\vdash_{L_{\omega_1\omega}} \varphi$ if and only if there is a countable sequence $\varphi_0, \dots, \varphi_\alpha, \dots, \varphi_\beta$ such that $\varphi_\beta = \varphi$, and for each $\alpha \le \beta$, φ_α is either an axiom of $L_{\omega_1\omega}$ or it is inferred from earlier formulas $\varphi_\gamma, \gamma < \alpha$, by a rule of inference in Definition 15. In this event, the sequence $\langle \varphi_\alpha : \alpha \le \beta \rangle$ is called a proof of φ.*

The following lemma is crucial for the completeness theorem. We speculate that it can be modified appropriately for the context of the capacity logic of Section 4.

Lemma 4. *Let S be the set of all finite sets s of sentences of $M_{\omega_1\omega}$ such that only finitely many $c \in C$ occur in s, and $\nvdash_{M_{\omega_1\omega}} \neg \bigwedge_{\psi \in s} \psi$. Then S is a consistency property.*

The following is the completeness theorem for the logic $L_{\omega_1\omega}$, which is due to Carol Karp. This is Theorem 3 of [Kei71].

Theorem 7 (Completeness Theorem for $L_{\omega_1\omega}$). *If φ is a sentence of $L_{\omega_1\omega}$, then $\vdash_{L_{\omega_1\omega}} \varphi$ if and only if $\vDash \varphi$.*

2.3. Fragments of $\mathcal{L}_{\omega_1\omega}$ and Completeness theorem for Fragments

Definition 20 ([Kei71]). *Let $L_{\omega_1\omega}$ denote the set of formulas of the language $\mathcal{L}_{\omega_1\omega}$. For any set $\mathcal{A}$ we let $L_{\mathcal{A}} = L_{\omega_1\omega} \cap \mathcal{A}$. The set of formulas $L_{\mathcal{A}}$ is called a fragment of $\mathcal{L}_{\omega_1\omega}$ if and only if*

(i) *$\mathcal{A}$ is a transitive set.*
(ii) *If $a, b \in \mathcal{A}$, then $\{a, b\} \in \mathcal{A}, a \cup b \in \mathcal{A}$, and $a \times b \in \mathcal{A}$.*
(iii) *If $a \in \mathcal{A}$ and α is the least ordinal that is not in the transitive closure of a, then $\alpha \in \mathcal{A}$.*
(iv) *If $\varphi(x \dots) \in L_{\mathcal{A}}$ and $t \in \mathcal{A}$ is a term of $\mathcal{L}_{\omega_1\omega}$, then $\varphi(t \dots) \in L_{\mathcal{A}}$.*

Theorem 8 (Completeness Theorem for $L_{\mathcal{A}}$). *If φ is a sentence of a countable fragment $L_{\mathcal{A}}$ of $L_{\omega_1\omega}$, then $\vdash_{L_{\mathcal{A}}} \varphi$ if and only if $\vDash \varphi$.*

2.4. Admissible Sets

In this section, we will give a brief introduction to admissible sets. For a detailed treatment, we direct the reader to [Bar17]. For an informal summary, the reader may look at Appendix A. A formula of set theory is a formula in the first order logic with identity and the binary relation symbol $\in$.

Definition 21. *The collection of Δ_0-formulas is the smallest collection of formulas containing the atomic formulas which is closed under the Boolean operations $\neg, \wedge, \vee$, and closed under bounded quantification.* [1]

Thus a Δ_0 formula of set theory is built up from atomic formulas and their negations using only the four operations $\wedge, \vee, (\forall x \in y)$, and $(\exists x \in y)$. If φ is Δ_0 and u, v are variables, then $(\exists u \in v)\varphi$ and $(\forall u \in v)\varphi$ are Δ_0-formulas.

Definition 22. *A Σ-formula is a formula of set theory built up from atomic formulas and their negations using only the four operations $\wedge, \vee, (\forall x \in y)$, and $(\exists x \in y)$ and $(\exists x)$. If, instead of $(\exists x)$, we use $(\forall x)$ we get a Π-formula.*

Now we are ready to define *admissible* sets.

Definition 23. *A set $\mathscr{A}$ is* admissible *if and only if :*

(i) *$\mathcal{A}$ is non-empty.*

(ii) *$\mathcal{A}$ is transitive.*

(iii) *If $x \in \mathcal{A}$, then the transitive closure $TC(x)$ of x belongs to $\mathcal{A}$.*

(iv) (Δ_0-*Separation Axiom*) *If $\varphi(x, y_1, \ldots, y_n)$ is a Δ_0-formula and $b_1, \ldots, b_n, c \in \mathcal{A}$, then*

$$\{a \in c : \langle \mathcal{A}, \in \rangle \vDash \varphi[a, b_1, \ldots, b_n]\} \in \mathcal{A}.$$

(v) (Σ-*Reflection Axiom*) *If $\varphi(y_1, \ldots, y_n)$ is a Σ–formula, $b_1, \ldots, b_n \in \mathcal{A}$, and $\langle \mathcal{A}, \in \rangle \vDash \varphi[b_1, \ldots, b_n]$, then there's a transitive set $a \in \mathcal{A}$ so that $b_1, \ldots, b_n \in a$ and $\langle a, \in \rangle \vDash \varphi[b_1, \ldots, b_n]$.*

[1]Bounded quantifiers are of the form $\exists u \in v[\cdots]$ or $\forall u \in v[\cdots]$. The former means $\exists u[u \in v \wedge \cdots]$, and the latter means $\forall u[u \in v \rightarrow \cdots]$

Next, we will define what it means to be (parametrically) definable in a first order structure.

Definition 24. *Let $\mathcal{L}$ be a countable first order language. And let $\mathfrak{A} = \langle A, \dots \rangle$ be an $\mathcal{L}$-structure. We say a relation $R \subseteq A^n$ is parametrically definable if there is some $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_k)$ and $b_1, \dots, b_k \in A$ so that $R = \{(a_1, \dots, a_n) : \mathfrak{A} \models \varphi(a_1, \dots, a_n, b_1, \dots, b_k)\}$. Here the parameters are the elements $b_1, \dots, b_k$. The relation R is said to be definable if we do not need any parameters.*

Definition 25. *A set $X \subseteq \mathscr{A}$ is Σ (Π) on $\mathscr{A}$ if X is definable by a Σ-formula (Π-formula) with parameters in $\mathscr{A}$. If $X \subseteq \mathscr{A}$ is both Σ and Π on $\mathscr{A}$, then X is said to be Δ on $\mathscr{A}$.*

3. Probability Quantifiers as Motivation

Our work, which we shall present in Section 4, is motivated by Hoover and Keisler's work on *probability quantifiers*. In this section we look at the main definitions and theorems that motivated our work. Our main source of references will be [Kei85] and [Hoo78a]. The structure of this section is as follows. First, we will introduce the notion of probability quantifiers. Then, we will look at probability models and some examples discussing the expressive power of this particular logic called $L_{\mathbb{A}P}$. Finally, we will look at how the so-called weak models are constructed as a way to prove a completeness theorem for $L_{\mathbb{A}P}$. In the due course of this section, we will need some ideas from Nonstandard Analysis. We shall not include all the details that we borrow from there as it may hinder the flow of the ideas presented in this section. Nonetheless, the reader is directed to Appendix B for a detailed summary of all the nonstandard analysis that we use here.

3.1. Probability Quantifiers

This section aims to introduce probability quantifiers. The logic $L_{\mathbb{A}P}$ is pretty similar to the logic $L_{\mathcal{A}}$ introduced in Section 2. The main difference is that $L_{\mathbb{A}P}$ has probability quantifiers $(Px \geq r)$ instead of the ordinary quantifiers $(\forall x)$ and $(\exists x)$. The formula $(Px \geq r)\varphi(x)$

denotes the assertion that the set $\{x : \varphi(x)\}$ has measure at least r, that is "the probability of the event $\{x : \varphi(x)\}$ is at least r".

Let $\mathbb{A}$ be an admissible set with $\omega \in \mathbb{A}$ and each $a \in \mathbb{A}$ countable. Thus, $\mathbb{A}$ is a subset of the set of the set of hereditarily countable sets (**HC**). Let $\mathcal{L}$ be a countable $\mathbb{A}$-recursive set of finitary relation and constant symbols only, so there are no function symbols in $\mathcal{L}$.

Definition 26. *The logic $L_{\mathbb{A}P}$ has the following logical symbols.*

(1) A countable list of variables $v_0, v_1, \ldots, v_n, \ldots; n < \omega$.

(2) The connectives $\neg$ and $\bigwedge$.

(3) The quantifiers $(P\overline{x} \geq r)$, where $\overline{x} = \langle x_1, \ldots, x_n \rangle$ with x_i's distinct and $r \in \mathbb{A} \cap [0, 1]$.

(4) The equality symbol $=$.

The formulas of $L_{\mathbb{A}P}$ are constructed set theoretically, so we have $L_{\mathbb{A}P} \subseteq \mathbb{A}$. See Remark 2. $L_{\omega_1 P}$ denotes $L_{\mathbb{A}P}$, where $\mathbb{A} = \mathbf{HC}$, so $L_{\mathbb{A}P} = \mathbb{A} \cap L_{\omega_1 P}$.

Definition 27. *The set of formulas of $L_{\mathbb{A}P}$ is the least set such that :*

(1) Each atomic formula of first order logic is a formula of $L_{\mathbb{A}P}$.

(2) $L_{\mathbb{A}P}$ is closed under $\neg$.

(3) If $\Phi \in \mathbb{A}$ is a set of formulas of $L_{\mathbb{A}P}$ with only finitely many free variables, then $\bigwedge \Phi$ is a formula of $L_{\mathbb{A}P}$.

(4) If φ is a formula of $L_{\mathbb{A}P}$ and $(P\overline{x} \geq r)$ is a quantifier of $L_{\mathbb{A}P}$, then $(P\overline{x} \geq r)\varphi$ is a formula of $L_{\mathbb{A}P}$.

3.2. Probability Models

We will list some definitions and theorems that we need to define probability structures.

Definition 28. *Let $\langle M, S, \mu \rangle$ be a probability space such that each singleton is measurable. Then for each $n \in \mathbb{N}^+$, $\langle M^n, S^{(n)}, \mu^{(n)} \rangle$ is the probability space such that $S^{(n)}$ is the σ-algebra generated by the measurable rectangles and the diagonal sets $D_{ij} = \{\overline{x} \in M^n : x_i = x_j\}$, and $\mu^{(n)}$ is the unique extension of μ^n to $S^{(n)}$ such that $\mu^{(n)}(D_{ij}) = \sum_{x \in M} \mu(\{x\})^2$.*

In particular, $\langle M^2, S^{(2)}, \mu^{(2)}\rangle$ has $S^{(2)}$ as the σ-algebra generated by the measurable rectangles and the diagonal D of M^2, and $\mu^{(2)}$ is the unique extension of the product measure μ^2 to $S^{(2)}$ so that the $\mu^{(2)}$-measure of the diagonal is the sum of $\mu(\{x\})^2$ over M. In fact, given $X = (Y \cap D) \cup (Z \setminus D) \in S^{(2)}$ with $Y, Z \in S^2$, $\mu^{(2)} : S^{(2)} \to [0,1]$ is given by

$$\mu^{(2)}(X) = \sum_{\langle x,x\rangle \in Y} \mu(\{x\})^2 + \mu^2(Z) - \sum_{\langle x,x\rangle \in Z} \mu(\{x\})^2.$$

Definition 29. *A probability structure for $\mathcal{L}$ is a structure*

$$\mathfrak{M} = \langle M, R_i^{\mathfrak{M}}, c_j^{\mathfrak{M}}, \mu\rangle_{i\in I,\ j\in J},$$

where μ is a countably additive probability measure on M such that each singleton is measurable, each $R_i^{\mathfrak{M}}$ is $\mu^{(n_i)}$-measurable, and each $c_j^{\mathfrak{M}} \in M$.

Theorem 9. *Let $\mathfrak{M}$ be a probability structure for $\mathcal{L}$. The satisfaction relation is defined inductively the same way it is defined for $L_{\mathbb{A}}$, except for the following:*

$\mathfrak{M} \vDash (P\overline{y} \geq r)\varphi(\overline{x},\overline{y})[\overline{a}]$ *iff* $\{\overline{b} \in M^n : \mathfrak{M} \vDash \varphi[\overline{a},\overline{b}]\}$ *is $\mu^{(n)}$-measurable and has measure at least r,*

where $\varphi(\overline{x}) \in L_{\mathbb{A}P}$ and $\overline{a} \in M$.

Theorem 10. *For each probability structure $\mathfrak{M}$, formula $\varphi(\overline{x},\overline{y}) \in L_{\mathbb{A}P}$, and tuple $\overline{a} \in M$, the set*

$$\{\overline{b} \in M^n : \mathfrak{M} \vDash \varphi[\overline{a},\overline{b}]\}$$

is $\mu^{(n)}$-measurable.

3.3. Expressive Power

We will look at the expressive power of the language $\mathcal{L}_{\mathbb{A}P}$ using some examples from the section 1.3 of [Kei85]. To get a flavor of these $\mathcal{L}_{\mathbb{A}P}$-sentences, we will include a proof found on page 21 of [Kei77].

Example 11. "There is a countable set of measure one" is expressed by $(Px \geq 1)(Py > 0)x = y$.

Consider an arbitrary probability space $\mathfrak{M} = (M, S, \mu)$. Let $\mathcal{A}$ be the set $\{a \in S : \mu(a) > 0\}$. Put $\mathcal{A} = \{a \in \mathcal{A} : \mu(a) > \frac{1}{n}\}$. Then $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Now by countable additivity of μ on S, and by $\mu(M) = 1$, we see that $\mathcal{A}$ is at most countable. So $\{x \in S : \mu(x) > 0\}$ is at most countable iff $\{x \in S : (Py > 0)(y = x)\}$ is at most countable iff $(Px \geq 1)(Py > 0)(y = x)$. The sentence $(Px \geq 1)(Py > 0)x = y$ roughly means that "*Almost all x have positive measure*". We will make the following remark about the two probability quantifiers $(Px \geq 1)$ and $(Py > 0)$.

Remark 3. $(Px \geq 1)$ is a weak analog of $(\forall x)$ in the sense that $\forall x$ here should be interpreted as *"for almost every x"*. The probability quantifier $(Py > 0)$ however is pretty much the same as saying that *"there is a bunch of y which makes a set of positive measure"*, so it can be interpreted as "*there exist some y's*", and so on.

Example 12. "There are no point masses", that is there are no singletons of positive measure, is expressed by $(Px \geq 1)(Py \geq 1)x \neq y$.

This is Proposition 2.8 of [Kei77]. First we note that the sentence $(Px \geq 1)(Py \geq 1)x \neq y$ is equivalent to the sentence $\neg(Px > 0)(Py > 0)x = y$. Now, consider a probability model $\langle \mathfrak{A}, \mu \rangle$. For each $a \in A$ we have $\mu(a) > 0$ if and only if $\langle \mathfrak{A}, \mu \rangle \models (Py > 0)(x = y)[a]$. Therefore, if $\langle \mathfrak{A}, \mu \rangle$ does not have point masses, then $\{a : \langle \mathfrak{A}, \mu \rangle \models (Py > 0)(x = y)[a]\} = \varnothing$, so $\langle \mathfrak{A}, \mu \rangle \models \neg(Px > 0)(Py > 0)x = y$. Conversely, if $\langle \mathfrak{A}, \mu \rangle$ has point masses, say b with $\mu(b) > 0$, then $\mu(\{a : \langle \mathfrak{A}, \mu \rangle \models (Py > 0)(x = y)[a]\}) \geq \mu(b) > 0$, so we have $\langle \mathfrak{A}, \mu \rangle (Px > 0)(Py > 0)x = y$.

3.4. Completeness Theorems for $L_{\mathbb{A}P}$

In the following definitions, φ and ψ range over $L_{\mathbb{A}P}$ and r, s range over $\mathbb{A} \cap [0, 1]$, and Φ and Ψ are sets in $L_{\mathbb{A}P}$ which are at most countable.

Definition 30. *The axioms for weak $L_{\mathbb{A}P}$ are the following.*

A1. All axioms of $L_{\mathbb{A}}$ without quantifiers (See Definition 14).

A2. Monotonicity. $(P\overline{x} \geq r)\varphi \rightarrow (P\overline{x} \geq s)\varphi$ *whenever* $r \geq s$.

A3. $(P\overline{x} \geq r)\varphi(\overline{x}) \rightarrow (P\overline{y} \geq r)\varphi(\overline{y})$.

A4. $(P\overline{x} \geq 0)\varphi$.

A5. Finite additivity.

(i) $(P\overline{x} \leq r)\varphi \wedge (P\overline{x} \leq s)\psi \rightarrow \big((P\overline{x} \leq r + s)(\varphi \vee \psi)\big)$.

(ii) $(P\overline{x} \geq r)\varphi \wedge (P\overline{x} \geq s)\psi \wedge (P\overline{x} \leq 0)(\varphi \wedge \psi) \rightarrow (Px \geq r + s)(\varphi \vee \psi)$.

A6. The Archimedean property.

$$(P\overline{x} > r)\varphi \leftrightarrow \bigvee_{n \in \mathbb{N}} \Big(P\overline{x} \geq r + \frac{1}{n}\Big)\varphi.$$

Definition 31. *The axioms for full $L_{\mathbb{A}P}$ consist of the axioms for weak $L_{\mathbb{A}P}$ and the following axioms.*

B1. Countable additivity.

$$\bigwedge_{\Psi \subseteq \Phi} (P\overline{x} \geq r) \bigwedge \Psi \rightarrow (P\overline{x} \geq r) \bigwedge \Phi,$$

where Ψ ranges over the finite subsets of Φ.

B2. Symmetry.

$$(Px_1 \cdots x_n \geq r)\varphi \leftrightarrow P(x_{\pi 1} \cdots x_{\pi n} \geq r)\varphi,$$

where π is a permutation of $\{1, \ldots, n\}$.

B3. Product independence.

$$(P\overline{x} \geq r)(P\overline{y} \geq s)\varphi \rightarrow (P\overline{xy} \geq rs)\varphi,$$

provided all variables in $\overline{x}, \overline{y}$ *are distinct.*

B4. Product measurability. *For each* $r < 1$,

$$(P\overline{x} \geq 1)(P\overline{y} > 0)(P\overline{z} \geq r)(\varphi(\bar{x}\bar{z}) \leftrightarrow \varphi(\bar{y}\bar{z}))),$$

whenever all variables in $\bar{x}, \bar{y}$, *and* $\bar{z}$ *are distinct.*

Definition 32. *The rules of inference for* $L_{\mathbb{A}P}$ *are the following.*

R1. Modus Ponens. $\varphi, \varphi \to \psi \vdash \psi$.

R2. Conjunction. $\{\varphi \to \psi : \psi \in \Psi\} \vdash \varphi \to \bigwedge \Psi$.

R3. Generalization. $\varphi \to \psi(\bar{x}) \vdash \varphi \to (P\bar{x} \geq 1)\psi(\bar{x})$.

We want to sketch a proof of a completeness theorem for the probability logic $L_{\mathbb{A}P}$, which appears in [Hoo78a]. The main idea is to first build a "weak model" and then transform it into a "strong model", using the techniques of consistency property and Loeb process. We shall restrict ourselves to the countable admissible fragment $L_{\mathbb{A}P}$ of $L_{\omega_1 P}$, where $\omega \in \mathbb{A}$. The interested reader is directed to [Hoo78a] for a complete treatment of $L_{\omega_1 P}$.

Let us first describe the Loeb process, which is the bridge between a weak model and a strong model. We start with an internal set Ω and an internal algebra $\mathcal{A}$ on Ω. Suppose $\mu : \mathcal{A} \to {}^*\mathbb{R}$ is an internal set-function such that:

(i) $\mu(\varnothing) = 0$.

(ii) $\mu(\Omega) = 1$.

(iii) μ is finitely additive, i.e., $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, whenever $A, B \in \mathcal{A}$.

Since for each $A \in \mathcal{A}$ $\mu(A)$ is finite, define ${}^\circ\mu(A) = {}^\circ\Big(\mu(A)\Big)$ for each $A \in \mathcal{A}$. That is, ${}^\circ\mu(A) = \mathrm{st}(\mu(A))$ for each $A \in \mathcal{A}$. Since the standard part function st is distributive over addition, we see that ${}^\circ\mu$ is finitely additive on $\mathcal{A}$. We claim that the notions algebra and *algebra coincide for internal sets. We only need to show that if $\mathcal{A}$ is a *algebra of internal sets then $\mathcal{A}$ is a standard algebra. So we see that $(\Omega, \mathcal{A}, {}^\circ\mu)$ is a *bona fide* finitely additive measure space. Now our goal is to extend $\mathcal{A}$ to a σ-algebra $\mathcal{A}_L$ and $^\circ\mu$ to a measure μ_L on $\mathcal{A}_L$. This is Theorem 2.1 of [Ros97]. For good measure, we will record it as our next theorem.

Theorem 11. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal finitely additive probability space. Then there is a standard σ-additive probability space $(\Omega, \mathcal{A}_L, \mu_L)$ such that:*

(i) $\mathcal{A}_L$ is a σ-algebra with $\mathcal{A} \subseteq \mathcal{A}_L \subseteq \mathcal{P}(\Omega)$.

(ii) $\mu_L = {}^\circ\mu$ on $\mathcal{A}$.

(iii) For every $A \in \mathcal{A}_L$ and standard $\epsilon > 0$ there are $A_i, A_o \in \mathcal{A}$ so that $A_i \subseteq A \subseteq A_o$ and $\mu(A_o \smallsetminus A_i) < \epsilon$.

(iv) For every $A \in \mathcal{A}_L$, there is some $B \in \mathcal{A}$ so that $\mu_L(A \Delta B) = 0$.

In this construction, we get a complete measure space $(\Omega, \mathcal{A}_L, \mu_L)$ called a Loeb space. Now, one may inquire: what is the exact procedure for measuring $A \in \mathcal{A}_L$ in the sense of the measure μ_L? First, A is *Loeb measurable* if $A \in \mathcal{A}_L$. Second, for a Loeb measurable set A, we have $\mu_L(A) = {}^\circ\mu(B)$, where B is the set from $\mathcal{A}$ which satisfies the condition $\mu_L(A \Delta B) = 0$. See part (iv) of the above theorem. We will call this procedure of passing from a finitely additive measure to a σ-additive measure *the Loeb process*.

Definition 33. *A weak structure for $L_{\mathbb{A}P}$ is a structure $\mathfrak{M} = \langle M, R_i^{\mathfrak{M}}, c_j^{\mathfrak{M}}, \mu_n \rangle_{i \in I, j \in J, n \in \mathbb{N}}$ such that the following two conditions are satisfied by μ_n.*

(1) Each μ_n is a finitely additive probability measure on M^n with each singleton measurable.

(2) The set $\{\bar{b} \in M^n : \mathfrak{M} \models \varphi[\bar{a}, \bar{b}]\}$ is μ_n-measurable for each $\varphi(\bar{x}, \bar{y})$ in $L_{\mathbb{A}P}$ and $\bar{a} \in M$.

Remark 4. Every probability structure induces a weak structure for $L_{\omega_1 P}$ with $\mu_n = \mu^{(n)}$.

Definition 34. *The formal negation $(\varphi\neg)$ of an $L_{\mathbb{A}P}$-formula is defined as before excluding the definitions for the $\exists$ and $\forall$ quantifiers. Instead, we have the following two definitions.*

(1) $\big((P\bar{x} \geq r)\varphi\big)\neg$ is $(P\bar{x} > 1 - r)\neg\varphi$.

(2) $\big((P\bar{x} > r)\varphi\big)\neg$ is $(P\bar{x} \geq 1 - r)\neg\varphi$.

Now we will define consistency properties for $L_{\mathbb{A}P}$. The idea is the same as in Section 1. The language $\mathcal{L}_{\mathbb{A}P}$ is augmented by a countable set C of new constant symbols. We will follow Hoover's notation. The infinitary logic corresponding to the augmented language will be denoted by $L_{\mathbb{A}P}(C)$.

Definition 35. *A consistency property for $L_{\mathbb{A}P}(C)$ has the same rules as in definition 18 excluding the $\forall$-rule and $\exists$-rule. Instead, the following two rules must be satisfied.*

(1) $((P\bar{x} > 0)$-Rule) $(P\bar{x} > 0)\varphi(\bar{x}) \in s$ implies $s \cup \{\varphi(\bar{c})\} \in S$ for some $\bar{c} \in C$.

(2) (Axiom-Rule) If $\varphi(\bar{x}) \in L_{\mathbb{A}P}(C)$ is an axiom, then each of the following holds.

- (α) *$s \cup \{(P\bar{x} \geq 1)\varphi(\bar{x})\} \in S$.*
- (β) *For any sequence $\bar{t}$ of closed terms of $L_{\mathbb{A}P}(C)$, $s \cup \{\varphi(\bar{t})\} \in S$.*

Theorem 12 (Model Existence Theorem for $L_{\mathbb{A}P}$)**.** *If S is a consistency property for $L_{\mathbb{A}P}$, then any $s_0 \in S$ has a probability model.*

Hoover proves the model existence theorem using two lemmas as the reader may have already guessed; first, a lemma that gives a weak model, and second, a strong model that proves the theorem.

Lemma 5 (Weak Model Lemma for $L_{\mathbb{A}P}$)**.** *If S is a consistency property for $L_{\mathbb{A}P}$ and $s_0 \in S$, then there exists a weak structure for $L_{\mathbb{A}P}$ which is a model of s_0.*

Proof. The proof is more or less similar to the proof of the model existence theorem we saw earlier, except for a few modifications, which involve the probability quantifiers. As before, first, enumerate the $L_{\mathbb{A}P}$-sentences as $\langle \varphi_n : n < \omega \rangle$. Next, construct the increasing sequence $\langle s_n : n < \omega \rangle$ of sets in S as before, but excluding the one which involves $\exists x\varphi(x)$, with the following additional property.

- If $s \cup \{\varphi_n\} \in S$ and φ_n is $(P\bar{x} > 0)\varphi(\bar{x})$, then for some $\bar{c} \in C$, $\varphi(\bar{c}) \in s_{n+1}$.

Put $s_\omega = \bigcup_{n<\omega} s_n$ as before. Let T be the set of all closed terms of $L_{\mathbb{A}P}(C)$. Define a relation $\sim$ on T by declaring $t_1 \sim t_2$ if and only if $(t_1 = t_2) \in s_\omega$. Then $\sim$ is an equivalence relation on

T. Let the model $\mathfrak{A}$ have the universe $A = \{[t] : t \in T\}$, where $[t]$ is the equivalence class of t, and have the relation $R([t_1], \ldots, [t_n])$ hold if and only if $R(t_1, \ldots, t_n) \in s_\omega$.

What follows is the new step of the proof. A measure μ_n is defined on the subsets A^n definable by $L_{\mathbb{A}P}$-formulas with parameters from A. Put

$$\mu_n\big(\{\bar{x} : \varphi(\bar{x}, \bar{t})\}\big) = \sup\{r : (P\bar{x} \geq r)\varphi(\bar{x}, \bar{t}) \in s_\omega\}$$

for each $\varphi(\bar{x}, \bar{t})$ and $\bar{t}$. By axioms *A1-A5*, μ_n is well-defined and is a finitely additive probability measure. By axiom *A6*, the supremum is always attained. Now $\langle \mathfrak{A}, \mu_n \rangle_{n<\omega} \vDash \varphi([t_1], \ldots, [t_n])$, whenever $\varphi(t_1, \ldots, t_n) \in s_\omega$. Thus, $\langle \mathfrak{A}, \mu_n \rangle_{n<\omega}$ is a weak model for $L_{\mathbb{A}P}$, whence a model for s_0.

$\dashv$

The next lemma is the crux of the proof of the completeness theorem for $L_{\mathbb{A}P}$. As the reader will see in Section 4, we do not mimic the proof of the lemma, but we merely speculate that a similar kind of proof will work in our favor in the context of capacity quantifiers. The reader is directed to Hoover's paper [Hoo78a] for a proof of the following lemma.

Lemma 6. *Let $\langle \mathfrak{A}, \mu_n \rangle_{n<\omega}$ be a weak model for $L_{\mathbb{A}P}$ so that every μ_n-measurable set is definable in $L_{\mathbb{A}P}$. Let $\mathbb{V} = \mathbb{V}(\mathbb{A}, \mathbb{R})$. Then the model $\langle {}^*\mathfrak{A}, \mu_{Ln} \rangle_{n<\omega}$ obtained by applying the Loeb process to $\langle {}^*\mathfrak{A}, {}^*\mu_n \rangle_{n<\omega}$ is a probability model, and for each $L_{\mathbb{A}P}$-formula $\varphi(\bar{x})$ and $\bar{a} \in A$,*

$$\langle \mathfrak{A}, \mu_n \rangle_{n<\omega} \vDash \varphi(\bar{a}) \text{ if and only if } \langle {}^*\mathfrak{A}, \mu_{Ln} \rangle_{n<\omega} \vDash \varphi(\bar{a}).$$

4. Capacity Quantifiers

In this section, we will introduce the notion of *capacity quantifiers*. We shall focus on strongly subadditive capacities. Our goal is to lay down a set of axioms and prove a completeness theorem for the logic $L_{\mathbb{A}\mathscr{I}}$, which we shall call *capacity logic*. Analogous to Hoover and Keisler's probability quantifiers [Kei85], we shall interpret $(\mathscr{I}x \geq r)\varphi(x)$ as "the set $\{x : \varphi(x)\}$ has capacity $\geq r$." We have the following abbreviations.

(a) $(\mathscr{I}x < r)\varphi$ for $\neg(\mathscr{I}x \geq r)\varphi$.

(b) $(\mathscr{I}x \leq r)\varphi$ for $\bigwedge_{n=1}^{\infty}(\mathscr{I}x < r + \frac{1}{n})\varphi$.

(c) $(\mathscr{I}x > r)\varphi$ for $\bigvee_{n=1}^{\infty}(\mathscr{I}x \geq r + \frac{1}{n})\varphi$.

(d) $(\mathscr{I}x = r)\varphi$ for $(\mathscr{I}x \geq r)\varphi \wedge (\mathscr{I}x \leq r)\varphi$.

Let $\mathbb{A}$ be a countable admissible set with $\omega \in \mathbb{A}$ and each $a \in \mathbb{A}$ countable. Suppose that L is a countable $\mathbb{A}$-recursive set of unary relation and constant symbols. Then the logic $L_{\mathbb{A}\mathscr{I}}$ has the following logical symbols.

(1) Variables v_n for each $n \in \mathbb{N}$.

(2) Connectives $\neg$ and $\bigwedge$.

(3) Quantifiers $(\mathscr{I}x \geq r)$, where $r \in \mathbb{A} \cap [0,1]$.

(4) Equality symbol $=$.

The set of formulas of $L_{\mathbb{A}\mathscr{I}}$ is the least set of formulas such that

(1) Each atomic formula of first order logic is a formula of $L_{\mathbb{A}\mathscr{I}}$.

(2) If φ is a formula of $L_{\mathbb{A}\mathscr{I}}$, then $\neg\varphi$ is a formula of $L_{\mathbb{A}\mathscr{I}}$.

(3) If $\Phi \in \mathbb{A}$ is a set of formulas of $L_{\mathbb{A}\mathscr{I}}$ with only finitely many free variables, then $\bigwedge \Phi$ is a formula of $L_{\mathbb{A}\mathscr{I}}$.

(4) If φ is a formula of $L_{\mathbb{A}\mathscr{I}}$ and $(\mathscr{I}x \geq r)$ is a quantifier of $L_{\mathbb{A}\mathscr{I}}$, then $(\mathscr{I}x \geq r)\varphi$ is a formula of $L_{\mathbb{A}\mathscr{I}}$.

Definition 36. *We say a relation $R \subseteq A$ is* parametrically definable *if there is some $\mathcal{L}_{\mathbb{A}\mathscr{I}}$-formula $\varphi(x, y_1, \dots, y_k)$ and $b_1, \dots, b_k \in A$ so that $R = \{a : \mathfrak{A} \models \varphi(a, b_1, \dots, b_k)\}$. Here the parameters are the elements $b_1, \dots, b_k$. The relation R is said to be* definable *if we do not need any parameters.*

Definition 37. *A* weak capacity structure *for $\mathcal{L}$ is a quadruple $\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, T\rangle_{i\in I, j\in J}$, where $R_i^{\mathfrak{A}} \subseteq A$ for each $i \in I$, $c_j^{\mathfrak{A}} \in A$ for each $j \in J$, where I and J are indexing sets, and T is a $\mathcal{K}$-precapacity, where $\mathcal{K}$ is the regular paving on A generated by formulas of $\mathcal{L}_{\mathbb{A}\mathscr{I}}$.*

Definition 38. *A* strong capacity structure *for $\mathcal{L}$ is the same as a weak capacity structure except that T is a normalized strongly subadditive $\mathcal{K}$-capacity on $\mathcal{P}(A)$.*

Now we record the following analogue of Theorem 1.2.4 of [Kei85]. Unlike probability quantifiers, we do not have to worry about capacitability when defining satisfaction as a capacity is a set function on the entire power set.

Definition 39. *Suppose $\langle\mathfrak{A},T\rangle$ is a capacity structure. The satisfaction relation $\langle\mathfrak{A},T\rangle \vDash \varphi[\overline{a}]$, where $\varphi(x) \in L_{\mathbb{A}\mathscr{I}}$ and $a \in A$ is defined recursively as in $L_{\mathbb{A}}$ with the following modification.*

$$\langle\mathfrak{A},T\rangle \vDash (\mathscr{I}y \geq r)\varphi(y,\overline{x})[\overline{a}] \text{ if and only if } T(\{b \in A : \mathfrak{A} \vDash \varphi[b,\overline{a}]\}) \geq r.$$

And $\langle\mathfrak{A},T\rangle$ is a model of a sentence φ if $\langle\mathfrak{A},T\rangle \vDash \varphi$.

Let $\mathbb{A}$ be a countable admissible set with $\omega \in \mathbb{A}$ and each $a \in \mathbb{A}$ countable as before. Suppose that L is a countable $\mathbb{A}$−recursive set of unary relation and constant symbols. The logic $L_{\mathbb{A}\mathscr{I}}$ has the following axioms, where $\varphi,\psi \in L_{\mathbb{A}\mathscr{I}}$, $\Phi \subseteq L_{\mathbb{A}\mathscr{I}}$ is countable, and $r,s,t \in \mathbb{A}\cap[0,1]$. First, we axiomatize the properties that correspond to a weak capacity model. We shall call them the *weak axioms*.

$W0$. All axioms of $L_{\mathbb{A}}$ without quantifiers.

$W1$. (*Normalized nonnegativity*) $(\mathscr{I}x \leq 1)(x = x)$ and $(\mathscr{I}x \geq 0)(x \neq x)$.

$W2$. (*Monotonicity of the quantifier*) $(\mathscr{I}x \geq r)\varphi \to (\mathscr{I}x \geq s)\varphi$ whenever $r \geq s$.

$W3$. (*Monotonicity of the capacity*)

(a) $\big((\mathscr{I}x \geq 1)(\varphi(x) \to \psi(x)) \wedge (\mathscr{I}x \leq r)\psi(x)\big) \to (\mathscr{I}x \leq r)\varphi(x)$.

(b) $\big((\mathscr{I}x \geq 1)(\varphi(x) \to \psi(x)) \wedge (\mathscr{I}x \geq r)\varphi(x)\big) \to (\mathscr{I}x \geq r)\psi(x)$.

$W4$. (*Strong subadditivity*)

$$(\mathscr{I}x \leq r)\varphi \ \wedge\ (\mathscr{I}x \leq s)\psi \ \wedge\ (\mathscr{I}x \leq t)(\varphi \wedge \psi) \to (\mathscr{I}x \leq r+s-t)(\varphi \vee \psi), \text{ whenever } t \leq \min\{r,s\}.$$

$W5$. (*Archimedean property*)

$$(\mathscr{I}x < r)\varphi \leftrightarrow \bigvee_{n=1}^{\infty}(\mathscr{I}x \leq r - \frac{1}{n})\varphi.$$

Now, the following axioms together with the weak axioms will be called the *strong axioms.*

$S1$. $\bigwedge_{\Psi\subseteq\Phi}(\mathscr{I}x \le r)\bigwedge\Psi \to (\mathscr{I}x \le r)\bigvee\Phi$, where Ψ ranges over the finite subsets of Φ.

$S2$. $\bigvee_{\Psi\subseteq\Phi}(\mathscr{I}x \ge r)\bigwedge\Psi \to (\mathscr{I}x \ge r)\bigwedge\Phi$, where Ψ ranges over the finite subsets of Φ.

The following lemma justifies the roles of the axioms $S1$ and $S2$. We want continuity of the capacity on monotonic sequences of definable subsets.

Lemma 7. *Let $\langle\mathfrak{A}, T\rangle$ be a weak capacity structure. Suppose $\langle\mathfrak{A}, T\rangle \vDash (S1 \wedge S2)$. Further, suppose each A_n below is definable. Then:*

(1) If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$, and $\bigcup_{i=1}^{\infty} A_i$ is also definable, then $T\big(\bigcup_{i=1}^{\infty} A_i\big) \le \sup_i T(A_i)$.
(2) If $A \supseteq A_1 \supseteq A_2 \supseteq \cdots$, then $T\big(\bigcap_{i=1}^{\infty} A_i\big) \ge \inf_i T(A_i)$.

Proof. We will prove (1), and (2) will follow by a similar argument. Suppose $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$. Let $i \in \mathbb{N}^+$. Then for some appropriate formula $\varphi_i(x, \underline{a}_i)$ we have $A_i = \{x : \varphi_i(x, \underline{a}_i)\}$. Let $\Phi = \{\varphi_i(x, \underline{a}_i) : i \in \mathbb{N}^+\}$. Now for any finite subset Ψ of Φ, we have $\langle\mathfrak{A}, T\rangle \vDash (\mathscr{I}x \le \sup_i T(A_i))\bigwedge\Psi$. Therefore, by $S1$, it follows that $\langle\mathfrak{A}, T\rangle \vDash (\mathscr{I}x \le \sup_i T(A_i))\bigvee\Phi$. Hence $T(\bigcup_{i=1}^{\infty} A_i) \le \sup_i T(A_i)$. ⊣

As an example of the expressive power of the language $\mathcal{L}_{\mathbb{A}\mathscr{I}}$, recall the definition of a *maxitive capacity* T. We require that T satisfies $T(A_1 \cup A_2) = \max\{T(A_1), T(A_2)\}$ in its domain. We claim that we can axiomatize the above as

$$\mathsf{MAX} : \Big((\mathscr{I}x \le r)\varphi(x) \wedge (\mathscr{I}x \le s)\psi(x)\Big) \to (\mathscr{I}x \le \max\{r, s\})\big(\varphi(x) \vee \psi(x)\big).$$

The following proposition justifies our claim.

Proposition. *Let $\langle\mathfrak{A}, T\rangle$ be a weak capacity model and suppose $\langle\mathfrak{A}, T\rangle \vDash \mathsf{MAX}$. Then $T(A_1 \cup A_2) = \max\{T(A_1), T(A_2)\}$ for all definable subsets A_1 and A_2 of A.*

Proof. Let $A_1 = \{x : \varphi_1(x, \underline{a}_1)\}$ and $A_2 = \{x : \varphi_2(x, \underline{a}_2)\}$ for appropriate formulas φ_1 and φ_2. Then $\langle\mathfrak{A}, T\rangle \vDash (\mathscr{I}x \le T(A_1))\varphi_1(x, \underline{a}_1)$ and $\langle\mathfrak{A}, T\rangle \vDash (\mathscr{I}x \le T(A_2))\varphi_2(x, \underline{a}_2)$. Therefore, by $W2$, $\langle\mathfrak{A}, T\rangle \vDash (\mathscr{I}x \le \max\{T(A_1), T(A_2)\})(\varphi_1(x, \underline{a}_1) \vee \varphi_2(x, \underline{a}_2))$, that is,

$$T(A_1 \cup A_2) = T\big(\{x : \varphi_1(x, \underline{a}_1) \vee \varphi_2(x, \underline{a}_2)\}\big) \le \max\{T(A_1), T(A_2)\}.$$

But by $W3$, we have $T(A_1 \cup A_2) \geq \max\{T(A_1), T(A_2)\}$, whence $T(A_1 \cup A_2) = \max\{T(A_1), T(A_2)\}$.

$\dashv$

The logic $L_{\mathbb{A}\mathscr{I}}$ is not compact. The example on [Kei85] works in this context too.

Example 13. Let R be a unary relation symbol and let Φ be the set of sentences

$$\{(\mathscr{I}x > 0)R(x)\} \cup \{(\mathscr{I}x \leq \frac{1}{n})R(x) : n \in \mathbb{N}^+\}.$$

Then any finite subset of Φ has a capacity model but Φ does not, owing to the axiom $W5$.

Now we aim to prove a "weak" model existence theorem for capacity logic, and then use the ideas from nonstandard capacity theory to pass to a "strong" model existence theorem. Prior to that, we will make a note about the following definition. The negation of a sentence with one or more capacity quantifiers, the way we have defined it here, is not the most general definition one may obtain. Indeed, for instance, the reader may observe that in the presence of only strong subadditivity, we only get one direction of the equivalence $\big((\mathscr{I}x \geq r)\varphi\big)\neg \equiv (\mathscr{I}x > 1 - r)\neg\varphi$. But for now, we make ourselves content with the following definition which covers only finitely additive capacities; $I(A \cup B) = I(A) + I(B)$ for disjoint A, B in the paving.

Definition 40.

$$\varphi\neg \equiv \neg\varphi \text{ } \textit{if } \varphi \textit{ is atomic.}$$

$$(\neg\varphi)\neg \equiv \varphi.$$

$$\big(\bigwedge_n \varphi_n\big)\neg \equiv \bigvee_n \neg\varphi_n.$$

$$\big(\bigvee_n \varphi_n\big)\neg \equiv \bigwedge_n \neg\varphi_n.$$

$$\big((\mathscr{I}x \geq r)\varphi\big)\neg \equiv \big(\mathscr{I}x > 1 - r\big)\neg\varphi.$$

$$\big((\mathscr{I}x > r)\varphi\big)\neg \equiv \big(\mathscr{I}x \geq 1 - r\big)\neg\varphi.$$

Let $C = \{c_n : n \in \mathbb{N}\}$ be a countable set of new constant symbols. Let $\mathcal{L}(C)$ be the augmented language. Let $L_{\mathbb{A}\mathscr{I}}(C)$ be the set of $\mathcal{L}(C)_{\omega_1\mathscr{I}}$-formulas consisting of all formulas derived from formulas in $L_{\mathbb{A}\mathscr{I}}$ by substituting finitely many $c \in C$ for free variables. A *consistency property* for $L_{\mathbb{A}\mathscr{I}}$ is a set S of sets s, where each s is a set of sentences of $L_{\mathbb{A}\mathscr{I}}(C)$ satisfying each of the following properties.

$C1$. (Consistency Rule) Either $\varphi \notin s$ or $(\neg\varphi) \notin s$.

$C2$. ($\neg$-Rule) $(\neg\varphi) \in s$ implies $s \cup \{(\varphi\neg)\} \in S$.

$C3$. ($\bigwedge$-Rule) $(\bigwedge \Phi) \in s$ implies $s \cup \{\varphi\} \in S$ for all $\varphi \in \Phi$.

$C4$. ($\bigvee$-Rule) $(\bigvee \Phi) \in s$ implies $s \cup \{\varphi\} \in S$ for some $\varphi \in \Phi$.

$C5$. (Equality Rules) Let $b, c, d \in C$.

(α) $(c = d) \in s$ implies $s \cup \{d = c\} \in S$.

(β) $c = b, \varphi(b) \in s$ imply $s \cup \{\varphi(c)\} \in S$.

(γ) $s \cup \{e = b\} \in S$ for some $e \in C$.

$C6$. ($(\mathscr{I} x > 0)$-Rule) $(\mathscr{I} x > 0)\varphi(x) \in s$ implies $s \cup \{\varphi(c)\} \in S$ for some $c \in C$.

$C7$. (Axiom-Rule) If $\varphi(x) \in L_{\mathbb{A}\mathscr{I}}(C)$ is an axiom, then each of the following holds.

(α) $s \cup \{(\mathscr{I} x \geq 1)\varphi(x)\} \in S$.

(β) For any constant c of $\mathcal{L}(C)$, $s \cup \{\varphi(c)\} \in S$.

Theorem 13 (Weak Model Existence Theorem for $L_{\mathbb{A}\mathscr{I}}$)**.** *Suppose S is a consistency property for $L_{\mathbb{A}\mathscr{I}}$. Then any $s_0 \in S$ has a weak capacity model.*

Proof. Let $\mathcal{S} = \{\varphi_n : n \in \mathbb{N}\}$ be an enumeration of the sentences of $L_{\mathbb{A}\mathscr{I}}(C)$, and let $\mathcal{D} = \{d_n : n \in \mathbb{N}\}$ be an enumeration of the constants of $L_{\mathbb{A}\mathscr{I}}(C)$. We wish to construct a sequence $s_0 \subseteq s_1 \subseteq \cdots s_n \subseteq \cdots$ of sets in S in the following manner. We will modify the proof of Theorem 2.5 of [Bar17]. Assume s_n is given. We will define s_{n+1}.

(1) Get the first constant symbol c in $\mathcal{D}$ such that $s_n \cup \{c = d_n\} \in S$. This is possible by $C5$. Put $s'_n = s_n \cup \{c = d_n\}$.

(2) If $s'_n \cup \{\varphi_n\} \notin S$, let $s_{n+1} = s'_n$. If $s'_n \cup \{\varphi_n\} \in S$, let $s''_n = s'_n \cup \{\varphi_n\}$.

(3) (i) If φ_n does not begin with $\bigvee$, let $s_{n+1} = s''_n$.

(ii) If φ_n is of the form $(\mathscr{I} x > 0)\psi$, get the first $d \in \mathcal{D}$ such that $s''_n \cup \{\psi(d)\} \in S$, and put $s_{n+1} = s''_n \cup \{\psi(d)\}$. This is possible by $C6$.

(iii) If φ_n is $\bigvee \Phi$, use $C4$ to find the least $\psi \in \Phi$ in the enumeration $\mathcal{S}$ such that $s''_n \cup \{\psi\} \in S$, and put $s_{n+1} = s''_n \cup \{\psi\}$.

Now let $s_\omega = \bigcup_{n\in\mathbb{N}} s_n$. We define an equivalence relation $\approx$ on $\mathcal{D}$ as follows. Let $c \approx d$ if and only if $(c = d) \in s_\omega$. Next put $M = \{[c] : c \in \mathcal{D}\}$, where $[c]$ is the $\approx$-equivalence class of $c \in \mathcal{D}$. Let $\mathfrak{M} = \langle M, R_i^{\mathfrak{M}}, c^{\mathfrak{M}}\rangle_{i\in I, c\in\mathcal{D}}$. The interpretation of the relation symbols is the following. For each relation symbol R_i of L, $R_i^{\mathfrak{M}}([c])$ if and only if $R_i([c]) \in s_\omega$. Notice that this interpretation is well-defined by $C5$. We interpret the constant symbols as $c^{\mathfrak{M}} = [c]$.

Next, we wish to construct a precapacity on the family $\mathcal{K}$ of all the definable subsets of M with parameters from M. We will follow the construction found in [Hoo78a]. Define, for any $\varphi(x, \underline{d}) \in L_{\mathbb{A}\mathscr{I}}$,

$$T\big(\{c^{\mathfrak{M}} : \varphi(c, \underline{d})\}\big) = \inf\{r : (\mathscr{I}x \leq r)\varphi(x, \underline{d}) \in s_\omega\}.$$

First of all, the Archimedean property, that is axiom $W5$, guarantees the infimum is attained. The axiom $W2$ guarantees that T is well-defined. We will show that T has the desired properties. Suppose X, Y are definable subsets of M with parameters from M. Then $X = \{c^{\mathfrak{M}} : \varphi(c, \underline{d})\}$ and $Y = \{c^{\mathfrak{M}} : \psi(c, \underline{e})\}$ for appropriate formulas φ and ψ, and suitable tuples $\underline{d}$ and $\underline{e}$ of elements of M.

Let us show that T is monotone nondecreasing. To that end, suppose $X \subseteq Y$. Then by Monotonicity axiom $W3$, we have

$$\{r : (\mathscr{I}x \leq r)\psi(c, \underline{d}) \in s_\omega\} \subseteq \{r : (\mathscr{I}x \leq r)\varphi(c, \underline{e}) \in s_\omega\}.$$

Therefore, $T(X) \leq T(Y)$ as desired. To prove that T is strongly subadditive, let $n \in \mathbb{N}^+$ be arbitrary. Then there exists r_1 such that $r_1 < T(X) + \frac{1}{n}$ with $(\mathscr{I}x \leq r_1)\varphi(c, \underline{d}) \in s_\omega$. Also, there exists r_2 such that $r_2 < T(Y) + \frac{1}{n}$ with $(\mathscr{I}x \leq r_2)\psi(c, \underline{e}) \in s_\omega$. Similarly, since $X \cap Y$ is definable by the conjunction of φ and ψ, there exists r_3 such that $r_3 < T(X \cap Y) + \frac{1}{n}$ with $(\mathscr{I}x \leq r_3)(\varphi(c, \underline{d}) \wedge \psi(c, \underline{e})) \in s_\omega$. Now by the strong subadditivity axiom $W4$, it follows that

$$T(X \cup Y) \leq T(X) + \frac{1}{n} + T(Y) + \frac{1}{n} - \Big(T(X \cap Y) + \frac{1}{n}\Big) = T(X) + T(Y) - T(X \cap Y) + \frac{1}{n}.$$

Since $n \in \mathbb{N}^+$ was arbitrary we obtain $T(X \cup Y) + T(X \cap Y) \leq T(X) + T(Y)$, which proves that T is strongly subadditive on $\mathcal{K}$. With the above considerations we see that $\langle \mathfrak{M}, T\rangle = \langle M, R_i^{\mathfrak{M}}, c^{\mathfrak{M}}, T\rangle_{i\in I, c\in\mathcal{D}}$ is a weak capacity structure as $\mathcal{K}$ is a regular paving on M and T is a $\mathcal{K}$-precapacity on M.

Finally, we want to verify that $\langle \mathfrak{M}, T\rangle \models \varphi$ whenever $\varphi \in s_\omega$. We will prove this by an induction on complexity argument. Suppose $\varphi \in s_\omega$ is atomic. Then by the definition of $R_i^{\mathfrak{M}}$ and $c_i^{\mathfrak{M}}$, we see that $\langle \mathfrak{M}, T\rangle \models \varphi$. By $C1$, it follows that $\langle \mathfrak{M}, T\rangle \models \neg\varphi$ whenever $\neg\varphi \in s_\omega$. The properties $C3$ and $C4$

handle conjunctions and disjunctions respectively. For instance, if $\varphi = \bigwedge \Phi$ and $\langle \mathfrak{M}, T\rangle \models \phi$ whenever $\phi \in s_\omega$ for each $\phi \in \Phi$, then by $C3$ we have $\langle \mathfrak{M}, T\rangle \models \varphi$. It remains to verify the result for φ of the form $(\mathscr{I}x \leq r)\psi(x)$. Suppose $\inf\{t : (\mathscr{I}x \leq t)\psi(x) \in s_\omega\} \leq r$. Then $T(\{c^{\mathfrak{M}} : \mathfrak{M} \models \psi[c]\}) \leq r$ by the induction hypothesis, whence $\langle \mathfrak{M}, T\rangle \models (\mathscr{I}x \leq r)\psi(x)$. This completes the induction argument. Thus $\langle \mathfrak{M}, T\rangle$ is a weak capacity model of s_ω. Hence $\langle \mathfrak{M}, T\rangle$ is a weak capacity model of s_0. This completes the proof. $\dashv$

Now we want to pass to a strong capacity model using the ideas found in [Ros90a]. His idea is to consider an internal mapping $J : \mathcal{F} \to {}^*[0,1]$ which is a precapacity, where $(F, \mathcal{F})$ is a paved set with both F and $\mathcal{F}$ internal sets, and then take the standard part of J. It turns out that a mapping induced by the standard part of J is a *bona fide* strongly subadditive $\mathcal{F}$-capacity.

Before passing to a strong capacity model, the following comment is due. One might inquire why we really want to construct a weak capacity model first. The answer is the following. In our proof of the Weak Model Lemma, the precapacity T that we constructed on the definable subsets of A may not necessarily satisfy the continuity conditions of a capacity. Consequently, it makes sense to first construct a weak capacity model and then pass to a strong capacity structure.

Definition 41. *Let F be a set with a regular paving $\mathcal{F}$. Let $T : \mathcal{F} \to [0,1]$ be a precapacity on $\mathcal{F}$. Define $L(T) : \mathcal{P}(F) \to [0,1]$ by*

$$L(T)(E) = \inf_{\substack{E \subseteq D \\ D \in \mathcal{F}_\sigma}} \sup_{\substack{X \subseteq D \\ X \in \mathcal{F}}} T(X).$$

Let F be an internal set and let $\mathcal{F}$ be an internal (standardly) regular paving on F. And let $T : \mathcal{F} \to {}^*[0,1]$ be internal precapacity. Let ${}^\circ T(A)$ be the standard part of $T(A)$ for each $A \in \mathcal{F}$. Thus we get a mapping ${}^\circ T : \mathcal{F} \to [0,1]$. It is easy to see that ${}^\circ T$ is an $\mathcal{F}$-precapacity on F. We have the following theorem.

Theorem 14 ([Ros90a])**.** *The mapping $L({}^\circ T)$ induced by T is a strongly subadditive $\mathcal{F}$-capacity on F.*

We wish to prove the following *strong model existence theorem* which is a description of the conversion of a weak capacity model into a strong capacity model. We shall denote $L({}^\circ T)$ by $\widetilde{T}$. Our superstructure will have $\mathbb{V} = \mathbb{V}(M, \mathbb{R})$ with $\mathbb{V}_0$ containing the transitive closure of M, and $\mathbb{R}$ as urelements, and we will assume it is $\aleph_1$-saturated. Note that M is the universe of the weak capacity model we obtain in the weak model lemma.

In our attempt at a proof of the theorem we face the following two issues as in [Hoo78a]. In the nonstandard universe the quantifier $(\mathscr{I}x \leq r)$ takes the values $r \in {}^*\mathbb{R}$, and for formulas of the kind $\varphi \equiv \bigvee_{n\in\mathbb{N}} \varphi_n$, the $*$-transform yields formulas of the kind ${}^*\varphi \equiv \bigvee_{n\in{}^*\mathbb{N}} {}^*\varphi_n$. However, we can fix the second issue by a saturation argument by proving that ${}^*\varphi$ is equivalent to a finite disjunction of ${}^*\psi_n{}'$s. We will leave the following as a conjecture.

Conjecture (Strong Model Existence Theorem)**.** *Let $\langle\mathfrak{M}, T\rangle$ be a weak model for $L_{\mathbb{A}\mathscr{I}}$. Then $\langle{}^*\mathfrak{M}, \widetilde{T}\rangle$ is a strong model. Moreover, for each $\varphi(x) \in L_{\mathbb{A}\mathscr{I}}$ and $a \in M$, we have*

$$\langle\mathfrak{M}, T\rangle \vDash \varphi(a) \text{ if and only if } \langle{}^*\mathfrak{M}, \widetilde{T}\rangle \vDash \varphi(a).$$

4.1. Proof Theory

We prove a "weak" completeness theorem for $L_{\mathbb{A}\mathscr{I}}$. Let $\mathbb{A}$ be a countable admissible set with $\omega \in \mathbb{A}$ and each $a \in \mathbb{A}$ countable. Then *weak* $L_{\mathbb{A}\mathscr{I}}$ is the capacity logic with only the weak axioms. Let φ, ψ be formulas of weak $L_{\mathbb{A}\mathscr{I}}$ and let Ψ be a set of formulas of weak $L_{\mathbb{A}\mathscr{I}}$. The rules of inference for weak $L_{\mathbb{A}\mathscr{I}}$ are the following.

R1. *Modus Ponens.* $\varphi, \varphi \to \psi \vdash \psi$.

R2. *Conjunction.* $\{\varphi \to \psi : \psi \in \Psi\} \vdash \varphi \to \bigwedge \Psi$, where Ψ is at most countable.

R3. *Generalization.* $\varphi \to \psi(x) \vdash \varphi \to (\mathscr{I}x \geq 1)\psi(x)$, provided x is not free in φ.

We define *a proof* in weak $L_{\mathbb{A}\mathscr{I}}$ and *validity* as in [Kei71].

Definition 42. *The set of* theorems *of weak $L_{\mathbb{A}\mathscr{I}}$ is the least set of formulas of weak $L_{\mathbb{A}\mathscr{I}}$ containing all the weak axioms so that it is closed under the rules of inference. $\vdash_{L_{\mathbb{A}\mathscr{I}}} \varphi$ denotes that φ is a theorem of weak $L_{\mathbb{A}\mathscr{I}}$. A formula φ of weak $L_{\mathbb{A}\mathscr{I}}$ is* valid*, denoted $\vDash_{L_{\mathbb{A}\mathscr{I}}} \varphi$, if and only if φ is satisfied in every weak model by every interpretation of the free variables of φ.*

We clearly see that our axioms are valid. For a completeness theorem, it remains to verify that our rules of inference preserve validity.

Theorem 15 (Weak Soundness Theorem)**.** *The weak axioms of $L_{\mathbb{A}\mathscr{I}}$ are sound and the rules of inference preserve validity.*

Proof. Obviously, any weak capacity model satisfies each of the weak axioms. It remains to prove that the rules of inference preserve validity. Indeed it suffices to prove that the rule R3 preserves validity as the proof for R1 and R2 is the same as in $L_{\omega\omega}$ as we only deal with formulas of at most finitely many free variables. Let $\langle \mathfrak{A}, T\rangle$ be an arbitrary weak model for $L_{\mathbb{A}\mathscr{I}}$, and suppose that $\langle \mathfrak{A}, T\rangle \models \varphi \to \psi(x)$, $\langle \mathfrak{A}, T\rangle \models \varphi$, and x is not free in φ. Then by R1 we have $\langle \mathfrak{A}, T\rangle \models \psi(x)$ and $T\big(\{a \in A : \mathfrak{A} \models \psi(x)[a]\}\big) = T(A) = 1 \geq 1$. Hence $\langle \mathfrak{A}, T\rangle \models (\mathscr{I}x \geq 1)\psi(x)$. This completes the proof. ⊣

Recall the definition of $L_{\mathbb{A}\mathscr{I}}(C)$. It is the set of $\mathcal{L}(C)_{\omega_1\mathscr{I}}$-formulas consisting of all formulas derived from formulas in $L_{\mathbb{A}\mathscr{I}}$ by substituting only finitely many $c \in C$ for free variables. Thus a formula of $L_{\mathbb{A}\mathscr{I}}(C)$ is of the form $\varphi(x_1, \ldots, x_n, c_1, \ldots, c_m)$, where $\varphi(x_1, \ldots, x_n, y_1, \cdots, y_m)$ is a formula of $L_{\mathbb{A}\mathscr{I}}$, where we have replaced each free occurrence of y_i by $c_i \in C$ for $1 \leq i \leq m$. Moreover, if $z_1, \ldots, z_m$ are variables in $L_{\mathbb{A}\mathscr{I}}$ not occurring in $\varphi(x_1, \ldots, x_n, c_1, \ldots, c_m)$, then $\varphi(x_1, \ldots, x_n, z_1, \ldots, z_m)$ is in $L_{\mathbb{A}\mathscr{I}}$. Thus, if φ is a sentence of $L_{\mathbb{A}\mathscr{I}}$ and $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \varphi$, then $\vdash_{L_{\mathbb{A}\mathscr{I}}} \varphi$. We have the following lemma which is an essential first step towards a weak completeness theorem.

Lemma 8. *Let S be the set of all finite sets s of sentences in $L_{\mathbb{A}\mathscr{I}}(C)$ such that $\nvdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg \bigwedge s$. Then S is a consistency property.*

Proof. It suffices to prove the properties $C6$ and $C7$ as everything else is proved in [Kei71] and [Mar16]. Let us prove $C6$ first, that is, if $(\mathscr{I}x > 0)\varphi(x) \in s$ then $s \cup \{\varphi(c)\} \in S$ for some $c \in C$. Suppose $(\mathscr{I}x > 0)\varphi(x) \in s \in S$ but for all $c \in C$ we have $s \cup \{\varphi(c)\} \notin S$. Let $c \in C$ be such that c does not occur in s. This is possible since s is a finite set of sentences in $L_{\mathbb{A}\mathscr{I}}(C)$. Then $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg\bigwedge(s \cup \{\varphi(c)\})$. Thus, $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg(\bigwedge s \wedge \varphi(c))$. Therefore, $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \to \neg\varphi(c)$. Now let y be a variable not occurring in s. Replacing c by y in the proof we see that $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \to \neg\varphi(y)$. Therefore, we have $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \to (\mathscr{I}y \geq 1)\neg\varphi(y)$, or rather $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \to \neg(\mathscr{I}y > 0)\varphi(y)$. Hence $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg \bigwedge s$, which is a contradiction. This proves $C6$.

To prove $C7$-(α), let us suppose that $\varphi(x) \in L_{\mathbb{A}\mathscr{I}(C)}$ is an axiom but $s \cup \{(\mathscr{I}x \geq 1)\varphi(x)\} \notin S$. Then $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg\bigwedge(s \cup \{(\mathscr{I}x \geq 1)\varphi(x)\})$. Thus, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg(\bigwedge s \wedge (\mathscr{I}x \geq 1)\varphi(x))$. Therefore, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \bigwedge s \to \neg(\mathscr{I}x \geq 1)\varphi(x)$, or rather $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} (\mathscr{I}x \geq 1)\varphi(x) \to \neg\bigwedge s$. Since $\varphi(x)$ is an axiom, by R3 we obtain $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} (\mathscr{I}x \geq 1)\varphi(x)$. Therefore, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg\bigwedge s$, which is a contradiction.

Finally, let us prove $C7$-(β), that is, for an axiom $\varphi(x) \in L_{\mathbb{A}\mathscr{I}(C)}$, if c is a constant of $\mathcal{L}(C)$, then $s \cup \{\varphi(c)\} \in S$. For the sake of a contradiction, suppose that c is a constant of $\mathcal{L}(C)$ such that $s \cup \{\varphi(c)\} \notin S$. Then $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg \bigwedge(s \cup \{\varphi(c)\})$. Thus, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg(\bigwedge s \wedge \varphi(c))$. Therefore, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \bigwedge s \to \neg\varphi(c)$. Now let y be a variable not occurring in s. Then replacing c in the proof by y we get $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \bigwedge s \to \neg\varphi(y)$. Therefore, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \bigwedge s \to (\mathscr{I} y \geq 1)\neg\varphi(y)$ by R3. Thus, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \bigwedge s \to \neg(\mathscr{I} y > 1)\varphi(y)$. Equivalently, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg \bigwedge s \vee \neg(\mathscr{I} y > 1)\varphi(y)$. Therefore, $\vdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg(\bigwedge s \wedge (\mathscr{I} y > 1)\varphi(y))$. Hence, $s \cup \{(\mathscr{I} y \geq 1)\varphi(y)\} \notin S$, which is a contradiction by $C7$-(α). This completes the proof. $\dashv$

Now observe that $\nvdash_{L_{\mathbb{A}\mathscr{I}(C)}} \varphi$ if and only if $\nvdash_{L_{\mathbb{A}\mathscr{I}(C)}} \neg(\neg\varphi)$. Thus by the weak model existence theorem for $L_{\mathbb{A}\mathscr{I}}$, since S of Lemma 8 is a consistency property, we get a weak model $\langle \mathfrak{M}, T\rangle \vDash \neg\varphi$, from which it follows that $\nvDash \varphi$. Thus, by the considerations just above Lemma 8, and by the Weak Soundness Theorem, we obtain the following weak completeness theorem. Let us call it *weak* until we have proven our conjecture.

Theorem 16 (Weak Completeness Theorem for $L_{\mathbb{A}\mathscr{I}}$)**.** *If φ is a sentence of weak $L_{\mathbb{A}\mathscr{I}}$, then $\vdash_{L_{\mathbb{A}\mathscr{I}}} \varphi$ if and only if $\vDash \varphi$.*

Appendices

A. KP and Admissible Sets

We will start this section by outlining the procedure of building the formulas of a language set theoretically. Next, we will have a brief look at *Admissible Sets*. The reader should keep in mind that this is just a hand-wavy outline, and for a thorough treatment of the subject, [Bar17] or [Kei71] are recommended. We will skip a lot of details and jot down only the important steps.

A.1. Formulas, Set Theoretically ...

We can build the formulas of a language $\mathcal{L}$ set theoretically. We will outline the procedure here and for more details, we defer the reader to [Bar17]. Suppose our meta-language contains the relations

- $\texttt{Relation} - \texttt{symbol}(x)$

- $\texttt{Function} - \texttt{symbol}(x)$
- $\texttt{Constant} - \texttt{symbol}(x)$
- $\texttt{Variable}(x)$

and the symbols $\texttt{v}$ and $\#$, and $\neg, \bigwedge, \bigvee, \forall, \exists$, and $\equiv$. We will write v_α for $\texttt{v}(\alpha)$ and suppose that $\alpha \neq \beta$ implies $v_\alpha \neq v_\beta$. Here α, β are ordinals. Further, $\texttt{Variable}(x)$ is true if and only if $\exists\alpha(x = v_\alpha)$. The role of $\#$ is the following. If x is a relation symbol or a function symbol of $\mathcal{L}$, the $\#(x)$ is a positive natural number asserting the arity of x. Now the formulas of $\mathcal{L}$ are defined set theoretically in the following fashion.

- "t is a term of $\mathcal{L}$" is defined recursively on the transitive closure of t, that is, t is a term of $\mathcal{L}$ if and only if t is a variable of $\mathcal{L}$, or t is a constant symbol of $\mathcal{L}$, or t is the ordered pair $\langle h, y\rangle$, where h is a function symbol of $\mathcal{L}$ and y is the $\#(h)$–tuple $(y_1, \ldots, y_{\#(h)})$ and each y_i is a term of $\mathcal{L}$.
- For atomic formulas of $\mathcal{L}$:
  (a) $\langle \equiv, t_1, t_2\rangle$, where t_1, t_2 are terms of $\mathcal{L}$, represents $t_1 \equiv t_2$, or rather $t_1 = t_2$.
  (b) $\langle r, t_1, \ldots, t_n\rangle$, where r is a relation symbol of $\mathcal{L}$ with $n = \#(r)$ and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$, represents $r(t_1, \ldots, t_1)$.
- The finite formulas of $\mathcal{L}$ are
  (a) Atomic formulas of $\mathcal{L}$.
  (b) $\langle \neg, \varphi\rangle$, where φ is a finite formula of $\mathcal{L}$.
  (c) $\langle \bigwedge, \{\varphi, \psi\}\rangle$ or $\langle \bigvee, \{\varphi, \psi\}\rangle$, where φ and ψ are finite formulas of $\mathcal{L}$.
  (d) $\langle \exists, v, \varphi\rangle$ or $\langle \forall, v, \varphi\rangle$ where v is a variable and φ is a finite formula of $\mathcal{L}$.

  Thus $\langle \neg, \varphi\rangle$, $\langle \bigwedge, \{\varphi, \psi\}\rangle$, $\langle \bigvee, \{\varphi, \psi\}\rangle$, $\langle \exists, v, \varphi\rangle$, and $\langle \forall, v, \varphi\rangle$ represent $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists v\varphi$, and $\forall v\varphi$ respectively. Note that even though we do not have the symbols $\wedge$ and $\vee$ in our meta-language, we still write $\varphi \wedge \psi$ and $\varphi \vee \psi$ keeping in mind that they represent $\bigwedge\{\varphi, \psi\}$ and $\bigvee\{\varphi, \psi\}$ respectively.
- The infinitary formulas of $\mathcal{L}$ are
  (a) Finite formulas φ.
  (b) $\neg\varphi$, where φ is an infinitary formula.
  (c) $\exists v\varphi$ and $\forall\varphi$, where v is a variable and φ is an infinitary formula.

(d) $\langle\bigwedge,\Phi\rangle$ and $\langle\bigvee,\Phi\rangle$, where Φ is a non-empty set of infinitary formulas. Here, $\langle\bigwedge,\Phi\rangle$ and $\langle\bigvee,\Phi\rangle$ represent $\bigwedge\Phi$ and $\bigvee\Phi$ respectively.

We will restrict our attention to (proper) infinitary formulas which have only a finite number of free variables in them. As is the case with formulas of $\mathcal{L}$, the $\mathcal{L}$–structures are also defined set theoretically. An $\mathcal{L}$–structure $\mathfrak{M}$ is a pair $\langle M, f\rangle$ with $f(x)$ representing $x^{\mathfrak{M}}$ such that

(a) M is a nonempty set.

(b) f is function with domain $\mathcal{L}$.

(c) For relation symbols r of $\mathcal{L}$, we have $r^{\mathfrak{M}} \subseteq M^{\#(r)}$.

(d) For function symbols h of $\mathcal{L}$, we have domain of $h^{\mathfrak{M}} = M^{\#(h)}$, and range of $h^{\mathfrak{M}} \subseteq M$.

(e) For constant symbols c of $\mathcal{L}$, we have $c^{\mathfrak{M}} \in M$.

Next, we will see how "assignments" work in this setting. Consider an $\mathcal{L}$–structure $\mathfrak{M}$. Let s be a function with domain a finite set of variables, and range a subset of M. Such a function is called an assignment. Let t be a term of $\mathcal{L}$. Suppose all the variables of t are contained in the domain of s. Define the value of t in $\mathfrak{M}$ at s, denoted by $t^{\mathfrak{M}}(s)$, by recursion as follows.

- $t^{\mathfrak{M}}(s) = c^{\mathfrak{M}}$ if t is a constant symbol c.
- $t^{\mathfrak{M}}(s) = s(v)$ if t is a variable v.
- $t^{\mathfrak{M}}(s) = h^{\mathfrak{M}}(t_1^{\mathfrak{M}}(s), \dots, t_n^{\mathfrak{M}}(s))$ if t is $h(t_1,\dots,t_n)$, where $t_1,\dots,t_n$ are terms of $\mathcal{L}$.

We are now ready to define the notion of satisfaction $\mathfrak{M} \vDash \varphi[s]$ set theoretically. Here, $\mathfrak{M}$ is an $\mathcal{L}$–structure, φ is a formula of $\mathcal{L}$, and s is an assignment to the free variables of φ. We will denote the set of free variables of φ by $FV(\varphi)$. Define $G(\mathfrak{M},\varphi) = \{s : s$ is an assignment in $\mathfrak{M}$ and domain of $s = FV(\varphi)\}$. Then satisfaction $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M},\varphi) = \{s \in G(\mathfrak{M},\varphi) : \mathfrak{M} \vDash \varphi[s]\}$ is defined recursively as follows.

- $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, r(t_1,t_2)) = \{s \in G(\mathfrak{M}, r(t_1,t_2)) : \langle t_1^{\mathfrak{M}}(s), t_2^{\mathfrak{M}}(s)\rangle \in r^{\mathfrak{M}}\}$.
- $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \neg\varphi) = \{s \in G(\mathfrak{M}, \neg\varphi) : s \notin \mathsf{Sat}_{\mathcal{L}}(\mathfrak{M},\varphi)\}$.
- $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \bigwedge\Phi) = \{s \in G(\mathfrak{M}, \bigwedge\Phi) : \text{for all } \varphi \in \Phi, s \restriction FV(\varphi) \in \mathsf{Sat}_{\mathcal{L}}(\mathfrak{M},\varphi)\}$.
- $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \exists v\varphi) = \{s \in G(\mathfrak{M}, \exists \mathfrak{v}\varphi) : \text{ for some } x \in M, s \cup \{\langle v,x\rangle\} \in \mathsf{Sat}_{\mathcal{L}}(\mathfrak{M},\varphi)\}$ whenever $v \in FV(\varphi)$, and $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \exists v\varphi) = \mathsf{Sat}_{\mathcal{L}}(\mathfrak{M},\varphi)$ whenever $v \notin FV(\varphi)$.
- $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \bigvee\Phi)$ and $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \forall v\varphi)$ are defined similarly.

Thus, the relation $\mathfrak{M} \vDash \varphi[s]$ is defined by

$$\mathfrak{M} \vDash \varphi[s] \text{ if and only if } s \restriction FV(\varphi) \in \mathsf{Sat}_{\mathcal{L}}(\mathfrak{M},\varphi).$$

If φ is a sentence of $\mathcal{L}$, we have $\mathfrak{M} \models \varphi$ if and only if the empty function is in $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \varphi)$.

A.2. Admissible sets and Fragments

Recall the definition of a *fragment* of $L_{\omega_1\omega}$. [Kei71] Let $L_{\omega_1\omega}$ denote the set of formulas of the language $\mathcal{L}_{\omega_1\omega}$. For any set $\mathcal{A}$ we let $L_{\mathcal{A}} = L_{\omega_1\omega} \cap \mathcal{A}$. The set of formulas $L_{\mathcal{A}}$ is called a fragment of $\mathcal{L}_{\omega_1\omega}$ if and only if

(i) $\mathcal{A}$ is a transitive set.

(ii) If $a, b \in \mathcal{A}$, then $\{a, b\} \in \mathcal{A}, a \cup b \in \mathcal{A}$, and $a \times b \in \mathcal{A}$.

(iii) If $a \in \mathcal{A}$ and α is the least ordinal that is not in the transitive closure of a, then $\alpha \in \mathcal{A}$.

(iv) If $\varphi(x \ldots) \in L_{\mathcal{A}}$ and $t \in \mathcal{A}$ is a term of $\mathcal{L}_{\omega_1\omega}$, then $\varphi(t \ldots) \in L_{\mathcal{A}}$.

By the set theoretic formalization of formulas we saw in the previous section, we can define a fragment of $\mathcal{L}_{\omega_1\omega}$ in the following manner as well.

Definition 43 ([Bar17])**.** *A fragment of $\mathcal{L}_{\omega_1\omega}$ is a set $L_{\mathcal{A}}$ of infinitary formulas and variables such that*

(i) Every finite formula of $\mathcal{L}_{\omega\omega}$ is in $L_{\mathcal{A}}$.

(ii) $L_{\mathcal{A}}$ is closed under subformulas and variables, that is, if $\varphi \in L_{\mathcal{A}}$, then every subformula of φ is in $L_{\mathcal{A}}$ and every variable of φ is in $L_{\mathcal{A}}$.

(iii) $L_{\mathcal{A}}$ is closed under substitution, that is, if $\varphi(v) \in L_{\mathcal{A}}$ and t is a term of $\mathcal{L}$ such that all the variables of t are in $L_{\mathcal{A}}$, then $\varphi(t)$ is in $L_{\mathcal{A}}$, where we substitute t for all the occurrences of v.

(iv) $L_{\mathcal{A}}$ is closed under the logical operations, or more precisely, if φ, ψ, v are in $L_{\mathcal{A}}$, then so are $\neg\varphi, \varphi\neg, \exists v\varphi, \forall v\varphi, \varphi \vee \psi, \varphi \wedge \psi$.

At this point, we want to understand the whole purpose of this formalization. Let us see what happens when $\mathcal{A}$ is a so-called *admissible* set. Before proceeding further, note that we have not really specified which set theory we are working in. One problem with that lack of specification is that we do not know whether the sets and functions, especially the function G which appeared in $\mathsf{Sat}_{\mathcal{L}}(\mathfrak{M}, \varphi)$, are indeed sets, or not, in our set theory. Recall the following.

The collection of Δ_0-formulas is the smallest collection of formulas containing the atomic formulas which is closed under the Boolean operations $\neg, \wedge, \vee$, and closed under bounded quantification. Thus a Δ_0-formula of set theory is built up from atomic formulas and their negations using only the four operations $\wedge, \vee, (\forall x \in y)$, and $(\exists x \in y)$. If φ is Δ_0 and u, v are variables, then $(\exists u \in v)\varphi$ and $(\forall u \in v)\varphi$ are Δ_0-formulas. A Σ-formula is a formula of set theory built up from atomic formulas and their negations using only the four operations $\wedge, \vee, (\forall x \in y)$, and $(\exists x \in y)$, and $(\exists x)$. If, instead of $(\exists x)$, we use $(\forall x)$, we get a Π-formula. [Kei71] A set $\mathcal{A}$ is admissible if and only if :

(i) $\mathcal{A}$ is non-empty.

(ii) $\mathcal{A}$ is transitive.

(iii) If $x \in \mathcal{A}$, then the transitive closure $TC(x)$ of x belongs to $\mathcal{A}$.

(iv) (Δ_0-Separation Axiom). If $\varphi(x, y_1, \dots, y_n)$ is a Δ_0-formula and $b_1, \dots, b_n, c \in \mathcal{A}$, then

$$\{a \in c : \langle \mathcal{A}, \in \rangle \vDash \varphi[a, b_1, \dots, b_n]\} \in \mathcal{A}.$$

(v) (Σ-Reflection Axiom). If $\varphi(y_1, \dots, y_n)$ is a Σ-formula, $b_1, \dots, b_n \in \mathcal{A}$, and $\langle \mathcal{A}, \in \rangle \vDash \varphi[b_1, \dots, b_n]$, then there is a transitive set $a \in \mathcal{A}$ so that $b_1, \dots, b_n \in a$ and $\langle a, \in \rangle \vDash \varphi[b_1, \dots, b_n]$.

The above definition of an admissible set is due to Platek. There is an alternative definition of admissible sets which we shall see shortly.

Definition 44 ([Bar17])**.** *The theory* KPU *consists of the universal closures of the following formulas.*

Extensionality: $\forall x(x \in a \longleftrightarrow x \in b) \rightarrow a = b$.

Foundation: $\exists x\varphi(x) \rightarrow \exists x[\varphi(x) \wedge \forall y \in x \varphi(y)]$ *for all formulas* $\varphi(x)$ *in which* y *does not occur free.*

Pair: $\exists a(x \in a \wedge y \in a)$.

Union: $\exists b \forall y \in a \forall x \in y(x \in b)$.

Δ_0-separation: $\exists b \forall x(x \in b \longleftrightarrow x \in a \wedge \varphi(x))$ *for all* Δ_0*-formulas in which* b *does not occur free.*

Δ_0-collection: $\forall x \in a \exists y \varphi(x,y) \to \exists b \forall x \in a \exists y \in b \varphi(x,y)$ *for all* Δ_0*-formulas in which* b *does not occur free.*

The theory KPU *is called* Kripke-Platek set theory with urelements. *The theory* KP *is* KPU + $\forall x \exists a(x = a)$*, that is* KP *is Kripke-Platek set theory in which there are no urelements.*

Now we are ready to give an alternative definition of an admissible set and an admissible fragment.

Definition 45 ([Bar17])**.** *An* admissible set *is a set* $\mathcal{A}$ *such that* $\mathcal{A}$ *is transitive and* $\langle \mathcal{A}, \in \rangle \vDash \mathsf{KP}$*. Moreover,* $L_{\mathcal{A}}$ *is an* admissible fragment *of* $\mathcal{L}_{\omega_1 \omega}$ *if* $\mathcal{A}$ *is an admissible set.*

An example of an admissible set is **HF**, the set of all hereditarily finite sets. Indeed $\langle \mathbf{HF}, \in \rangle \vDash$ $(\mathsf{ZF} - \text{Axiom of Infinity})$. The set **HC** of hereditarily countable sets is another example.

It turns out that in the formalization of formulas inside KP, every relation between the entities that we formalized is Δ and the function G is Σ. We conclude this section with this excerpt from [Kei71], and recording *Barwise Compactness* theorem. We will also record the analogues of those theorems for Keisler's probability logic as stated in [Hoo78a]. Let $\mathcal{A}$ be an admissible set.

"In general, one may think of the property of belonging to $\mathcal{A}$ *as a generalization of the property of being finite, and* Δ *on* $\mathcal{A}$*,* Σ *on* $\mathcal{A}$ *as generalizations of the properties of recursiveness and recursive enumerability."*

Theorem 17 ([Bar17] Barwise Compactness Theorem)**.** *Let* $L_{\mathcal{A}}$ *be a countable admissible fragment of* $\mathcal{L}_{\omega_1,\omega}$*. Let* T *be a set of sentences of* $L_{\mathcal{A}}$ *which is* Σ_1 *on* $\mathcal{A}$*. If every* $T_0 \subseteq T$ *which is an element of* $\mathcal{A}$ *has a model, then* T *has a model.*

Theorem 18 ([Hoo78a] Barwise Compactness Theorem for $L_{\mathbb{A}P}$)**.** *If* T *is a set of sentences of* $L_{\mathbb{A}P}$ *which is* Σ_1 *on* $\mathbb{A}$*,* $\omega \cup L \in \mathbb{A}$ *countable admissible, and every* $T_0 \subseteq T$ *such that* $T_0 \in \mathcal{A}$ *has a model, then* T *has a model.*

B. Nonstandard Preliminaries

We give a brief introduction to Abraham Robinson's nonstandard analysis. Our goal is twofold; to equip the reader with the nonstandard tools indispensable for the understanding of nonstandard techniques in capacity theory, and to evoke an interest in nonstandard analysis in general. There are different approaches for nonstandard analysis as described in [BDNF+06]. But we choose the superstructure approach for our purposes. Our references for this section are [SL77], [ACH12], [DNGL19] and lecture notes from Math 649B - Nonstandard Analysis by David Ross. We avoid going down the rabbit hole of being too technical here. The reader is advised to take the ideas presented here with a grain of salt and refer the books cited above for a thorough treatment of the topics discussed.

B.1. Superstructures

By a urelement, we mean an object that is not assumed to be a set by itself. Let S be a set of urelements. We define $\mathbb{V}_n(S)$ inductively in the following manner.

$$\mathbb{V}_0(S) = S \text{ and } \mathbb{V}_{n+1}(S) = \mathbb{V}_n(S) \cup \mathcal{P}(\mathbb{V}_n(S)),$$

where $\mathcal{P}(\mathbb{V}_n(S))$ is the power set of $\mathbb{V}_n(S)$. Then $\mathbb{V}(S) = \bigcup_{n=0}^{\infty} \mathbb{V}_n(S)$ is called the *superstructure over* S. We will assume that the set of real numbers $\mathbb{R}$ is contained in S. From now on, we will drop S and write $\mathbb{V}$ for $\mathbb{V}(S)$. Given a superstructure $\mathbb{V}$, there exists a larger superstructure ${}^*\mathbb{V}$ and an embedding $* : \mathbb{V} \to {}^*\mathbb{V}$ preserving the mathematical structure of $\mathbb{V}$. More precisely,

(1) If A is a set in $\mathbb{V}$ and $a \in A$, then ${}^*a \in {}^*A$.

(2) If A is a set in $\mathbb{V}$, then ${}^*\{(x,x) : x \in A\} = \{(y,y) : y \in {}^*A\}$.

(3) ${}^*\{a_1, \dots, a_n\} = \{{}^*a_1, \dots, {}^*a_n\}$ for $a_1, \dots, a_n \in \mathbb{V}$.

(4) $*$ preserves the basic set operations, that is, ${}^*\varnothing = \varnothing, {}^*(A \cup B) = {}^*A \cup {}^*B, {}^*(A \cap B) = {}^*A \cap {}^*B, {}^*(A \times B) = {}^*A \times {}^*B$, where $A, B \in \mathbb{V}$.

(5) $*$ preserves domains and ranges of relations.

(6) ${}^*\{(x,y) : x \in y \in A\} = \{(z,w) : z \in w \in {}^*A\}$, where $A \in \mathbb{V}$.

(7) ${}^*A \supseteq {}^\sigma A = \{{}^*a : a \in A\}$ with equality if and only if A is finite.

Thus, the relations $\in$ and $=$ are preserved by the $*$-transform. The structure ${}^*\mathbb{V} = \mathbb{V}({}^*S)$ is called a *nonstandard extension* of $\mathbb{V}$, and the embedding $*$ is called its *$*$-transform*.

Recall that the language of set theory has $\in$ and $=$ as the relation symbols. We take $\mathcal{L}_{\mathbb{V}(\mathbb{S})}$ to be the language of set theory augmented by a set $\mathcal{C}$ of constant symbols, where $\mathcal{C}$ has a symbol for each urelement and each set in $\mathbb{V}(S)$. The terms and formulas of $\mathcal{L}_{\mathbb{V}(S)}$ are defined as usual starting with atomic terms and formulas and building upwards with increasing complexity. However, we only consider the bounded $\mathcal{L}_{\mathbb{V}(S)}$-formulas; the ones with quantifiers of the form $\forall x \in y$ or $\exists x \in y$ whenever a quantifier is present in them. Now the properties (1)-(7) dictate that the $*$-transform can be applied to the bounded formulas of $\mathcal{L}_{\mathbb{V}(S)}$ which in turn give rise to bounded formulas of $\mathcal{L}_{\mathbb{V}({}^*S)}$. Given a bounded formula φ of $\mathcal{L}_{\mathbb{V}(S)}$, the $*$-transform of φ is denoted by ${}^*\varphi$. Note that one may first define the $*$-transform of bounded $\mathcal{L}_{\mathbb{V}(S)}$-formulas and obtain the properties (1)-(7) as consequences of it. This process is known as the *transfer principle* in nonstandard analysis. We will illustrate the use of transfer using several examples.

Example 14. For $a_1, \ldots, a_n \in \mathbb{V}(S)$ we have ${}^*\{a_1, \ldots, a_n\} = \{{}^*a_1, \ldots, {}^*a_n\}$.

Proof. Let $A = \{a_1, \ldots, a_n\}$. Transfer the sentence

$$\forall x \in A\big(x \in \{a_1, \ldots, a_n\} \iff (x = a_1 \vee \cdots \vee x = a_n)\big)$$

and obtain

$$\forall x \in {}^*A\big(x \in {}^*\{a_1, \ldots, a_n\} \iff (x = {}^*a_1 \vee \cdots \vee x = {}^*a_n)\big).$$

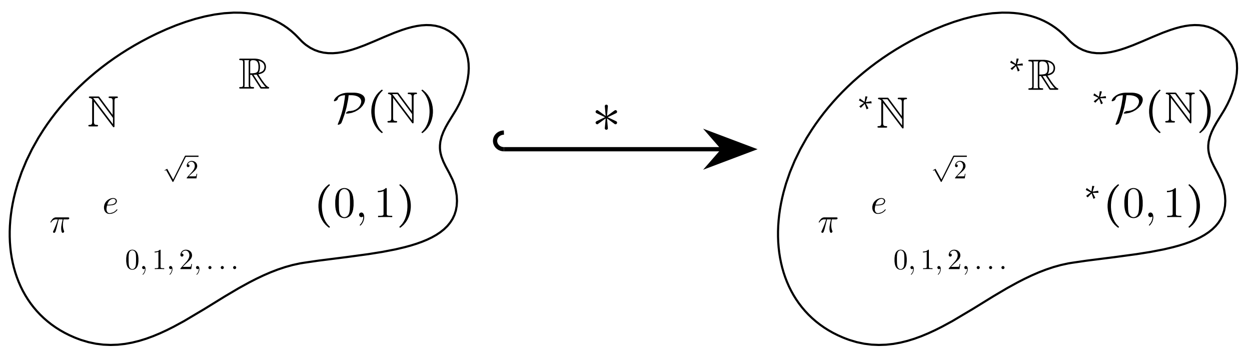
⊣

Example 15. ${}^*(A \cup B) = {}^*A \cup {}^*B$.

Proof. Let $C = A \cup B$. Then by applying transfer to $\forall x \in C\big(x \in A \vee x \in B\big)$ we obtain ${}^*(A \cup B) \subseteq {}^*A \cup {}^*B$. For the other inclusion, we transfer the sentences $\forall x \in A(x \in C)$ and $\forall x \in B(x \in C)$, whence ${}^*A \subseteq {}^*C$ and ${}^*B \subseteq {}^*C$ proving the result. ⊣

Example 16. ${}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$.

Proof. Transfer the sentence $\forall x \in \mathcal{P}(A) \forall y \in x\big(y \in A\big)$ and obtain $\forall x \in {}^*\mathcal{P}(A) \forall y \in x(y \in {}^*A)$, and hence ${}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$. ⊣



The above considerations can be summarized formally as follows. Let $\mathbb{V}(S)$ and $\mathbb{V}(T)$ be superstructures. Then the triple $\langle \mathbb{V}(S), \mathbb{V}(T), *\rangle$ is a nonstandard universe if and only if each of the following holds.

(1) S and T are infinite sets of urelements.

(2) $T = {}^*S$.

(3) $* : \mathbb{V}(S) \to \mathbb{V}(T)$ is an injection.

(4) For any bounded sentence φ of $\mathcal{L}_{\mathbb{V}(S)}$, $\mathbb{V}(S) \vDash \varphi$ if and only if $\mathbb{V}(T) \vDash {}^*\varphi$.

(5) ${}^*A \supsetneq \{{}^*a : a \in A\}$ whenever $A \in \mathbb{V}(S)$ is infinite.

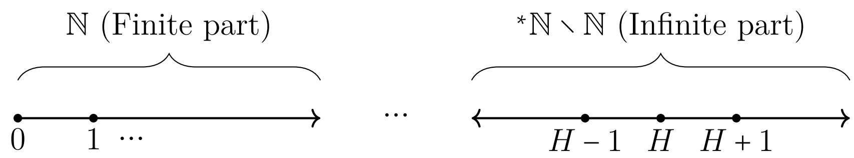
B.2. Standard, Internal, and External Sets

Let $\langle \mathbb{V}(S), \mathbb{V}(T), *\rangle$ be a nonstandard universe.

Definition 46. *A subset E of $\mathbb{V}(T)$ is called* standard *if $E = {}^*A$ for some $A \in \mathbb{V}(S)$. We say E is an* internal *set if $E \in {}^*A$ for some $A \in \mathbb{V}(S)$, and E is said to be an* external *set otherwise.*

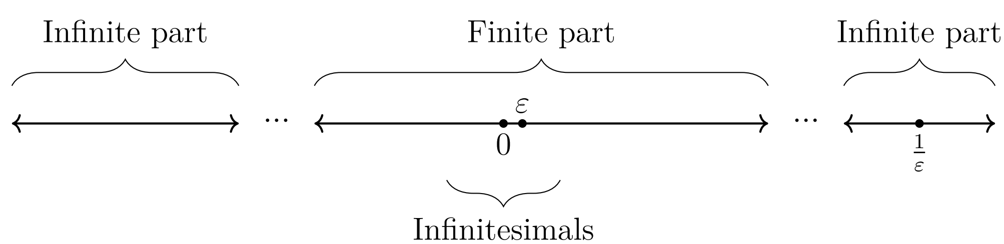
Observe that for any superstructure $\mathbb{V}(S)$, we have $\mathbb{V}_0(S) \subseteq \mathbb{V}_1(S) \subseteq \cdots \subseteq \mathbb{V}_n(S) \subseteq \cdots \in \mathbb{V}(S)$ and $\mathbb{V}_0(S) \in \mathbb{V}_1(S) \in \cdots \in \mathbb{V}_n(S) \in \cdots \in \mathbb{V}(S)$, where $n \in \mathbb{N}$. Thus we see that any standard subset E of $\mathbb{V}(T)$ is internal as well. In particular ${}^*\mathbb{N}$ and ${}^*\mathbb{R}$ are internal sets. Indeed the set of internal subsets of an internal set A is the set ${}^*\mathcal{P}(A)$. To see this, observe that by transfer, both $\mathcal{P}(A)$ and ${}^*\mathcal{P}(A)$ have the same properties.

The structure of ${}^*\mathbb{N}$ is somewhat fascinating in its own right. One finds bizarre entities in the nonstandard universe, especially when one steps into the realm of $\mathbb{V}(T)$. For instance, we know that ${}^*\mathbb{N} \supsetneq \mathbb{N}$. This tells us that there are numbers H, called *hypernatural numbers*, in ${}^*\mathbb{N}$ which are not in $\mathbb{N}$. Each such H has to be infinite for the following reasons. Suppose $n \in \mathbb{N}$ is arbitrary. Then the bounded sentence $\forall x \in \mathbb{N}\big((x \neq 0 \wedge x \neq 1 \wedge \cdots \wedge x \neq n) \to x > n\big)$ is true in $\mathbb{V}(S)$. Therefore by transfer, the sentence $\forall x \in {}^*\mathbb{N}\big((x \neq 0 \wedge x \neq 1 \wedge \cdots\ x \neq n) \to x > n\big)$ is true in $\mathbb{V}(T)$. Now since n was arbitrary, any $H \in {}^*\mathbb{N}$ which is not in $\mathbb{N}$ must be infinite. More is true about ${}^*\mathbb{N}$. The set $\mathbb{N}$ is in fact an initial segment of ${}^*\mathbb{N}$.



To see what the structure of ${}^*\mathbb{N}$ brings to the table, suppose $A \subseteq {}^*\mathbb{N}$ is an internal set containing arbitrarily small infinite numbers. Then A must have a finite number. Because if not, then as $A \subseteq {}^*\mathbb{N}$ is internal, we have $A \in {}^*\mathcal{P}(\mathbb{N})$, whence we apply the well-ordering principle to A to find a minimum element. Now, if this minimum element were infinite, then the assumption that A has arbitrarily small infinite numbers gives us a contradiction. Hence A must have a finite element. This property of ${}^*\mathbb{N}$ is called the *underflow principle*.

Since we included $\mathbb{R}$ in S, our nonstandard extension contains the set ${}^*\mathbb{R}$ which is called the set of *hyperreals*. Notice that for each $n \in \mathbb{N}^+$ the interval $(\frac{-1}{n}, \frac{1}{n})$ is in $\mathbb{V}(S)$. Thus we have ${}^*(-\frac{1}{n}, \frac{1}{n}) \in \mathbb{V}(T)$. We say $\varepsilon \in {}^*\mathbb{R}$ is *infinitesimal* if $\varepsilon \in \bigcap_{n \in \mathbb{N}^+} {}^*(-\frac{1}{n}, \frac{1}{n})$, or equivalently, for all positive $r \in \mathbb{R}$ we have $|x| < r$ [2]. Similarly, $x \in {}^*\mathbb{R}$ is *finite* if $|x| < r$ for some positive $r \in \mathbb{R}$, and x is *infinite* otherwise.



Before exposing the reader to some external sets, we describe this one theorem in nonstandard analysis called the *internal definition principle*. It is the following. Suppose we are given a bounded formula $\varphi(x, y_1, \ldots, y_n)$. Further suppose we have internal sets $B, b_1, \ldots, b_n$. Then the set

$$A = \{b \in B : \mathbb{V}(T) \models {}^*\varphi(b, b_1, \ldots, b_n)\}$$

[2] We have written $|x|$ for ${}^*|x|$ here.

is an internal set. On the other hand, any internal set can be defined this way. We can use the internal definition principle to prove that the collection of internal sets is closed under finite unions, finite intersections, and relative complements. For instance, if A and B are internal sets, then $A \cap B = \{x \in A : x \in B\}$ is internal by the internal definition principle.

B.3. Saturation

In nonstandard analysis $\aleph_1$-*saturation* is a handy tool. What it basically says is that if a countable set $\Sigma(x)$ of bounded sentences of $\mathcal{L}_{\mathbb{V}(S)}$ is finitely satisfiable by elements in $\mathbb{V}(X)$, then it is satisfiable by an element in $\mathbb{V}(T)$. We have the following formulation of κ-saturation.

Definition 47. *Let κ be an infinite cardinal. Suppose $\mathcal{A}$ is any collection of internal subsets of ${}^*\mathbb{V}$ with cardinality of $\mathcal{A} < \kappa$. Then the nonstandard extension ${}^*\mathbb{V}$ is κ-saturated if and only if $\bigcap \mathcal{A} \neq \varnothing$ whenever $\mathcal{A}$ satisfies the finite intersection property.*

More often than not, countable saturation, that is $\aleph_1$-saturation, is quite sufficient for various applications. Countable saturation is equivalent to the so-called *countable comprehension property*; if B is an internal set in $\mathbb{V}(T)$ and $\langle A_n \rangle$ is a sequence of elements of B, then there exists an internal function $g : {}^*\mathbb{N} \to B$ such that $g(n) = A_n$ for each $n \in \mathbb{N}$. The idea is the following. The sets of internal functions $F_n = \{g \mid g : {}^*\mathbb{N} \to B \text{ is internal with } g(n) = A_n\}$ is internal for each $n \in \mathbb{N}$ by the internal definition principle. Thus, if we show that this set of functions satisfies the finite intersection property, then we get a function as desired. To show the finite intersection property, let $n_0, n_1, \ldots, n_k \in \mathbb{N}$ be arbitrary, where $k \in \mathbb{N}$. We define a function $g : {}^*\mathbb{N} \to B$ as follows.

$$h(n) = \begin{cases} A_{n_i} & \text{if } \ n = n_i \ \text{ for some } \ i \in \{0, 1, \ldots, k\}, \\ \\ A_{n_0} & \text{otherwise.} \end{cases}$$

Now it is easy to see that $h \in F_{n_0} \cap F_{n_1} \cap \cdots \cap F_{n_k}$. Therefore $\{F_n : n \in \mathbb{N}\}$ satisfies the finite intersection property. Hence, by countable saturation, there exists $g : {}^*\mathbb{N} \to B$ such that $g(n) = A_n$ for each $n \in \mathbb{N}$.

To see how the countable comprehension property implies countable saturation, let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a countable collection of internal sets satisfying the finite intersection property. Let $g : {}^*\mathbb{N} \to \mathcal{A}$ be an internal function with $g(n) = A_n$ for each $n \in \mathbb{N}$. Consider the formula $\forall k \in \mathbb{N}\big(g(0) \cap g(1) \cap \cdots \cap g(k) \neq \varnothing\big)$, which is true as $\mathcal{A}$ satisfies the finite intersection property. Thus, by transfer we get $\forall k \in {}^*\mathbb{N}\big(g(0) \cap g(1) \cap \cdots \cap g(k) \neq \varnothing\big)$. In particular, for any $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ we have $g(0) \cap g(1) \cap \cdots \cap g(H) \neq \varnothing$, whence $\bigcap \mathcal{A} \neq \varnothing$.

Notice that in the previous subsection, we defined ε to be an infinitesimal if $\varepsilon \in \bigcap_{n \in \mathbb{N}^+} {}^*(-\frac{1}{n}, \frac{1}{n})$. But we have not justified that this intersection is nonempty yet. Put $\mathcal{A} = \{A_n : n \in \mathbb{N}^+\}$, where $A_n = {}^*(-\frac{1}{n}, \frac{1}{n})$. Then, since $\mathcal{A}$ has the finite intersection property, by countable saturation, it immediately follows that $\bigcap_{n \in \mathbb{N}^+} {}^*(-\frac{1}{n}, \frac{1}{n})$ is nonempty.

Now we give some examples of external sets. Prior to that, we make the following claim, the proof of which involves saturation. Suppose we have κ-saturation ($\kappa > \aleph_0$). If $A \in \mathbb{V}(T)$ is a set such that $\aleph_0 \leq \mathrm{Card}(A) < \kappa$, then A is external, where $\mathrm{Card}(A)$ is the (external) cardinality of A. For, suppose A is internal and $\aleph_0 \leq \mathrm{Card}(A) < \kappa$. Put $A_a = A \setminus \{a\}$. Then $\mathcal{A} = \{A_a : a \in A\}$ is a family of internal sets. Since $\mathrm{Card}(A) < \kappa$ we have $\mathrm{Card}(\mathcal{A}) < \kappa$. Moreover, if $a_0, \ldots, a_n \in A$, then $A_{a_0} \cap \cdots \cap A_{a_n} = A \setminus \{a_0, \ldots, a_n\} \neq \varnothing$ as $\mathrm{Card}(A) > n$ for each $n \in \omega$. So $\mathcal{A}$ has the finite intersection property. So by κ-saturation $\bigcap_{a \in A} A_a \neq \varnothing$. But this is a contradiction because if $x \in \bigcap_{a \in A} A_a$, then $x \in A_a$ for all $a \in A$, or rather $x \in A$ and $x \neq a$ for any $a \in A$. This proves the claim. Hence $\mathbb{N}$ and $\mathbb{Q}$ are examples of external sets.

Some more examples of external sets are the following. The set of infinitesimals is external because it does not have a supremum even though it is a bounded subset of $\mathbb{R}$. Note that the existence of a supremum can be expressed using bounded sentences of $L_{\mathbb{V}(S)}$, whence it is an "internal property". The set of infinite hypernatural numbers is external as it is a subset of ${}^*\mathbb{N}$ with no least element. We end this subsection by stating the fundamental theorem of nonstandard analysis.

Theorem 19 (Fundamental Theorem)**.** *Let $\mathbb{V}(S)$ and $\mathbb{V}(T)$ be superstructures and let κ be an infinite cardinal. Then the triple $\langle \mathbb{V}(S), \mathbb{V}(T), * \rangle$ is a κ-saturated nonstandard universe if and only if each of the following holds.*

(1) S and T are infinite sets of urelements.

(2) $T = {}^*S$.

(3) $* : \mathbb{V}(S) \to \mathbb{V}(T)$ is an injection.

(4) For any bounded sentence φ of $\mathcal{L}_{\mathbb{V}(S)}$, $\mathbb{V}(S) \models \varphi$ if and only if $\mathbb{V}(T) \models {}^*\varphi$.

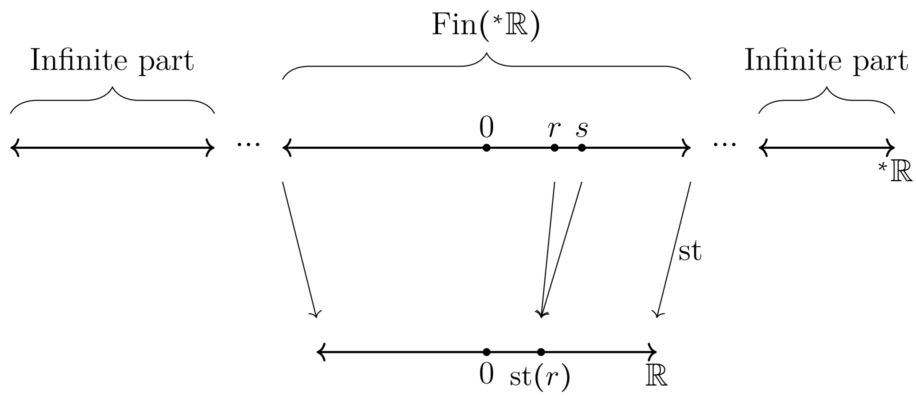
(5) ${}^*A \supsetneq \{{}^*a : a \in A\}$ whenever $A \in \mathbb{V}(S)$ is infinite.

(6) Suppose $\mathcal{A}$ is any collection of internal subsets of $\mathbb{V}(T)$ with cardinality of $\mathcal{A} < \kappa$. Then $\bigcap \mathcal{A} \neq \varnothing$ whenever $\mathcal{A}$ satisfies the finite intersection property.

B.4. The Standard Part Map

Theorem 20 (Standard Part Map Theorem). *If $x \in {}^*\mathbb{R}$ is finite, then there exists a unique $r \in \mathbb{R}$ so that $x - r$ is infinitesimal. The number r is called the standard part of x, and it is denoted by $^\circ x$ or $\mathrm{st}(x)$.*

The *finite part* of ${}^*\mathbb{R}$ is the set of all $x \in {}^*\mathbb{R}$ such that x is finite. Let us denote it by $\mathrm{Fin}({}^*\mathbb{R})$. The following picture summarizes the action of the standard part map on $\mathrm{Fin}({}^*\mathbb{R})$.



B.5. Proof of theorem 14

We prove theorem 14 as an illustration of the nonstandard techniques from the previous sections of this appendix. That is, we prove that the mapping $L(^\circ T)$ induced by T is a strongly subaddtive $\mathcal{F}$-capacity on F. We rephrase the theorem as in [Ros90a]. Let F be an internal set and let $\mathcal{F} \subseteq {}^*\mathcal{P}(F)$ be a regular paving on F. Suppose T is a precapacity on $\mathcal{F}$. Then $L(T) : \mathcal{P}(F) \to [0,1]$ given by

$$L(T)(E) = \inf_{\substack{E \subseteq D \\ D \in \mathcal{F}_\sigma}} \sup_{\substack{X \subseteq D \\ X \in \mathcal{F}}} T(X)$$

is an $\mathcal{F}$-capacity on F.

Proof. Note that by theorem 2, it is enough to verify the following two conditions.

(1) If $A_0 \subseteq A_1 \subseteq \cdots$ are elements of $\mathcal{F}$ with $\bigcup_n A_n \in \mathcal{F}$, then $T(\bigcup_n A_n) = \sup_n T(A_n)$.

(2) If $A_0 \supseteq A_1 \supseteq \cdots$ are elements of $\mathcal{F}$, then $L(T)(\bigcap_n A_n) = \inf_n T(A_n)$.

Let us prove that (1) holds. To that end, let $A_0 \subseteq A_1 \subseteq \cdots$ be elements of $\mathcal{F}$ with $\bigcup_n A_n \in \mathcal{F}$. By saturation, or rather countable comprehension, and applying the underflow principle to the set

$$\{H \in {}^*\mathbb{N} : \bigcup_n A_n \subseteq \bigcup_{n=1}^{H} A_n\},$$

we see that there exists some $m \in \mathbb{N}$ so that $\bigcup_n A_n \subseteq \bigcup_{n=1}^m A_n$. Hence $\bigcup_n A_n = \bigcup_{n=1}^m A_n$. Since $\bigcup_n A_n \in \mathcal{F}$, we have $T(\bigcup_{n=1}^m A_n) \leq T(A_k)$ for all $k \geq m$. Since $\langle A_n \rangle$ is monotonically increasing, we have $T(\bigcup_n A_n) = T(\bigcup_{n=1}^m A_n) \leq \sup_n T(A_n)$. By monotonicity of T we have $T(\bigcup_n A_n) \geq T(A_n)$ for each $n \in \mathbb{N}$. Therefore, $T(\bigcup_n A_n) \geq \sup_n T(A_n)$. Hence $T(\bigcup_n A_n) = \sup_n T(A_n)$ proving (1).

Finally, to prove (2), let $A_0 \supseteq A_1 \supseteq \cdots$ be elements of $\mathcal{F}$. Put $A = \bigcap_n A_n$. Since $L(T)$ is monotonically increasing, we at once have $L(T)(A) \leq \inf_n T(A_n)$. To prove the reverse inequality, assume that $A \subseteq D$ and D is $\mathcal{F}_\sigma$. Since $\mathcal{F}$ is a regular paving we may assume that $D_0 \subseteq D_1 \subseteq \cdots$, for we can take $D_0' = D_0$, $D_1' = D_0 \cup D_2, \ldots$ and each D_n' is in $\mathcal{F}$ as it is closed under finite unions. Again by saturation, and using underflow with the observation $A_M \subseteq \bigcap_n A_n \subseteq \bigcup_n D_n \subseteq D_M$ for arbitrarily small infinite $M \in {}^*\mathbb{N}$, we see that there exists $m \in \mathbb{N}$ such that $A_m \subseteq D_m$. Therefore,

$$T(A_m) \leq T(D_m) \leq \sup_{\substack{X \subseteq D \\ X \in \mathcal{F}}} T(X) = L(T)(D).$$

Hence $\inf_n T(A_n) \leq L(T)(A)$, and this completes the proof. $\dashv$

There is an alternative proof of the fact that if $A_0 \subseteq A_1 \subseteq \cdots$ are as above, then the sequence $\langle A_n \rangle$ must be eventually constant. We give that proof as another example of an application of saturation. Suppose $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \ldots$, where A_n is internal. First note that we can assume that each A_n is nonempty because the sequence $\langle A_n \rangle$ is strictly increasing and we can relabel starting with the index of the first nonempty set in the sequence. Put $A = \bigcup_{n \in \mathbb{N}} A_n$. Suppose A is internal. Then we have a strictly decreasing sequence $\langle A \smallsetminus A_n \rangle$ of non-empty internal sets. Fix $k \in \mathbb{N}$. Then $\bigcup_{j=k}^n A \smallsetminus A_j \supseteq A \smallsetminus A_n \neq \varnothing$ for each $n \geq k$. Therefore, the sequence $\langle A \smallsetminus A_n \rangle$ has the finite intersection property. Now by $\aleph_1$-saturation let $a \in \bigcap_{n=1}^\infty A \smallsetminus A_n$. Then $a \in A$, and $a \notin A_n$ for any $n \in \mathbb{N}$. But this is a contradiction as a must belong to some A_n as $a \in A = \bigcup_{n=1}^\infty A_n$. Hence $\bigcup_{n=1}^\infty A_n$ must be external.

Acknowledgements

I thank Professor David Ross for teaching me model theory and supervising this project. I would not have done this without his patience and vast knowledge across different fields of logic and mathematics. I am grateful to him for letting me use his LaTeX codes for the pictures in Appendix B. I thank Professor Bjørn Kjos-Hanssen and Dr. Kameryn J. Williams for teaching me logic at the University of Hawai'i at Mānoa, and all the professors who taught me mathematics. I thank Professor Kjos-Hanssen again for suggesting ways to improve this paper. Thank you Ikenna Nometa, Lance Ferrer and Shubham Joshi for being friends, and Sue for putting up with me. Lastly, but most importantly, I thank my parents Cyril and Princy, and brothers Rasika and Nisal, who complete my world.

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