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Optimal Controller Structure Reduction for Decentralized Control

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Abstract: This paper addresses the design of decentralized controllers for linear discrete-time systems. We consider state feedback control for networks of scalar interconnected systems. The structure of the decentralized controller is not specified in advance but included into the controller design. Decentralized controllers are designed based on local measurement with minimal number of additional measurement links between the subsystems and the controllers. We formulate this problem as one of maximizing the degree of decentralization subject to a given error performance in terms of the $\mathcal{H}_\infty$-norm between the system controlled by a centralized controller and the system controlled by the decentralized controller. For the resulting non-convex optimization problem, numerically tractable convex relaxations are provided and an example shows the effectiveness of this approach.

1. INTRODUCTION

Analysis and control of large groups of interconnected systems is one of the big challenges of modern engineering science. Most of conventional controller synthesis assumes that one centralized controller has access to all measurements. In a decentralized framework, one has multiple controllers each with access to different information. Figure 1(a) shows exemplarily a system of three interconnected subsystems (Σ_1, Σ_2, and Σ_3) with a centralized controller K that has access to and can influence all subsystems. In Figure 1(c), the system is controlled by three completely decentralized controllers k_{ii} which have only access to local measurements. In Figure 1(b), additional measurement links k_{ij} are allowed to improve the performance of the controller compared to the decentralized one. This is a partly decentralized control structure consisting of local measurements and minimal number of additional measurement links. In this article, we present a first step towards a new formulation of this partly decentralized control problem. We design $\mathcal{H}_\infty$ state feedback controllers for networks of scalar interconnected systems, where the number of measurement links is minimized.

In a standard control framework, the decentralization of the system manifests itself as pattern or sparsity structure on the centralized controller to be designed. For scalar interconnected systems, decentralization is equivalent to zeros in the controller matrix K. Our goal is to design a decentralized controller with as much decentralization as possible such that the performance degradation compared to the centralized controller is small. We do not specify the controller topology in advance but instead include it into the optimization problem. Therefore, we assume that all states are measurable in principle, but want as less measurement links as possible in the actual controller.

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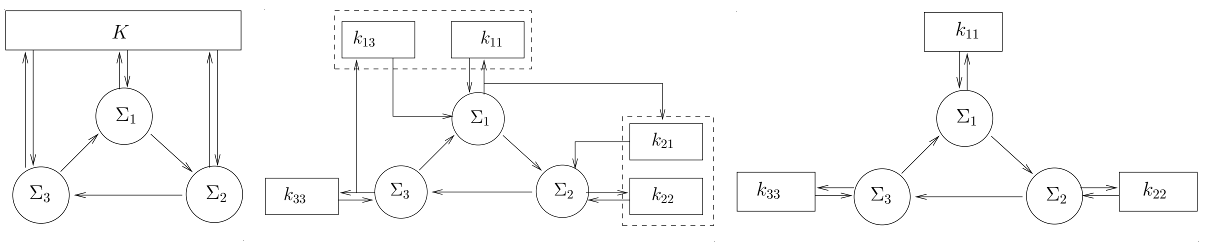
In the past, subclasses of decentralized controllers have been identified for which convex solutions exist, see Rotkowitz and Lall (2006a,b) or Shah and Parrilo (2008). Similar approaches can be found in Qi et al. (2004) or Scherer (2002) for example. When the controller to be designed does not belong to one of these subclasses, no convex solution exists, and the non-convex problem has to be solved by the iterative optimization of convex problems (see e.g. Wang et al., 2009). The existing approaches have in common that they are restricted to very special structures of network and controller topology and furthermore that the controller structure has to be specified in advance. However, it is not always easy to choose a suitable structure for the controller topology beforehand, especially in networked control of interconnected systems. This is the main contribution of this paper. In our approach, the degree of decentralization and the topology of the controller are not specified in advance but used as an optimization criterion. To achieve decentralization, we combine control theoretic insight with results from weighted ℓ_1-minimization (Candes et al., 2006; Donoho, 2006).

The article is organized as follows: After introducing some mathematical preliminaries in Section 2, the new formulation of the decentralized control problem is proposed in Section 3. We formulate the error system which describes the performance error between the system controlled by the optimal centralized controller and the system controlled by the decentralized controller. Furthermore, the optimization criterion for the controller topology is introduced. Since this formulation leads to a non-convex optimization problem, numerically tractable relaxations are provided in Section 4. A convex ℓ_1-minimization is used to find the controller topology and a linearization algorithm is proposed which shows local convergence. The paper concludes with an example which shows the effectiveness of the algorithm in Section 5 and a summary in Section 6.



(a) centr. control (b) partly decentralized control (c) completely decentralized control

Fig. 1. Control schemes with different degree of decentralization.

2. MATHEMATICAL PRELIMINARIES

The notation I and 0 denote the identity matrix and zero matrix, respectively, with size determined from context. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $M > 0$ (resp. $M \geq 0$) denotes M is symmetric and positive-definite (resp. positive-semidefinite), M^T and M^{-1} denote the transpose and inverse of a matrix, and $\mathrm{tr}(M)$ refers to the trace of M. A matrix with entries $a_1, \dots a_n$ on the diagonal is abbreviated with $\mathrm{diag}(a_1, \dots a_n)$. In symmetric block matrices, we are using the symbol $(\star)$ to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

Given $A \in \mathbb{C}^{m \times n}$, we may write A in terms of its columns as

$$A = [a_1 \dots a_n]$$

and then associate a vector $\mathrm{vec}(A) \in \mathbb{C}^{nm \times 1}$ defined by

$$\mathrm{vec}(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$ $A.B$ denotes the element-wise multiplication of A and B. A state-space realization of a transfer matrix $H(z)$ is written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(zI - A)^{-1}B + D = H(z).$$

The space ℓ_2^n is the Hilbert space of right-sided square summable real vector sequences of dimension n, with the ℓ_2-norm

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} \sum_{k=0}^{\infty} (x_i(k))^2}.$$

The ℓ_2-induced norm (or ℓ_2-gain) of a map $H : \ell_2^n \to \ell_2^m$ is

$$\|H\|_{\ell_2-\mathrm{ind}} := \sup_{w \in \ell_2^n \setminus \{0\}} \frac{\|Hw\|_2}{\|w\|_2},$$

and corresponds to the $\mathcal{H}_\infty$-norm of the transfer function $H(z)$. The ℓ_0-norm of a vector is defined as

$$\|x\|_{\ell_0} := |\{i : x_i \neq 0\}|$$

and counts the non-zero entries in x. A vector is called sparse if its ℓ_0-norm is small, i.e. if its elements are mainly zero.

3. FORMULATION OF DECENTRALIZED CONTROL PROBLEM

In this section we give a new formulation for decentralized controller design. Starting with classical centralized control, the problem of finding a decentralized controller is formulated by minimizing the number of measurement links between subsystems and controller subject to an upper bound on the $\mathcal{H}_\infty$ performance error achieved by the decentralized controller compared to the centralized one. This is motivated by the fact that the centralized controller achieves optimal performance and the decentralized controller should achieve almost the same performance.

3.1 Interconnected Systems

We consider a network consisting of scalar interconnected systems

$$x_i(k+1) = a_{ii}x_i(k) + \sum_{j=1}^{n} a_{ij}x_j(k) + b_{1i}w_i(k) + b_{2i}u_i(k)$$
$$z_i(k) = c_i x_i(k) + d_{1i}w_i(k) + d_{2i}u_i(k) \quad (1)$$

where $x_i(k) \in \mathbb{R}$ is the state of subsystem i and $a_{ij} \neq 0$ if and only if subsystem j influences subsystem i. $z_i(k) \in \mathbb{R}$ is the controlled output, $w_i(k)$ is the exogenous input, and $u_i(k)$ is the control input to each subsystem.

The model of the network consisting of n scalar subsystems (1) is then given by

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \quad (2a)$$
$$z(k) = Cx(k) + D_1 w(k) + D_2 u(k), \quad (2b)$$

with $A = [a_{ij}]$ and

$$B_1 = \mathrm{diag}(b_{1i}), \quad B_2 = \mathrm{diag}(b_{2i}), \quad C = \mathrm{diag}(c_i)$$
$$D_1 = \mathrm{diag}(d_{1i}), \quad D_2 = \mathrm{diag}(d_{2i}), \quad i = 1 \dots n,$$
$$w(k) = [w_1(k), \dots, w_n(k)]^T, \quad z(k) = [z_1(k), \dots z_n(k)]^T$$
$$u(k) = [u_1(k), \dots, u_n(k)]^T, \quad x(k) = [x_1(k), \dots, x_n(k)]^T.$$

We search for a controller $K \in \mathbb{R}^{n \times n}$ that minimizes the influence of the exogenous input $w(k)$ to the performance output $z(k)$ in terms of the $\mathcal{H}_\infty$-norm of the closed loop.

3.2 Centralized Control

In a centralized framework, where all controllers have access to all states, each subsystem is driven by a controller of the following type

$$u_i = \sum_{j=1}^{n} k_{ij} x_j.$$

All single controllers can be combined to the full state feedback controller

$$u(k) = Kx(k), \quad K \in \mathbb{R}^{n\times n}. \tag{3}$$

The closed loop is then given by

$$\Sigma: \quad \begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A+B_2K & B_1 \\ C+D_2K & D_1 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \tag{4}$$

This centralized controller uses all possible degrees of freedom and can be designed via convex optimization.

Assumption 1. The centralized controller K achieves

(i) closed loop stability
(ii) optimality in terms of the $\mathcal{H}_\infty$-norm of the closed loop.

In the following, Assumption 1 is assumed to hold. To design the decentralized controller, we use the centralized controller as reference and try to find a decentralized controller with the same performance or as little performance degradation as possible. Note, that only the centralized problem can be solved as a convex optimization problem (Gahinet and Apkarian, 1994) with standard LMI solvers (see e.g. Sturm, 1999; Löfberg, 2004).

3.3 Decentralized Control

Next, we describe a decentralized controller such that less measurement links between sensors and controllers are necessary. We especially consider that each controller does not have access to all subsystems but only knows the states of a few subsystems, i.e. we want to remove measurement links between subsystems and controllers. Therefore we search for a controller

$$u_i = \sum_{j=1}^{n} \hat{k}_{ij} x_j,$$

where $\hat{k}_{ij} = 0$ for as many (ij), $i \neq j$ as possible (see 1(b)). This controller is then given as

$$u(k) = \hat{K}x(k), \tag{5}$$

with decentralized (sparse) structure. As stated before, the degree of decentralization (DOD) is equivalent to the sparsity of $\hat{K}$, i.e. DOD $:= n^2 - \|\mathrm{vec}(\hat{K})\|_{\ell_0}$.

The closed loop with this decentralized controller is then written as

$$\hat{\Sigma}: \quad \begin{bmatrix} \hat{x}(k+1) \\ \hat{z}(k) \end{bmatrix} = \begin{bmatrix} A+B_2\hat{K} & B_1 \\ C+D_2\hat{K} & D_1 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ w(k) \end{bmatrix}. \tag{6}$$

In the considered state feedback case of scalar interconnected subsystems, the degree of decentralization is equivalent to the number of zeros in the controller matrix $\hat{K}$. In a fully decentralized control strategy, $\hat{K}$ would be a diagonal matrix, i.e. only local measurements. To achieve a better performance, we also allow additional exchange of information between the subsystems. Therefore, we search for a controller $\hat{K}$ with sparse structure that achieves almost the same performance as the centralized controller.

3.4 Error System

Performance degradation between the centralized and the decentralized controller can now be investigated by the analysis of the error system $E(z) = \Sigma(z) - \hat{\Sigma}(z)$

$$E: \quad \begin{bmatrix} \zeta(k+1) \\ e(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \zeta(k) \\ w(k) \end{bmatrix}, \tag{7}$$

where

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A+B_2K & 0 & B_1 \\ 0 & A+B_2\hat{K} & B_1 \\ \hline C+D_2K & -(C+D_2\hat{K}) & 0 \end{array}\right],$$

with $\zeta = [x \; \hat{x}]^T$ and $e(k) = z(k) - \hat{z}(k)$. In the centralized case, an optimal controller can be designed. We search for a decentralized controller which is close to the centralized one in terms of performance. To achieve this, we specify an performance error and minimize the ℓ_0-norm of the decentralized controller $\hat{K}$. Since the state variables of the error system E in (7) are composed of x and $\hat{x}$, and the system Σ in (4) is asymptotically stable by Assumption 1, the asymptotic stability of $\hat{\Sigma}$ is then equivalent to that of E. Thus, the requirement of the closed-loop system with the sparse controller to be stable is achieved by guaranteeing the stability of the error system.

3.5 Decentralized Control Problem

The considered decentralized controller design problem can be formulated as follows.

Problem 2. Given the linear system (2) and the centralized controller (3), determine a decentralized controller (5), such that

(i) the closed-loop system $\hat{\Sigma}$ given in (6) is asymptotically stable and
(ii) the ℓ_0-norm of the controller $\hat{K}$ is minimized subject to a given $\mathcal{H}_\infty$ performance degradation

$$\min_{\hat{K}\text{stabilizing}} \left\| \mathrm{vec}(\hat{K}) \right\|_{\ell_0} \tag{8a}$$

$$\text{subject to } \left\| \Sigma - \hat{\Sigma} \right\|_{\ell_2-\mathrm{ind}} < \gamma \tag{8b}$$

and $\gamma > 0$ small.

This implies that the closed loop, controlled by the decentralized controller, maintains the performance of the closed loop with the original centralized controller. The minimization over the ℓ_0-norm of $\hat{K}$ implies that the degree of decentralization is as large as possible. Problem 2 is non-convex and difficult to solve. Therefore, we show in the next section how this problem can be transformed into a numerically tractable optimization problem.

4. CONVEX RELAXATION FOR NUMERICAL SOLUTION

In this section, a solution to the decentralized controller design problem is presented. We introduce a weighted ℓ_1-minimization of the state space controller to relax the exhaustive enumeration of the ℓ_0-norm. The non-convex condition induced by the $\mathcal{H}_\infty$ problem is then solved using the Cone Complementarity Linearization algorithm. The following characterization of the $\mathcal{H}_\infty$ performance obtained from the Bounded Real Lemma is used.

Lemma 3. (Gahinet and Apkarian (1994)). Consider a discrete-time transfer function $H(z)$ with (not necessarily minimal) realization $H(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$. Then the following statements are equivalent:

(i) $\mathcal{A}$ is stable and $\|H\|_{\ell_2-\mathrm{ind}} < \gamma$.

(ii) There exists $P > 0$ such that

$$\begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B} & \mathcal{C}^T \\ \star & \mathcal{B}^T P \mathcal{B} - \gamma^2 I & \mathcal{D}^T \\ \star & \star & -I \end{bmatrix} < 0. \tag{9}$$

The inequalities in Lemma 3 will be transformed to compute a decentralized controller $\hat{K}$ that is the solution to the minimization problem defined in (8). The solution to the decentralized controller design (Problem 2) with predefined the $\mathcal{H}_\infty$-error performance is then formulated in the following theorem.

Theorem 4. Given a network consisting of scalar interconnected systems (2) and a centralized controller (3), an admissible decentralized controller $\hat{K}$ (5) exists such that the closed loop system $\hat{\Sigma}$ (6) is asymptotically stable and the $\mathcal{H}_\infty$-norm error performance in (8b) is guaranteed for given $\gamma > 0$, if there exist appropriately dimensioned matrices $L = L^T$, $P = P^T$, $\hat{K}$ such that

$$\min \quad \|\mathrm{vec}(\hat{K})\|_{\ell_0} \tag{10a}$$

$$\text{subject to} \quad \begin{bmatrix} -L & A_0 + F\hat{K}G & \mathcal{B} & 0 \\ \star & -P & 0 & (C_0 - J\hat{K})^T \\ \star & \star & -\gamma^2 I & \mathcal{D}^T \\ \star & \star & \star & -I \end{bmatrix} < 0 \tag{10b}$$

$$LP = I. \tag{10c}$$

where

$$A_0 = \begin{bmatrix} A + B_2 K & 0 \\ 0 & A \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad G = [0 \ I]$$

$$C_0 = [C + D_2 K \ -C], \quad J = [0 \ -D_2]$$

is feasible.

Proof. Inserting the equations of the error system (7) into Lemma 3 leads to a feasibility condition for the decentralized controller in the form of (5), such that the closed-loop system $\hat{\Sigma}$ is asymptotically stable and the $\mathcal{H}_\infty$ error norm is less than γ. By Schur complement, (9) is equivalent to

$$\begin{bmatrix} -P^{-1} & \mathcal{A} & \mathcal{B} & 0 \\ \star & -P & 0 & \mathcal{C}^T \\ \star & \star & -\gamma^2 I & \mathcal{D}^T \\ \star & \star & \star & -I \end{bmatrix} < 0. \tag{11}$$

In the following, we introduce the decompositions

$$\mathcal{A} = A_0 + F\hat{K}G \text{ and } \mathcal{C} = C_0 + J\hat{K} \tag{12}$$

with A_0, F, G, C_0 and J as defined in Theorem 4. The matrix inequality for the new controller $\hat{K}$ with sparse structure is then given by

$$\begin{bmatrix} -P^{-1} & A_0 + F\hat{K}G & \mathcal{B} & 0 \\ \star & -P & 0 & (C_0 - J\hat{K})^T \\ \star & \star & -\gamma^2 I & \mathcal{D}^T \\ \star & \star & \star & -I \end{bmatrix} < 0. \tag{13}$$

The last step will be to define a new decision variable $L = P^{-1}$ that gives the equality (10c). □

It is noted that the obtained conditions in (10) are not convex due to the minimization over the ℓ_0-norm (10a) and the nonlinear equality constraint in (10c). In the following sections we will provide convex relaxations both for the ℓ_0 minimization and also for the equality constraint.

4.1 Relaxation of the ℓ_0-minimization

As stated before, the degree of decentralization of the controller is equivalent to sparsity requirements on the controller matrix $\hat{K}$. This is structurally equivalent to finding a sparse vector $\xi \in \mathbb{R}^n$, by solving the optimization problem

$$\min_{\xi \in \mathbb{R}^n} \|\xi\|_{\ell_0} \quad \text{subject to} \quad \psi = \Phi\xi, \tag{14}$$

for given $\psi \in \mathbb{R}^n$ and $\Phi \in \mathbb{R}^{m \times n}$, with $m < n$. This optimization problem (14) is non-convex and (NP)-hard as its solution requires a combinatorial search which becomes intractable very fast for growing system size. As a relaxation of the combinatorial search, we instead consider an alternative convex problem

$$\min_{\xi \in \mathbb{R}^n} \|\xi\|_{\ell_1} \quad \text{subject to } \psi = \Phi\xi, \tag{15}$$

where $\|\xi\|_{\ell_1} = \sum_{i=1}^n |\xi_i|$ (see e.g Candes et al., 2006; Donoho, 2006). Unlike (14), this is convex and can be solved using linear programming. It is well known in the literature that ℓ_1-minimization leads to very good approximations of the non-tractable combinatorial search and gives sparse solutions (see e.g. Candes et al., 2008), since the ℓ_1-norm is the convex envelope of the ℓ_0-norm (see Fazel, 2002).

For the design of the decentralized controller, the ℓ_1-norm of the vectorized controller $\mathrm{vec}(\hat{K})$ will replace the ℓ_0-minimization. The ℓ_1-minimization removes sensor-controller links which contribute only little to the performance of the closed loop system. Here, the novelty of our approach can be seen: it is not necessary to specify the controller topology in advance, since it is now included into the optimization problem.

As described in Candes et al. (2008) "weighted" ℓ_1-minimization can be used if knowledge about the system is available to improve the result of the minimization. The weighted ℓ_1-minimization problem is defined as

$$\min_{\xi \in \mathbb{R}^n} \sum_{i=1}^{n} m_i |\xi_i| \quad \text{subject to } \psi = \Phi\xi, \tag{16}$$

where $m_1, m_2, \ldots, m_n$ are non-negative weights. One may think of the weights m_i as free parameters, whose values improve the solution of the convex optimization problem. They counteract the influence of the signal magnitude on the ℓ_1 penalty function. We can use these weights to include system and control theoretic insight into the optimization problem to improve the result. When dealing with multivariable systems which have to be controlled by SISO controllers the relative gain array (RGA) (see Bristol, 1966) gives very good intuition about which input influences which output and how the pairings should be chosen. To use the information from the relative gain array, we may choose m_i as the element wise inverse of the RGA. Furthermore if other knowledge about the system is available, e.g. some measurement links are very unattractive since they are related to high implementation costs or just impossible to implement, those measurement links can also be penalized by a large weight in the weighted ℓ_1-minimization. Furthermore, weights that correspond to the local measurement of each subsystem can be set to zero since they do not increase the communication cost. This allows also to include all possible a-priori knowledge about the subsystem interconnection and implementation constraints to improve the optimization results. The new optimization problem can now be rewritten as

$$\begin{aligned} \inf \quad & \|\mathrm{vec}(M.\hat{K})\|_{\ell_1} \\ \text{subject to} \quad & \text{(10b) and (10c)}, \end{aligned} \tag{17}$$

where $M = [m_{ij}]$ with m_{ij} the weight of k_{ij} and $m_{ii} = 0$.

With this ℓ_1-relaxation, the non-convex ℓ_0-minimization has been convexified. In the next section, we show how to deal with the non-convexity caused by the equality constraint $LP = I$.

4.2 Cone Complementarity Linearization Algorithm

The Cone Complementarity Linearization (CCL) algorithm was initially proposed by El Ghaoui as a method to obtain reduced order controllers (El Ghaoui et al., 1997), extensive numerical investigations were performed by the authors using the CCL method in the solution of reduced-order output feedback (ROF) and static output-feedback (SOF) problems. This numerical study shows how effective the method is not only in finding a solution, but also in a remarkable convergence speed. Since our decentralized controller design problem shares some similarities with the robust output feedback control problem in El Ghaoui et al. (1997) and in view of the effectiveness of the CCL to handle non-convex constraints as (10c), we adapt the method to solve the conditions stated in (17). The CCL is one possible option to handle the non-convexity of the optimization problem. Other possibilities include simple iteration over the Lyapunov matrix P and the decentralized controller $\hat{K}$ or more elaborate solutions as sequential linear programming as e.g. stated in Leibfritz (2001). We opted for the CCL because of its fast convergence and simplicity compared to the other options.

According to Gao et al. (2006), the basic idea in the CCL algorithm is that if the LMI

$$\begin{bmatrix} L & I \\ I & P \end{bmatrix} \geq 0 \tag{18}$$

is feasible in the $n \times n$ matrix variables $L > 0$ and $P > 0$, then $\mathrm{tr}(LP) = n$ if and only if $LP = I$ and $\mathrm{tr}(LP) > n$ otherwise. Therefore, it is possible to solve the matrix equality in (10c) by associating the inequality (18) to a trace minimization problem. Then we have an equivalent problem formulation

$$\begin{aligned} \inf \quad & \mathrm{tr}(LP) + \|\mathrm{vec}(M.\hat{K})\|_{\ell_1} \\ \text{subject to} \quad & \text{(10b) and (18)}. \end{aligned}$$

The objective function is still non-convex. In order to linearize it in the neighborhood of a local point, first a feasible point is found (L_0, P_0) and then a linear approximation of $\mathrm{tr}(LP)$ is defined as

$$\Phi_{lin}(L, P) = \mathrm{tr}(P_0 L + L_0 P).$$

This approach was proposed in Mangasarian and Pang (1995). Since LMI optimization does not allow multi-objective minimization to additionally deal with the trace minimization, a standard technique is used and a positively weighted sum of the objective functions is minimized. So we introduce a scalar weight α

$$\Phi_{lin}(L, P) = \mathrm{tr}(P_0 L + L_0 P) + \alpha \|\mathrm{vec}(M.\hat{K})\|_{\ell_1}.$$

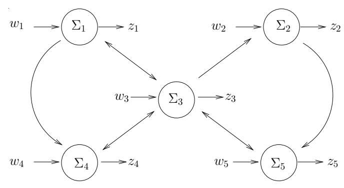


Fig. 2. Network of five interconnected subsystems.

Obviously, the choice of α has a major influence on how fast and effective the numerical algorithm is, since we introduce a trade-off between the trace and the ℓ_1-minimization. With this linearization, the problem of finding a decentralized controller can be expressed in the following algorithm.

Algorithm 1. Decentralized Controller Design

(1) Find a feasible set (L_0, P_0) satisfying (10b) and (18). If none, exit. Set $h = 0$.
(2) Find (L_{h+1}, P_{h+1}) that solve the LMI problem
$$\begin{aligned} \inf \quad & \mathrm{tr}(P_h L + L_h P) + \alpha \|\mathrm{vec}(M.\hat{K})\|_{\ell_1} \\ \text{subject to} \quad & \text{(10b) and (18)} \end{aligned}$$
for a predefined $\gamma > 0$.
(3) Substitute the obtained feasible set (L, $\hat{K}$) or (P^{-1}, $\hat{K}$) into (13). If the condition is satisfied, then $\hat{K}$ is a feasible solution. Otherwise, set $h = h + 1$, $(L_h, P_h) = (L, P)$ and go to Step 2.

This algorithm gives as a result a decentralized controller $\hat{K}$ with error performance less than γ and sparse structure. Since the convergence is only guaranteed locally by the CCL a good starting point for the iteration is necessary. This is given since the decentralized controller is always compared to the centralized controller which guarantees the best possible performance. In the following, the algorithm is applied to a simple network of interconnected subsystems to show its effectiveness.

5. EXAMPLE

We consider a system consisting of five interconnected subsystems as shown in Fig. 2. The model of the network is given as

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1.5 & 0 & 0.9 & 0 & 0 \\ 0 & 0.5 & 0.1 & 0 & 0 \\ 0.7 & 0 & 0.5 & 0.1 & 0.4 \\ 0.2 & 0 & 0.1 & 0.8 & 0 \\ 0 & 0.1 & 0.1 & 0 & 0.3 \end{bmatrix} x(k) + Iw(k) + Iu(k) \\ z(k) &= x(k). \end{aligned}$$

As can be seen easily, the first subsystem is unstable and some of the couplings between the subsystems are strong compared to their own dynamic, e.g. between the first and the third subsystem. First a centralized $\mathcal{H}_\infty$-controller is designed for this system. With the controller

$$K = \begin{bmatrix} -1.4573 & 0.0189 & -0.7600 & -0.0851 & -0.1389 \\ -0.0137 & -0.5035 & -0.0883 & -0.0092 & -0.0817 \\ -0.6470 & -0.0593 & -0.5266 & -0.0899 & -0.2333 \\ -0.1915 & -0.0130 & -0.1239 & -0.8000 & 0.0139 \\ -0.0485 & -0.0866 & -0.1734 & 0.0436 & -0.3013 \end{bmatrix}$$

a performance of $\|\Sigma\|_{\ell_2-\mathrm{ind}} < 1.17$ can be achieved. Since the controller can be computed using convex optimization we know that this is the best possible performance. Using results from the RGA analysis, we use the following weighting function for the design of the decentralized controller

$$M = \begin{bmatrix} 0 & 1000 & 1.44 & 12.12 & 1000 \\ 1000 & 0 & 1000 & 1000 & 158.73 \\ 1.26 & 158.73 & 0 & 36.5 & 31.95 \\ 1000 & 1000 & 12.03 & 0 & 1000 \\ 1000 & 1000 & 3.25 & 1000 & 0 \end{bmatrix}.$$

For $\gamma = 0.18$, which implies

$$\|\Sigma - \hat{\Sigma}\|_{\ell_2-\text{ind}} < \gamma$$

of the error system, the following decentralized controller is found

$$\hat{K} = \begin{bmatrix} -1.4472 & 0 & -0.7835 & -0.0588 & 0 \\ 0 & -0.4862 & 0 & 0 & -0.0359 \\ -0.6651 & 0 & -0.5215 & -0.0368 & -0.2390 \\ -0.0501 & 0 & -0.1531 & -0.7770 & 0 \\ 0 & 0 & -0.1714 & 0 & -0.2661 \end{bmatrix}.$$

As can be seen, a significant amount of measurement links were eliminated by the algorithm and much less communication between sensors and controllers is necessary. The ℓ_0-norm has been decreased from $\|K\|_{\ell_0} = 25$ to $\|\hat{K}\|_{\ell_0} = 14$ for the decentralized controller. The system controlled by the decentralized controller has a $\mathcal{H}_\infty$-norm of $\|\hat{\Sigma}\|_{\ell_2-\text{ind}} < 1.2$ which implies a performance degradation of less than 3%. As can be seen, the controller topology is neither equivalent to the subsystem interconnection nor to the very small elements of M. The weighting function M is an effective tool to include a-priori knowledge either from the RGA or from implementation constraints. For comparison a completely decentralized controller is designed using CCL and predefined structure on $\hat{K}_{\text{diag}}$. The diagonal controller is then given by

$$\hat{K}_{\text{diag}} = \text{diag}(-1.5117 - 1.0521 - 0.5082 - 1.4840 - 0.3404)$$

and achieves a performance of $\|\hat{\Sigma}_{\text{diag}}\|_{\ell_2-\text{ind}} < 5.74$. This implies an performance of $\gamma_{\text{diag}} = 4.7$ of the error system. Allowing exchange of information in addition to local measurement leads to significantly better performance.

6. CONCLUSIONS

This paper presents a new formulation to design decentralized controllers for a network of interconnected scalar systems. The performance error between the decentralized closed loop and the centralized closed loop is pre-specified in terms of the $\mathcal{H}_\infty$-norm and the ℓ_0-norm of the controller is minimized to achieve a sparse controller topology. In contrast to existing design methods for decentralized control, the structure of the decentralized controller and the degree of decentralization is not specified a-priori but included into the optimization algorithm. These conditions generally lead to a non-convex feasibility problem. It has been demonstrated in an example that the weighted ℓ_1-norm as a relaxation for the ℓ_0-norm and the Cone Complementarity Linearization algorithm are effective tools to handle this non-convex problem. Ongoing research deals with interconnected systems of higher order controlled via state and output feedback as a generalization. The presented approach can be extended to these block structured problems (see Schuler et al., 2010).

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