(5) operations & compositions on crisp relations

operations on crisp relations

Define R & S are two seperate relations on the cartessian universe $x \times y$

union

for $R \cup S$ union operation

$$\underbrace{\chi_{R \cup S}(x,y)}_{\text{crisp mem value of each ordered pair in } R \cup S} = \max\left[\chi_R(x,y), \chi_S(x,y)\right]$$

Ex

$R(x,y)$ & $S(x,y)$ are two relations defined over crisp sets X & Y such that $x \in X$ & $y \in Y$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$R \cup S = \begin{array}{c} \begin{matrix} (1,1) & (0,1) \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \begin{matrix} (0,0) & (1,0) \end{matrix} \end{array}$$

max value of each elem

pos

intersection

for $R \cap S$ intersection of relations

$$\chi_{R \cap S}(x, y) = \min\left[\chi_R(x, y),\ \chi_S(x, y)\right]$$

Ex

two relations $R(x, y)$ & $S(x, y)$ two relations defined over crisp sets X & Y such that $x \in X$ & $y \in Y$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$R \cap S = \begin{array}{c} \begin{array}{cc} (1,1) & (1,0) \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{array}{cc} (0,0) & (1,0) \end{array} \end{array}$$

minimum of at each elem pos

complement

for $\bar{R}$ complement of a relation

$$\bar{R} = \chi_{\bar{R}}(x,y) = 1 - \chi_R(x,y)$$

Ex

two relations $R(x,y)$ & $S(x,y)$ two relations defined over crisp sets X & Y such that $x \in X$ & $y \in Y$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\bar{R} = 1 - R = 1 - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Contamination / inclusion

for R inclusion of relation

$$R \subset S = \chi_R(x,y) \leq \chi_S(x,y)$$

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$\chi_M(x1, y1) < \chi_N(x1, y1)$ ✓

$\chi_M(x1, y2) = \chi_N(x1, y2)$ ✓

$\chi_M(x_2, y_1) = \chi_N(x_2, y_1)$ ✓

$\chi_M(x_2, y_2) = \chi_N(x_2, y_2)$ ✓

all satisfy $\chi_M(x,y) \leq \underline{\chi_S(x,y)}$

Prop of crisp relations

Commutative

Associative

Distributive

Idempotency

De Morgan's

law of excluded middle

Composition

Relation R map elem from univers X to Y

$R : X \to Y$

Relation S map elem from universe Y to Z

$S : Y \to Z$

then R composition S

$R \circ S : X \to Z$

Composition $\to$ Max - Min

$\to$ Max - product

① Max-min composition

$R \circ S \rightarrow$

$$\chi_{R \circ S}(x, z) = \max \{ \min [\chi_R(x, y), \chi_S(y, z)] \}$$

$\forall\, y \in Y$ ← universe

$\downarrow$

$$\chi_{R \circ S}(x, z) = \bigvee_{y \in Y}^{\max} \left[\chi_R(x, y) \wedge \chi_S(y, z) \right]$$

and

$\downarrow$

minimum

② Max-product composition

$R \circ S \rightarrow$

$$\chi_{R \circ S}(x, z) = \max \left[\chi_R(x, y) \cdot \chi_S(y, z) \right]$$

$\downarrow$ $\forall\, y \in Y$

$$= \bigvee_{\substack{y \in Y \\ \max}} \left[\chi_R(x, y) \cdot \chi_S(y, z) \right]$$

product

EX given universes

$$X = \{1, 2\} \qquad Y = \{2, 3\} \qquad Z = \{3, 4\}$$

relations btwn universes

$$R = \{(x, y) \mid y = x + 1,\ x \in X \ \& \ y \in Y\}$$

$$S = \{(y, z) \mid y < z,\ y \in Y \ \& \ z \in Z\}$$

find R composition S $\quad R \circ S$

relation R

$$X \times Y = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$

$$y = x + 1$$

$2 = 1 + 1$ ✓ $\qquad 2 = 2 + 1$ ✗

$3 = 1 + 1$ ✗ $\qquad 3 = 2 + 1$ ✓

$$R = \{(1, 2), (2, 3)\}$$

$$R = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

relation S

$$Y \times Z = \{(2,3),(2,4),(3,3),(3,4)\}$$

$$y < z$$

$2 < 3$ ✓ $\quad\quad$ $3 < 3$ ✗

$2 < 4$ ✓ $\quad\quad$ $3 < 4$ ✓

$$S' = \{(2,3),(2,4),(3,4)\}$$

$$S' = \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

(1,3)

$$\chi_{R \cdot S}(1,3) = \max\{\min(1,1), \min(0,0)\}$$

$$= \max\{1,0\} = \underline{\underline{1}}$$

(1,4)

$$\chi_{R \cdot S}(1,4) = \max\{\min(1,1), \min(0,1)\}$$

$$= \max\{1,0\}$$

$$= \underline{\underline{1}}$$