Tutorial 1

E/20/420

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Continuous Time Systems

1 (i) $\ddot{y} + 8\dot{y} + 12y = 12x$

a) Assuming $y(0)=0$ & $y'(0)=0$;
using laplace transform

$$\left[s^2Y(s) - s\underbrace{y'(0)}_{0} - \underbrace{y''(0)}_{0}\right] + 8\left[sY(s) - \underbrace{y'(0)}_{0}\right] + 12Y(s) = 12X(s)$$

$$s^2Y(s) + 8sY(s) + 12Y(s) = 12X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{12}{s^2+8s+12}$$

$$H(s) = \frac{12}{(s+6)(s+2)}$$

(b) No zeroes;
Poles at $s=-6$ & $s=-2$

(c) $y(t) = h(t) * u(t)$

$\therefore Y(s) = H(s) \cdot U(s)$

$$= \frac{12}{(s+6)(s+2)} \cdot \left(\frac{1}{s}\right) = \frac{12}{s(s+2)(s+6)}$$

Considering a general case

$$G(s) = \frac{1}{s(s-a)(s-b)} \qquad a, b \in \mathbb{C}$$

$$F(s) = \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b}$$

$$A = \frac{1}{ab} \qquad B = \frac{1}{a(a-b)} \qquad C = \frac{1}{b(b-a)}$$

$$\therefore F(s) = \frac{1/ab}{s} + \frac{1/a(a-b)}{s-a} + \frac{1/b(b-a)}{s-b}$$

(ILT ;

$$f(t) = A.u(t) + B\, e^{at} + C\, e^{bt}$$

but $Y(s) = 12\,F(s)$ where $a = -2$, $b = -6$

$\therefore$ from linearity

$$y(t) = 12 f(t)$$

$$= 12\left\{ A + Be^{at} + Ce^{bt} \right\} u(t)$$

where $A = \frac{1}{12}$ $B = \frac{1}{24}$ & $C = \frac{1}{8}$

(ii) $\ddot{y} + 8\dot{y} + 116y = 116x$

$\downarrow$ LT; where $y(0) = 0$ & $\dot{y}(0) = 0$

$s^2 Y(s) + 8sY(s) + 116 Y(s) = 116 X(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{116}{s^2 + 8s + 116}$$

$$= \frac{116}{(s+4)^2 + 116 - 16}$$

$$= \frac{116}{(s+4)^2 + 100}$$

$$= \frac{116}{(s+4-10j)(s+4+10j)}$$

$\therefore$ No zeroes

Poles; at $s = -4 + 10j$ & $s = -4 - 10j$

(c) $y(t) = u(t) * h(t)$

$\therefore Y(s) = U(s) H(s) \qquad U(s) = \frac{1}{s}$

$$= \frac{1}{s(s+4-10j)(s+4+10j)}$$

Considering a general case

$$F(s) = \frac{1}{s(s-a)(s-b)}$$

$$F(s) = \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b}$$

$$A = \frac{1}{ab} \; ; \; B = \frac{1}{a(a-b)} \qquad C = \frac{1}{b(b-a)}$$

$$Y(s) = 116 F(s) \quad \text{when} \quad \begin{aligned} a &= -4-10j \\ b &= -4+10j \\ a &= b^* \end{aligned}$$

$$A = \frac{1}{a \cdot a^*} = \frac{1}{|a|^2} = \frac{1}{116}$$

$$B = \frac{1}{a(a-a^*)} = \frac{1}{2a\,\mathrm{Im}\{a\}} = \frac{1}{2(-4+10j)(10)} = \frac{1}{-80+200j}$$

$$C = \frac{1}{a^*(a^*-a)} = \frac{1}{-2a^*\,\mathrm{Im}\{a\}} = \frac{1}{80+200j}$$

$$\therefore y(t) = 116 f(t)$$

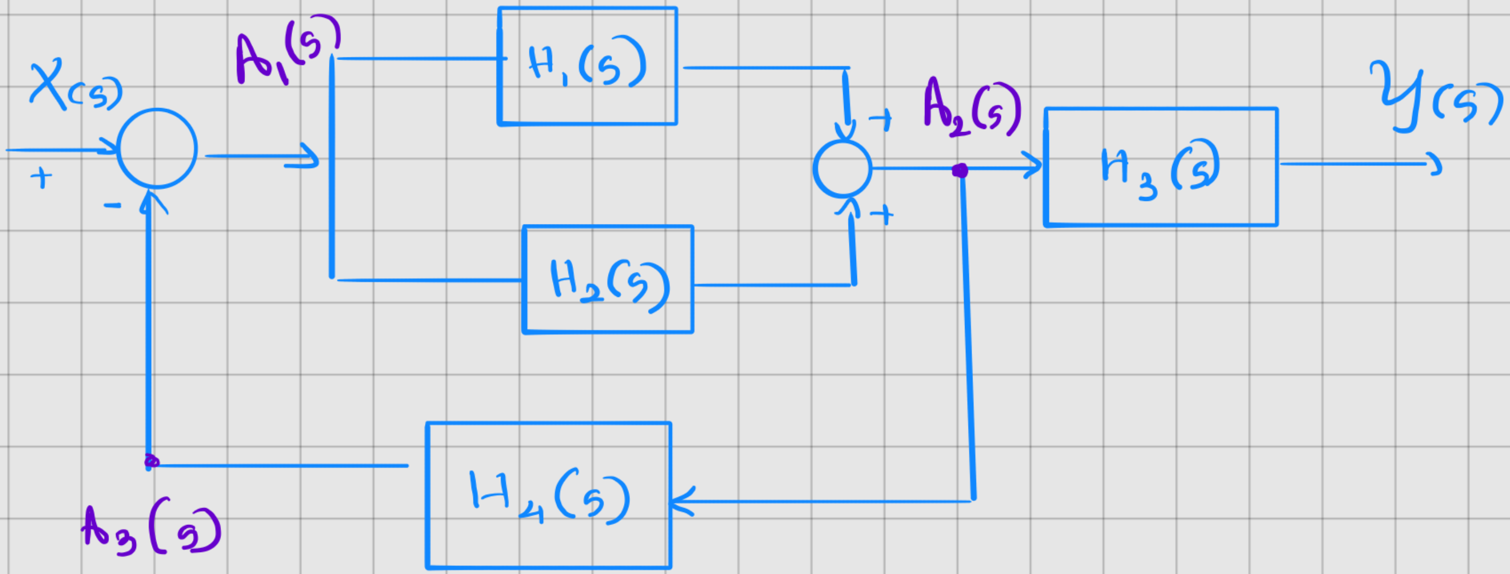
$$= 116\,u(t) \left\{ \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b} \right\}$$

where A, B, C are found above

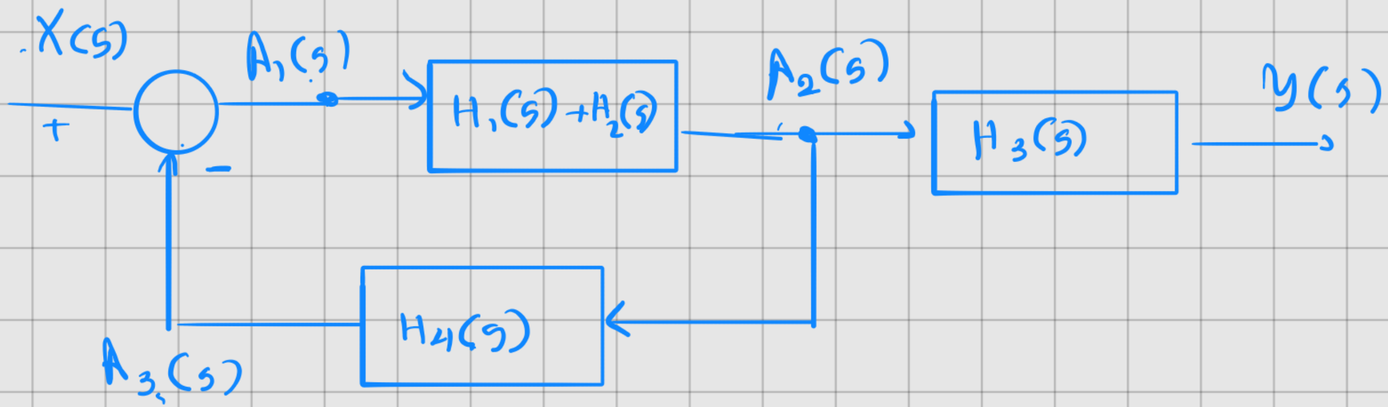
② Parallel Systems; $H_1(s)$, $H_2(s)$ $\Rightarrow$ $\boxed{H_1(s) + H_2(s)}$

Series systems $\Rightarrow$ Convolution

$\therefore$ in s domain $\Rightarrow$ multiplication



$\Downarrow$



$$A_1(s) = X(s) - A_3(s) \quad \text{—— ①}$$

$$A_2(s) = \left(H_1(s) + H_2(s)\right) A_1(s) \quad \text{—— ②}$$

$$A_3(s) = H_4(s) \cdot A_2(s) \quad \text{—— ③}$$

$$Y(s) = A_2(s) \cdot H_3(s) \quad \text{—— ④}$$

from 3 to 1

$$A_1(s) = X(s) - H_4(s)A_2(s) \longrightarrow 5$$

from 2 to 1;

$$A_1(s) = X(s) - H_4(s)\{H_1(s) + H_2(s)\}A_1(s)$$

$$\therefore A_1(s) = \frac{X(s)}{1 + H_4(s)\{H_1(s) + H_2(s)\}}$$

$$A_2(s) = \frac{X(s)\{H_1(s) + H_2(s)\}}{1 + H_4(s)\{H_1(s) + H_2(s)\}}$$

$$Y(s) = \frac{X(s)\{H_1(s) + H_2(s)\}H_3(s)}{1 + H_4(s)\{H_1(s) + H_2(s)\}}$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{\left(2 + \frac{10}{s}\right)\left(\frac{0.1}{s+20}\right)}{1 + \left(\frac{2}{s+4}\right)\left(\left(2 + \frac{10}{s}\right)\right)}$$

$$= \frac{\frac{(2s+10)(0.1)}{\cancel{s}(s+20)}}{\frac{s^2+4s+4s+20}{(s+4)(\cancel{s})}}$$

$$H(s) = \frac{(0.2s+1)(s+4)}{(s+20)(s^2+8s+20)} \,//$$

1) Discrete Time Systems

③ $x[n] \xrightarrow{z} X(z) = \dfrac{z^{-1}}{(1-z^{-1})^2}$

at $a = 1$

$x[n] = n(1)^n \cdot u[n]$

ROC $= |z| > 1$

$x[n] = n\,u[n]$

a) $x[n-3] \xrightarrow{z\ t/f} z^{-3} X(z)$

$x[n-3] \xleftrightarrow{z\ t/f} \dfrac{z^{-4}}{(1-z^{-1})^2}$ ROC $= |z| > 1$

b) $x[n] * \delta[n-3] \longleftrightarrow X[z]\,\Delta[z]$

$\delta[n] \xleftrightarrow{z\ t/f} 1$ ROC: $\forall z$

$\delta[n-3] \xleftrightarrow{z\ t/f} z^{-3}$ ROC; $\forall z - \{0\}$

$\therefore x[n] * \delta[n-3] \xleftrightarrow{z\ t/f} \dfrac{z^{-1}}{(1-z^{-1})^2} \cdot z^{-3}$

$x[n] * \delta[n-3] \xleftrightarrow{z\ t/f} \dfrac{z^{-4}}{(1-z^{-1})^2}$

ROC = At least the intersection of $|z| > 1$ and all z except $z = 0$

c) $x[n] - x[n-1] \xleftrightarrow{z\text{-}t/f} X[z] - z^{-1}X[z]$
(first differen

$$\longleftrightarrow (1 - z^{-1})\left(\frac{z^{-1}}{(1-z^{-1})^2}\right)$$

$$x[n] - x[n-1] \xleftrightarrow{z\text{-}t/f} \frac{z^{-1}}{1 - z^{-1}} \quad ROC = |z| > 1$$

(d) $x[n] * (\delta[n] - \delta[n-1])$

$$\xleftrightarrow{z\text{-}t/f} X(z)\left\{1 - z^{-1}\right\}$$

$$x[n] * (\delta[n] - \delta[n-1]) \xleftrightarrow{z\text{-}t/f} \frac{z^{-1}}{(1-z^{-1})^{\cancel{2}}} \cancel{(1 - z^{-1})} = \frac{z^{-1}}{(1-z^{-1})}$$

$$x[n] * (\delta[n] - \delta[n-1]) \xleftrightarrow{z\text{-}t/f} \frac{z^{-1}}{(1-z^{-1})}$$

ROC = At least the intersection of
$|z| > 1$ and all z except $z = 0$

(e) $5x[n-1] + 4\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z\text{-}t/f} 5z^{-1}X(z) + 4 \cdot \frac{1}{1 - (-1/3)/z}$

$$5x[n-1] + 4\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z\text{-}t/f} \frac{5z^{-1} \cdot z^{-1}}{(1-z^{-1})^2} - \frac{4}{1 + \frac{1}{3}z^{-1}}$$

$$5x[n-1] + 4\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z\text{-}t/f} \frac{5z^{-2}}{(1-z^{-1})^2} - \frac{4}{1 + \frac{1}{3}z^{-1}}$$

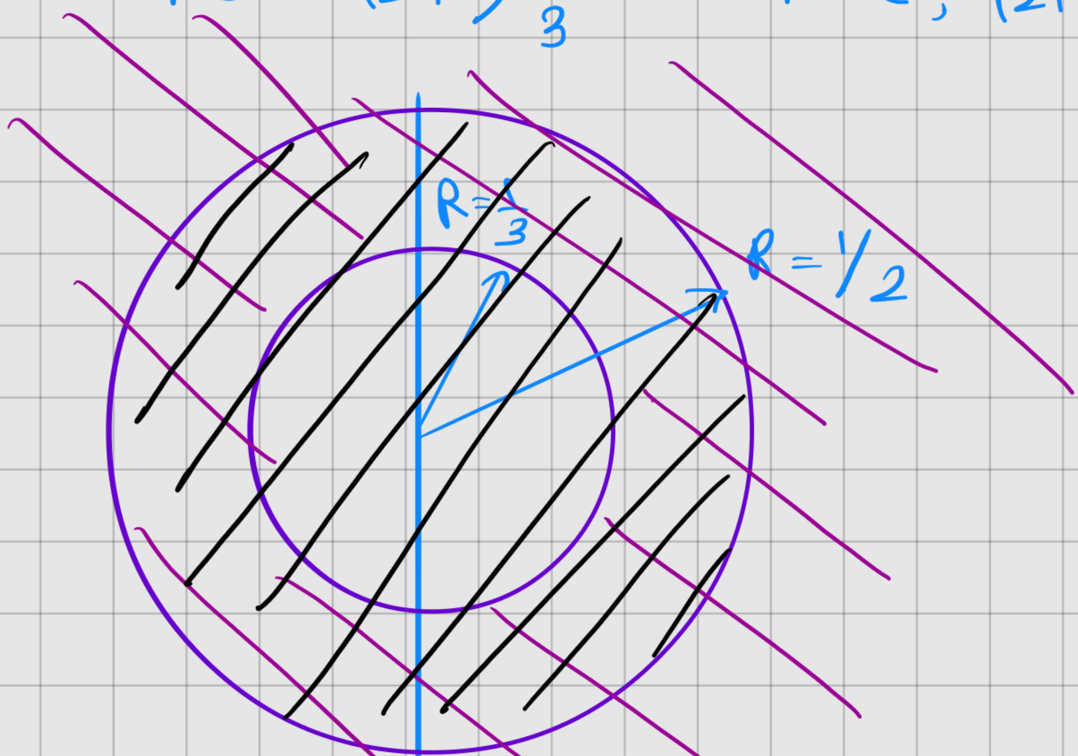
ROC = At least intersection of $|z| > 1$ and
$|z| > \frac{1}{3}$

4

a) a. $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{-n} u[-n-1]$

$\downarrow$ Z t/f

$$X(z) = \underbrace{\frac{1}{1 - \frac{1}{3} z^{-1}}}_{ROC ;\ |z| > \frac{1}{3}} - \underbrace{\frac{1}{1 - \frac{1}{2} z^{-1}}}_{ROC ;\ |z| < \frac{1}{2}}$$



$\therefore$ ROC $= \frac{1}{3} < |z| < \frac{1}{2}$

$$X(z) = \frac{\cancel{1} - \frac{1}{2} z^{-1} - (\cancel{1} - \frac{1}{3} z^{-1})}{(1 - \frac{1}{3} z^{-1})(1 - \frac{1}{2} z^{-1})}$$

$$= \frac{-\frac{1}{6} z^{-1}}{(1 - \frac{1}{3} z^{-1})(1 - \frac{1}{2} z^{-1})} \quad //$$

b. $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$

$$= n\left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$$

$(z \mid f' f$

$$X(z) = \underbrace{\frac{\frac{1}{4} z^{-1}}{\left(1 - \frac{1}{4} z^{-1}\right)^2}}_{ROC = |z| > \frac{1}{4}} + \underbrace{\frac{1}{1 - \frac{1}{4} z^{-1}}}_{ROC = |z| > \frac{1}{4}}$$

$\therefore ROC = |z| > \frac{1}{4}$

$$X(z) = \frac{\cancel{\frac{1}{4} z^{-1}} + 1 - \cancel{\frac{1}{4} z}^{-2}}{\left(1 - \frac{1}{4} z^{-1}\right)^2}$$

$$= \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)^2}$$

$$X(z) = \frac{16}{\left(4 - z^{-1}\right)^2}$$

(b) a. $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1]$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4} z^{-1}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

$$= \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$ROC = |z| > \frac{1}{2}$

$$= \frac{4z-1}{4z-2}$$

$y[n] = \left(\frac{1}{2}\right)^n u[n]$ $\quad ROC = |z| > \frac{1}{2}$

$z+f$

$$Y(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) = \frac{2z}{2z-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{2z}{\cancel{2z-1}}}{\frac{4z-1}{\cancel{4z-2}}\,2}$$

$$H(z) = \frac{4z}{4z-1}$$

$ROC = |z| > \frac{1}{4}$

$$= \frac{1}{1-\frac{1}{4}z^{-1}} \quad //$$

b. from tables; assuming causality

IZT

$$h[n] = \left(\frac{1}{4}\right)^n u[n]$$

c. Difference Eq^n of LTI system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k]$$

$\downarrow$ z transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{m} b_k z^{-k} X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{m} b_k z^{-k}}{\sum_{k=0}^{n} a_k z^{-k}}$$

in the given LTI Causal System,

$$H(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$\therefore$ from corresponding z polynomial coefficients

$a_0 = 1 \quad a_1 = -\frac{1}{4} \quad b_0 = 1$

$a_i = 0 \ \forall i \geq 2 \quad b_i = 0 \ \forall i \geq 1$

∴ Difference Eqⁿ,

$$1\, y[n] + \frac{-1}{4} y[n-1] = 1\, x[n]$$

$$y[n] - \frac{1}{4} y[n-1] = x[n]$$

d. Stability : stable iff the unit circle ($|z|=1$) is in the ROC,

but for causal systems,
H(z) is stable if and only if all poles of H(z) lie inside $|z|=1$

pole; $\frac{1}{4}$ ⇒ ∴ $\left|\frac{1}{4}\right| < |1|$

∴ The system is stable