

Tutorial ①

E1201420

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Continuous Time Systems

① (i) $\ddot{y} + 8\dot{y} + 12y = 12x$

a) Assuming $y(0) = 0$ & $y'(0) = 0$;
using Laplace transform

$$\left[s^2 Y(s) - s y'(0) - y''(0) \right] + 8 \left[s Y(s) - y(0) \right] + 12 Y(s) = 12 X(s)$$

$$s^2 Y(s) + 8s Y(s) + 12 Y(s) = 12 X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{12}{s^2 + 8s + 12}$$

$$H(s) = \frac{12}{(s+6)(s+2)} //$$

(b) No zeroes ;

Poles at $s = -6$ & $s = -2$

—

(c) $y(t) = h(t) * u(t)$

$$\therefore Y(s) = H(s) \cdot U(s)$$

$$= \frac{12}{(s+6)(s+2)} \cdot \left(\frac{1}{s} \right) = \frac{12}{s(s+2)(s+6)}$$

Considering a general case

$$G(s) = \frac{1}{s(s-a)(s-b)} \quad a, b \in \mathbb{C}$$

$$F(s) = \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b}$$

$$A = \frac{1}{ab}$$

$$B = \frac{1}{a(a-b)}$$

$$C = \frac{1}{b(b-a)}$$

$$\therefore f(s) = \frac{\cancel{1}_{ab}}{s} + \frac{\cancel{1}_{a(a-b)}}{s-a} \rightarrow \frac{\cancel{1}_{b(b-a)}}{s-b}$$

(ILT)

$$f(t) = A \cdot u(t) + B e^{at} + C e^{bt}$$

but $y(s) = 12 F(s)$ where $a = -2$
 \therefore from linearity $b = -6$

$$y(t) = 12 f(t)$$

$$= 12 \left\{ A + B e^{at} + C e^{bt} \right\} y_u(t)$$

where $A = \frac{1}{12}$ $B = \frac{1}{24}$ & $C = \frac{1}{8}$

$$(i) \ddot{y} + 8\dot{y} + 116y = 116x$$

$\downarrow LT$; where $y(0) = 0$ & $\dot{y}(0) = 0$

$$3Y(s) + 8sY(s) + 116Y(s) = 116X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{116}{s^2 + 8s + 116}$$

$$= \frac{116}{(s+4)^2 + 116 - 16}$$

$$= \frac{116}{(s+4)^2 + 100}$$

$$= \frac{116}{(s+4-10j)(s+4+10j)}$$

\therefore No zeroes

Poles at $s = -4 + 10j$ & $s = -4 - 10j$

$$(c) y(t) = u(t) * h(t)$$

$$\begin{aligned} \therefore Y(s) &= U(s) H(s) \quad U(s) = \frac{1}{s} \\ &= \frac{1}{s(s+4-10j)(s+4+10j)} \end{aligned}$$

Considering a general case

$$F(s) = \frac{1}{s(s-a)(s-b)}$$

$$F(s) = \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b}$$

$$A = \frac{1}{ab}; B = \frac{1}{a(a-b)}; C = \frac{1}{b(b-a)}$$

$$Y(s) = 116 F(s) \text{ when } a = -4 - 10j \\ b = -4 + 10j \\ a^* = b^*$$

$$A = \frac{1}{a \cdot a^*} = \frac{1}{(a)^2} = \frac{1}{116}$$

$$B = \frac{1}{a(a-a^*)} = \frac{1}{2a \operatorname{Im}\{a\}} = \frac{1}{2(-4+10j)(10)} \\ = \frac{1}{-80 + 200j}$$

$$C = \frac{1}{a^*(a^*-a)} = \frac{1}{-2a^* \operatorname{Im}\{a\}} = \frac{1}{80 + 200j}$$

$$\therefore y(t) = 116 f(t)$$

$$= 116 u(t) \left\{ \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b} \right\} y$$

where A, B, C are found above

1) d.

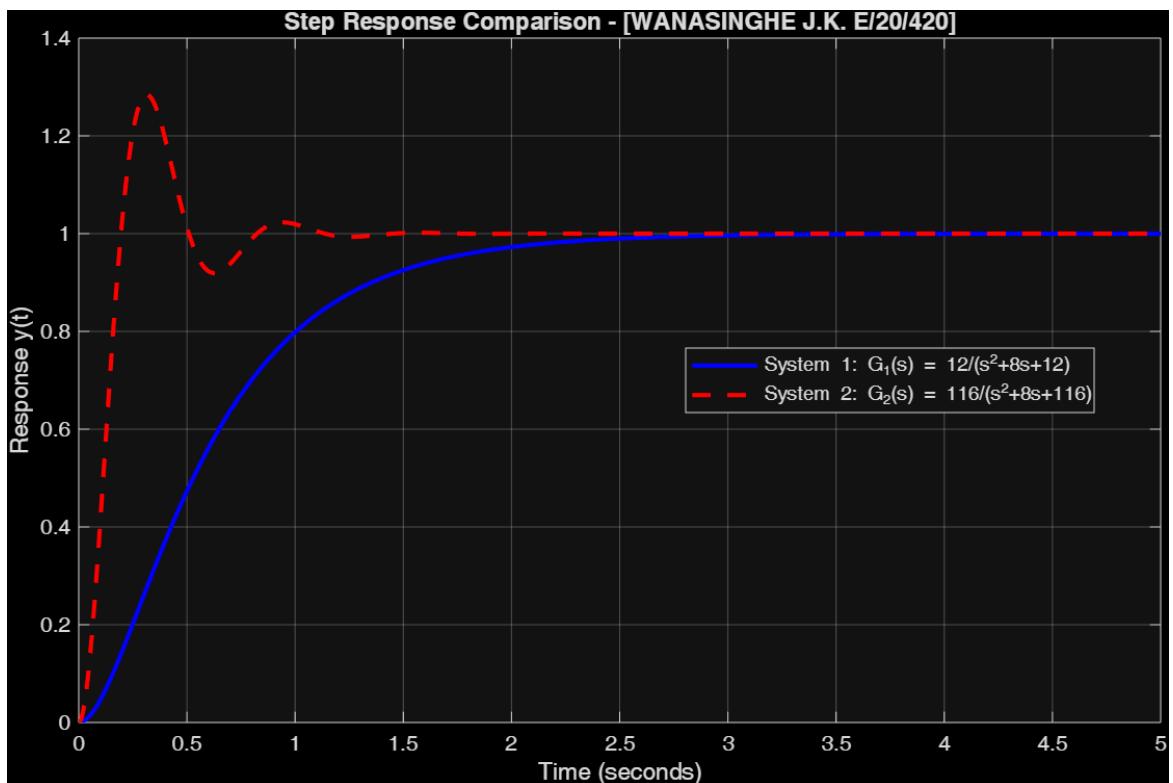


Figure 1:STEP RESPONSE COMPARISON

```
% Define transfer functions for both systems
% System 1: G1(s) = 12/(s^2 + 8s + 12)
num1 = 12;
den1 = [1 8 12];
G1 = tf(num1, den1);

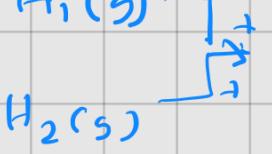
% System 2: G2(s) = 116/(s^2 + 8s + 116)
num2 = 116;
den2 = [1 8 116];
G2 = tf(num2, den2);

% Create time vector for plotting
t = 0:0.01:5; % 5 seconds with 0.01 step size

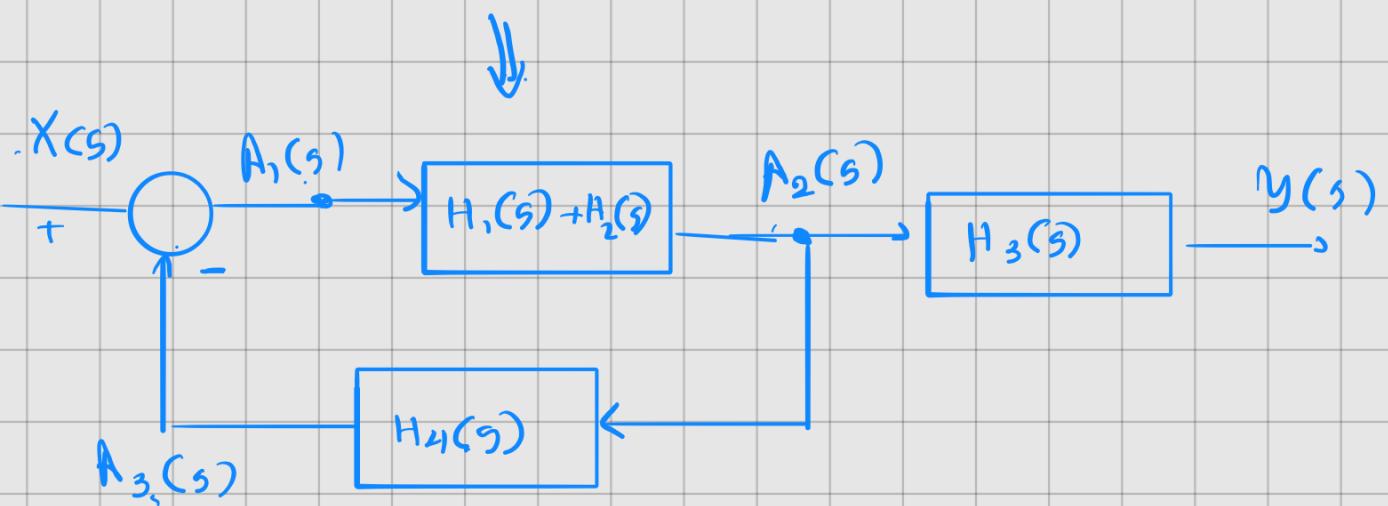
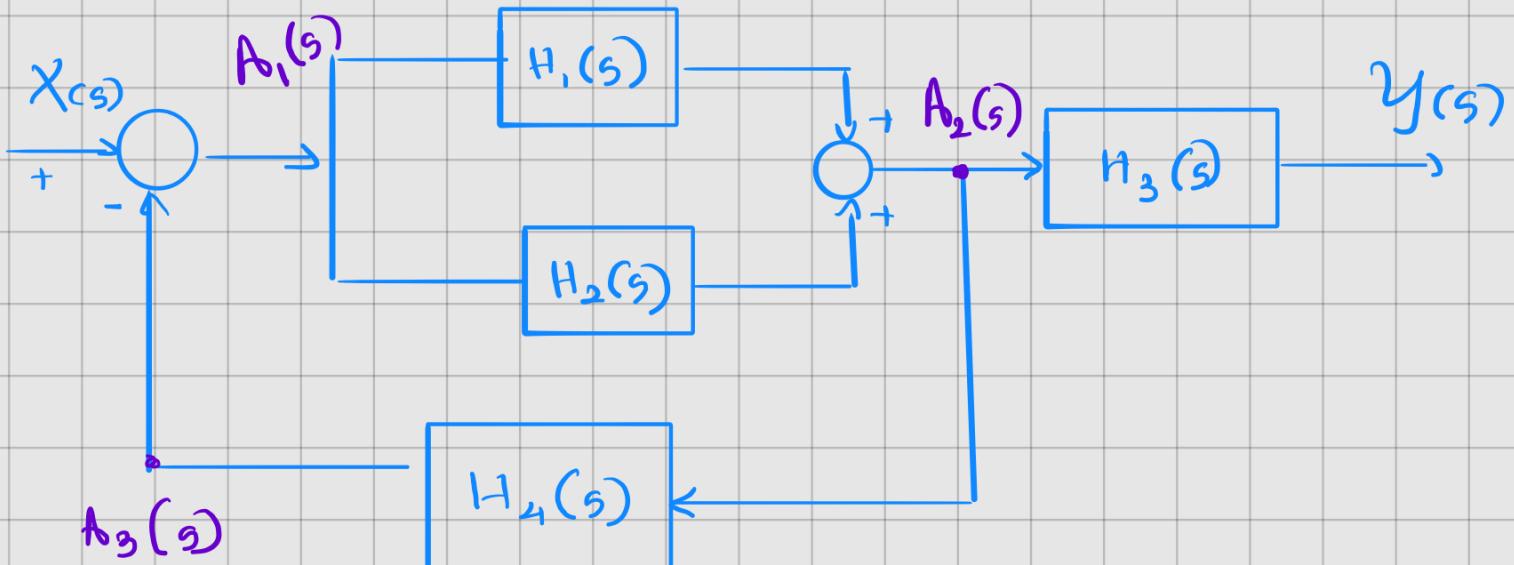
% Calculate step responses
[y1, t1] = step(G1, t);
[y2, t2] = step(G2, t);

% Create the plot
figure;
plot(t1, y1, 'b-', 'LineWidth', 2);
hold on;
plot(t2, y2, 'r--', 'LineWidth', 2);
grid on;

xlabel('Time (seconds)');
ylabel('Response y(t)');
title('Step Response Comparison - [WANASINGHE J.K. E/20/420]');
legend('System 1: G_1(s) = 12/(s^2+8s+12)', 'System 2: G_2(s) = 116/(s^2+8s+116)', 'Location', 'best');
```

② Parallel Systems: $H_1(s)$  $\Rightarrow H_1(s) + H_2(s)$

Series systems \Rightarrow Convolution
 \therefore in S domain \Rightarrow multiplication



$$A_1(s) = X(s) - A_3(s) \quad \text{--- } ①$$

$$A_2(s) = (H_1(s) + H_2(s)) A_1(s) \quad \text{--- } ②$$

$$A_3(s) = H_4(s) \cdot A_2(s) \quad \text{--- } ③$$

$$Y(s) = A_2(s) \cdot H_3(s) \quad \text{--- } ④$$

From ③ to ①

$$A_1(s) = X(s) - H_4(s) A_2(s) \quad \text{---} ④$$

From ④ to ①;

$$A_1(s) = X(s) - H_4(s) \left\{ H_1(s) + H_2(s) \right\} Y A_1(s)$$

$$\therefore A_1(s) = \frac{X(s)}{1 + H_4(s) \left\{ H_1(s) + H_2(s) \right\} Y}$$

$$A_2(s) = \frac{X(s) \left\{ A_1(s) + H_2(s) \right\} Y}{1 + H_4(s) \left\{ H_1(s) + H_2(s) \right\} Y}$$

$$Y(s) = \frac{X(s) \left\{ H_1(s) + H_2(s) \right\} Y H_3(s)}{1 + H_4(s) \left\{ A_1(s) + H_2(s) \right\} Y}$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{\left(2 + \frac{10}{s}\right) \left(\frac{0.1}{s+20}\right)}{1 + \left(\frac{2}{s+4}\right) \left(\left(2 + \frac{10}{s}\right)\right)}$$

$$= \frac{(2s+10)(0.1)}{\cancel{s}(s+20)}$$

$$\frac{s^2 + 4s + 4s + 20}{(s+4)(2)}$$

$$H(s) = \frac{(0.2s+1)(s+4)}{(s+20)(s^2+8s+20)} //$$

Discrete Time Systems

$$③ x[n] \xrightarrow{z} X(z) = \frac{z^{-1}}{(1 - z^{-1})^2}$$

at $a = 1$
 $x[n] = u(1)^n \cdot u[n]$

$$\text{ROC} = |z| > 1$$

$$x[n] = n u[n]$$

$$a) x[n-3] \xrightarrow{z+1/f} z^{-3} X(z)$$

$$x[n-3] \xrightarrow[z+1/f]{\text{---}} \frac{z^{-4}}{(1 - z^{-1})^2} \quad \text{ROC} = |z| > 1$$

$$b) x[n] * f[n-3] \longleftrightarrow X(z) \Delta(z)$$

$$f[n] \xrightarrow{z+1/f} 1 \quad \text{ROC: } \forall z$$

$$f[n-3] \xrightarrow{z+1/f} z^{-3} \quad \text{ROC: } \forall z - \{0\}$$

$$\therefore x[n] * f[n-3] \xrightarrow[z+1/f]{\text{---}} \frac{z^{-1}}{(1 - z^{-1})^2} \cdot z^{-3}$$

$$x[n] * f[n-3] \xrightarrow[z+1/f]{\text{---}} \frac{z^{-4}}{(1 - z^{-1})^2}$$

$\text{ROC} = \text{At least the intersection of } |z| > 1 \text{ and all } z \text{ except } z = 0$

c) $x[n] - x[n-1] \xrightarrow{z+1/f} X(z) - z^{-1}X(z)$
 (first diff.)

$$\xleftarrow{\quad} (1 - z^{-1}) \left(\frac{z^{-1}}{(1 - z^{-1})^2} \right)$$

$\frac{x[n] - x[n-1]}{z+1/f} \xrightarrow{z+1/f} \frac{z^{-1}}{1-z} \Rightarrow \text{ROC } = |z| > 1$

(d) $x[n] + (f[n] - f[n-1])$

$$\xleftarrow{z+1/f} X(z) \left\{ 1 - z^{-1} \right\}$$

$x[n] + (f[n] - f[n-1]) \xrightarrow{z+1/f} \frac{z^{-1}}{(1 - z^{-1})^2} \left(\frac{1 - z^{-1}}{z^{-1}} \right) \frac{z^{-1}}{(1 - z^{-1})}$

$\text{ROC} = \text{At least } \frac{1}{2} > 1$ and all 2 except 2 = 0

$$(e) 5x[n-1] + 4\left(\frac{-1}{3}\right)^n u[n] \xrightarrow{Z^{-1} f} 5z^{-1} X(z) + 4 \cdot \frac{1}{1 - \left(\frac{-1}{3}\right)z^{-1}}$$

$$5x[n-1] + 4\left(\frac{-1}{3}\right)^n u[n] \xrightarrow{Z^{-1} f} \frac{5z^{-1} \cdot z^{-1}}{\left(1 - z^{-1}\right)^2} - \frac{4}{1 + \frac{1}{3}z^{-1}}$$

$$5x[n-1] + 4\left(\frac{-1}{3}\right)^n u[n] \xrightarrow{Z^{-1} f} \frac{5z^{-2}}{\left(1 - z^{-1}\right)^2} - \frac{4}{1 - \frac{1}{3}z^{-1}}$$

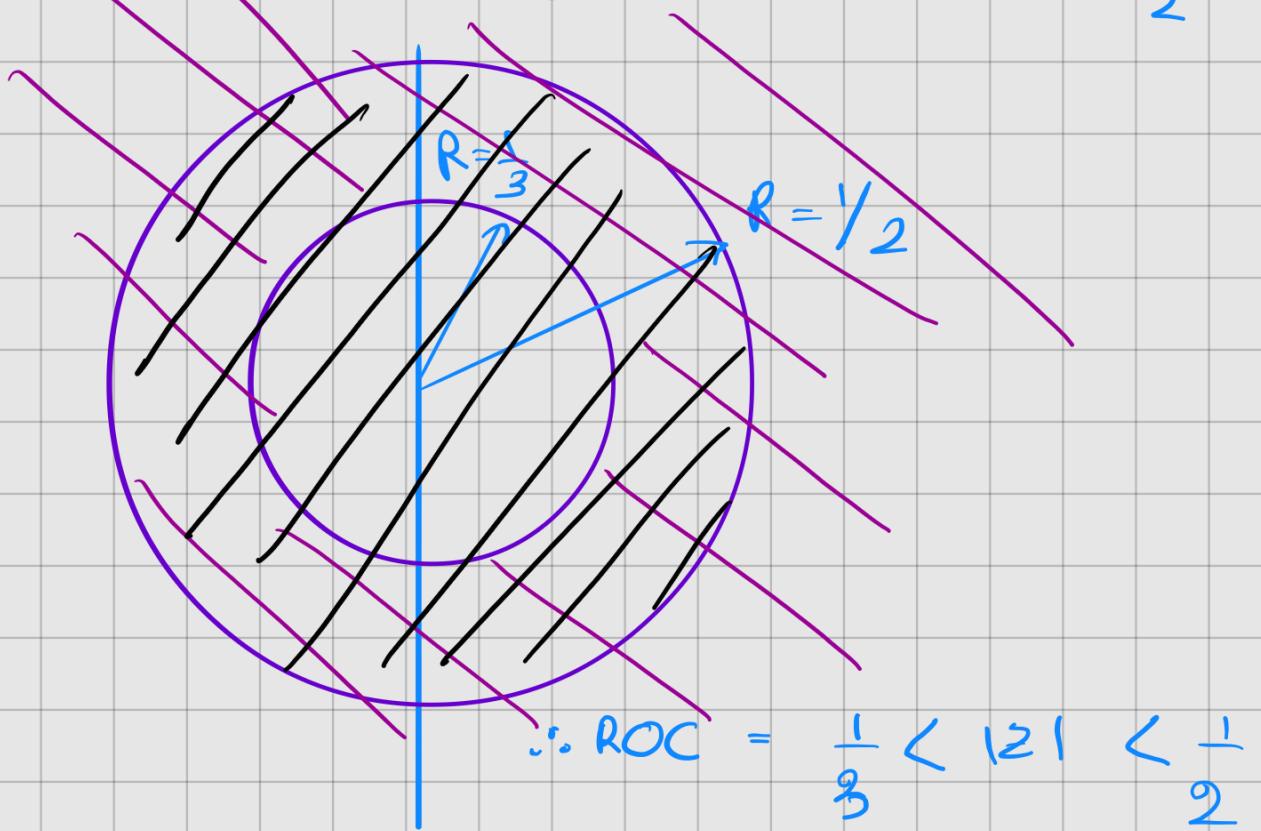
$R6C = \text{At least intersection of } |z_1\rangle_1 \text{ and } |z_1\rangle_3$

$$④ \text{ a. } a_n = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{-n} u[-n-1]$$

\downarrow
 $(z+1)$

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$\underbrace{\quad}_{\text{ROC: } |z| > \frac{1}{3}}$ $\underbrace{\quad}_{\text{ROC: } |z| < \frac{1}{2}}$



$$\begin{aligned} X(z) &= \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{3}z^{-1}} \\ &= \frac{-\frac{1}{6}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \end{aligned}$$

$$b_n[n] = (n+1) \left(\frac{1}{4}\right)^n u[n]$$

$$= n \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{\frac{1}{4}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)^2} + \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$\underbrace{\left(1 - \frac{1}{4}z^{-1}\right)^2}_{ROC = |z| > \frac{1}{4}}$

$\underbrace{1 - \frac{1}{4}z^{-1}}_{ROC = |z| > \frac{1}{2}}$

$\therefore ROC = |z| > \frac{1}{2}$

$$X(z) = \frac{\cancel{\frac{1}{4}z^{-1}} + \cancel{1 - \frac{1}{4}z^{-2}}}{\left(1 - \frac{1}{4}z^{-1}\right)^2}$$

$$= \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)^2}$$

$$X(z) = \frac{16}{(4 - z^{-1})^2}$$

↓

$$\begin{aligned}
 (b) a_n u[n] &= \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right) \left(\frac{1}{2}\right)^{n-1} u[n-1] \\
 &= \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{4} z^{-1} \left(\frac{1}{1 - \frac{1}{2}z^{-1}} \right) \\
 &= \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}
 \end{aligned}$$

$$ROC = |z| > \frac{1}{2}$$

$$= \frac{4z - 1}{4z - 2}$$

$$g[n] = \left(\frac{1}{2}\right)^n u[n] \quad ROC = |z| > \frac{1}{2}$$

$$Y(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}} \right) = \frac{2z}{2z - 1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{2z}{2z-1}}{\frac{4z-1}{4z-2}} = \frac{2z}{2z-1}$$

$$\begin{aligned}
 H(z) &= \frac{4z}{4z-1} \quad ROC = |z| > \frac{1}{4} \\
 &= \frac{1}{1 - \frac{1}{4}z^{-1}}
 \end{aligned}$$

bo from tables; assuming causality

$$h[n] = \left(\frac{1}{4}\right)^n u[n]$$

c. Difference Eq $\in \mathbb{R}$ LTI system

$$\sum_{k=0}^n a_k y[n-k] = \sum_{k=0}^m b_k x[n-k]$$

z transform

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^m b_k z^{-k} X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^m b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

in the given LTI Causal System;

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

∴ from corresponding z polynomial coefficients

$$a_0 = 1 \quad a_1 = -\frac{1}{4} \quad b_0 = 1$$

$$a_i = 0 \text{ if } i > 1 \quad b_i = 0 \text{ if } i > 1$$

\therefore Difference Eqⁿ,

$$1y[n] + \frac{-1}{4}y[n-1] = 1x[n]$$

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

d. Stability : Stable iff the unit circle ($|z|=1$) is in the ROC.

but for causal systems;

$H(z)$ is stable if and only if all poles of $H(z)$ lie inside $|z|=1$

$$\text{pole: } \frac{1}{4} \Rightarrow | \frac{1}{4} | < 1 \quad |$$

\therefore The system is stable