STA 331 2.0 Stochastic Processes

5. Continuous Parameter Markov Chains

Dr Thiyanga S. Talagala

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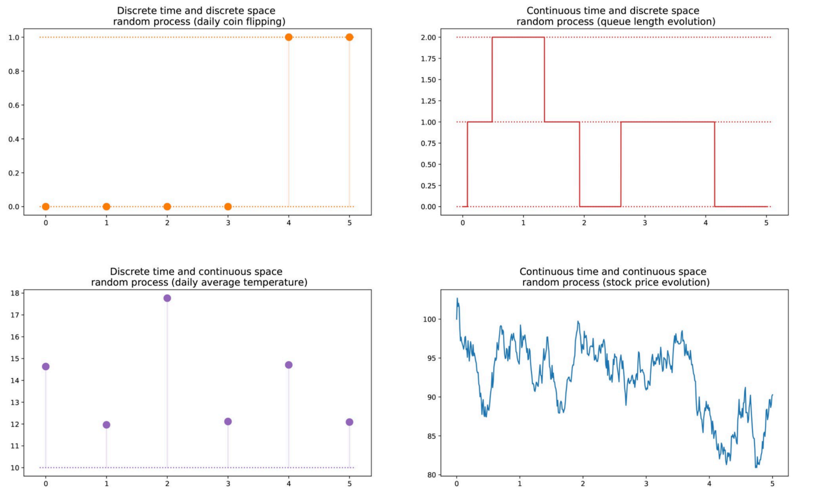
Department of Statistics, University of Sri Jayewardenepura

Goals

1. Explain the Markov property in the continuous-time stochastic processes.
2. Explain the difference between continuous time and discrete time Markov chains.
3. Learn how to apply continuous Markov chains for modelling stochastic processes.

Stochastic Processes



Different kind of random processes (discrete/continuous in space/time).

parameter = time

Continuous Parameter Markov Chains

Suppose that we have a continuous-time (continuous-parameter) stochastic process $\{N(t); t \geq 0\}$ taking on values in the set of nonnegative integers. The process $\{N(t); t \geq 0\}$ is called a **continuous parameter Markov chain** if for all $u, v, w > 0$ such that $0 \leq u < v$ and nonnegative integers i, j, k,

$$P[N(v+w) = k | N(v) = j, N(u) = i, 0 \leq u < v] = P[N(v+w) = k | N(v) = j].$$

Continuous Parameter Markov Chains (cont.)

If in addition.

$$P[N(v+w)=k|N(v)=j]$$

is independent of v, then the continuous parameter Markov chain is said to have stationary or homogeneous transition probabilities.

In other words, a continuous-time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future $N(v+w)$ given the present $N(v)$ and the past $N(u)$, $0 \leq u < s$, depends only on the present and is independent of the past.

Discrete Time versus Continuous Time (In class)

diagram

DTMC: Jump at discrete times: 1, 2, 3, …

CTMC: Jump can occur at any time $t \geq 0$.

Transition Probabilities

Recap: P_{ij}^n - transition probability of discrete Markov chains

Transition probability of continuous Markov chains

$$p_{ij}(t,s) = P[N(t) = j | N(s) = i], \text{ s < t.}$$

- If the transition probabilities do not explicitly depend on s or t but only depend on the length of the time interval $t - s$, they are called **stationary** or **homogeneous**.
- Otherwise, they are nonstationary or nonhomogeneous.
- We'll assume the transition probabilities are stationary (unless stated otherwise).

Homogeneous transition probabilities

$$p_{jk}(w) = P[N(v+w) = k | N(v) = j]$$

$p_{jk}(w)$ represents the probability that the process presently in state j will be in start k a time w later.

Poisson Process

Let $N(t)$ be the total number of **events** that have occurred up to time t. Then, the stochastic process $\{N(t); t \geq 0\}$ is said to be a Poisson process with rate λ if

1. $N(0) = 0$,
2. The process has independent increments,
3. For any $t \geq 0$ and $h \rightarrow 0_+$,

$$P[N(t+h) - N(t) = k] = \begin{cases} \lambda h + o(h), & \text{k=1} \\ o(h), & k \geq 2 \\ 1 - \lambda h + o(h), & k = 0 \end{cases}$$

- The function $f(.)$ is said to be $o(h)$ if $lim_{h \rightarrow 0} \frac{f(h)}{h} = 0$.
- The third condition implies that the process has stationary increments.

Theorem

Suppose $\{N(t); t \geq 0\}$ is a Poisson process with rate λ. Then $\{N(t); t \geq 0\}$ is a Markov process.

Theorem

Suppose that $\{N(t); t \geq 0\}$ is a Poisson process with rate λ. Then, the number of events in any interval of length t has a Poisson distribution with mean λt. That is for all $s, t \geq 0$,

$$P[N(t+s) - N(s) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

For a Poisson process with rate λ, the transition probability $p_{ij}(t)$ is given by

$$p_{ij}(t) = \frac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!}$$

Acknowledgement

The contents in the slides are mainly based on Introduction to Probability Models by Sheldon M. Ross.