STA 331 2.0 Stochastic Processes

8. Pure Death Process

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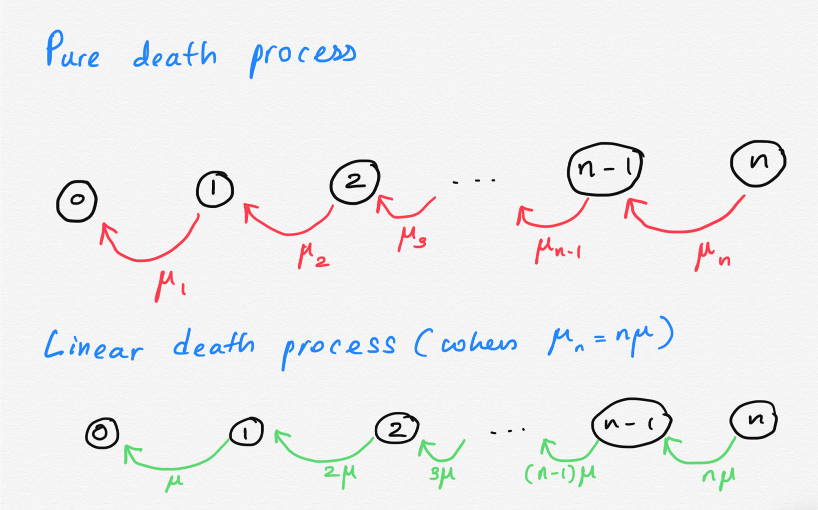
Pure Death Process

- Individuals persist only until they die and there are no births.
- We assume initially, there are n_0 number of individuals at time $t = 0$.

The state transition diagram.



Pure Death Process

Let us consider a death process whose total number of individuals at time t is denoted by a discrete random variable $N(t)$. As parameter t varies $\{N(t) : t \geq 0\}$ represent a stochastic process with a continuous parameter space and a discrete state space.

We assume that the individuals of a death process with initial size n_0 die at a certain rate, which depends on the present size of the population eventually reducing the size to the zero.

Pure Death Process

Condition 1

$$P[N(t+h) = n_0 - k | N(t) = n_0] = \begin{cases} 1 - \mu_n h + o(h), & k = 0 \\ \mu_n h + o(h), & \text{k=1} \\ o(h), & k \geq 2. \end{cases}$$

where μ_n is the rate at which the births occur at time t and n_0 being the size of the population at time t.

Condition 2

In non-overlapping time intervals deaths occur independently each other.

Condition 3

$N(0) = n_0$

Notation

Let $N(t)$ be the number of individuals alive at time t. Suppose initially, there are n_0 individuals, that is $N(0) = n_0$.

$$P_n(t) = P[N(t) = n | N(0) = n_0]$$

Linear Death Process

Suppose $\mu_i = i\mu$, where $i = 1, 2, 3, ...$ and initially, $N(t) = n_0$.

Then we can show that,

$$X_i \sim Binomial(n_0, p)$$

where $p = e^{-\mu t}$. That is,

$$P_n(t) = \frac{n_0!}{(n_0 - n)!n!}(e^{-\mu t})^n(1 - e^{-\mu t})^{n_0 - n},$$

for $0 \leq n \leq n_0$.

Question 1

Suppose that a population has an average death rate of μ_i. Let $P_n(t)$ be the probability that there are n individuals in the population at time t. Assume that initially, there are n_0 number of individuals at time $t = 0$. Derive the following system of differential equations for $P_n(t)$.

$P'_{n_0}(t) = -\mu_n P_n(t)$ and

$P'_n(t) = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t)$ for $0 \leq n < n_0$.

Note: These system of differential equations can be solved subject to the conditions $P_{n_0}(0) = 1$ and $P_n(0) = 0$ for $0 \leq n < n_0$.

Hint: You can obtain a system of differential equations similar to the pure birth process.

Question 2: Linear Death Process - PMF

When $\mu_n = n\mu$, i.e. when the death rate is linear in the present size of the population, the pure death process is said to be a **linear death process**. Let us assume that there are n_0 individuals in the population initially.

i) When $\mu_n = n\mu$, obtain the system of differential equations of the linear death process.

ii) Based on the system of differential equations show that

$$P_n(t) = \frac{n_0!}{(n_0 - n)!n!}(e^{-\mu t})^n(1 - e^{-\mu t})^{n_0 - n},$$

for $0 \leq n \leq n_0$.

Question 3: The mean and variance of the pure death process

Show that the mean of the pure linear death process is

$E(N(t)) = n_0 e^{-\mu t}$

and the variance is

$V(X(t)) = n_0 e^{-\mu t}(1 - e^{-\mu t}).$

Question 4: Extinction

In the pure death process the population either remains constant or it decreases. It may eventually reach zero in which case we say that the population has gone **extinct**. Show that the probability the population is extinct at time t is given by

$$P(N(t) = 0 | N(0) = n_0) = (1 - e^{-\mu t})^{n_0}.$$