Quicksort (2)

Friday, January 31, 2025 9:47 AM

Last Time:
- Master Theorem
- Quicksort

Today
- Quicksort
- Selection
- **HW2 due**
- **HW3 assigned!** → due Friday

```
Qsort (A[1...n])
  if n ≤ 1:
      sorted
  otherwise:
    choose a pivot index p
    r ← partition (A[1...n], p)
    Qsort (A[1...r-1])
    Qsort (A[r+1...n])
```

```
Partition (A[1...n], p)
  swap A[n] and A[p]
  l ← 0
  for i from 1 to (n-1):
    if A[i] < A[n]:
      l ← l+1
      swap A[l] and A[i]
  swap A[l+1] and A[n]
  return (l+1)
```

worst-case → $O(n^2)$

① Randomized Quicksort → randomly choose a pivot
↳ uniform

Observations

① Partition does most of the work.

② We add up the "contributions" of partition.

③ Partition is run at most $(n-1)$ times

red part contribute at most $\boxed{O(n)}$ to the running time red pair contribute at most

$\boxed{O(n)}$ to the running time.

purple part → contribute equal to the number of comparisons

X

Quicksort is $O(n + X)$

For analysis, let $A = z_1, z_2, \ldots \ldots z_n$

Let Z_{ij} be $\{z_i, z_{i+1} \ldots z_j\}$

(all items from z_i through z_j)

Let X_{ij} be a random variable:

$$X_{ij} = \begin{cases} 1 & \text{if alg. compares } z_i \text{ v.s. } z_j \\ 0 & \text{if alg. never compares } z_i \text{ v.s. } z_j. \end{cases}$$

. Each z_i v.s. z_j can only be compare one time.

$$X = \sum_{\substack{\text{all } i,j \\ i \neq j}} X_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[z_i \text{ v.s } z_j \text{ happens}].$$

Probability of comparing z_i v.s. z_j

$$Z_{ij} = \{z_i, \underbrace{z_{i+1} \ldots z_{j-1}}_{\text{pivot in here}}, z_j\}$$

① If a pivot in $\{z_{i+1} \ldots z_{j-1}\}$ is chosen, then, z_i will never be compared to z_j.

② If z_i or z_j is chosen, then a comparison <u>will</u> happen between z_i, z_j.

So: $Pr[z_i \text{ v.s. } z_j \text{ happens}]$

$= Pr[z_i \text{ is chosen as pivot or } z_j \text{ is chosen as pivot}]$

$= Pr[z_i \text{ chosen}] + Pr[z_j \text{ chosen}]$

$= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

let $k = j - i$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}$$

$$= 2 \sum_{i=1}^{n-1} \ln n = 2n \cdot \ln n$$

$$\boxed{\Theta(n \log n)}$$

$E[X]$ is $\boxed{\Theta(n \log n)}$

Quicksort is in expectation, $\Theta(n \log n)$

$$O(n + n\log n) \longrightarrow O(n \log n)$$

If you want $\Theta(n \log n)$ worst case for quicksort,

then, choose pivot of rank $n/2$!

Select

Input: $A[1 \ldots n]$

Input: $A[1 \ldots \ldots n]$

$r \rightarrow$ target rank

Output: The item of rank r.

① If $r=1$, then, find the min. $\rightarrow O(n)$

n

② If $r=2 \longrightarrow 2n$

⋮

If $r = n/2 \longrightarrow \frac{n}{2} \cdot n \rightarrow \frac{n^2}{2}$ $\times$

Quickselect

subroutine

① Partition.

Quickselect $(A[1 \ldots n], r)$

① Pick a pivot, p. $\rightarrow A[p]$

② Partition $(A[1 \ldots n, p])$.)

$\hat{p}$

$[\; < A[p] \;] \; A[p] \; | \; > A[p] \;]$

$\uparrow$

$\hat{p}$

③ If $r < \hat{p}$, then Quickselect$(A[1 \ldots \hat{p}-1], r)$

otherwise $r > \hat{p}$, then Quickselect$(A[\hat{p}+1, n], r - \hat{p})$

or $r = \hat{p}$, done!

Efficiency:

$$T(n) = \theta(n) + \max\{T(n-\hat{p}), T(\hat{p}-1)\}$$

worst-case: $\boxed{T(n) = \theta(n) + T(n-1)}$

$\boxed{\theta(n^2)}$

If you do randomized quickselect

Expected $\boxed{\theta(n)}$ running time.

It's easy to improve this!

→ pick a good pivot!

It is possible to solve select in $\boxed{\theta(n)}$ worst case.

$$T(n) = \theta(n) + T(n/2) \Longrightarrow \theta(n)$$

n

$\log_2 1 \to 0$

n

$n/2$

$n/4$

$n/8$

decreasing

$\log_2 1 \rightarrow 0$

n^0 v.s. $\Theta(n)$