An Efficient Algorithm for the "Optimal" Stable Marriage

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Abstract. In an instance of size n of the stable marriage problem, each of n men and n women ranks the members of the opposite sex in order of preference. A stable matching is a complete matching of men and women such that no man and woman who are not partners both prefer each other to their actual partners under the matching. It is well known [2] that at least one stable matching exists for every stable marriage instance. However, the classical Gale–Shapley algorithm produces a marriage that greatly favors the men at the expense of the women, or vice versa. The problem arises of finding a stable matching that is optimal under some more equitable or egalitarian criterion of optimality. This problem was posed by Knuth [6] and has remained unsolved for some time. Here, the objective of maximizing the average (or, equivalently, the total) "satisfaction" of all people is used. This objective is achieved when a person's satisfaction is measured by the position of his/her partner in his/her preference list. By exploiting the structure of the set of all stable matchings, and using graph-theoretic methods, an $O(n^4)$ algorithm for this problem is derived.

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1. *Introduction*

In an instance of the stable marriage problem, each of n men and n women lists the members of the opposite sex in order of preference. A *stable marriage* or *matching* is defined as a complete matching of men and women with the property that there are no two couples (m, w) and (m', w') such that m prefers w' to w and

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w' prefers m to m'. For obvious reasons, such a situation would be regarded as unstable.

An elementary discussion of the problem is given by Polya et al. [10], and a more detailed study of a wide range of associated issues is presented by Knuth [6].

Gale and Shapley [2] proved that at least one stable matching exists for all instances of the problem and gave an efficient algorithm for finding such a matching. An alternative recursive formulation of essentially the same algorithm was given by McVitie and Wilson [7], together with an extension to find all stable matchings for a given problem instance.

As we shall see in Section 2 below, this classical algorithm normally yields what is called the *male optimal* solution, with the property that every man has the best partner that he can have in any stable matching. If applied with the roles of men and women interchanged, the algorithm will yield the female optimal solution, which similarly favors the women. However, it turns out that the achievement of best possible partners by the members of one sex results in the members of the opposite sex having their worst possible partners, so that the question has been raised, for example by Knuth [6], Polya et al. [10], and others [7, 14], as to whether there exists an efficient algorithm to find the optimal stable marriage under a more equitable measure of optimality. We solve this problem, giving an $O(n^4)$ algorithm to find a stable marriage that maximizes the total satisfaction of all the men and women, where a person's satisfaction is measured by the position of his/her partner in his/her preference list.

An instance of the stable marriage problem may be specified by the male and female *ranking matrices*. Relative to arbitrary but fixed numberings of males and females, these are defined by

$$\begin{aligned} mr(i, k) &= j \quad \text{if woman } k \text{ is the } j\text{th choice of man } i, \\ wr(i, k) &= j \quad \text{if man } k \text{ is the } j\text{th choice of woman } i. \end{aligned}$$

In this paper, we use the egalitarian measure of optimality, introduced by McVitie and Wilson [7] under which total satisfaction is maximized. The problem of how to find a stable marriage maximizing total satisfaction was unsolved until now.

Suppose that, for a given stable marriage instance,

$$S = \{(m_1, w_1), \ldots, (m_n, w_n)\}$$

is a stable matching. We define the *value* $c(S)$ of S by

$$c(S) = \sum_1^n mr(m_i, w_i) + \sum_1^n wr(w_i, m_i),$$

and we say that a stable matching S is *optimal* if it has minimum possible value $c(S)$. It is clear that a stable matching that is optimal in this sense maximizes the total (or average) satisfaction of the $2n$ individuals.

Such an optimal stable matching can certainly be found by generating all stable matchings and comparing their values. However, it has been shown [5, Corollary 2.1] that the maximum number of stable matchings for an instance of size n (i.e., involving n men and n women) grows exponentially with n, a result that was already known to Knuth [6]. It follows that such an exhaustive search algorithm is doomed to be of exponential worst-case time complexity. In this paper, we give an $O(n^4)$-time algorithm to find an optimal stable marriage.

Example. We find it convenient to have an illustrative example for use throughout the paper, and the example of size 8 with the preference lists shown in Figure 1 is chosen for this purpose:

In this example, the person in position j of i's list is person i's jth choice.

FIG. 1. Male and female preference lists.

male preference lists	female preference lists
1: 3 1 5 7 4 2 8 6	1: 4 3 8 1 2 5 7 6
2: 6 1 3 4 8 7 5 2	2: 3 7 5 8 6 4 1 2
3: 7 4 3 6 5 1 2 8	3: 7 5 8 3 6 2 1 4
4: 5 3 8 2 6 1 4 7	4: 6 4 2 7 3 1 5 8
5: 4 1 2 8 7 3 6 5	5: 8 7 1 5 6 4 3 2
6: 6 2 5 7 8 4 3 1	6: 5 4 7 6 2 8 3 1
7: 7 8 1 6 2 3 4 5	7: 1 4 5 6 2 8 3 7
8: 2 6 7 1 8 3 4 5	8: 2 5 4 3 7 8 1 6

2. *Male and Female Optimal Solutions*; *The Shortlists*

The classical algorithm for a solution to a stable marriage instance [2, 7] is based on a sequence of "proposals" from the men to the women. Each man proposes, in order, to the women on his preference list, pausing when a woman agrees to consider his proposal, but continuing if a proposal is either immediately or subsequently rejected. When a woman receives a proposal, she rejects it if she already holds a better proposal, but otherwise agrees to hold it for consideration, simultaneously rejecting any poorer proposal that she may currently hold. (A "better" proposal means a proposal from some man higher in the woman's preference list.)

It is not difficult to show, as in [7], that the sequence of proposals so specified ends with every woman holding a unique proposal, and that the proposals held constitute a stable matching. (A similar outcome results if the roles of males and females are reversed, in which case the resulting stable matching may or may not be the same as that obtained from the male proposal sequence.)

Two fundamental implications of this initial proposal sequence, implicit in [7], are

(i) if m proposes to w, then there is no stable matching in which m has a better partner than w;
(ii) if w receives a proposal from m, then there is no stable matching in which w has a worse partner than m.

These observations suggest that we should explicitly remove m from w's list, and w from m's, if w receives a proposal from someone she likes better than m. We refer to the resulting lists as the (male-oriented) *shortlists* for the given problem instance. The following properties are either immediate, or are explicit or implicit in [7]:

Property 1. If w does not appear on m's shortlist, then there is no stable matching in which m and w are partners.

Property 2. w appears on m's shortlist if and only if m appears on w's, and is first on m's shortlist if and only if m is last on w's.

Property 3. If every man is paired with the first woman on his shortlist, then the resulting matching is stable; it is called the *male optimal solution*, for no man can have a better partner than he does in this matching, and indeed no woman can have a worse one.

Property 4. If the roles of males and females are interchanged, and if every woman is paired with the first man on her (female-oriented) shortlist, then the resulting matching is stable; it is called the *female optimal solution*, for no woman can have a better partner than she does in this matching, and indeed no man can have a worse one.

male shortlists	female shortlists
1: 3 1 5 7 4	1: 4 3 8 1 2
2: 1 3 4 8 7	2: 3 7 5 8
3: 7 4 3 1 2 8	3: 7 5 8 3 6 2 1
4: 5 8 6 1 4 7	4: 6 4 2 7 3 1 5
5: 4 2 8 7 3 6 5	5: 8 7 1 5 6 4
6: 6 5 7 4 3	6: 5 4 7 6
7: 8 6 2 3 4 5	7: 1 4 5 6 2 8 3
8: 2 7 1 3 5	8: 2 5 4 3 7

FIG. 2. Male and female shortlists.

Example. In the case of the instance of size 8 specified above, the male proposal sequence proceeds as follows (where m/w means m proposes to w):

$$1/3,\ 2/6,\ 3/7,\ 4/5,\ 5/4,\ 6/6,\ 2/1,\ 7/7,\ 7/8,\ 8/2$$

and the shortlists are as shown in Figure 2.

Hence the male optimal solution is

$$\{(1, 3), (2, 1), (3, 7), (4, 5), (5, 4), (6, 6), (7, 8), (8, 2)\}.$$

3. *Rotations*

The concept of a rotation, first introduced in [4] in the context of the related stable roommates problem and studied in detail in the context of the stable marriage problem in [5], is crucial to the understanding of the structure of the set of solutions to a stable marriage instance.

In the context of the shortlists, a rotation ρ is a sequence

$$\rho = (m_0, w_0), \ldots, (m_{r-1}, w_{r-1})$$

of man/woman pairs such that, for each i $(0 \le i \le r - 1)$,

(i) w_i is first in m_i's shortlist;
(ii) w_{i+1} is second in m_i's shortlist ($i + 1$ taken modulo r).

Such a rotation is said to be *exposed* in the shortlists (or relative to the male optimal stable marriage), and it is not difficult to prove that, provided the first entries in the shortlists do not specify the female optimal solution, then at least one rotation must be exposed.

Example. In our example of size 8, there are three rotations exposed in the shortlists, namely,

$$\rho_1 = (1, 3), (2, 1), \qquad \rho_2 = (3, 7), (5, 4), (8, 2), \qquad \rho_3 = (4, 5), (7, 8), (6,6).$$

The chief significance of such a rotation lies in the fact that if, in the male optimal solution, each m_i exchanges his partner w_i for w_{i+1} ($i + 1$ mod r) then the resulting matching is also stable.

If, given rotation $\rho = (m_0, w_0), \ldots, (m_{r-1}, w_{r-1})$, each successor x of m_{i-1} in w_i's shortlist is removed, together with the corresponding appearance of w_i in x's list, for each i $(0 \le i \le r - 1$, $i - 1$ taken modulo r), then the rotation ρ is said to have been *eliminated.* The rotation ρ is called the *eliminating rotation* for any pair thereby deleted from each other's lists.

The concept of an exposed rotation, and of eliminating such an exposed rotation, can be extended to any set of (reduced) preference lists obtained from the shortlists by a sequence of zero or more rotation eliminations. Any such rotation

$\rho = (m_0, w_0), \ldots, (m_{r-1}, w_{r-1})$ will, in general, be the eliminating rotation for pairs of two kinds:

(a) members of the rotation, that is, (m_i, w_i) $(0 \le i \le r-1)$;
(b) pairs (m, w_i) for some m with $wr(w_i, m_{i-1}) < wr(w_i, m) < wr(w_i, m_i)$.

As proved in [5], the former pairs appear in some stable matching, the latter in none. Also, the pair (m, w) can appear in at most one rotation [5, Lemma 4.7], and since any rotation contains at least two pairs, the total number of rotations in any problem instance of size n is $O(n^2)$. Furthermore, it follows easily that no pair is eliminated by more than one rotation.

Once an exposed rotation has been identified and eliminated, then one or more rotations may be exposed in the resulting further reduced lists. This process may be repeated, and after any such sequence of rotation eliminations, pairing each man with the first woman in his (reduced) list yields a stable matching. It is proved in [5] that every stable matching for a given stable marriage instance may be obtained in this way, starting from the shortlists and eliminating some sequence of zero or more exposed rotations. Further, the set R_M of rotations that must be eliminated to reveal a particular stable matching M is fixed, though the order of their elimination need not be. We can thus use the term *lists reduced for stable matching M* for the lists obtained by eliminating the rotations in R_M. For brevity, we say that a rotation exposed in the lists reduced for M is exposed in M.

A rotation π is said to be an *explicit predecessor* of rotation $\rho = (m_0, w_0), \ldots, (m_{r-1}, w_{r-1})$ if, for some i $(0 \le i \le r-1)$ and some woman x $(\neq w_i)$, π is the eliminating rotation for (m_i, x) and m_i prefers x to w_{i+1}. It should be clear that a rotation cannot become exposed until all of its explicit predecessors have been eliminated. Further, it is shown in [5] that the reflexive transitive closure $\le$ of the explicit predecessor relation is a partial order on the set of rotations, called the *rotation poset* for the particular problem instance, and that $\pi < \rho$ if and only if π must be eliminated before ρ becomes exposed.

A *closed set* in a poset P, $\le$ is a subset C of P such that

$$\rho \in \mathrm{C}, \quad \pi < \rho \Rightarrow \pi \in \mathrm{C}.$$

The following theorem of [5] is crucial for our present purposes.

THEOREM 3.1. *The stable matchings of a given stable marriage instance are in one-to-one correspondence with the closed subsets of the rotation poset.*

In this correspondence, each closed subset C represents the stable matching obtained by eliminating the rotations in C from the shortlists.

Example. A complete list of rotations for our example is shown in Figure 3, each with its list of immediate predecessors (π is an *immediate predecessor* of ρ if $\pi < \rho$ and there is no σ such that $\pi < \sigma < \rho$).

An alternative representation for the rotation poset is in the form of an acyclic directed graph, with the rotations as nodes and an arc from π to ρ if and only if π is an immediate predecessor of ρ, as shown in Figure 4. The number on each node gives the weight of the corresponding rotation, as defined in Figure 4.

Careful examination of closed sets in this digraph reveals a total of 23 (including the empty set and the complete node set), so that this is the total number of stable matchings for this problem instance.

Rotation	*Immediate Predecessors*
$\rho_1 = (1, 3), (2, 1)$	—
$\rho_2 = (3, 7), (5, 4), (8, 2)$	—
$\rho_3 = (4, 5), (7, 8), (6, 6)$	—
$\rho_4 = (1, 1), (6, 5), (8, 7)$	ρ_1, ρ_2, ρ_3
$\rho_5 = (2, 3), (3, 4)$	ρ_1, ρ_2
$\rho_6 = (4, 8), (7, 6), (5, 2)$	ρ_2, ρ_3
$\rho_7 = (3, 3), (8, 1)$	ρ_4, ρ_5
$\rho_8 = (2, 4), (5, 8), (6, 7)$	ρ_4, ρ_5, ρ_6
$\rho_9 = (1, 5), (5, 7), (8, 3)$	ρ_7, ρ_8
$\rho_{10} = (3, 1), (7, 2), (5, 3), (4, 6)$	ρ_9

FIG. 3. List of rotations.

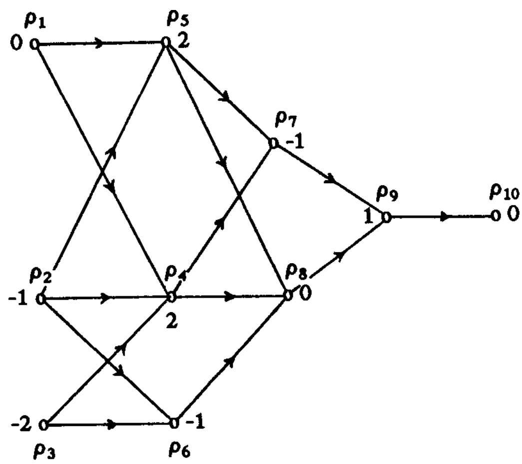

FIGURE 4

4. *Optimal Stable Marriages*

Given a rotation $\rho = (m_0, w_0), \ldots, (m_{r-1}, w_{r-1})$ in a stable marriage instance, we define the *weight* $w(\rho)$ of that rotation by

$$w(\rho) = \sum_{0}^{r-1} (mr(m_i, w_i) - mr(m_i, w_{i+1})) + \sum_{0}^{r-1} (wr(w_i, m_i) - wr(w_i, m_{i-1}))$$

$$(i - 1, i + 1 \text{ taken mod } r).$$

The reason for this definition becomes apparent in the following Lemma:

LEMMA 4.1. *Let S be the stable matching obtained by starting from the shortlists and eliminating a particular sequence of rotations. Suppose that ρ is a rotation that is exposed in S, and let S′ be the stable matching obtained from S by eliminating ρ. Then*

$$c(S') = c(S) - w(\rho).$$

PROOF. If $\rho = (m_0, w_0), \ldots, (m_{r-1}, w_{r-1})$, then only men $m_0, \ldots, m_{r-1}$ and women $w_0, \ldots, w_{r-1}$ have different partners in S′ and S. The result now follows since, for each i, m_i and w_i are partners in S, and m_i and w_{i+1} are partners in S′ $(0 \le i \le r - 1, i + 1 \text{ taken mod } r)$. □

COROLLARY 4.1.1. *If, for a given stable marriage instance, S is the stable matching obtained from the shortlists by the elimination of rotations $\rho_1, \ldots, \rho_t$,* *then*

$$c(S) = c(S_0) - \sum_{1}^{t} w(\rho_i),$$

where S_0 is the male optimal solution.

Note that the rotations $\rho_1, \ldots, \rho_t$ must form a closed subset of the rotation poset.

We use the term *weighted poset* for a poset P, $\leq$ in which each element has an associated integer weight.

From the rotation poset for a stable marriage instance we form the *weighted rotation poset* by attaching the weight $w(\rho)$ to each rotation ρ. We also refer to the sum of the weights of the elements in some subset of this poset as the weight of the subset itself.

Given Theorem 3.1 and Corollary 4.1.1, it follows that in order to find an optimal stable matching, it suffices to develop an algorithm that will yield a closed subset of a weighted poset with largest possible weight. In the example, the maximum-weight closed subset has weight +1. The set $\{\rho_1, \rho_2, \rho_5\}$ is one of several closed subsets with this weight, and the corresponding optimal stable marriage, obtained by eliminating these rotations, is $\{(1, 1), (2, 4), (3, 3), (4, 5), (5, 2), (6, 6), (7, 8), (8, 7)\}$.

The general problem of finding a maximum-weight closed subset of a weighted poset (or of a digraph) has arisen in other contexts, and there are several known methods [8, 9, 11] that solve this problem using network flow or related algorithms. However, use of these methods to find the maximum-weight closed subset of the rotation poset P (even ignoring the time needed to construct P) gives an $O(n^6)$-time solution for the optimal stable marriage problem. Further, the best practical time we see for explicitly constructing P is $O(n^6)$, although $O(n^5)$ time is possible using fast transitive closure [1]. In the next section, we give a practical $O(n^4)$-time solution to the optimal stable marriage problem. We use a known solution to the maximum-weight closed subset problem, but we speed up the computation by taking advantage of special properties of the weighted rotation poset.

5. *An $O(n^4)$-Time Solution*

The general outline of the method is essentially the same as above: find all the rotations; construct a directed graph P′ representing (in some way) the weighted rotation poset P; use P′ to find the maximum-weight closed subset of P; and eliminate the rotations in that closed subset to obtain the optimal stable marriage. The central idea in this section is to observe and exploit the fact that the maximum-weight closed subset of P can be obtained from a very sparse subgraph P′ of P. This subgraph P′ can be constructed from the rotations much faster than P can, and, being sparse, the subsequent computation of maximum network flow is also much faster. These two observations will lead to an $O(n^4 \log n)$-time solution for the optimal stable marriage problem and weighted generalizations; one additional observation will reduce this to $O(n^4)$ time for the optimal stable marriage problem.

Consider P as a directed acyclic graph containing one node for every rotation, and a directed edge from rotation π to rotation ρ if and only if $\pi < \rho$ in P. The name of each node in P will be the name of the rotation associated with it.

We denote by P′ the subgraph of P consisting of the same node set as P, but only of those edges defined by the following two rules, which are applied for each man m:

Rule 1. If (m, w) is a member of a rotation, say π, and w' is the first woman below w in m's list such that (m, w') is a member of some other rotation, say ρ, then P′ contains a directed edge from π to ρ.

Rule 2. If (m, w') is not a member of any rotation, but is eliminated by some rotation, say π, and w is the first woman above w' in m's list such that (m, w) is a member of some rotation, say ρ, then P′ contains a directed edge from π to ρ.

LEMMA 5.1. *P′ has at most $O(n^2)$ edges, and, given the rotations, P′ can be constructed in $O(n^2)$-time.*

PROOF. In the above rules, each creation of an edge in P′ is associated with a pair (m, w'). But no pair is associated more than once, so that P′ can have at most $O(n^2)$ edges.

To construct P′, assuming the rotations are known, we first scan each rotation π, noting the pairs that are in it, and the pairs that are eliminated by it, labeling each such pair by the name of the rotation. If w is a woman in rotation π, then π eliminates the pairs (m, w) corresponding to a contiguous subsequence of men in w's preference list; hence the pairs that are eliminated by π can be found in time proportional to the number of them. Then, since no pair is eliminated by more than one rotation, the time to collect the eliminated pairs and to set the labels is $O(n^2)$. Now, P′ can be constructed by scanning down each man m's preference list, keeping track of the most recently encountered pair contained in some rotation, and applying the above two rules. Each scan takes $O(n)$ time; hence $O(n^2)$ time overall. □

LEMMA 5.2. *The transitive closure of P′ is P; hence P′ preserves the closed subsets of P.*

PROOF. Certainly P′ is a subgraph of P; hence we need only show that a closed subset in P is also closed in P′. For this, it suffices to show that if rotation π explicitly precedes rotation ρ, then there is a directed path from π to ρ in P′. So, following the definition of "explicitly precedes", let (m_i, x) be a pair eliminated by π such that (m_i, w_i) is a pair in ρ and m_i prefers x to w_{i+1}. Now, if there is a woman w such that m_i prefers w to x, and (m_i, w) is a pair in some rotation, then let w be the first such woman above x in m_i's list, and let (m_i, w) be in ρ_1. Then successive applications of rule 1 give a directed path of zero or more edges in P′ from ρ_1 to ρ, and rule 2 gives a directed edge from π to ρ_1. So the lemma is proved, unless there is no such woman w. But then, m_i must prefer x to his mate in every other stable marriage, in particular in any marriage M in which π is exposed. But since (m_i, x) is eliminated by π, x must prefer m_i to her mate in M, so M cannot be stable. □

Note that P′ is not necessarily the transitive reduction of P; however, if it could be constructed from the rotations quickly enough, the transitive reduction of P could be used in place of P′, as it preserves the closed subsets of P, and is sparser than P′, although not in worst case order of magnitude.

We still have to describe how to find efficiently the set of all rotations. The method given here takes $O(n^3)$ time. In [3] this method is refined to find all rotations in $O(n^2)$ time.

Let S be a stable marriage, and assume that the preference lists have been reduced for S. Then for each man m we denote by $s(\mathrm{S}, m)$ the woman who is second (if there is one) on m's reduced list, and by $s'(\mathrm{S}, m)$ her mate in S. We further denote by G(S) the directed graph consisting of n nodes, one for each man, where for every man m_i, there is a directed edge from node m_i to node $s'(\mathrm{S}, m_i)$.

LEMMA 5.3. *All cycles in $G(S)$ can be identified in $O(n)$ time.*

PROOF. Note that each node has outdegree at most 1; it follows that G(S) has at most n edges, and that no node, and hence no edge, is in more than one cycle. We can find the cycles in G(S) by depth-first search: whenever a back edge $\langle x, y\rangle$ is found, a cycle is closed, and we can output the cycle by following the unique out-edges from y to x. Since the cycles are edge disjoint, an edge is traversed exactly three times if it is in some cycle (twice by the depth-first search, and once to output the cycle), and all other edges are traversed twice. □

It is clear that each directed cycle in G(S) specifies a rotation exposed in S, and vice versa. In particular, any directed cycle in G(S) exactly specifies a set of men in some rotation ρ exposed in S; rotation ρ is completely defined by these men and their mates in S, where the pairs are ordered by the circular order of the men in the cycle. Hence, all the rotations exposed in S can be found in $O(n)$ time.

We define a node ρ in P′ to be at *level* 0 if and only if ρ has no predecessors in P′, and otherwise to be at level i if and only if it has at least one predecessor at each of levels $0, \ldots, i-1$ but none at level i.

Clearly, the rotations exposed in S_0 (the male optimal marriage) are exactly the rotations at level 0 in P′. When these are eliminated, the rotations exposed in the resulting stable marriage S_1 are exactly the rotations at level 1 in P′. Continuing in this way, we can find all the rotations, level by level. The stable marriage obtained by eliminating all the rotations in levels $0, \ldots, i-1$ will be called S_i.

THEOREM 5.1. *All the rotations can be found in $O(n^3)$ time.*

PROOF. Over the entire computation, the total time needed to eliminate all the rotations, reduce the resulting lists, and update $s(\mathrm{S}_i, m)$ and $s'(\mathrm{S}_i, m)$ is $O(n^2)$. Each graph $\mathrm{G}(\mathrm{S}_i)$ is built and all cycles identified in $O(n)$-time. Then since there can be at most $O(n^2)$ levels, the theorem follows. □

5.1 FINDING THE MAXIMUM-WEIGHT CLOSED SUBSET OF P′. We first show how network flow can be used to find the maximum-weight closed subset of P′. The method is essentially that in [8], although the proof here is simpler; the details are included for completeness. We then use the sparsity of P′ and a bound on the size of the minimum cut to accelerate the time for the general solution.

Given the graph P′, we define the following capacitated $s-t$ flow graph $\mathrm{P}'(s, t)$. Source node s and sink node t are added to the graph P′. A directed edge is added from s to every negative node (i.e., every node ρ_i such that $w(\rho_i) < 0$); the capacity of edge $\langle s, \rho_i\rangle$ is $|w(\rho_i)|$. A directed edge is added to node t from every positive node (i.e., every node ρ_j such that $w(\rho_j) > 0$); the capacity of edge $\langle \rho_j, t\rangle$ is $w(\rho_j)$. Note that $\mathrm{P}'(s, t)$ contains neither edge $\langle s, \rho\rangle$ nor $\langle \rho, t\rangle$ if $w(\rho) = 0$. The capacity of every original edge in P′ is set to infinity.

THEOREM 5.2. *Let X be the set of edges crossing a minimum $s-t$ cut in $P'(s, t)$, and denote the weight or capacity of X by $w(X)$. The positive nodes in the maximum-weight closed subset of P' are exactly the positive nodes whose edges* *into t are uncut by X. These nodes, and all nodes that reach them in P' (their predecessors), define a maximum-weight closed subset in P'.*

PROOF. Denote by V^+ and V^- the sets of positive and negative nodes respectively in P', and by $N(W)$ the set of all negative predecessors of nodes in any subset W of V^+.

Any negative node in a maximum-weight closed subset C of P' must precede at least one positive node in C, and hence a maximum-weight closed subset of P' can be defined as a subset $W \subseteq V^+$ which maximizes the quantity $w(W) - |w(N(W))|$ over all subsets W of V^+. But then the same subset *minimizes* $w(V^+) - (w(W) - |w(N(W))|)$ over all subsets W of V^+. Hence the problem of finding a maximum weight closed subset can be solved by minimizing $w(V^+ - W) + |w(N(W))|$ over all subsets W of V^+.

Now let W be an arbitrary subset of V^+, and consider graph $P'(s, t)$. If every edge from s to a node in $N(W)$ is cut, and every edge from a node in $V^+ - W$ to t is also cut, then all paths from s to t are cut. Hence $w(X) \le w(V^+ - W) + |w(N(W))|$ for any $W \subseteq V^+$. Conversely, if we let $W^* \subseteq V^+$ consist of the positive nodes whose edges to t are uncut by X, then, by definition, X cuts all edges to t from nodes in $V^+ - W^*$, and X must certainly cut all the edges from s to nodes in $N(W^*)$, since X is an $s - t$ cut of finite capacity, and all original edges in P' have infinite capacity. Hence

$$w(X) = w(V^+ - W^*) + |w(N(W^*))| \le w(V^+ - W) + |w(N(W))|$$

for any arbitrary $W \subseteq V^+$, and the theorem is proved. □

LEMMA 5.4. *Given the positive nodes W^* in a maximum-weight closed subset of P', an optimal stable marriage can be constructed in time $O(n^2)$.*

PROOF. The predecessors in P' of W^* can be found and marked by scanning backwards from the nodes in W^*, marking any unmarked nodes that are reached, scanning backwards from these nodes, but not from any marked nodes that are reached. In this way, each edge is scanned at most once, so that only $O(n^2)$ time is needed, since P' has only $O(n^2)$ edges. Once all the predecessors and the nodes in W^* are marked, we simply start with the male optimal marriage and traverse P' top down, by levels, eliminating the marked rotations in turn. When all marked rotations have been eliminated, the resulting stable marriage is optimal. □

Note that $P'(s, t)$ has $N = O(n^2)$ nodes and $E = O(n^2)$ edges. The maximum flow algorithm of Sleator and Tarjan [12] finds both the maximum flow and minimum cut X in time $O(NE \log N)$; hence a minimum cut in $P'(s, t)$ can be found in $O(n^4 \log n)$ time.

To reduce the upper bound to $O(n^4)$ we use the following lemma.

LEMMA 5.5. *The minimum cut in $P'(s, t)$ has capacity bounded by $O(n^2)$.*

PROOF. The minimum cut is certainly bounded by $\sum_{i \in V^+} w(\rho_i)$, and we shall show that this is $O(n^2)$. Each term $w(\rho_i)$ consists of a negative part representing the net change of the men when rotation ρ_i is eliminated, and a positive part representing the net change of the women. For each rotation ρ_i, let $W(\rho_i)$ denote the positive (women's) part. Then $\sum_{i \in V^+} w(\rho_i) < \sum_i W(\rho_i)$. But $W(\rho_i)$ is simply the number of pairs eliminated by rotation ρ_i, and since no pair is eliminated by more than one rotation, and there are only n^2 pairs, $\sum_i W(\rho_i)$ is $O(n^2)$, and the lemma is proved. □

THEOREM 5.3. *The maximum flow and minimum cut X in P′(s, t) can be found in $O(n^4)$-time.*

PROOF. When all the capacities in P′(s, t) are integral, the running time of the Ford–Fulkerson algorithm (or any of the many subsequent algorithms) is $O(EK)$, see [13], where K is the maximum $s - t$ flow value. All the capacities in P′(s, t) are integral, and both E and K are $O(n^2)$, hence the theorem follows. □

Note that, if m is the number of rotations, then the time for the flow computation is also bounded by $O(m^3)$. Often m will be considerably smaller than n^2 (m can be much smaller than n), in which case this time bound may be tighter.

Summarizing, we can find all the rotations in $O(n^3)$ time (in $O(n^2)$ time in [3]), build P′(s, t) in time $O(n^2)$, find a minimum $s - t$ cut X in P′(s, t) in $O(n^4)$ time, use X to find a maximum-weight closed subset of P′ in time $O(n^2)$, and then eliminate each rotation in that subset in $O(n^2)$ total time to obtain an optimal stable marriage. Note that other than the $O(n^4)$ time network flow computation, the method can be made to run in $O(n^2)$ time; hence any speed up in the flow computation would immediately speed up the overall solution. Such an improvement seems quite likely, as the $O(n^4)$ time bound holds even for the Ford–Fulkerson algorithm; perhaps more sophisticated flow algorithms can be shown to run faster than this on P′(s, t), or perhaps faster specialized flow algorithms can be developed for P′(s, t) or for other sparse subgraphs of P that preserve the closed subsets of P.

6. *Weighted Preference Lists*

The optimal stable marriage problem can be generalized so that each person i not only has a rank ordering of the people of the other sex, but a numerical preference weight $p(i, j)$ for each such person j. These weights define a rank ordering, so the notion of a stable marriage is unchanged. We can then ask for a stable marriage $S = (m_1, w_1), \ldots, (m_n, w_n)$ that minimizes (or maximizes) $\sum_1^n p(m_i, w_i) + \sum_1^n p(w_i, m_i)$. This problem can be solved in $O(m^3)$ or $O(n^2 m \log m) = O(n^4 \log n)$-time; the method is the same as in the optimal marriage problem, but for the second bound, the minimum cut is found with the Sleator–Tarjan algorithm. The weighted maximization problem can be converted to a minimization problem by subtracting each weight from a large constant. This works because the weight of each stable marriage involves exactly $2n$ of these constant terms.

The weighted version of the optimal marriage problem allows each person to specify the structure of their preferences in more detail, and hence may give more useful solutions. In addition, there are several problems that can be solved by the appropriate selection of weights. For example, if S is a stable marriage, and V(S) is an n-vector where the ith element indicates the number of people who get their ith choice in S, then we can choose weights so that the solution to the resulting minimum-weighted stable marriage problem solves the problem of finding a stable marriage S such that V(S) is lexicographically maximum, that is, S maximizes the number of people who get their first-choice mate, and within that maximizes the number of people who get their second-choice mate, and within that ... etc. This is done by the standard trick of assigning a weight of n^{n-i} for each person's ith choice; the resulting maximum weight stable marriage is lexicographically maximum. The number of arithmetic operations needed to solve this problem is still $O(n^4 \log n)$, but unless a (nonrealistic) unit time model is used, the cost of each operation must now be counted as $n \log n$ instead of the (implicit) cost of $\log n$ for the unweighted optimal stable marriage problem. Hence the total time to find the lexicographic maximum stable marriage is $O(n^5 \log n \log n)$.

We can also choose weights to solve the minimum regret stable marriage problem [6], where the quality of the marriage is measured by the person who is worst off in it. However, a faster method for this problem appears in [3].

REFERENCES

1. AHO, A. V., HOPCROFT, J. E., AND ULLMAN, J. D. *The Design and Analysis of Computer Algorithms.* Addison-Wesley, Reading, Mass., 1974.
2. GALE, D., AND SHAPLEY, L. S. College admissions and the stability of marriage. *Am. Math. Monthly 69* (1962), 9–15.
3. GUSFIELD, D. Three fast algorithms for four problems in stable marriage. *SIAM J. Comput.* 16 (1987), 111–128.
4. IRVING, R. W. An efficient algorithm for the "stable room-mates" problem. *J. Algorithms 6* (1985), 577–595.
5. IRVING, R. W., AND LEATHER, P. The complexity of counting stable marriages. *SIAM J. Comput.* 15 (1986), 655–667.
6. KNUTH, D. E. *Mariages Stables.* Les Presses de l'Université de Montréal, Montréal, Quebec, Canada 1976.
7. MCVITIE, D., AND WILSON, L. B. The stable marriage problem. *Commun. ACM 14* (1971), 486–492.
8. PICARD, J. Maximum closure of a graph and applications to combinatorial problems. *Manage. Sci. 22* (1976), 1268–1272.
9. PICARD, J., AND QUEYRANNE, M. Selected applications of minimum cuts in networks, *INFOR—Can. J. Oper. Res. Inf. Process. 20* (1982), 394–422.
10. POLYA, G., TARJAN, R. E., AND WOODS, D. R. *Notes on Introductory Combinatorics.* Birkhauser Verlag, Boston, Mass., 1983.
11. RHYS, J. A selection problem of shared fixed costs and network flows. *Manage. Sci. 17* (1970), 200–207.
12. SLEATOR, D., AND TARJAN, R. E. A data structure for dynamic trees, *J. Comput. Syst. Sci. 26* (1983), 362–391.
13. TARJAN, R. E. *Data Structures and Algorithms.* SIAM, Philadelphia, Pa., 1983.
14. WIRTH, N. *Algorithms + Data Structures = Programs*, Prentice-Hall, Englewood Cliffs, N.J., 1976.

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