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Exactly Solvable Model of Correlated Lattice Electrons in Any Dimensions

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We present an exactly solvable Hamiltonian which consists of nearest neighbor hopping and long-range interaction. The ground state and the thermodynamic quantities are analytically obtained in any dimensions. In a repulsive case, the Luttinger's theorem breaks down and the hypothesis of the adiabatic continuation no longer holds. At half filling, we show the existence of a metal-insulator transition as a function of the strength of the interaction. The zero energy excitation disappears at the critical value of the interaction when we increase the interaction. In this case, the system is an insulator. The charge compressibility behaves as $\kappa \approx |1-\rho|^{1-2/d}$ near half filling in d dimensions where ρ is the density of electrons. At other fillings, there is always a zero energy excitation and the system is metallic. When the interaction is attractive, the ground state is composed of pairs of electrons with opposite spins. The low energy excitations, however, is not a particle-hole type of a non-interacting case but a pairwise excitation across the Fermi surface.

§1. Introduction

The correlated electron system is one of the main problems in condensed matter physics. It is an old one and there are a large number of works in this field. In the strongly correlated systems, however, the validity of usual approximations is not so clear. Thus it is important to study models with a reliable method. One of the simplest models of correlated electrons is the Hubbard model.[1] Even in this simple model, the exact solution is known only in one dimension.[2] At this stage it is worthwhile to investigate exactly solvable models even though they are somewhat unrealistic. There are several successful studies of this kind. A notable example is the Sherrington-Kirpatrick model for spin glass.[3] Although the interaction range is of infinite in this model which is unrealistic, it has many interesting characteristics of the spin glass phase. Similar interesting but unrealistic limits were investigated by many authors, for example, an infinite degeneracy limit of the Anderson model,[4] the Hubbard model in infinite dimensions[5] and the Hubbard model with infinite-range hopping.[6]

In this paper we propose a Hamiltonian which includes infinite-range interaction between electrons. On this point, our model is also somewhat unrealistic but it shows many interesting many-body effects as discussed later. It is not bilinear and the two-body interaction of electrons are treated exactly. The essential point of the solvability of our Hamiltonian is that it becomes a sum of decoupled Hamiltonians for each k mode in the momentum space. There is no interaction between different k modes and the interaction works only between electrons with the same momentum. Since the Hamiltonian is a sum of that of each k mode, the eigenstates are given by a tensor product of each k mode state. We can obtain all the thermodynamic quantities in the grand canonical ensemble. The explicit ground state is also obtained rather easily. Using the ground state wave function, we can demonstrate the break down of the Luttinger's theorem[7] in the repulsive case, that is, the phase space volume of the momentum space enclosed by a sharp step in the momentum distribution is modified from that of a noninteracting case. This means that the assumption of adiabatic continuation[8] no longer holds and the concept of Fermi liquid theory breaks down. We can show the ex-

istence of a metal-insulator transition as a function of the strength of the interaction. When the filling is away from half filled, the system is always metallic. At half filling, there is a metal insulator transition at the critical value of the interaction which is equal to the band width. Since the interaction is of infinite range, our Hamiltonian necessarily tends to represent strongly correlated electrons. Also the dimensionality only enters as the number of integrations in the expressions for the physical quantities. Thus it does not alter the properties of the system qualitatively.

A brief summary of this work was presented in ref. 9 and we discuss the model in detail in this paper. In §2, we present the hamiltonian. In §3, we discuss the ground state and excitations of the model. In §4, we show the thermodynamics. In §3 and §4, numerical results will be mostly presented for one dimension. A summary is given in §5.

§2. Model Hamiltonian

We consider the following Hamiltonian as a simplified model for correlated lattice electrons in any dimensions

$$H=-t\sum_{\langle i,j\rangle,\sigma} c_{i\sigma}^{\dagger}c_{j\sigma}+h.c. +\frac{U}{L^d}\sum_{j_1j_2j_3j_4}\delta_{j_1+j_3,j_2+j_4}c_{j_1\uparrow}^{\dagger}c_{j_2\uparrow}c_{j_3\downarrow}^{\dagger}c_{j_4\downarrow}, \quad (2.1)$$

where the first term represents hopping of electrons and the sum is taken over all nearest-neighbor pairs on a d-dimensional simple cubic lattice. In the second term, L^d is the number of sites and $\delta_{j_1+j_3,j_2+j_4}$ means that the sum is taken over the sites which satisfy a condition $\boldsymbol{r}_{j_1}+\boldsymbol{r}_{j_3}=\boldsymbol{r}_{j_2}+\boldsymbol{r}_{j_4}$, namely, the center of mass of the two particles is conserved in scattering process. Thus the interaction is of infinite range.[10]

Let us introduce momentum representations of the operators as

$$c_{j\sigma}=\frac{1}{L^{d/2}}\sum_k e^{ik\cdot r_j}c_{k\sigma}, \quad (2.2)$$

$$c_{j\sigma}^{\dagger}=\frac{1}{L^{d/2}}\sum_k e^{-ik\cdot r_j}c_{k\sigma}^{\dagger}, \quad (2.3)$$

where Σ_k represents a sum over $\boldsymbol{k}=((2\pi/L)l_1, (2\pi/L)l_2,\cdots,(2\pi/L)l_d)$, $l_i=1,\cdots,L$ and we can replace it by an integral as

$$\sum_k \to L^d\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\times\cdots\times\int_{-\pi}^{\pi}\frac{dk_d}{2\pi}\equiv L^d\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}, \quad (2.4)$$

in the thermodynamic limit. In the momentum representation, Hamiltonian (2.1) is written

$$H=\sum_k H_k, \quad (2.5)$$

$$H_k=\varepsilon_k(n_{k\uparrow}+n_{k\downarrow})+Un_{k\uparrow}n_{k\downarrow}, \quad (2.6)$$

$$n_{k\sigma}=c_{k\sigma}^{\dagger}c_{k\sigma}, \quad (2.7)$$

and

$$\varepsilon_k=-2t\sum_{l=1}^{d}\cos(k_l). \quad (2.8)$$

Thus the Hilbert space of H is a direct product of those of H_k's. A state is given by a tensor product of that of H_k

$$|\Psi\rangle=\bigotimes_k|\Phi_k\rangle, \quad (2.9)$$

where $|\Phi_k\rangle$ is spanned by the following four states

$$|0\rangle,\quad c_{k\uparrow}^{\dagger}|0\rangle,\quad c_{k\downarrow}^{\dagger}|0\rangle,\quad c_{k\uparrow}^{\dagger}c_{k\downarrow}^{\dagger}|0\rangle. \quad (2.10)$$

We can investigate the ground state and thermodynamics from this property.[10] Here we notice that the interaction term of the hamiltonian commute with the kinetic term. This is the origin of the solvability. However, this does not necessarily mean that there is no correlation effect. As we shows below, for example, there is a metal insulator transition (without impurities) which is a consequence of typical correlation effects. Although our hamiltonian is not invariant under a particle-hole transformation $c_{j\sigma}\to c_{j\sigma}^{\dagger}$ and $c_{j\sigma}^{\dagger}\to c_{j\sigma}$, a modified Hamiltonian

$$\tilde{H}=\sum_k\varepsilon_k(n_{k\uparrow}+n_{k\downarrow}) +U\sum_k\left(n_{k\uparrow}-\frac{1}{2}\right)\left(n_{k\downarrow}-\frac{1}{2}\right), \quad (2.11)$$

is. Since the modified hamiltonian differs from the original one only by the origin of the chemical potential and a constant, it is sufficient to consider only $\rho \leq 1$ case due to the particle-hole symmetry of $\tilde{H}$.

§3. Ground State and Excitation

First we consider an attractive case ($U<0$). The ground state is obtained by filling each k state with a pair of electrons with opposite spins

$$|\Psi_G\rangle = \prod_{\varepsilon_k \leq \varepsilon_{k_F}} (c_{k\uparrow}^{\dagger} c_{k\downarrow}^{\dagger}) |0\rangle, \tag{3.1}$$

where ε_{k_F} is the Fermi energy of a noninteracting system. The configuration of electrons is shown in Fig.1 for one dimensional case. The momentum distribution is the same as that of a noninteracting system. The ground state is not degenerate and the lowest excitation is a pair excitation across the Fermi surface. There is a zero energy excitation in the thermodynamic limit for any filling. There also exists a pair breaking excitation with energy scale $|U|$.

In the repulsive case ($U>0$), let us define $U_c(\rho)$ by

$$\rho = \int_{[-\pi,\pi]^d} \frac{dk^d}{(2\pi)^d} \theta(U_c - \varepsilon_k), \quad \text{for } \rho < 1, \tag{3.2}$$

$$U_c = W = 2td, \quad \text{(band width)}, \quad \text{for} \quad \rho = 1, \tag{3.3}$$

where $\theta(x)$ is a step function and ρ is the density of electrons. We have to consider two cases separately depending on the value of U with respect to U_c. If U is smaller than U_c, the ground state is given by

$$|\Psi_G; \{\sigma_k\}\rangle = \prod_{\varepsilon_0 < \varepsilon_k \leq \varepsilon_0 + U} (c_{k\sigma_k}^{\dagger}) \prod_{\varepsilon_k \leq \varepsilon_0} (c_{k\uparrow}^{\dagger} c_{k\downarrow}^{\dagger}) |0\rangle, \tag{3.4}$$

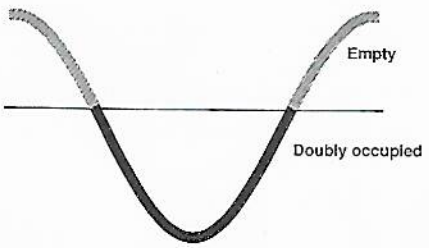

Fig. 1. Ground state configuration in one dimension with a negative interaction ($U<0$).

where ε_0 is determined by

$$\rho = \int_{[-\pi,\pi]^d} \frac{dk^d}{(2\pi)^d} \{\theta(\varepsilon_0 + U - \varepsilon_k) + \theta(\varepsilon_0 - \varepsilon_k)\}. \tag{3.5}$$

The ground state is degenerate due to the spin configuration $\{\sigma_k\}$ of the singly occupied k states (see Fig. 2(a) for a one dimensional example). The number of the degeneracy is enormous in this case. If we take one ground state, the momentum distribution is strange like Fig. 2(b) (the figure is for a one dimensional example). The physical momentum distribution of these degenerate states is given by taking an average over the states (see Fig. 2(c)). (Note

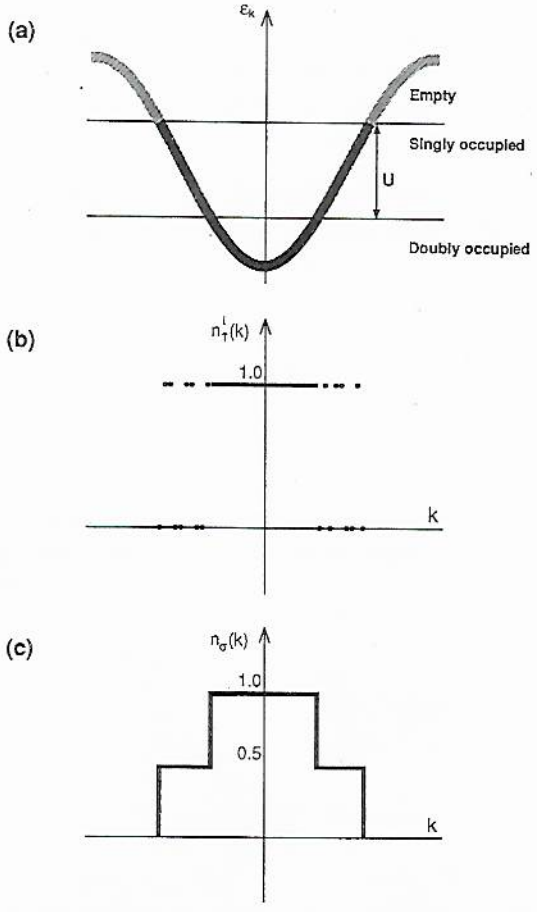

Fig. 2. (a): Ground state configuration in one dimension with a positive (not so strong) interaction ($0<U<U_c$). (b): Corresponding momentum distribution for up spin electrons in one of the ground states. (c): Physical momentum distribution of one dimensional solvable model with a positive interaction ($0<U<U_c$).

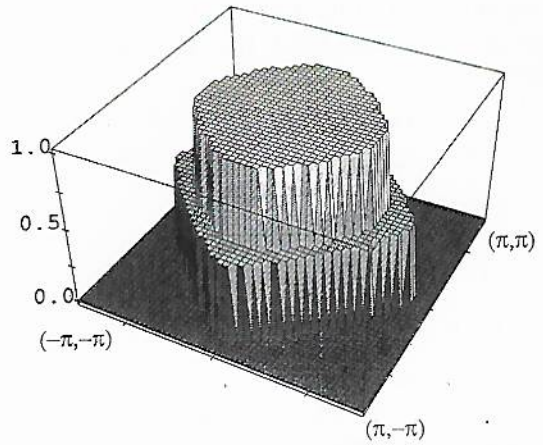

Fig. 3. Physical momentum distribution in two dimensions with a positive (not so strong) interaction ($0<U<U_c$). $U=t$ and $\rho\approx 0.73$.

that it is not necessary to take an average if we consider $n_k=n_{k\uparrow}+n_{k\downarrow}$ as a momentum distribution.) Then the physical momentum distribution is

$$\langle n_{k\uparrow}\rangle=\langle n_{k\downarrow}\rangle=\begin{cases}1, & \varepsilon_k\leq\varepsilon_0\\ \frac{1}{2}, & \varepsilon_0<\varepsilon_k\leq\varepsilon_0+U\\ 0, & \varepsilon_0+U<\varepsilon_k.\end{cases} \tag{3.6}$$

In Fig. 3, we show a typical momentum distribution in two dimensions. There are two sharp Fermi-like surfaces and the low energy excitation is given by an particle-hole excitation across these surfaces (zero energy).

If U is larger than U_c, there is no doubly occupied k state and only one Fermi surface exists. There also exists a enormous number of degeneracy in the ground state and the momentum distribution is given by

$$\langle n_{k\uparrow}\rangle=\langle n_{k\downarrow}\rangle=\begin{cases}\frac{1}{2}, & k\in R_k\\ 0, & \text{otherwise},\end{cases} \tag{3.7}$$

where R_k is the region of a Brillouin zone corresponding to the noninteracting Fermi sea which contains electrons with density 2ρ. This implies that the phase volume enclosed by the sharp step of the momentum distribution is not conserved when the interaction is introduced, that is, there exists a Fermi surface but the Luttinger's theorem breaks down.

Here it is interesting to compare the momentum distribution of our model to that of the half-filled Hubbard model. In the strong coupling limit of the Hubbard model, it is believed that the system is a Mott insulator. The momentum distribution behaves[11] as

$$\langle n_{k\uparrow}\rangle=\langle n_{k\downarrow}\rangle=\frac{1}{2}-C\frac{t}{\tilde{U}}\varepsilon(k)+o\left(\frac{t}{\tilde{U}}\right), \tag{3.8}$$

where $\tilde{U}$ represents the on-site Coulomb repulsion and C is a positive constant which is determined from the spin-spin correlation of the Heisenberg model. This behavior is similar to our case as $\tilde{U}\to\infty$.

Next let us discuss the excitation gap and the discontinuity of the chemical potential. The excitation gap is defined by

$$\Delta E=E_2(N)-E_1(N), \tag{3.9}$$

and the discontinuity of the chemical potential is defined by

$$\Delta\mu=\mu_+-\mu_-, \tag{3.10}$$

where

$$\begin{aligned}\mu_+&=E_1(N+1)-E_1(N),\\ &=\mu(\rho+0,\,T),\quad T\to 0\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\mu_-&=E_1(N)-E_1(N-1),\\ &=\mu(\rho-0,\,T).\quad T\to 0\end{aligned} \tag{3.12}$$

In these equations, $E_j(N)$ is the j-th eigenenergy of the N particle system. (In case the states are constructed from one particle states, ΔE coincides with $\Delta\mu$.) We can obtain these quantities explicitly for our model. When the filling is away from half-filled, there is always zero energy particle-hole pair excitation from inside of R_k to outside and

$$\Delta\mu=\Delta E=0. \tag{3.13}$$

Thus, the system remains always metallic. At half filling, there are two cases. When $0<U\leq U_c$, we have $\Delta\mu=\Delta E=0$ and the situation is essentially the same as less-than-half-filled case. When $U>U_c$, the zero energy excitation disappears. There is an energy gap between the ground state and the lowest excited state. The magnitude of the gap is

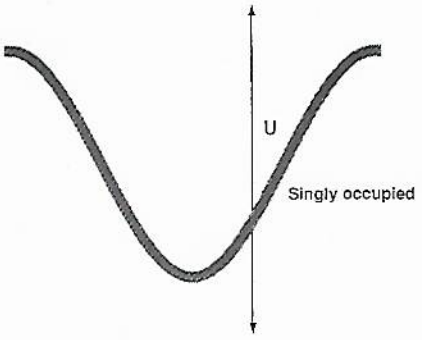

Fig. 4. Ground state configuration in one dimension with a sufficiently strong positive interaction ($U_c > U$) and half-filled electrons.

given by

$$\Delta E = U - W, \tag{3.14}$$

where $W=2td$ is the band width. In this case, the ground state is given by filling all k states singly as shown in Fig. 4. We have to make a doubly occupied state to make an excited state. This is the origin of the energy gap. When we want to add or remove one electron to this half-filled ground state, it costs a finite energy

$$\Delta\mu = U - W, \tag{3.15}$$

$$\mu_+ = U - \frac{W}{2}, \tag{3.16}$$

$$\mu_- = \frac{W}{2}. \tag{3.17}$$

Here it is interesting to compare these quantities to the Lieb-Wu[2] results for the one-dimensional Hubbard model. The above quantities in the half-filled Hubbard model are given by

$$\Delta\mu = \tilde{U} - 4t + 2g(\tilde{U}), \tag{3.18}$$

$$\mu_+ = \tilde{U} - 2t + g(\tilde{U}), \tag{3.19}$$

$$\mu_- = 2t - g(\tilde{U}), \tag{3.20}$$

$$g(\tilde{U}) = 4\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{1+\frac{1}{4}n^2\tilde{U}^2}-\frac{1}{2}n\tilde{U}\right). \tag{3.21}$$

Note that the band width in one-dimension is $W=4t$. In a strong coupling limit, we have $g(\tilde{U})\to 0$ and the results exactly coincide with ours for $U>U_c$ case. It is rather interesting that our model gives similar results to the strong coupling limit of the Hubbard model, although the interactions in the two models U and $\tilde{U}$ have totally different nature, namely, U gives long-range interaction while $\tilde{U}$ gives on-site interaction. The interaction range of our model is of infinite and it tends to represent strong coupling even if U is finite.

Here we have demonstrated that there occurs a metal-insulator transition at $U=U_c$ when we change the strength of the interaction. This metal insulator transition occurs only in the half-filled case. The system is always metallic in other fillings. The existence of the *metal-insulator* transition is explicitly shown in *any* dimensions in our model.

§4. Thermodynamics

The partition function $Z(\beta,\mu)$ of our model in the grand canonical ensemble is

$$Z(\beta,\mu) = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \prod_k \mathrm{Tr}_k \exp\left[-\beta\{H_k - \mu(n_{k\uparrow}+n_{k\downarrow})\}\right],$$

$$= \prod_k f_k, \tag{4.1}$$

$$f_k = 1 + 2z_k + z_k^2 e^{-\beta U} \tag{4.2}$$

$$z_k = \exp\left[-\beta(\varepsilon_k - \mu)\right], \tag{4.3}$$

where N is the number operator, $\beta=1/T$ and μ is the chemical potential. The grand potential $\Omega(\beta,\mu)$ is

$$\Omega(\beta,\mu) = -\frac{1}{\beta}\ln Z(\beta,\mu) = -\frac{1}{\beta}\sum_k \ln f_k. \tag{4.4}$$

Once $\Omega(\beta,\mu)$ is obtained, all the thermodynamic quantities are calculated in the grand canonical ensemble. Here we show explicit expressions for some physical quantities. The particle density ρ is given as

$$\rho(\beta,\mu)=-\frac{1}{L^d}\frac{\partial}{\partial\mu}\Omega(\beta,\mu),$$
$$=2\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{z_k(1+z_k e^{-\beta U})}{f_k}. \tag{4.5}$$

By solving (4.5) for μ, we have

$$\mu=\mu(\beta,\rho). \tag{4.6}$$

In the following, we consider physical quantities as a function of β and ρ. For a $T=0$ case, we can get an analytical evaluation for (4.5) as

$$\rho=\begin{cases}\displaystyle\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}[\theta(\mu-\varepsilon_k)+\theta(\mu-\varepsilon_k-U)], & \text{for} \quad U\geq 0 \quad (4.7)\\ \displaystyle\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\left[2\theta\left(\mu-\varepsilon_k-\frac{U}{2}\right)\right]. & \text{for} \quad U<0 \quad (4.8)\end{cases}$$

So we can evaluate a ground state charge compressibility as

$$\kappa\equiv\left(\frac{d\mu}{d\rho}\right)^{-1}=\frac{d\rho}{d\mu}=\begin{cases}D_d(\mu)+D_d(\mu-U), & \text{for} \quad U\geq 0 \quad (4.9)\\ 2D_d\left(\mu-\dfrac{U}{2}\right), & \text{for} \quad U<0, \quad (4.10)\end{cases}$$

where

$$D_d(\mu)=\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\delta(\mu-\varepsilon_k), \tag{4.11}$$

is the density of states for non-interacting lattice electrons in d dimensions. When U is positive, we can interpret (4.9) as the existence of upper and lower Hubbard bands separated by U. The behavior of the chemical potential discussed in the previous section is also seen from this expression. The behavior of the charge compressibility κ is seen from the knowledge of the behavior of $D_d(\mu)$. When $U>U_c$ and half filled, for example, κ behaves

$$\kappa\approx|1-\rho|^{1-(2/d)}, \quad (\rho\approx 1, \rho\neq 1),$$
$$\kappa=0, \quad (\rho=1). \tag{4.12}$$

For $d=2$ case, there is a logarithmic divergence at quarter filling due to the Van Hove singularity. In one dimension, we can evaluate ρ and κ analytically as

$$\rho=2-\frac{1}{\pi}\left(E\left(\frac{\mu-U}{2t}\right)\cos^{-1}\frac{\mu-U}{2t}+E\left(\frac{\mu}{2t}\right)\cos^{-1}\frac{\mu}{2t}\right), \tag{4.13}$$

$$\kappa=\frac{1}{\pi}\left(E\left(\frac{\mu-U}{2t}\right)\frac{1}{\sqrt{4t^2-(\mu-U)^2}}+E\left(\frac{\mu}{2t}\right)\frac{1}{\sqrt{4t^2-\mu^2}}\right), \quad (U\geq 0) \tag{4.14}$$

and

$$\rho=2-\frac{2}{\pi}E\left(\frac{2\mu-U}{4t}\right)\cos^{-1}\frac{2\mu-U}{4t}, \tag{4.15}$$

$$\kappa=\frac{2}{\pi}\frac{1}{\sqrt{4t^2-\left(\mu-\dfrac{U}{2}\right)^2}}E\left(\frac{2\mu-U}{4t}\right), \quad (U<0), \tag{4.16}$$

where $E(x)$ is

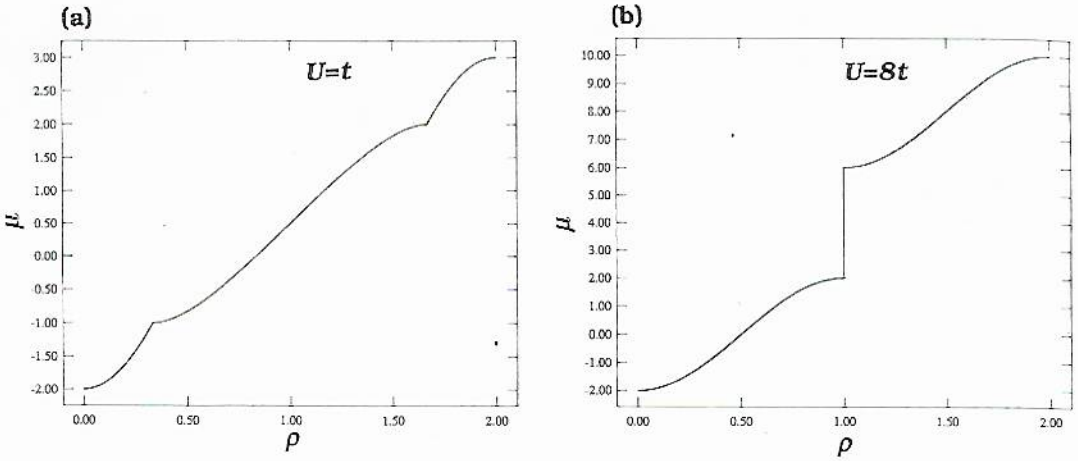

Fig. 5. Ground state chemical potential as a function of filling in one dimension. (a): $U=t$, (b): $U=8t$.

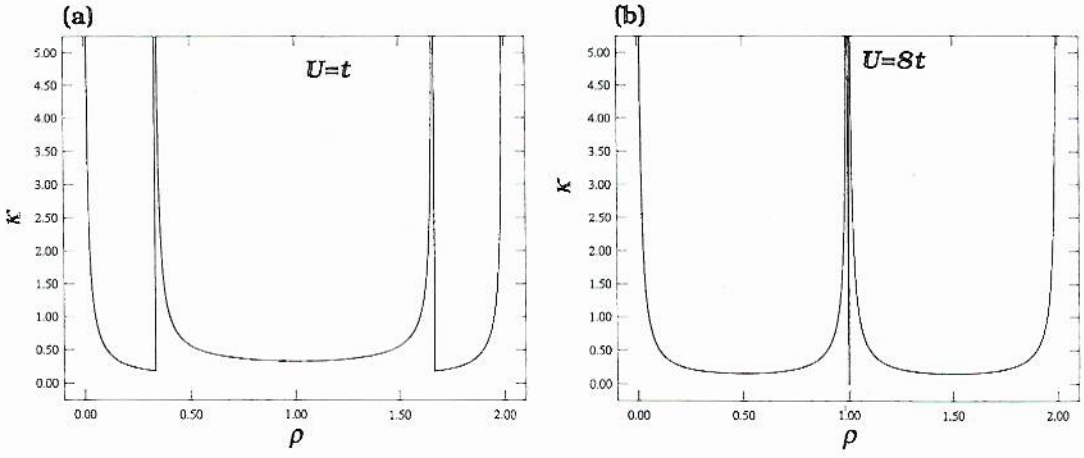

Fig. 6. Ground state charge compressibility as a function of filling in one dimension. (a): $U=t$, (b): $U=8t$.

$$E(x)=\begin{cases}1 & \text{for} \quad 0\leq x\leq 1\\ 0 & \text{otherwise.}\end{cases}$$

In Figs. 5 and 6, we show μ and κ as a function of ρ for $U=t$ and $U=8t$ cases.

Energy per site $e(\beta, \rho)$ is given as

$$e(\beta,\rho)=\frac{1}{L^d}\left[\frac{\partial}{\partial\beta}(\beta\Omega)|_{\mu}+\mu\langle N\rangle\right],$$

$$=2\int_{[-\pi,\pi]^d}\frac{\mathrm{d}k^d}{(2\pi)^d}\frac{z_k\left\{\varepsilon_k+\left(\varepsilon_k+\dfrac{U}{2}\right)z_k\,\mathrm{e}^{-\beta U}\right\}}{f_k}. \tag{4.17}$$

When the interaction is repulsive and sufficiently strong $U\geq U_c$, there is a symmetry in the energy. In the strong coupling case, all the k modes are singly occupied in its ground state. We can see that the energies of systems with filing ρ and $1-\rho$ is the same by a simple energetic consideration.

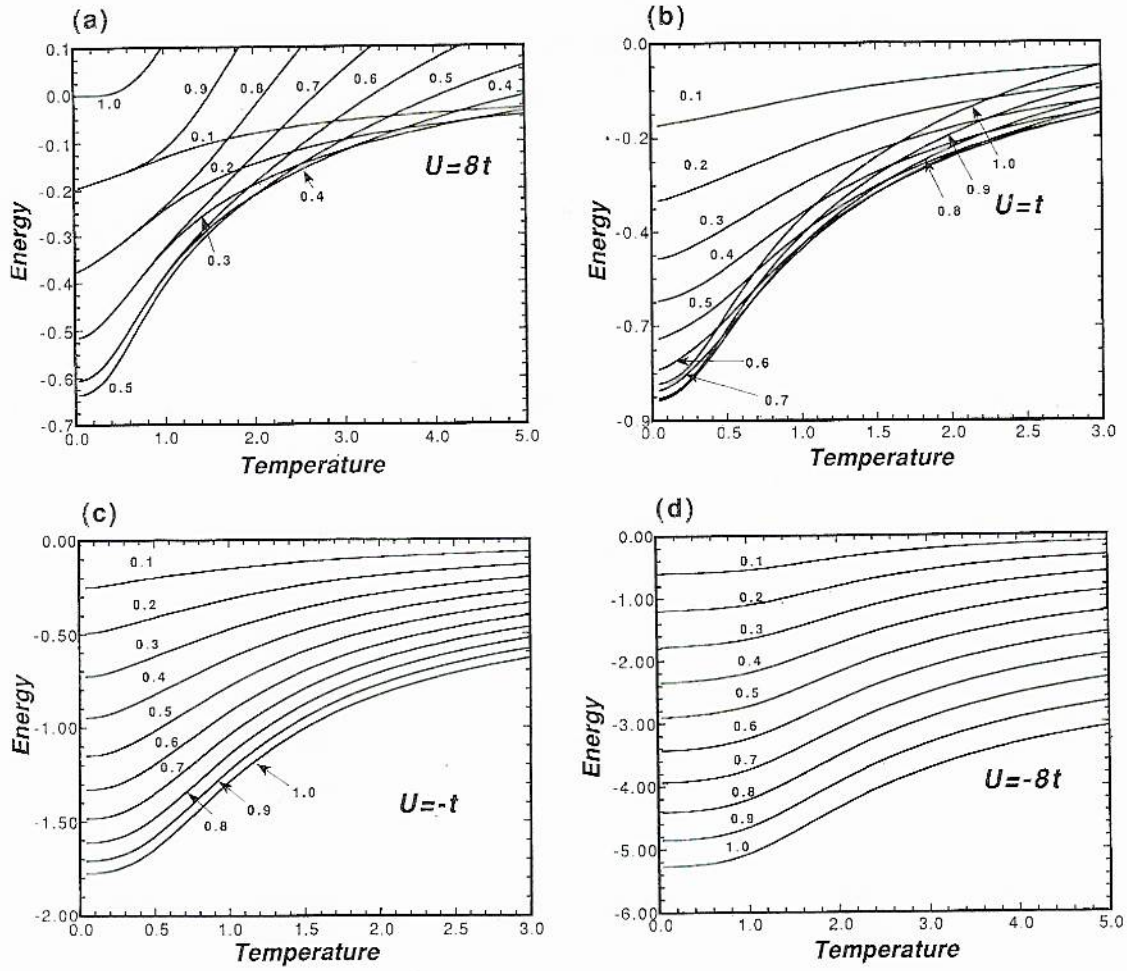

Fig. 7. Internal energy per site as a function of temperature with various filling ρ in one dimension. (a): $U=8t>U_c$, (b): $U=t<U_c$, (c): $U=-t$, and (d): $U=-8t$.

$$e(\rho)=e(1-\rho). \quad (T\to 0,\ U>U_c) \tag{4.18}$$

In Figs. 7, we show numerical results of the energy in one dimension for various U with various fillings. (In the following, we show numerical data for various physical quantities in one dimension. In evaluating the momentum integration, we use a discretization of the momentum space which corresponds to that of 500 sites system. There is no essential difficulty in performing a calculation even in higher dimensions.) From Figs. 7, one can see that $e(\rho)$ satisfy (4.18) for the case (a): $U=8$ and $T\to 0$.

Specific heat is calculated from

$$C(\beta,\rho)=\left.\frac{\partial e(\beta,\mu(\beta,\rho))}{\partial T}\right|_{\rho}=\left.\frac{\partial e(\beta,\mu)}{\partial T}\right|_{\mu}+\left.\frac{\partial \mu(\beta,\rho)}{\partial T}\right|_{\rho}\left.\frac{\partial e(\beta,\mu)}{\partial \mu}\right|_{\beta}, \tag{4.19}$$

where $(\partial\mu/\partial T)|_{\rho}$ is given by the condition

$$\left.\frac{\partial \rho(\beta,\mu(\beta,\rho))}{\partial T}\right|_{\rho}=0=\left.\frac{\partial \rho(\beta,\mu)}{\partial T}\right|_{\mu}+\left.\frac{\partial \mu(\beta,\rho)}{\partial T}\right|_{\rho}\left.\frac{\partial \rho(\beta,\mu)}{\partial \mu}\right|_{\beta}. \tag{4.20}$$

Then $C(\beta,\rho)$ is expressed as

$$C(\beta,\rho)=2\beta^2\left(C_1-\frac{C_2\tilde{C}_2}{C_3}\right), \tag{4.21}$$

$$C_1=\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{1}{f_k^2}z_k\left\{\varepsilon_k\tilde{\varepsilon}_k+2\left(\varepsilon_k+\frac{U}{2}\right)\left(\tilde{\varepsilon}_k+\frac{U}{2}\right)z_k\,e^{-\beta U}+(\varepsilon_k+U)(\tilde{\varepsilon}_k+U)z_k^2\,e^{-\beta U}\right\}, \tag{4.22}$$

$$C_2=\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{1}{f_k^2}z_k\left\{\varepsilon_k+2\left(\varepsilon_k+\frac{U}{2}\right)z_k\,e^{-\beta U}+(\varepsilon_k+U)z_k^2\,e^{-\beta U}\right\}, \tag{4.23}$$

$$\tilde{C}_2=\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{1}{f_k^2}z_k\left\{\tilde{\varepsilon}_k+2\left(\tilde{\varepsilon}_k+\frac{U}{2}\right)z_k\,e^{-\beta U}+(\tilde{\varepsilon}_k+U)z_k^2\,e^{-\beta U}\right\}, \tag{4.24}$$

$$C_3=\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{1}{f_k^2}z_k(1+2z_k\,e^{-\beta U}+z_k^2\,e^{-\beta U}), \tag{4.25}$$

$$\tilde{\varepsilon}_k=\varepsilon_k-\mu. \tag{4.26}$$

In Figs. 8, we show numerical results of specific heat for $U=8$, 1 -1, and -8 with various fillings. For $U=8$, 1 -1, and -8 case, the results is consistent with the metallic behavior, that is $C(\beta,\rho)\propto T$, $(T\to 0)$. On the other hand, when the interaction is repulsive and strong enough ($U\geq U_c$), and filling is half-filled, $C(\beta,\rho)$ shows an exponential behavior in low temperature which is characteristic for the existence of an excitation gap (see Fig. 8(a)). In the infinite-range hopping model of Dongen and Vollhardt,[6] specific heat always behaves exponentially as a function of $\beta=1/T$ which seems unphysical for a metallic case. By using the particle-hole symmetry, we obtain

$$C(\beta,1+\rho)=C(\beta,1-\rho), \tag{4.27}$$

and when $U\geq U_c$, we get

$$C(\beta,1-\rho)=C(\beta,\rho),\quad (T\to 0) \tag{4.28}$$

due to (4.18).

Entropy per site $s(\beta,\rho)$ is given by

$$s(\beta,\rho)=-\frac{1}{L^d}\frac{\partial\Omega(\beta,\mu(\rho))}{\partial T}=\beta\{e(\beta,\rho)-\rho\mu(\beta,\rho)-\Omega(\beta,\mu(\rho))\}. \tag{4.29}$$

For negative U, entropy goes to zero at $T=0$, because there is no degeneracy in the ground state. On the other hand, there is a finite entropy even in the ground state for positive U. This is due to the macroscopic degeneracy of the ground state. In Fig. 9, we show numerical results of entropy for $U=8$, 1 -1, and -8 with various fillings. From the particle-hole symmetry, we obtain

$$s(\beta,1+\rho)=s(\beta,1-\rho), \tag{4.30}$$

because the entropy do not depend on the origin of energy.

Pressure is given by

$$P(\beta,\rho)=\frac{1}{L^d}\Omega(\beta,\mu(\rho))=\frac{1}{\beta}\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\ln f_k. \tag{4.31}$$

In Fig. 10, we show numerical results of pressure for $U=8$, 1 -1, and -8 with various fillings.

Next let us calculate the ratio of doubly occupied sites $\bar{d}=D/L^d$ where D is defined by

$$D=\sum_j\langle n_{j\uparrow}n_{j\downarrow}\rangle. \tag{4.32}$$

In the momentum representation D is written as

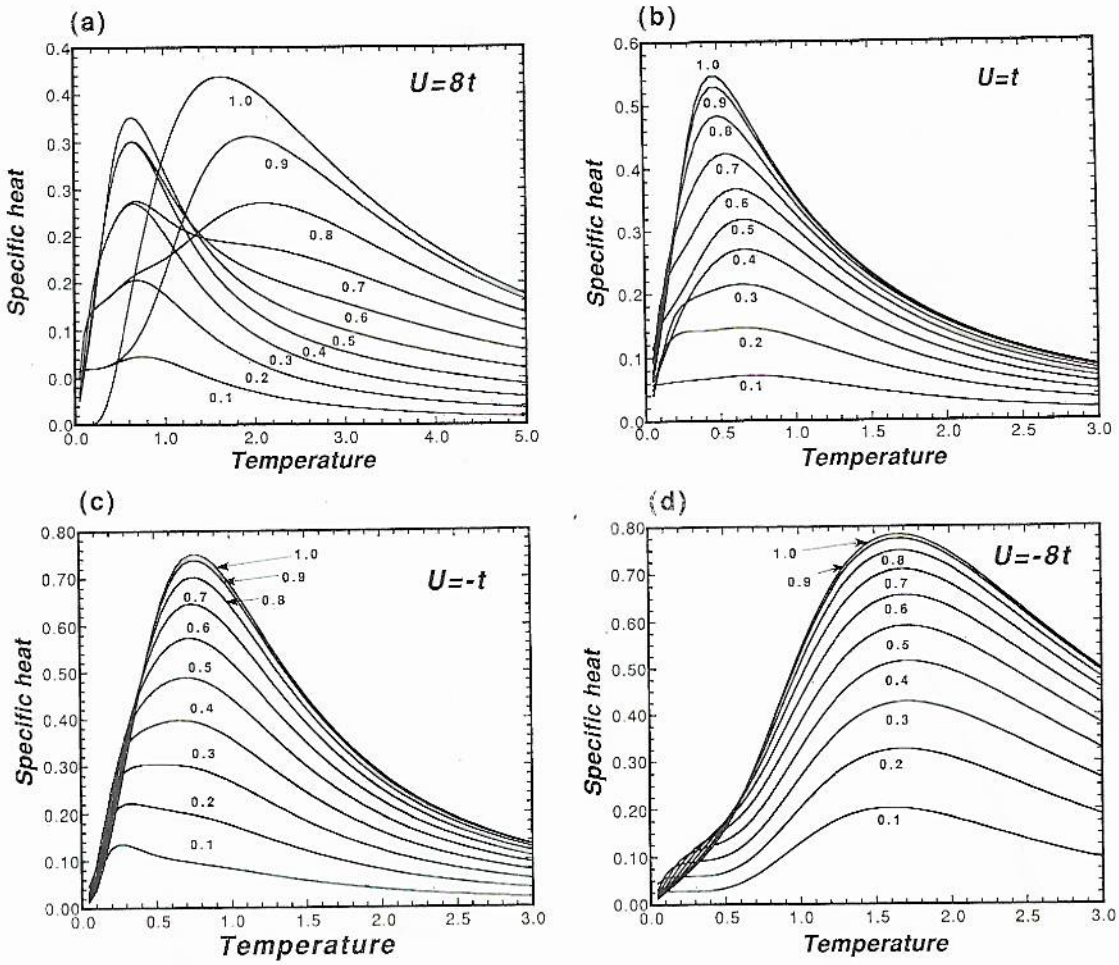

Fig. 8. Specific heat as a function of temperature with various filling ρ in one dimension. (a): $U=8t>U_c$, (b): $U=t<U_c$, (c): $U=-t$, and (d): $U=-8t$.

$$D=\frac{1}{L^d}\sum_{k,K}\langle n_{k\uparrow}n_{K\downarrow}\rangle,$$
$$=\frac{1}{L^d}\{\sum_k\langle n_{k\uparrow}n_{k\downarrow}\rangle+\sum_k\langle n_{k\uparrow}\rangle\sum_k\langle n_{k\uparrow}\rangle-\sum_k\langle n_{k\uparrow}\rangle\langle n_{k\downarrow}\rangle\}, \tag{4.33}$$

so $\bar{d}$ is given by

$$\bar{d}=\left(\frac{\rho}{2}\right)^2-\frac{1}{L^d}\bar{D}, \tag{4.34}$$

$$\bar{D}=\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{1}{f_k^2}z_k^2(1-e^{-\beta U}), \tag{4.35}$$

where we have used following relations:

$$\sum_k\langle n_{k\uparrow}n_{k\downarrow}\rangle=\frac{\partial\Omega}{\partial U}=L^d\int_{[-\pi,\pi]^d}\frac{dk^d}{(2\pi)^d}\frac{1}{f_k}z_k e^{-\beta U}, \tag{4.36}$$

$$\langle n_{k\uparrow}\rangle=\langle n_{k\downarrow}\rangle=\frac{1}{2}\frac{\partial\Omega}{\partial\varepsilon_k}=\frac{1}{f_k}(1+z_k e^{-\beta U})z_k. \tag{4.37}$$

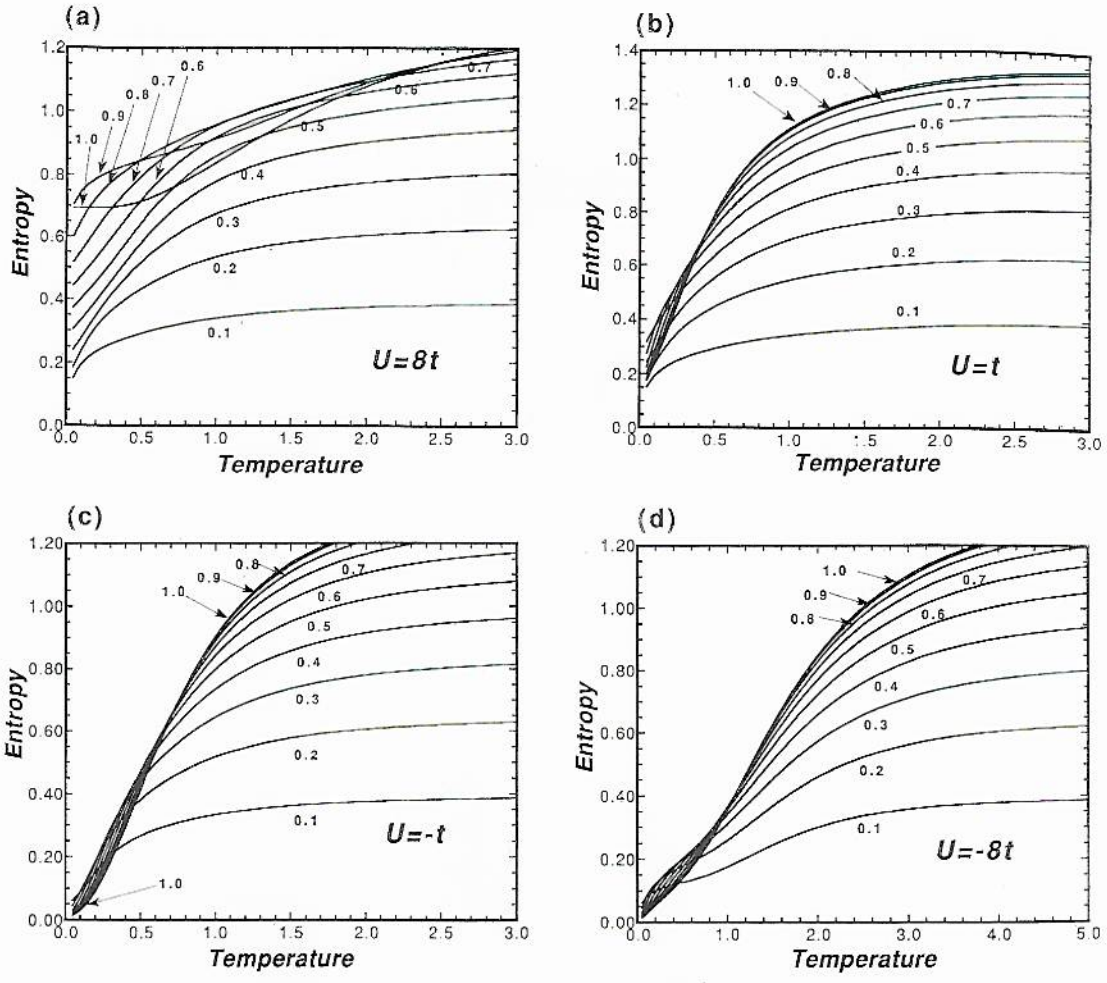

Fig. 9. Entropy per site as a function of temperature with various filling ρ in one dimension. (a): $U=8t>U_c$, (b): $U=t<U_c$, (c): $U=-t$, and (d): $U=-8t$.

The ratio of doubly occupied sites $\bar{d}$ is independent of U in the thermodynamic limit ($L\to\infty$) as seen from (4.34). This seemingly unphysical result is due to the infinite range interaction of our Hamiltonian. When we consider a finite size system, we can interpret that $\bar{D}$ shows correlations in the real space because the sign of $\bar{D}$ is determined by the sign of U in (4.35). In Figs.11, we show numerical results of $\bar{D}$ for $U=8$, 1 -1, and -8 with various fillings. The particle-hole symmetry gives

$$\bar{D}(1+\rho)=\bar{D}(1-\rho). \tag{4.38}$$

Similarly we can calculate correlation functions for any order parameters. For example, we calculate an equal time correlation function of superconducting singlet pairing. The order parameter O_j is given by

$$O_j=c_{j\uparrow}c_{j\downarrow}. \tag{4.39}$$

We define the equal time correlation function as

$$S_O(k)=\frac{1}{L^d}\sum_j e^{-ik\cdot r_j}\langle O_0 O_j^\dagger\rangle. \tag{4.40}$$

If there is a long range-order in O_j, $S_O(k)$ is finite for some k in the thermodynamic limit. In our case, (4.40) is evaluated as

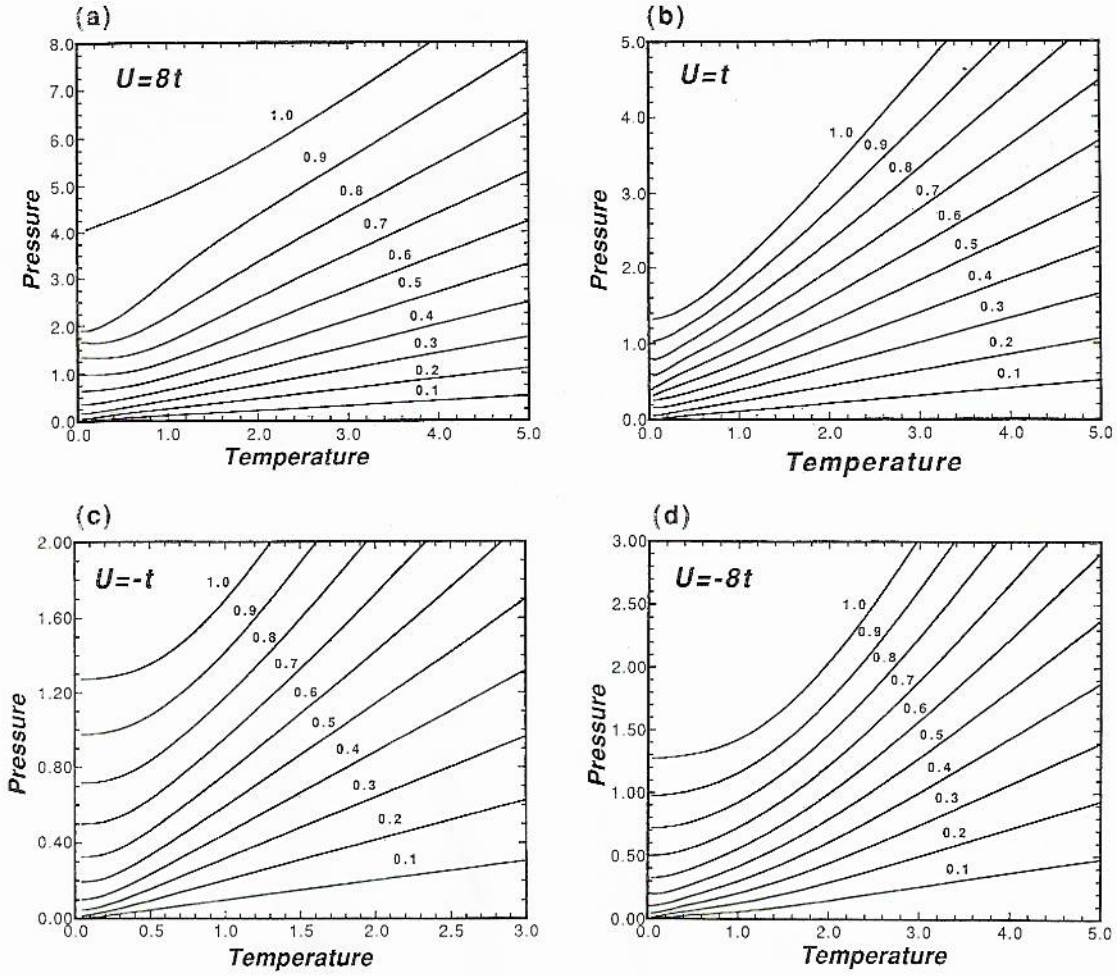

Fig. 10. Pressure as a function of temperature with various filling ρ in one dimension. (a): $U=8t>U_c$, (b): $U=t<U_c$, (c): $U=-t$, and (d): $U=-8t$.

$$S_O(k)=\left(\frac{1}{L^d}\right)^2\sum_K\{1-\langle n_{K\uparrow}\rangle-\langle n_{K\downarrow}\rangle+\langle n_{K\downarrow}n_{-K+k\downarrow}\rangle,$$
$$=\frac{1}{L^d}\left\{1-\rho+\frac{1}{L^d}\sum_K\langle n_{K\downarrow}n_{-K+k\downarrow}\rangle\right\}. \tag{4.41}$$

We can evaluate it by using (4.36) and the difference from that of $U=0$ case is given by

$$\delta S_O(k)=S_O(k)-S_O(k)|_{U=0}=\frac{1}{L^{2d}}\frac{z_{k/2}^2}{f_{k/2}^2}(e^{-\beta U}-1). \tag{4.42}$$

This implies that O_j is of short range and there is no superconductive instability. It merely shows that sign of $\delta S_O(k)$ depends on the sign of U.

§5. Summary

In this paper we have presented an exactly solvable model for correlated lattice electrons in any dimensions and have studied the ground state and thermodynamics. Our

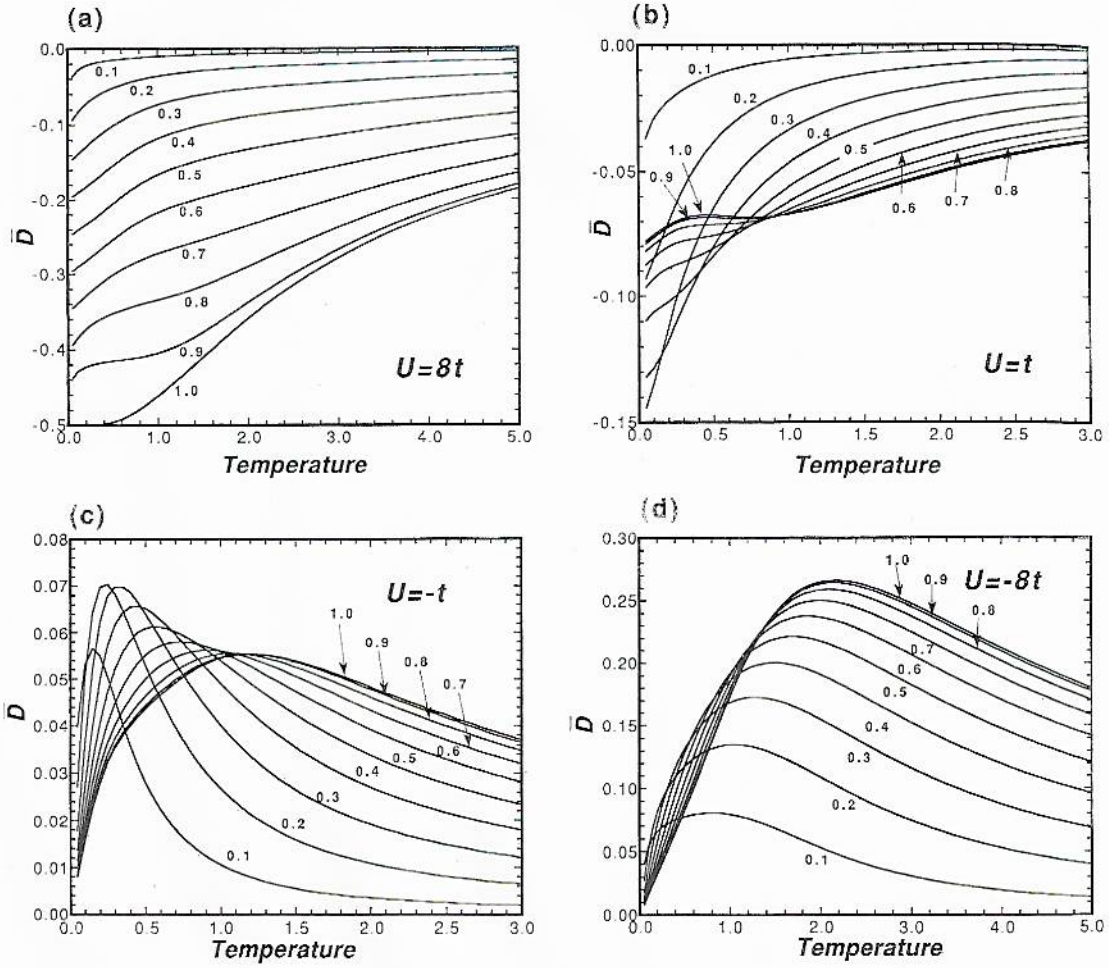

Fig. 11. Tendency of the double occupation $\bar{D}$ from a noninteracting case as a function of temperature with various filling ρ in one dimension. Note that the double occupation does not depend on the strength of the interaction (see text). (a): $U=8t>U_c$, (b): $U=t<U_c$, (c): $U=-t$, and (d): $U=-8t$.

Hamiltonian includes nearest-neighbor hopping and infinite range four-body interaction. In the momentum space, the Hamiltonian H is decomposed into decoupled Hamiltonian H_k for each k modes. There are four eigenstates for H_k, $|0\rangle$, $c_{k\uparrow}^{+}|0\rangle$, $c_{k\downarrow}^{+}|0\rangle$, and $c_{k\uparrow}^{+}c_{k\downarrow}^{+}|0\rangle$. We were able to evaluate the grand potential in the grand canonical ensemble and the thermodynamic quantities analytically since the eigenstates of H are tensor products of the eigenstates of H_k.

Using the explicit ground state wave function which is obtained by energy consideration, we evaluated the momentum distribution function. In an attractive case, the Fermi surface of a noninteracting system is unchanged and the ground state is composed of pairs of electrons with opposite spins filled up to the Fermi energy in the momentum space. Low energy excitations is not particle-hole excitation but a pairwise excitation across the Fermi surface. On the other hand, the momentum distribution of a repulsive case is not continuously connected to a noninteracting system. There are one or two sharp discontinuities in the momentum distribution and the phase space volume enclosed by the discontinuity continuously changes with the strength of interaction. Thus the Luttinger's theorem does not hold. The concept of adiabatic con-

tinuation and the Fermi liquid theory break down.

As for the metal-insulator transition, we have found that the system is always metallic away from half filling. In a repulsive case at half filling, there is a phase transition from a metallic phase to an insulating phase when the strength of the interaction is increased. We have found that the critical value of the metal-insulator transition is equal to the band width in any dimensions. Similar phenomena occur in the excitation gap. At half filling, the zero energy excitation disappears when the strength of the interaction exceeds the band width when the interaction is repulsive. In other cases, there are always gapless excitations. The behavior is seen in specific heat at low temperatures obtained numerically (see Fig. 8).

Although our model is somewhat artificial, we can get all the physical quantities including correlation functions and momentum distribution functions which are usually difficult to treat, for example, by the Bethe Ansaz technique. Furthermore, this model has many physical properties expected from approximate studies of the correlated electron system such as the metal insulator-transition at the strength of the band width.

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Note added—After this work was completed, we noticed that a mathematically similar model was discussed in the context of the Anderson localization.[12] The structure of the Hilbert space is similar but the physical quantities treated are totally different from these in the present work. Recently Baskaran also discussed a similar model in a continuum space.[13] In the model, the interaction term is given by $J\Sigma S_k^2-J(\Sigma S_k)^2$ where $S_k=(1/2)\Sigma_{ss'}c_{ks}^{+}[\sigma]_{ss'}c_{ks'}$ is a spin operator in the momentum space. The first term is equivalent to our interaction $U\Sigma n_{k\uparrow}n_{k\downarrow}$ if we set $U=(3/2)J$. When $J>0$, that is, an antiferromagnetically coupled case, doubly occupied states are not preferable. In this case, it gives qualitatively similar results to our repulsive ones.