# Intelligent Systems Programming

Lecture 5: **Boolean Expression Representations & Binary Decision Diagrams (BDDs)**

# Today's Program

- **[12:00-12:45] Classical representations**
  - Boolean expressions and Boolean functions
  - Desirable properties of representations of Boolean functions
  - Classical representations of Boolean expressions
    - Truth tables
    - Two-level normal forms: CNF, DNF
- **[12:55-14:00] Binary Decision Diagrams**
  - If-then-else normal form (INF)
  - Decision trees
  - Ordered Binary Decision Diagrams (OBDDs)
  - Reduced Ordered Binary Decision Diagrams (ROBDDs / BDDs)

# Boolean Expressions

- Boolean Expressions

  $t ::= x \mid 0 \mid 1 \mid \neg t \mid t \wedge t \mid t \vee t \mid t \Rightarrow t \mid t \Leftrightarrow t$

- Literals

  $l ::= x \mid \neg x$

- Precedence

  $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

- Terminology
  - **Boolean expression** = Boolean formula/Propositional formula/Sentence in propositional logic
  - **Boolean variable** = Propositional symbol/letter/variable

# Boolean Functions

- **Definition**

  An $n$-ary function $f : \boldsymbol{B}^n \rightarrow \boldsymbol{B}$ ($\boldsymbol{B} = \{0,1\}$)

  $$f(x_1,x_2,\ldots,x_n) = E(x_1,x_2,\ldots,x_n),$$

  where $E$ is a Boolean expression

- **Example**

  $$f(x_1,x_2,x_3) = x_1 \Leftrightarrow \neg x_2$$

# Properties of Boolean Functions

- Equality

  $f = g \quad \text{iff} \quad \forall \boldsymbol{x} \,.\, f(\boldsymbol{x}) = g(\boldsymbol{x})$

- Several expressions may represent a function

  $f(x,y) = x \Rightarrow y = \neg x \vee y = (\neg x \vee y) \wedge (\neg x \vee x) = \ldots$

- Order of arguments matter

  $f(x,y) = x \Rightarrow y \quad \neq \quad g(y,x) = x \Rightarrow y$

- Number of Boolean functions $f : \boldsymbol{B}^n \rightarrow \boldsymbol{B}$

$$2^{\left(2^n\right)}$$

# Desirable properties of a representation

1. Compact
2. Equality check easy
3. Easy to evaluate the truth-value of an assignment
4. Boolean operations efficient
5. SAT check efficient
6. Tautology check efficient
7. Canonicity: exactly one representation of each Boolean function
   – Solves 2, 5, and 6, why?

# Compact representations are rare

- $2^{\left(2^n\right)}$ boolean functions in *n* variables…
  - How do we find a single compact representation for them all?
- The fraction of Boolean functions of *n* variables with a polynomial size in $n \rightarrow 0$ for $n \rightarrow \infty$

**Curse of Boolean function representations**:

This problem exists for all representations we know!

# Classical Representations of Boolean Functions

# Truth tables

- Compact table size $2^n$
- Equality check easy canonical
- Easy to evaluate the truth-value of an assignment
  log m or constant
- Boolean operations efficient linear
- SAT check efficient linear
- Tautology check efficient linear

| $x$ | $y$ | $z$ | $x \wedge y \vee z$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

# Two-level normal forms: DNF CNF

- Is there a DNF and CNF of every expression?
- Given a truth table representation of a Boolean formula, can we easily define a DNF and CNF of the formula?

| $x$ | $y$ | $z$ | $e$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

# Two-level normal forms: DNF CNF

- Example DNF of $e$

| $x$ | $y$ | $z$ | $e$ | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 0 | 0 | 1 | 1 | $\neg x \wedge \neg y \wedge z \ \vee$ |
| 0 | 1 | 0 | 0 | |
| 0 | 1 | 1 | 1 | $\neg x \wedge y \wedge z \ \vee$ |
| 1 | 0 | 0 | 0 | |
| 1 | 0 | 1 | 1 | $x \wedge \neg y \wedge z \ \vee$ |
| 1 | 1 | 0 | 1 | $x \wedge y \wedge \neg z \ \vee$ |
| 1 | 1 | 1 | 1 | $x \wedge y \wedge z$ |

# Two-level normal forms: DNF CNF

- Example CNF of *e*

| $x$ | $y$ | $z$ | $e$ | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\neg(\neg x \wedge \neg y \wedge \neg z) \ \wedge$ |
| 0 | 0 | 1 | 1 | |
| 0 | 1 | 0 | 0 | $\neg(\neg x \wedge y \wedge \neg z) \ \wedge$ |
| 0 | 1 | 1 | 1 | |
| 1 | 0 | 0 | 0 | $\neg(x \wedge \neg y \wedge \neg z) \ \wedge$ |
| 1 | 0 | 1 | 1 | |
| 1 | 1 | 0 | 1 | |
| 1 | 1 | 1 | 1 | |

# Two-level normal forms: DNF CNF

- Example CNF of *e*

| $x$ | $y$ | $z$ | $e$ | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $(x \vee y \vee z) \ \wedge$ |
| 0 | 0 | 1 | 1 | |
| 0 | 1 | 0 | 0 | $(x \vee \neg y \vee z) \ \wedge$ |
| 0 | 1 | 1 | 1 | |
| 1 | 0 | 0 | 0 | $(\neg x \vee y \vee z)$ |
| 1 | 0 | 1 | 1 | |
| 1 | 1 | 0 | 1 | |
| 1 | 1 | 1 | 1 | |

# Two-level normal forms: DNF CNF

Every Boolean formula has a DNF and CNF representation

- The special version DNF and CNF representations produced from *on* and *off*-tuples are canonical and called cDNF and cCNF
- Are cDNF and cCNF minimum size DNF and CNF representations?

# Two-level normal forms: DNF CNF

- Symmetry properties of DNF and CNF

| | SAT | Tautology |
|---|---|---|
| **CNF** | NP complete | Polynomial (exercise) |
| **DNF** | Polynomial (exercise) | Co-NP complete |

- Idea: Solve CNF-SAT by conversion to DNF-SAT
  - Problem: conversion between CNF and DNF may be exponential

# Two-level normal forms: DNF CNF

- Example
  - CNF $\left(x_0^1 \vee x_1^1\right) \wedge \left(x_0^2 \vee x_1^2\right) \wedge \cdots \wedge \left(x_0^n \vee x_1^n\right)$
  - Corresponding DNF blows up

$$
\begin{gathered}
\left(x_0^1 \wedge x_0^2 \wedge \cdots \wedge x_0^{n-1} \wedge x_0^n\right) \vee \\
\left(x_0^1 \wedge x_0^2 \wedge \cdots \wedge x_0^{n-1} \wedge x_1^n\right) \vee \\
\vdots \\
\left(x_1^1 \wedge x_1^2 \wedge \cdots \wedge x_1^{n-1} \wedge x_0^n\right) \vee \\
\left(x_1^1 \wedge x_1^2 \wedge \cdots \wedge x_1^{n-1} \wedge x_1^n\right)
\end{gathered}
$$

# Binary Decision Diagrams

# If-then-else operator

- The *if-then-else* Boolean operator is defined by

$$x \to y_1, y_0 \equiv (x \land y_1) \lor (\neg x \land y_0)$$

- We have

$$(x \to y_1, y_0)[1/x] \equiv (1 \land y_1) \lor (0 \land y_0) \equiv y_1$$

$$(x \to y_1, y_0)[0/x] \equiv (0 \land y_1) \lor (1 \land y_0) \equiv y_0$$

# If-then-else operator

- All operators in propositional logic can be expressed using only $\rightarrow$ operators with:
  - $\rightarrow$ expressions and 0 and 1 for $y_1$ and $y_0$
  - tests on un-negated variables
- What are *if-then-else* expressions for
  - $x$, $\neg x$
  - $x \wedge y$
  - $x \vee y$
  - $x \Rightarrow y$

# If-then-else Normal Form (INF)

An *if-then-else* Normal Form (INF) is a Boolean expression build entirely from the if-then-else operator and the constants 0 and 1 such that all test are performed only on un-negated variables

- **Proposition**: any Boolean expression $t$ is equivalent to an expression in INF

  Proof:

  $t \equiv x \rightarrow t[1/x], t[0/x]$ (Shannon expansion of $t$)

  Apply the Shannon expansion recursively on $t$. The recursion must terminate in 0 or 1, since the number of variables is finite

# Shannon Expansion

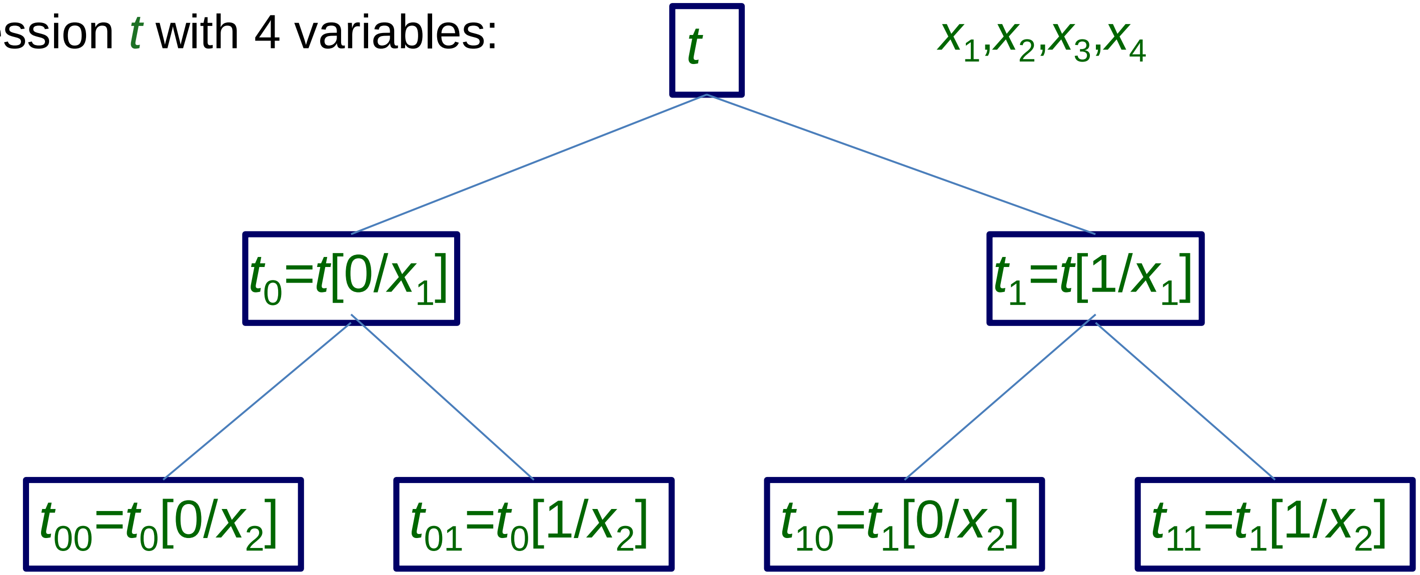
Expression $t$ with 4 variables: $\boxed{t}$ $x_1, x_2, x_3, x_4$

$\boxed{t_0 = t[0/x_1]}$ $\boxed{t_1 = t[1/x_1]}$

$$t = x_1 \rightarrow t_1,\ t_0$$

# Shannon Expansion

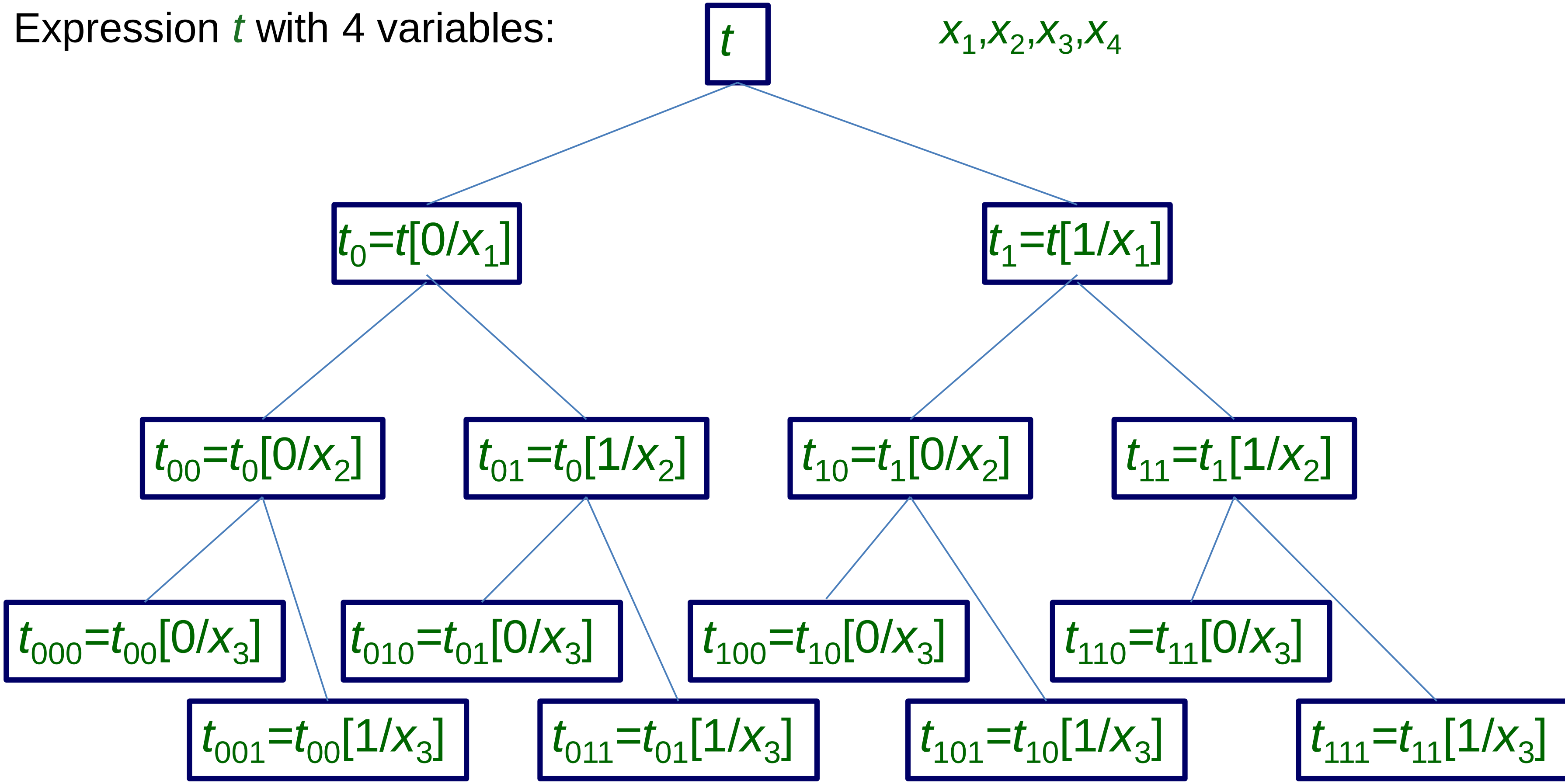
Expression $t$ with 4 variables:




$$t = x_1 \rightarrow t_1,\ t_0$$
$$t_0 = x_2 \rightarrow t_{01},\ t_{00}$$
$$t_1 = x_2 \rightarrow t_{11},\ t_{10}$$

# Shannon Expansion

Expression $t$ with 4 variables:

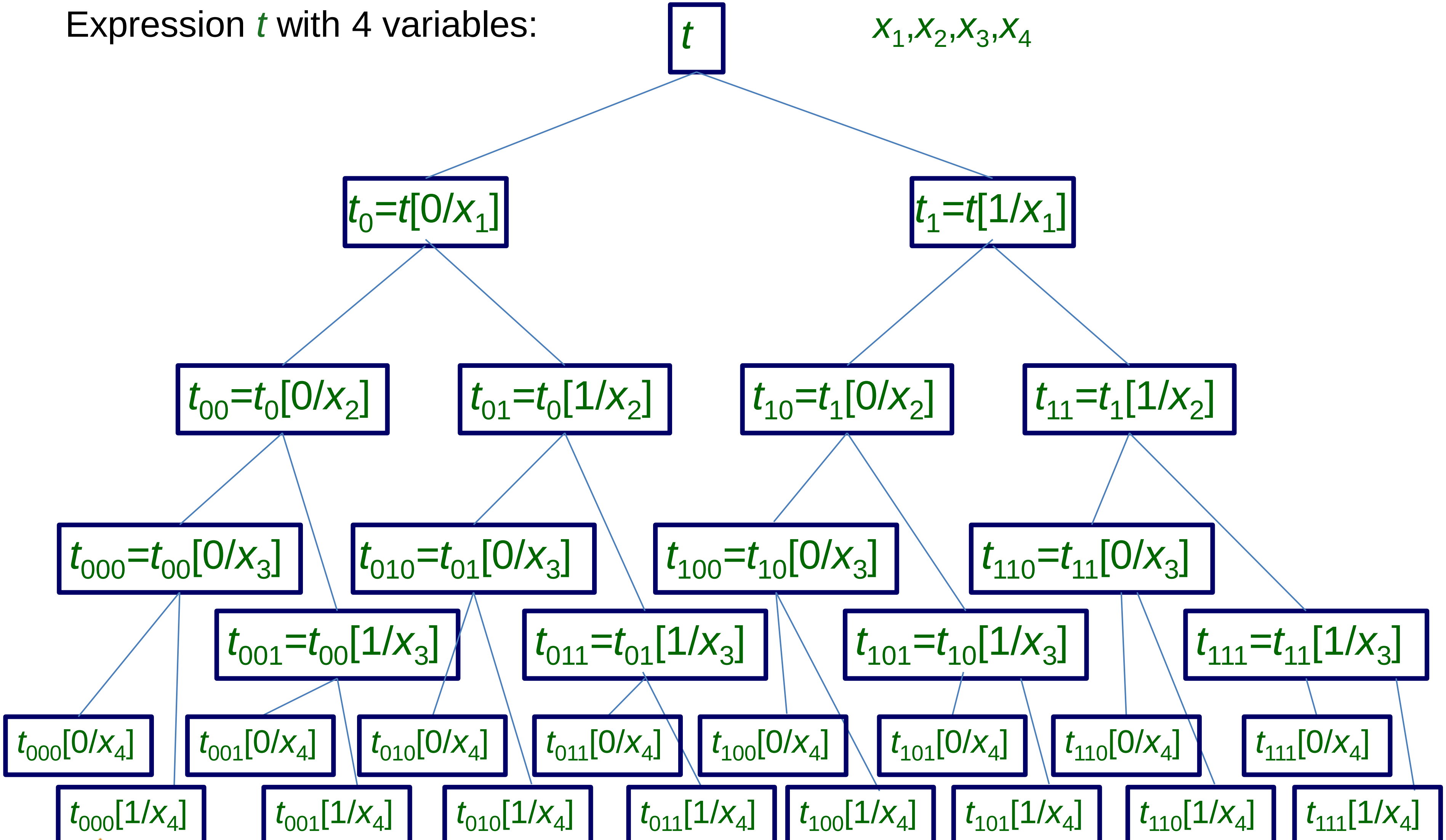



$$t = x_1 \to t_1,\ t_0$$
$$t_0 = x_2 \to t_{01},\ t_{00}$$
$$t_1 = x_2 \to t_{11},\ t_{10}$$

$$t_{00} = x_3 \to t_{001},\ t_{000}$$
$$t_{01} = x_3 \to t_{011},\ t_{010}$$
$$t_{10} = x_3 \to t_{101},\ t_{110}$$
$$t_{11} = x_3 \to t_{111},\ t_{110}$$

# Shannon Expansion

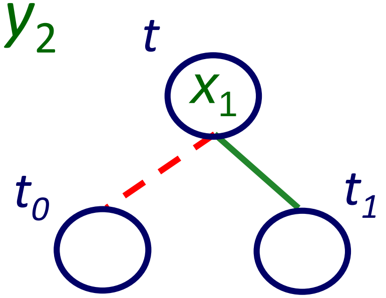
Expression *t* with 4 variables:



# Example

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of $t$ in order $x_1,y_1,x_2,y_2$

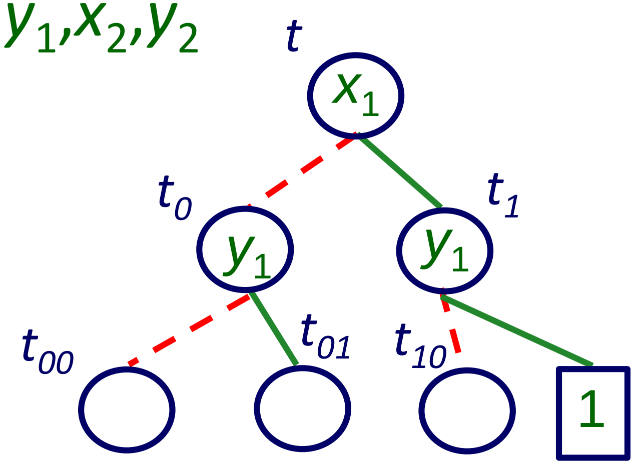
  $t \quad = x_1 \rightarrow t_1, t_0$



# Example

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of $t$ in order $x_1, y_1, x_2, y_2$

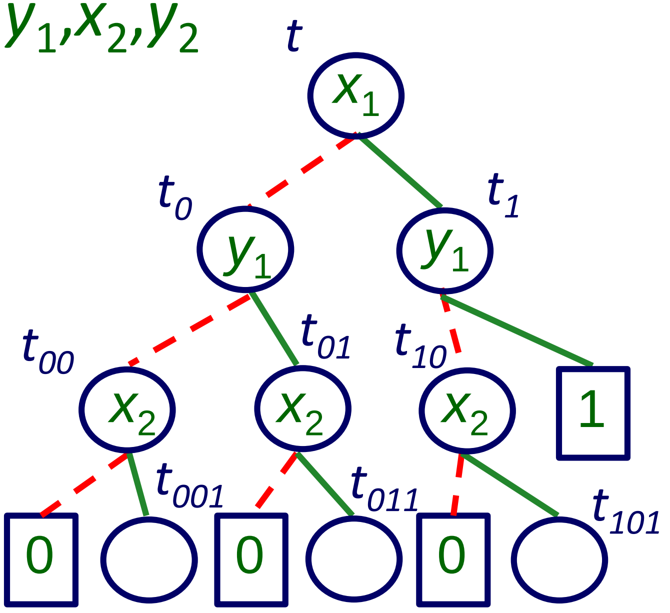
$$t = x_1 \rightarrow t_1, t_0$$
$$t_0 = y_1 \rightarrow t_{01}, t_{00}$$
$$t_1 = y_1 \rightarrow 1, t_{10}$$



# Example

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of $t$ in order $x_1,y_1,x_2,y_2$

$$
\begin{aligned}
t &= x_1 \rightarrow t_1, t_0 \\
t_0 &= y_1 \rightarrow t_{01}, t_{00} \\
t_1 &= y_1 \rightarrow 1, t_{10} \\
t_{01} &= x_2 \rightarrow t_{011}, 0 \\
t_{00} &= x_2 \rightarrow t_{001}, 0 \\
t_{10} &= x_2 \rightarrow t_{101}, 0
\end{aligned}
$$



# Decision Tree

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of $t$ in order $x_1, y_1, x_2, y_2$

$t \quad = x_1 \rightarrow t_1, t_0$

$t_0 \quad = y_1 \rightarrow t_{01}, t_{00}$

$t_1 \quad = y_1 \rightarrow 1, t_{10}$

$t_{01} \quad = x_2 \rightarrow t_{011}, 0$

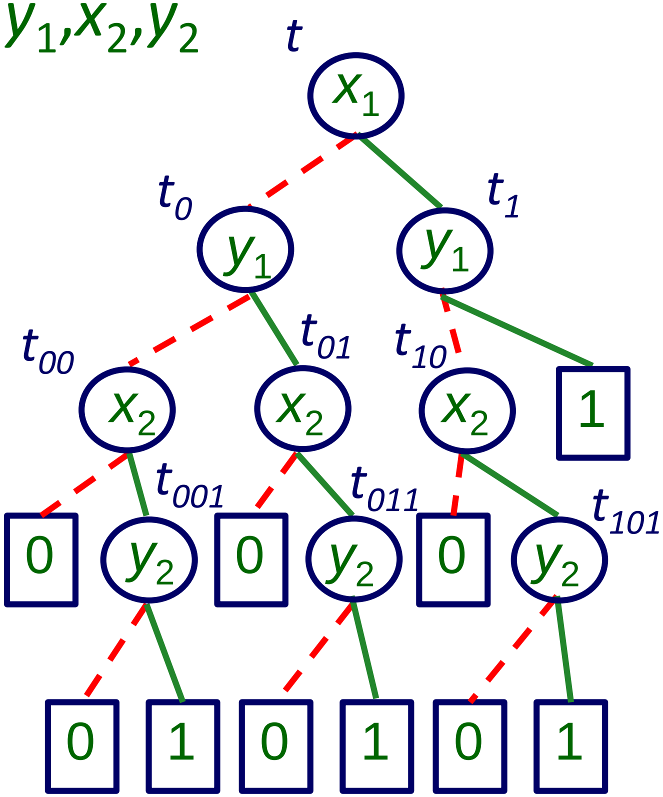
$t_{00} \quad = x_2 \rightarrow t_{001}, 0$

$t_{10} \quad = x_2 \rightarrow t_{101}, 0$

$t_{011} \quad = y_2 \rightarrow 1,0$

$t_{001} \quad = y_2 \rightarrow 1,0$

$t_{101} \quad = y_2 \rightarrow 1,0$

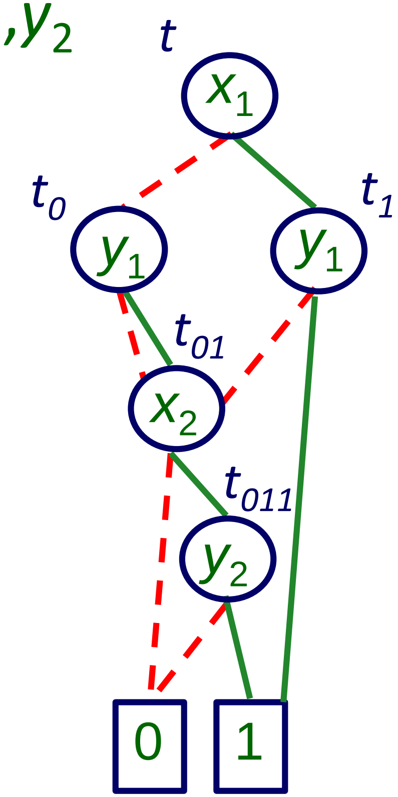



$$t \quad = x_1 \rightarrow (y_1 \rightarrow 1,(x_2 \rightarrow (y_2 \rightarrow 1,0), 0)),$$
$$(y_1 \rightarrow (x_2 \rightarrow (y_2 \rightarrow 1,0), 0), (x_2 \rightarrow (y_2 \rightarrow 1,0), 0))$$

# Reduction I: substitute identical subtrees

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of $t$ in order $x_1, y_1, x_2, y_2$

$$
\begin{aligned}
t &= x_1 \rightarrow t_1, t_0 \\
t_0 &= y_1 \rightarrow t_{01}, t_{01} \\
t_1 &= y_1 \rightarrow 1, t_{01} \\
t_{01} &= x_2 \rightarrow t_{011}, 0 \\
t_{011} &= y_2 \rightarrow 1,0
\end{aligned}
$$

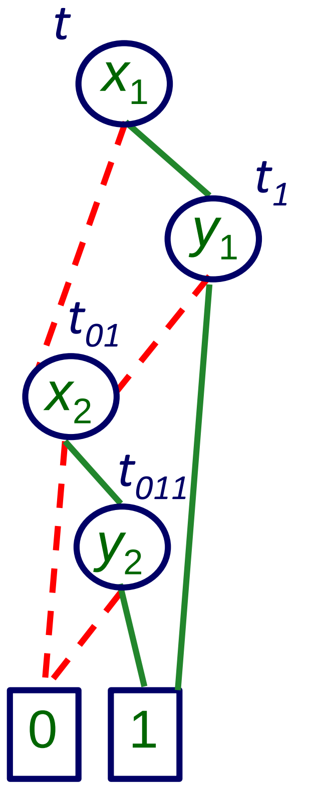



Result: an Ordered Binary Decision Diagram (OBDD)

# Reduction II: remove redundant tests

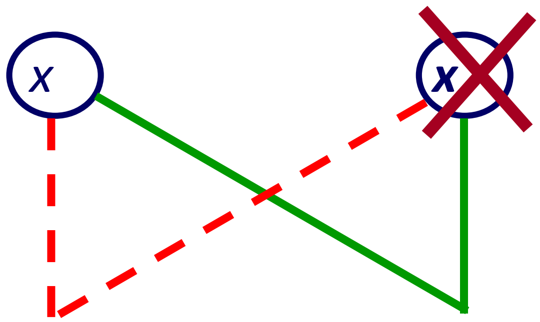
- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of $t$ in order $x_1,y_1,x_2,y_2$

$$t = x_1 \rightarrow t_1, t_{01}$$
$$t_1 = y_1 \rightarrow 1, t_{01}$$
$$t_{01} = x_2 \rightarrow t_{011}, 0$$
$$t_{011} = y_2 \rightarrow 1,0$$

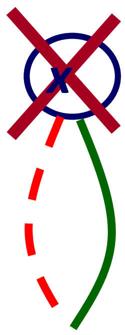



Result: a Reduced Ordered Binary Decision Diagram (ROBDD)
[often called a BDD]

# Reductions

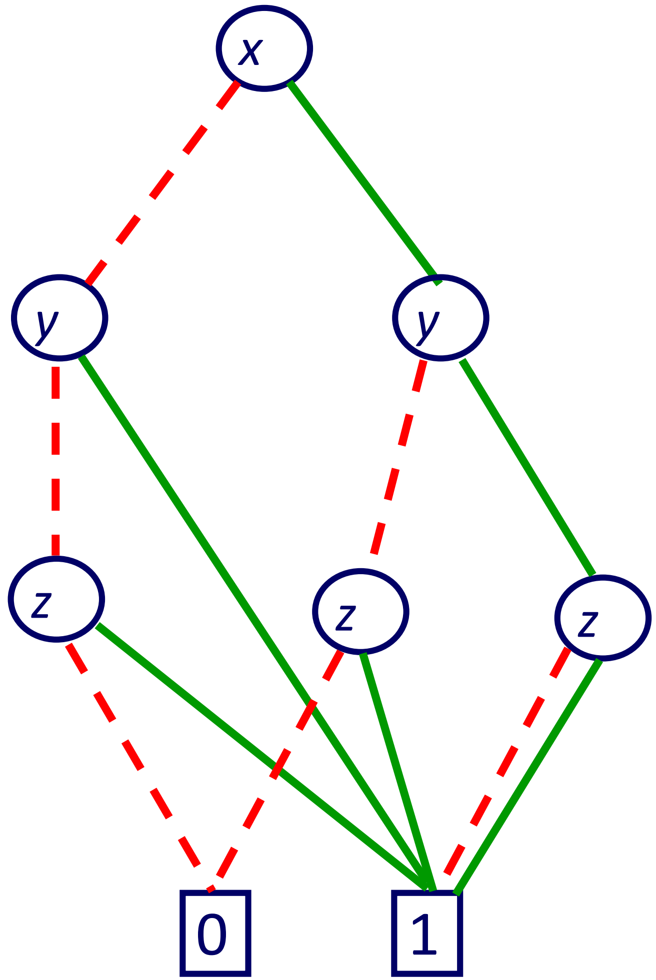



*Uniqueness requirement*

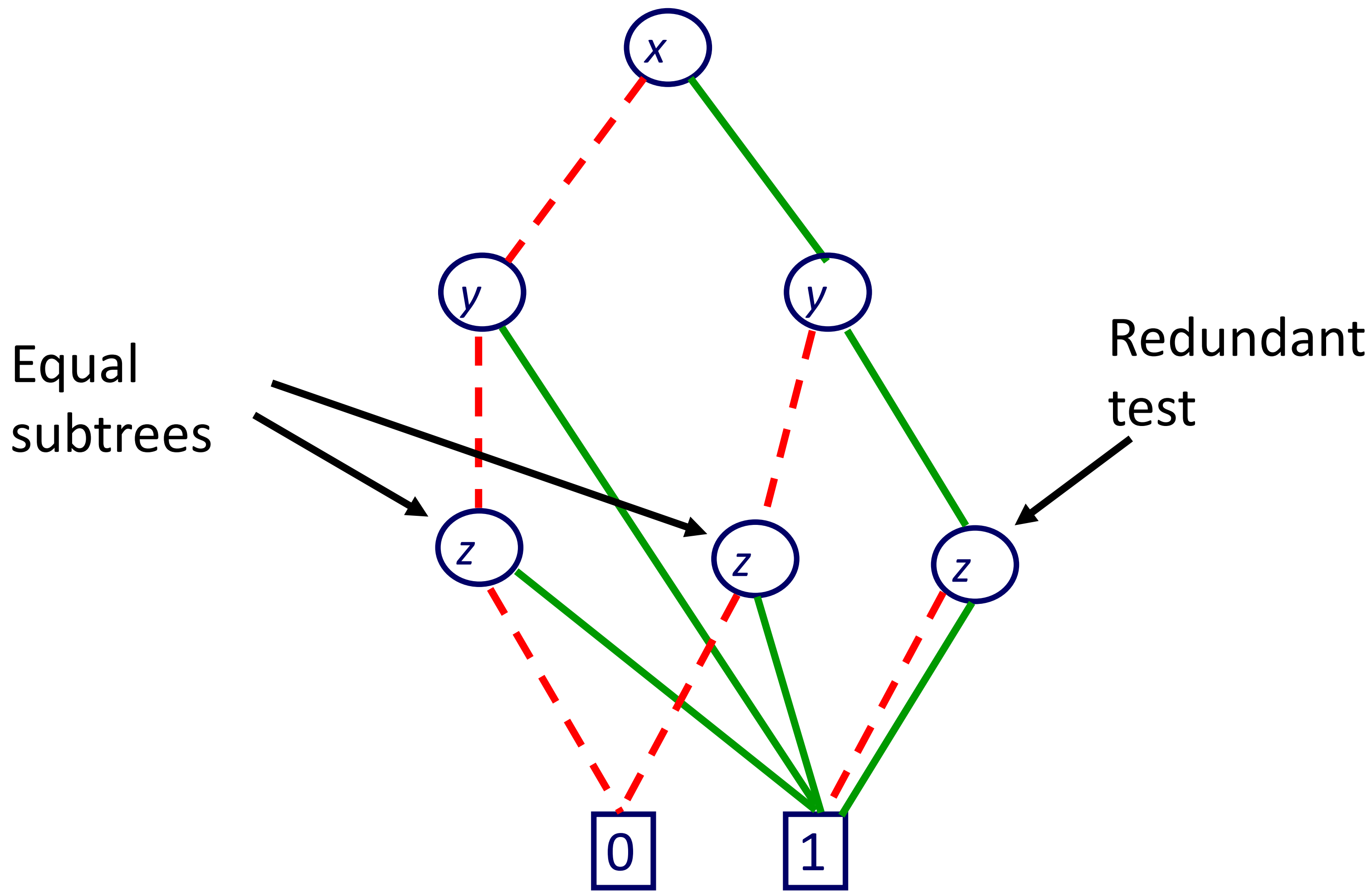



*Non-redundant tests requirement*

# Another reduction example

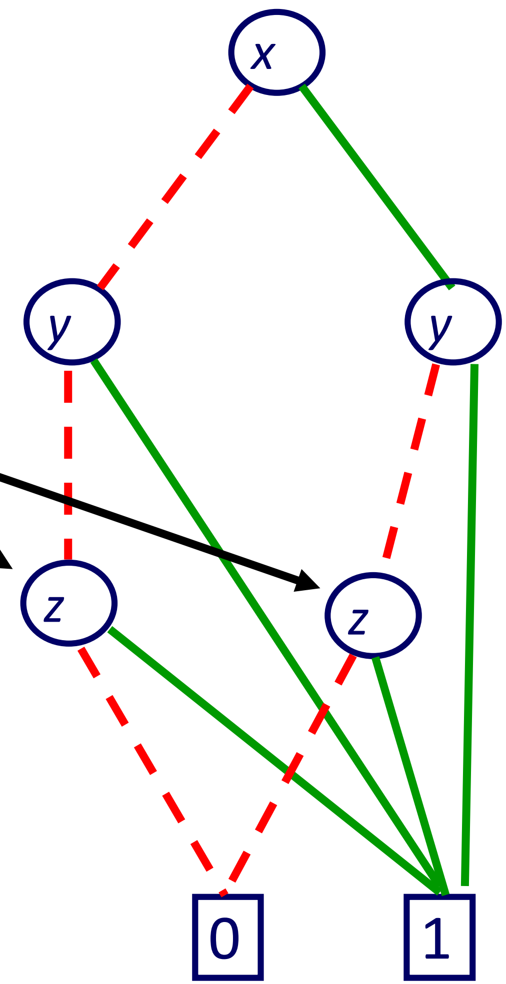


# Another reduction example

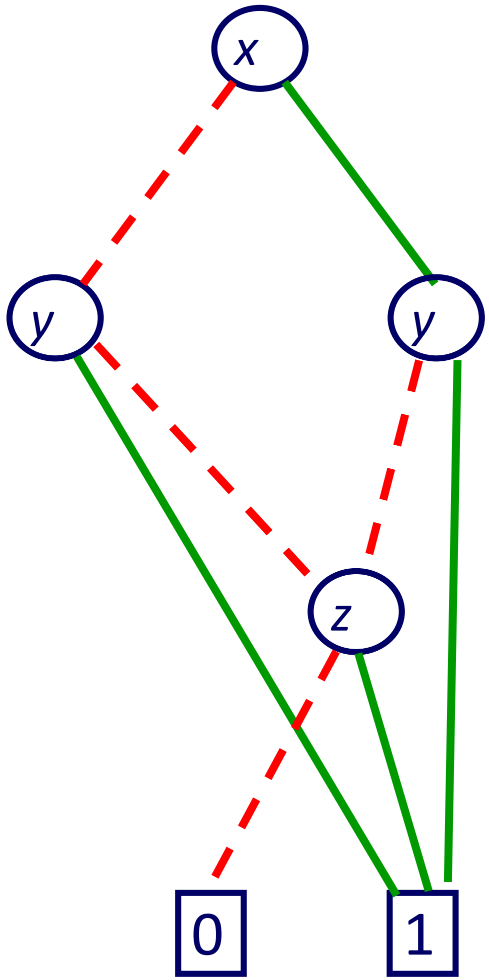


# Another reduction example

Equal
subtrees



# Another reduction example



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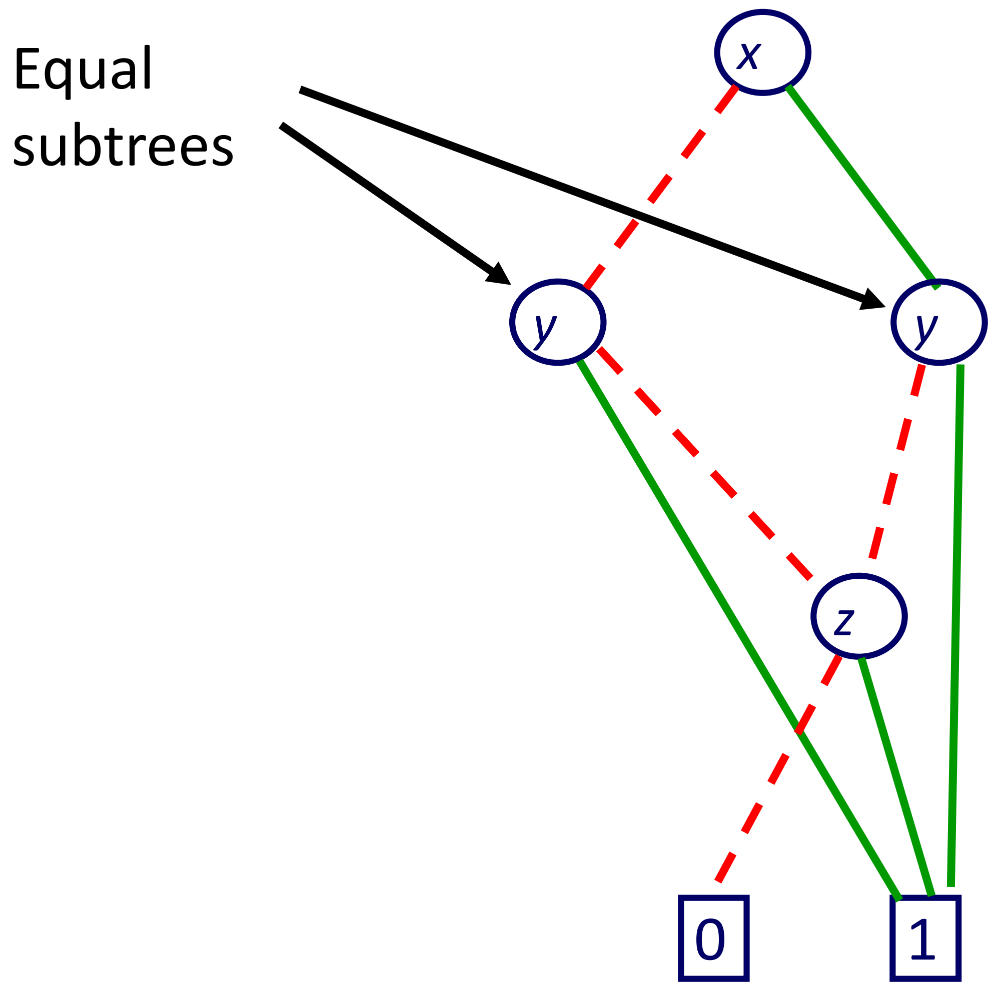

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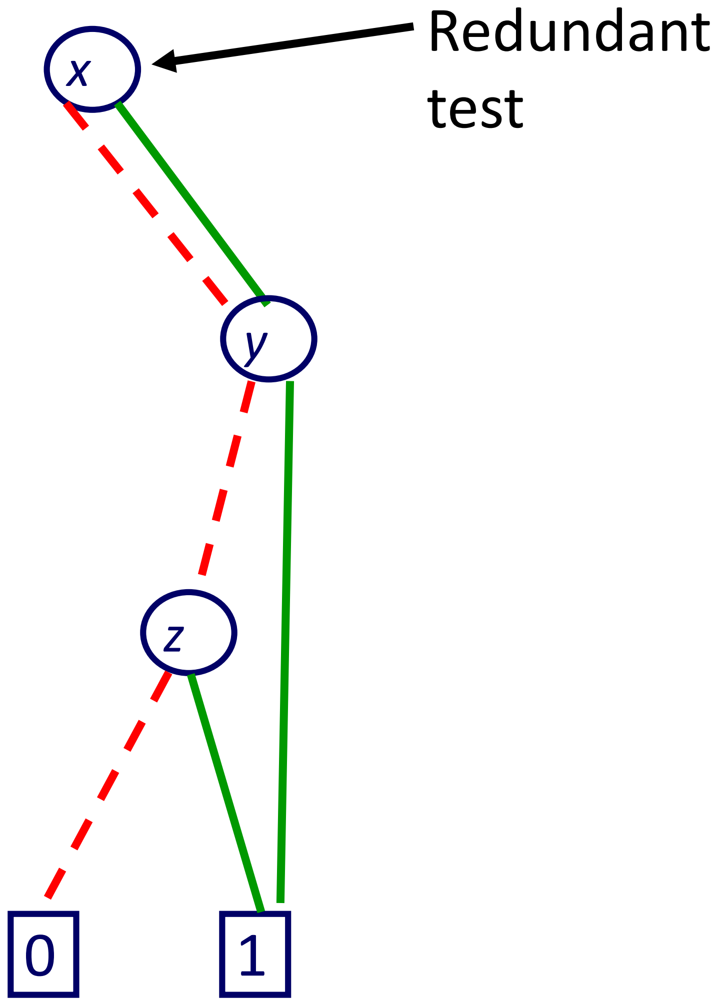
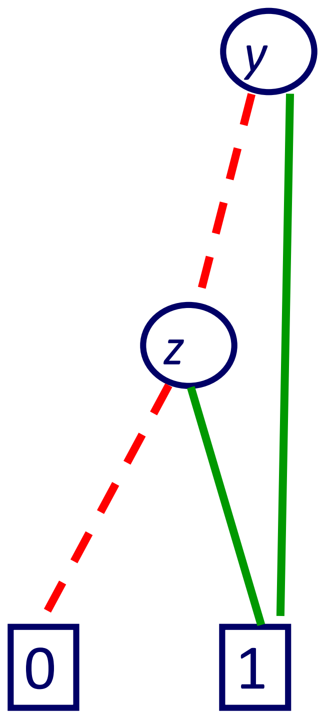


# Another reduction example



# Canonicity of ROBDDs

- **Canonicity Lemma**: for any function $f : \boldsymbol{B}^n \rightarrow \boldsymbol{B}$ there is exactly one ROBDD $u$ with a variable ordering $x_1 < x_2 < \ldots < x_n$ such that $f_u = f(x_1,\ldots,x_n)$

Proof (by induction on $n$)

Read on your own!

# Practice

- What are the ROBDDs of
  - $x$
  - $1$
  - $0$
  - $x \wedge y$ order $x, y$
  - $(x \Rightarrow y) \wedge z$ order $x, y, z$

# Size of ROBDDs

- ROBDDs of many practically important Boolean functions are small
- Do all functions have polynomial ROBDD size? NO
  - ROBDDs do not escape the curse of Boolean function representation

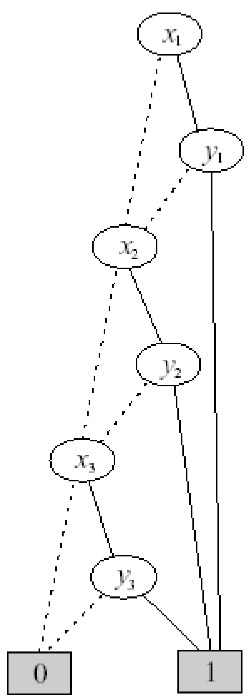
# Size of ROBDDs

- The size of an ROBDD depends heavily on the variable ordering
- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Build ROBDD of $t$ in order $x_1, x_2, y_1, y_2$

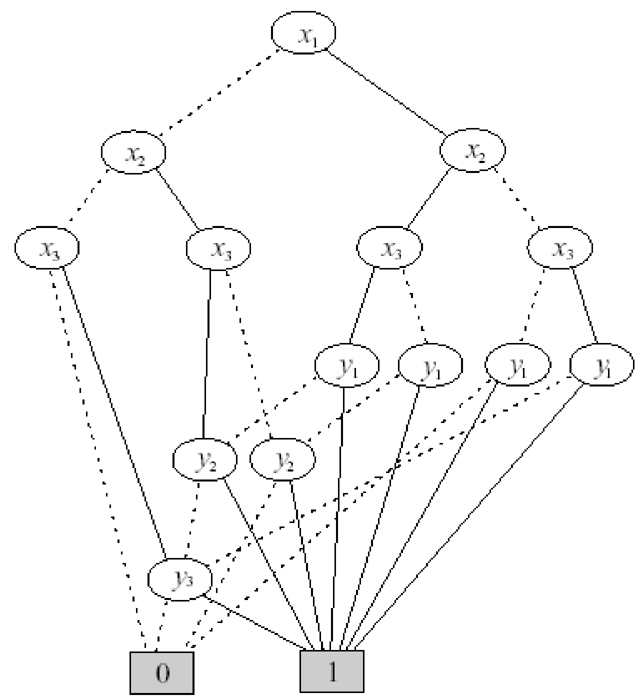
# Size of ROBDDs

- The size of an ROBDD depends heavily on the variable ordering

$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$




$x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n$




$x_1 < x_2 < \ldots < x_n < y_1 < x_2 < \ldots < y_n$