Coupled Pendulums

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Abstract

In this document I investigate the dynamic behavior of two coupled pendulums across three levels of numerical models. While the classic textbook model predicts a perfect, periodic exchange of energy, I demonstrate that as the initial energy of the system increases, the linear assumptions fail. By implementing numerical solutions in MATLAB, we visualize the transition from ordered resonance to geometric non-linearity and, eventually, potential chaos. This document is a summary of the results for the homework for the course called Computer Solution of Technical and Physical Problems.

1 Introduction

Imagine two identical pendulums hanging in a void, linked by a single spring. In a world of small movements, they are like two dancers in perfect sync, passing a ball back and forth. This is the resonance we teach in introductory physics.

However, what happens when the "push" is no longer a gentle nudge, but a violent impulse? At high velocities, the geometry of the system changes. The spring is no longer a horizontal link; it becomes a diagonal tether pulling in two dimensions. This study explores the "Hierarchy of Complexity" to see where our mathematical shortcuts stop working.

2 Methodology

We define the system using the following physical constants:

- L = 1.0 m
- g = 9.81 m/s^2
- k = 30.0 N/m
- m1 = m2 = 1 kg
- pivot separation: $a = L/2$
- spring rest length: $L_0 = L/2$

For all the following models the state vector describing the system can be written as:

$$\mathbf{y} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{1}$$

Then using Lagrangian of the system we can arrive to the following 3 models with 3 different assumptions.

2.1 I. The Idealized World (Linear model)

Here, we assume the pendulums barely move. We use the small-angle approximation, and arrive to the following set of ODEs:

$$\dot{y}_1 = y_2$$

$$
\begin{aligned}
\dot{y}_2 &= -\frac{g}{L}y_1 + \frac{k}{m_1}(y_3 - y_1) \\
\dot{y}_3 &= y_4 \\
\dot{y}_4 &= -\frac{g}{L}y_3 - \frac{k}{m_2}(y_3 - y_1)
\end{aligned}
$$

2.2 II. The Intermediate World (Semi-Nonlinear model)

In this model we acknowledge that gravity is non-linear ($\sin\theta$), but we still pretend the spring only pulls horizontally. This is the bridge between textbook theory and reality.

$$
\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\frac{g}{L}\sin(y_1) + \frac{k}{m_1}\left(\sin(y_3) - \sin(y_1)\right)\cos(y_1) \\
\dot{y}_3 &= y_4 \\
\dot{y}_4 &= -\frac{g}{L}\sin(y_3) - \frac{k}{m_2}\left(\sin(y_3) - \sin(y_1)\right)\cos(y_3)
\end{aligned}
$$

2.3 III. The Real World (Fully Non-Linear model)

We abandon all shortcuts. The spring length is calculated via the Pythagorean theorem based on the (x, y) coordinates of both masses. This accounts for the vertical tug the spring exerts when the pendulums are at different heights.

First displacement has to be calculated:

$$
\begin{aligned}
\Delta x &= \frac{L}{2} + L\sin(y_3) - L\sin(y_1) \\
\Delta y &= L\cos(y_1) - L\cos(y_3) \\
d &= \sqrt{\Delta x^2 + \Delta y^2}
\end{aligned}
$$

Than the ODE can be expressed as:

$$
\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\frac{g}{L}\sin(y_1) + \frac{F_s}{m_1 L d}\left[\Delta x\cos(y_1) + \Delta y\sin(y_1)\right] \\
\dot{y}_3 &= y_4 \\
\dot{y}_4 &= -\frac{g}{L}\sin(y_3) - \frac{F_s}{m_2 L d}\left[\Delta x\cos(y_3) + \Delta y\sin(y_3)\right]
\end{aligned}
$$

2.4 MATLAB implementation

I implemented the three model levels in MATLAB as a single ODE right-hand side with a `modelLevel` switch:

- `coupledPendulumsODE.m`: $\dot{\mathbf{y}} = f(t, \mathbf{y})$ for Model 1/2/3
- `simulateCoupledPendulums.m`: wrapper around `ode113` that also computes energies
- `coupledPendulumsEnergy.m`: energy diagnostics $E_{\mathrm{tot}}(t)$ and partitions

All simulations were run in SI units. Angles are integrated in radians, but plots are displayed in degrees. The numerical solver was `ode113` with tolerances `RelTol = 1e-9`, `AbsTol = 1e-11`.

Instead of specifying the initial push as v_0, the scripts use an angular velocity ω_0 and set:

$$\dot{\theta}_1(0) = \omega_0, \qquad \dot{\theta}_2(0) = 0, \qquad \theta_1(0) = \theta_2(0) = 0.$$

Since $v_0 = \omega_0 L$, the interpretation is identical.

3 Results

To present the behavior as a narrative, I generated a small set of "storyline" figures using `make_storyline_figures.m`. The corresponding PNGs are saved into `story_figs/`. For Model 3, I also generated MP4 animations using `make_model3_animations.m` into `model3_anims/`.

3.1 Phase 1: Harmony (small energy, models agree)

With a small push ($\omega_0 = 0.50$ rad/s), all three models produce nearly identical $\theta(t)$ traces. The linear assumptions are valid: $\sin\theta \approx \theta$, and the spring geometry does not deviate strongly from the horizontal approximation.

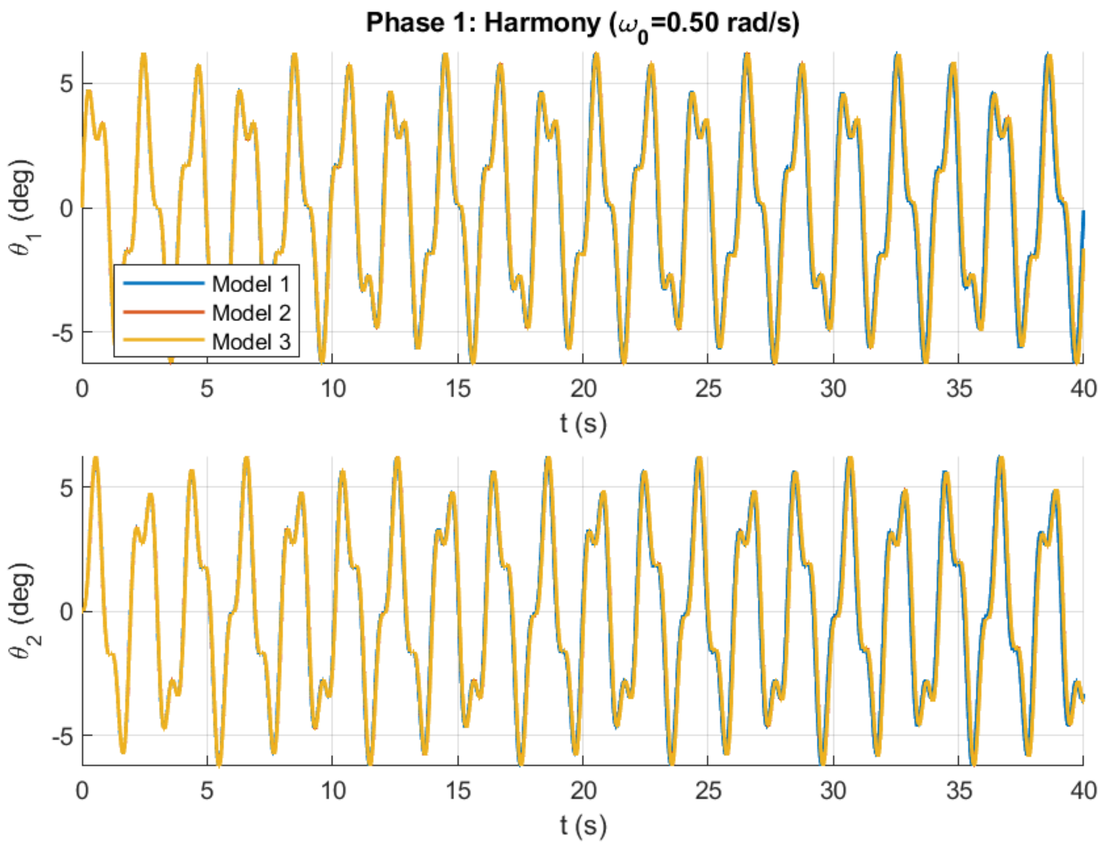
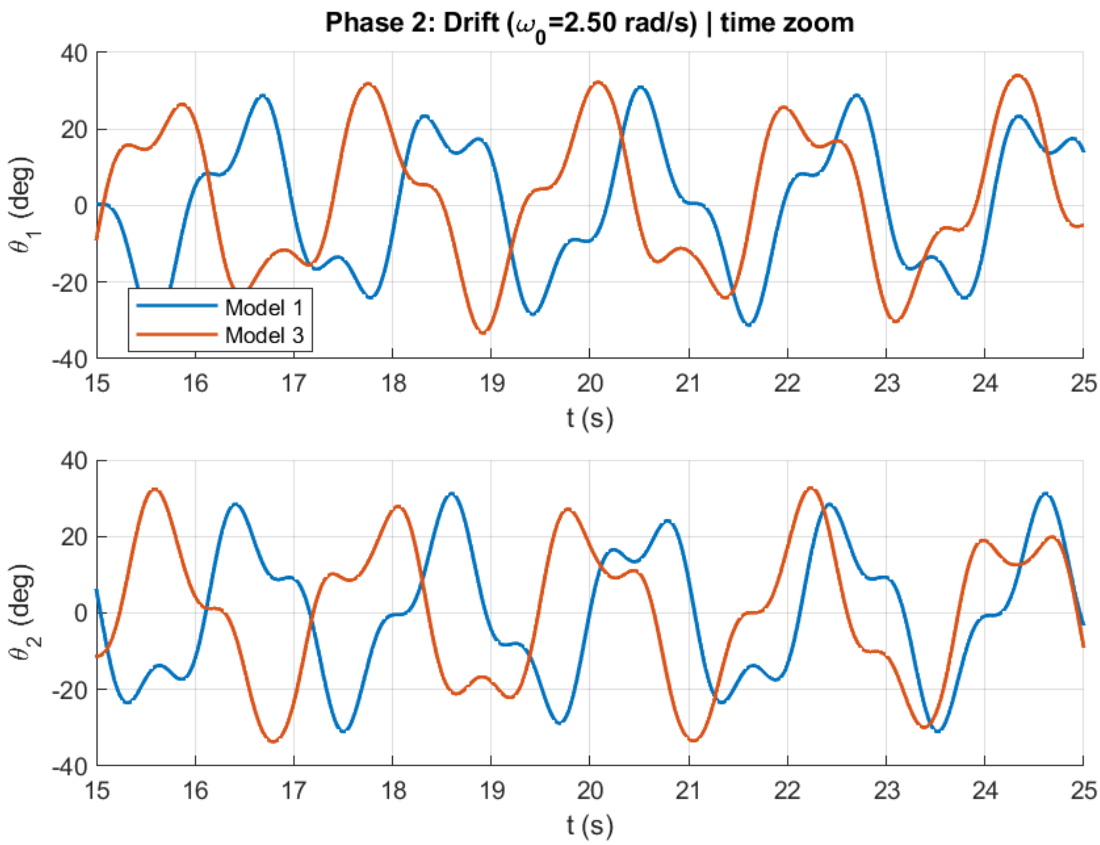


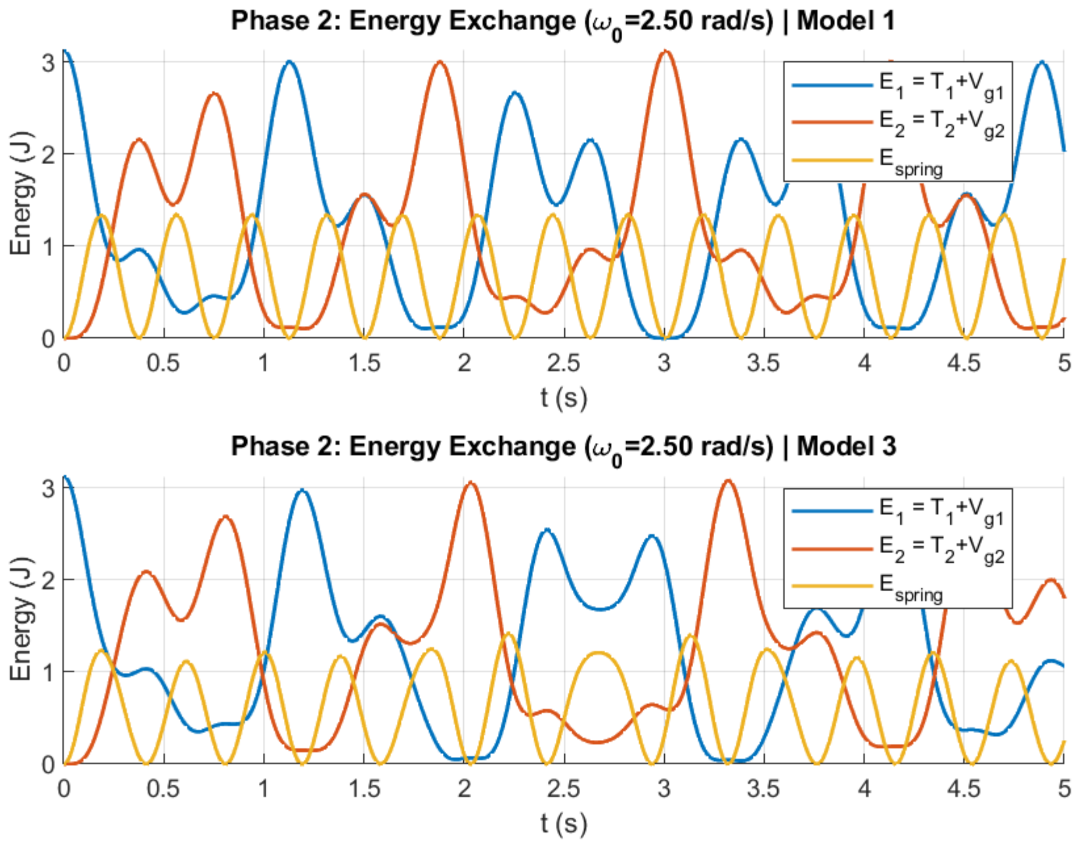
Figure 1: Phase 1: all three models overlap for small excitation ($\omega_0 = 0.50$ rad/s).

3.2 Phase 2: Drift (moderate energy, Model 1 runs ahead)

At moderate energy ($\omega_0 = 2.50$ rad/s), the linear model begins to drift relative to Model 3. The difference is clearly visible in a zoomed time window. In the energy partition plot, Model 3 shows a different (and less "perfect") exchange pattern because the true geometry introduces additional coupling terms that do not exist in Model 1.



(a) Time zoom: Model 1 vs Model 3.

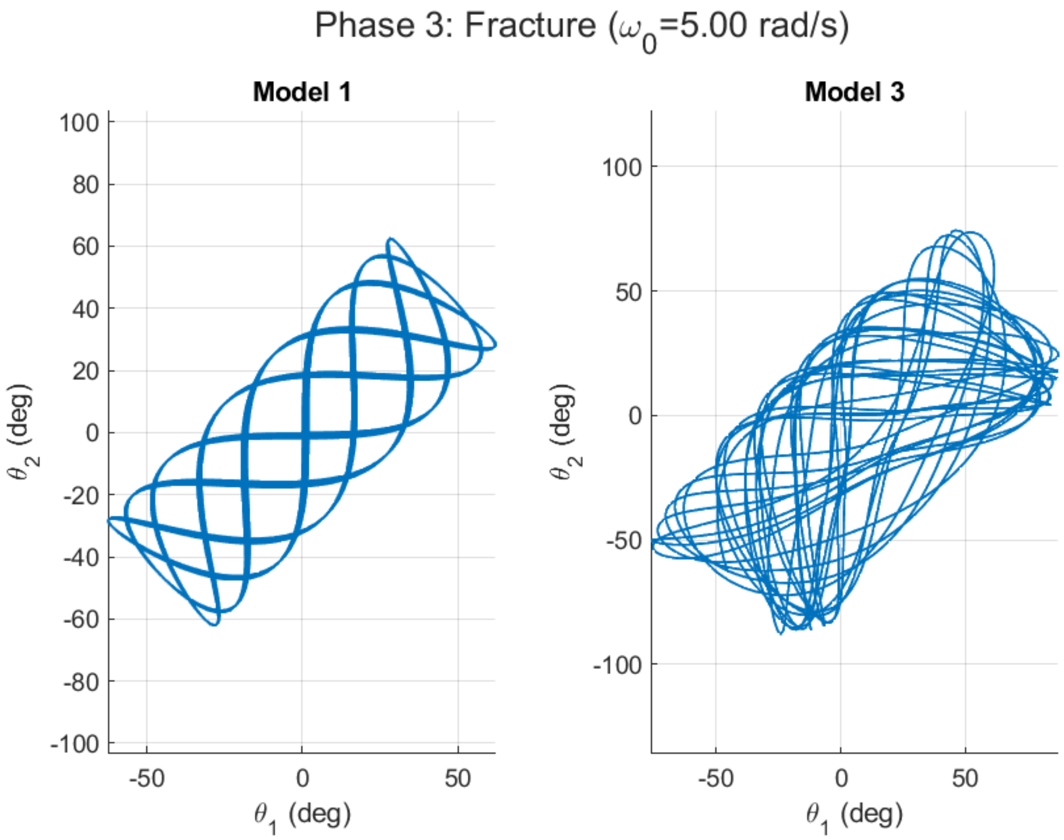


(b) Energy partition: Model 1 vs Model 3.

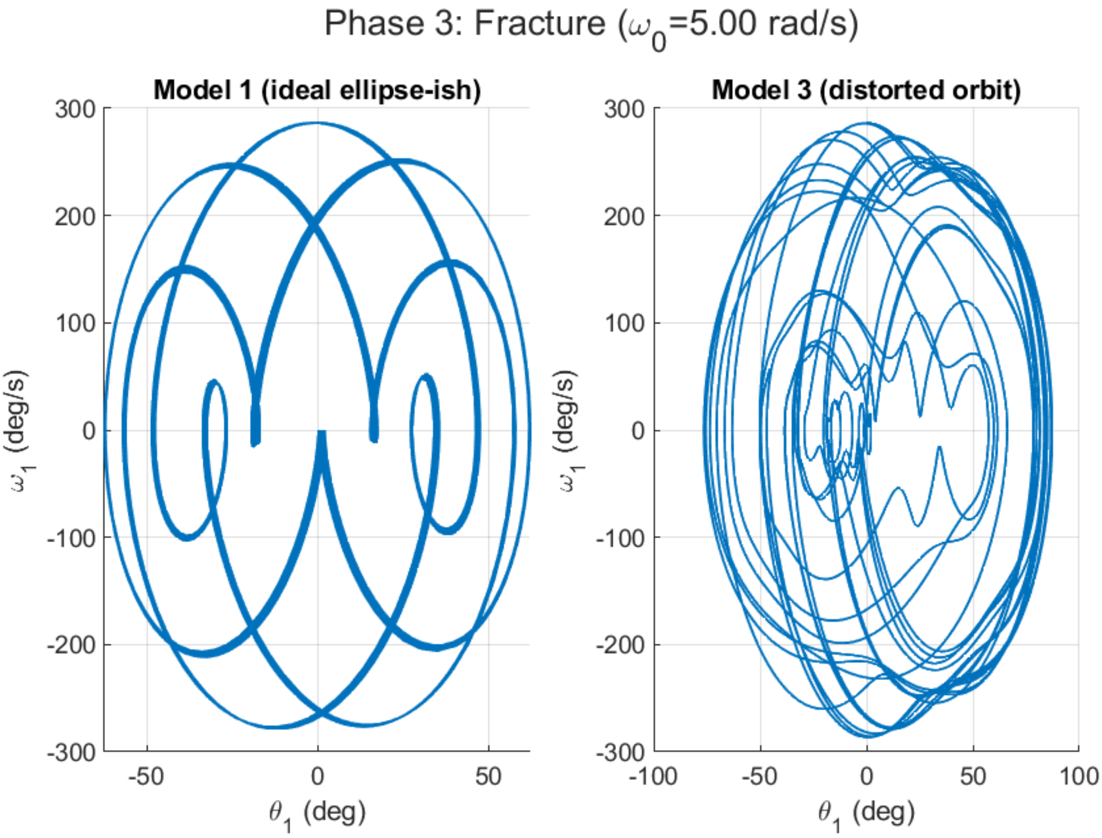
Figure 2: Phase 2: drift and energy-exchange differences emerge at $\omega_0 = 2.50$ rad/s.

3.3 Phase 3: Fracture (high energy, geometry dominates)

At higher excitation ($\omega_0 = 5.00$ rad/s), the "flat-spring" simplification fails strongly. In configuration space (θ_1, θ_2) and in the phase portrait (θ_1, ω_1), Model 3 trajectories become visibly distorted compared to the more idealized structure predicted by Model 1.



(a) Configuration space: Model 1 vs Model 3.



(b) Phase portrait: Model 1 vs Model 3.

Figure 3: Phase 3: geometric nonlinearity dominates at $\omega_0 = 5.00$ rad/s.

3.4 Phase 4: Rotor regime (separatrix crossing)

For very high excitation, the system enters a rotor-like regime where wrapping angles becomes necessary. The linear model is conceptually wrong here: it cannot represent rotations (it continues to treat θ as a small oscillation variable). In contrast, Model 3 shows a rotor-like wrapped phase portrait.

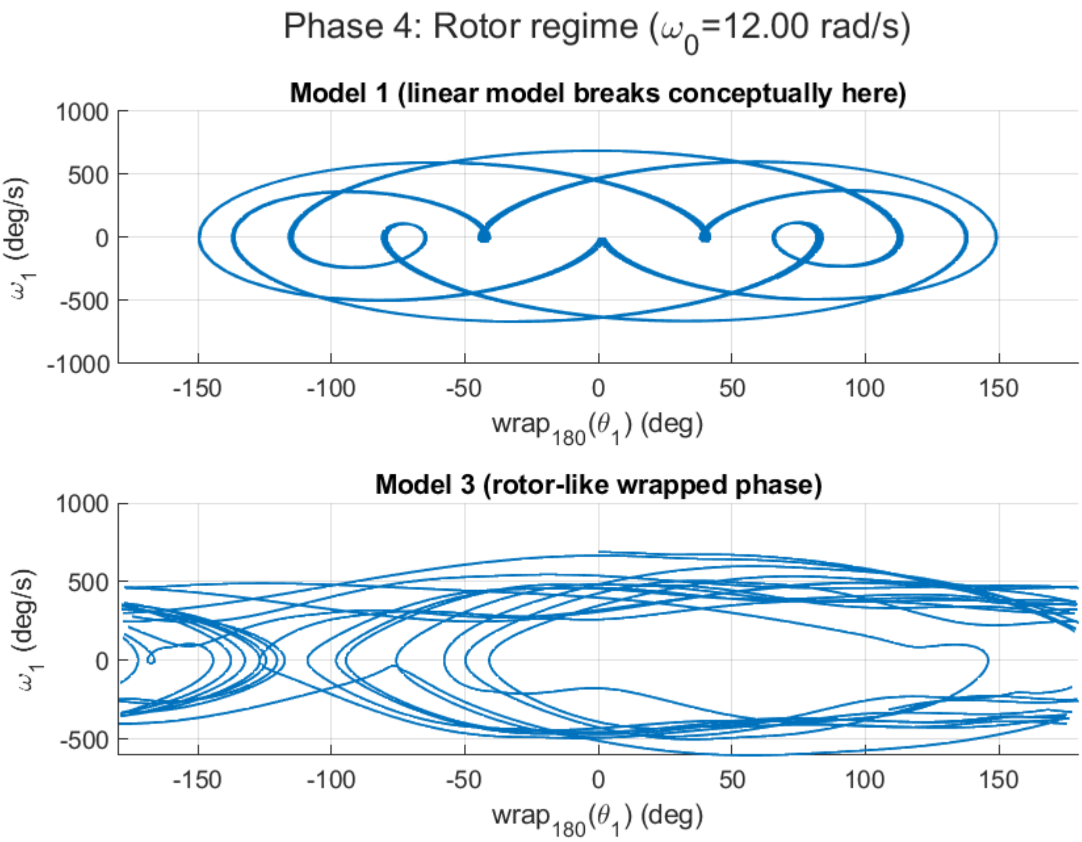


Figure 4: Phase 4: wrapped phase portrait. The wrap is split at 180° to avoid artificial line segments across the discontinuity.

4 Conclusions

- For small excitations, the linearized textbook model is accurate and predicts the expected periodic energy exchange.
- At moderate energy, the linear model begins to drift in phase and timing; the semi-nonlinear and fully nonlinear models capture amplitude-dependent effects.
- At high energy, the exact spring geometry (Model 3) becomes essential: configuration-space structure and phase portraits deviate strongly from Model 1.

- In the rotor regime, Model 1 is not just inaccurate but conceptually invalid; Model 3 naturally represents separatrix crossing and rotations.

Model 3 animations (MP4)

Generated by `make_model3_animations.m` into `model3_anims/`:

- `model3_phase1_harmony_omega0_0.50.mp4`
- `model3_phase2_drift_omega0_2.50.mp4`
- `model3_phase3_fracture_omega0_5.00.mp4`
- `model3_phase4_rotor_omega0_12.00.mp4`

Overall, the "Hierarchy of Complexity" provides a practical workflow: use Model 1 for intuition and quick checks, then move to Model 3 when the excitation is large enough that geometry and rotations matter.