Given the following results for the M/M/1/K queue:

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 for n \geq 1 and p_0 = \left[\sum_{n=0}^{K} p_n\right]^{-1},$$

and applying them to them M/M/1/1 queue, we have

$$p_0 = \frac{1}{1+\rho} and p_1 = \frac{\rho}{1+\rho}.$$

The M/M/1/1 example has application to the so-called machine breakdown problem, wherein a machine (or machine component) fails and is subsequently repaired. State 0 is used to represent the machine in its working condition while state 1 represents the machine undergoing repair. To qualify as a birth-death process, it is necessary that both failures and repairs occur according to exponential distributions with rates λ and μ, respectively. Thus the mean time to failure (MTTF) is equal to $1/\lambda$, the mean time repair (MTTR) is equal to $1/\mu$, and the steady-state availability of the machine is just the steady-state probability of the system being in state 0. The availability is therefore given by

$$\frac{1}{1+\rho} = \frac{1}{1+\lambda/\mu} = \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{MTTF}{MTTF + MTTR}.$$

Performance Measures for the M/M/1/K Queue

We consider the case when $\rho \neq 1$, and let L be the expected system size. Then

$$L = \sum_{n=0}^{K} np_n and p_n = p^n p_0$$

We have

$$L = p_0 \rho \sum_{n=0}^{K} n\rho^{n-1} = p_0 \rho \sum_{n=0}^{K} \frac{d}{d\rho}\Big(\rho^n\Big) = p_0 \rho \frac{d}{d\rho}\left(\sum_{n=0}^{K} p_n\right)$$

$$= p_0 \rho \left(\frac{d}{d\rho}\left[\frac{1-\rho^{K+1}}{1-\rho}\right]\right) = p_0 \rho \left(\frac{[1-\rho][-(K+1)\rho^K] + 1 - \rho^{K+1}}{(1-\rho)^2}\right)$$

$$= p_0 \rho \left(\frac{-(K+1)\rho^K + (K+1)\rho^{K+1} + 1 = \rho K + 1}{(1-\rho)^2}\right)$$

$$= p_0 \rho \left(\frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)^2}\right).$$

Using

$$p_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

we find

$$L = \frac{\rho[1-(K+1)\rho^K + K\rho^{K+1}]}{(1-\rho)^2} \frac{1-\rho}{1-\rho^{K+1}}$$

$$= \frac{\rho[1-(K+1)\rho^K + K\rho^{K+1}]}{(1-\rho)(1-\rho^{K+1}}$$