Graph Theory Tutorial

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Q1. Check this document on Graph Theory.
Yes
Q2. Summarize this document and report the contents.

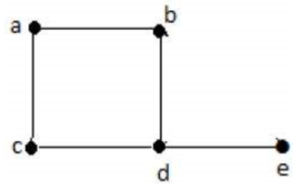
! I have read "this document" in Q1, but just now I notice that I summarized "this document" in Q2 by my mistake. I am sorry.

Q3.Get the igraph package in R ready for use on June 5. Will use this document.
https://www.tutorialspoint.com/graph_theory/graph_theory_introduction.htm
Yes

1 Introduction

Graph theory is the study of graphs that concerns with the relationship among edges and vertices.

1-1. What is a Graph?



A graph is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, connecting the pairs of vertices.
V: vertices 頂点, V = {a, b, c, d, e}
E: edges 辺, E = {ab, ac, bd, cd, de}

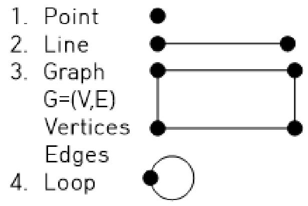
1-2. Applications of Graph Theory

Graph theory has its applications in diverse fields of engineering: Electrical Engineering, Computer Science, Computer Network, Science , Linguistics, and General.

2 Fundamentals

2-0. Graph

A graph is a diagram of points and lines connected to the points. It has at least one line joining a set of two vertices with no vertex connecting itself.



1. Point: a point is a particular position in a one-dimensional, two-dimensional, or three-dimensional space.

2. Line: a Line is a connection between two points.
3. Vertex / Node: a vertex is a point where multiple lines meet. It is also called a node.
4. Edge: an edge is the mathematical term for a line that connects two vertices.
5. Graph: a graph 'G' is defined as G = (V, E) Where V is a set of all vertices and E is a set of all edges in the graph.
6. Loop: in a graph, if an edge is drawn from vertex to itself, it is called a loop.

2-1. Degree of Vertex − deg(V).

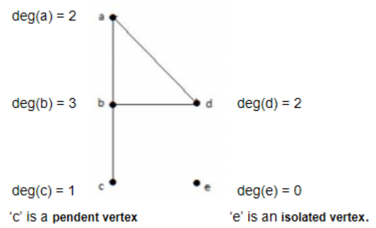
It is the number of vertices adjacent to a vertex V.

$$deg\,(v) \leq n - 1 \forall \vee \in G$$

A vertex can form an edge with all other vertices except by itself. So the degree of a vertex will be up to the number of vertices in the graph minus 1. This 1 is for the self-vertex as it cannot form a loop by itself. If there is a loop at any of the vertices, then it is not a Simple Graph.

2-1-1. Degree of Vertex in an Undirected Graph

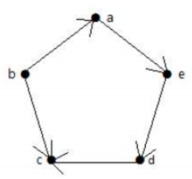
An undirected graph has no directed edges.



2-1-2. Degree of Vertex in a Directed Graph

In a directed graph, each vertex has an indegree and an outdegree.

- Indegree of a Graph − deg − (V).: Indegree of vertex V is the number of edges which are coming into the vertex V.
- Outdegree of a Graph − deg+(V).:Outdegree of vertex V is the number of edges which are going out from the vertex V.



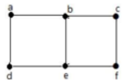
The indegree and outdegree of other vertices are shown in the following table −

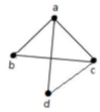
Vertex	Indegree	Outdegree
a	1	1
b	0	2
c	2	0
d	1	1
e	1	1

2-1-3. Pendent Vertex → 2-1-1
2-1-4. Isolated Vertex → 2-1-1
2-1-5. Adjacency

Adjacency

•In a graph, two vertices are said to be **adjacent,** if there is an edge between the two vertices.
•In a graph, two edges are said to be adjacent, if there is a common vertex between the two edges.





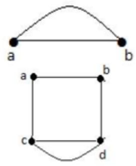
2-1-6.Parallel Edges
2-1-7.Multi Graph

Parallel Edges

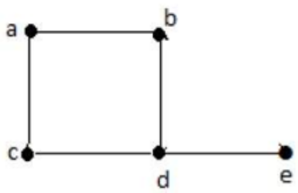
In a graph, if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.

Multi Graph

A graph having parallel edges is known as a Multigraph.

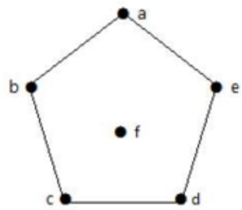


2-1-8.Degree Sequence of a Graph
If the degrees of all vertices in a graph are arranged in descending or ascending order, then the sequence obtained is known as the degree sequence of the graph.



Vertex	A	b	c	d	e
Connecting to	b,c	a,d	a,d	c,b,e	d
Degree	2	2	2	3	1

In the above graph, for the vertices {d, a, b, c, e}, the degree sequence is {3, 2, 2, 2, 1}.



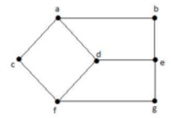
Vertex	A	b	c	d	e	f
Connecting to	b,e	a,c	b,d	c,e	a,d	-
Degree	2	2	2	2	2	0

In the above graph, for the vertices {a, b, c, d, e, f}, the degree sequence is {2, 2, 2, 2, 2, 0}.

3 Basic Properties

3-1. Distance between Two Vertices − d(U,V)
The shortest path is considered as the distance between the two vertices.

3-2. Eccentricity of a Vertex − e(V)
The maximum distance between a vertex to all other vertices is considered as the eccentricity of vertex.

Distance between Two Vertices	
	•de (It is considered for distance between the vertices) The distance from vertex 'd' to vertex 'e' or simply 'de' is 1 as there is one edge between them.
Eccentricity of a Vertex	In the above graph, the eccentricity of 'a' is 3. The distance from 'a' to 'b' is 1 ('ab'), from 'a' to 'g' is 3 ('ac'-'cf'-'fg') or ('ad'-'df'-'fg').
Radius of a Connected Graph	r(G) = 2, which is the minimum eccentricity for 'd'.
Diameter of a Graph	d(G) = 3; which is the maximum eccentricity.
Central Point	'd' is the central point of the graph. e(d) = r(d) = 2
Centre	{'d'} is the centre of the Graph.
Circumference	The circumference is 6, which we derived from the longest cycle a-c-f-g-e-b-a or a-c-f-d-e-b-a.
Girth	The Girth of the graph is 4, which we derived from the shortest cycle a-c-f-d-a or d-f-g-e-d or a-b-e-d-a.

3-3. Radius of a Connected Graph − r(G)

The minimum eccentricity from all the vertices is considered as the radius of the Graph G. The minimum among all the maximum distances between a vertex to all other vertices is considered as the radius of the Graph G.

3-4. Diameter of a Graph − d(G)

The maximum among all the distances between a vertex to all other vertices is considered as the diameter of the Graph G.

3-5. Central Point

If the eccentricity of a graph is equal to its radius, then it is known as the central point of the graph. If e(V) = r(V), then 'V' is the central point of the Graph 'G'.

3-6. Centre

The set of all central points of 'G' is called the centre of the Graph.

3-7. Circumference

The number of edges in the longest cycle of 'G' is called as the circumference of 'G'.

3-7. Girth − g(G).

The number of edges in the shortest cycle of 'G' is called its Girth.

3-7. Sum of Degrees of Vertices Theorem

If G = (V, E) be a non-directed graph with vertices V = {V1, V2,⋯Vn} then

$$\sum_{i=1}^{n} deg\left(V_i\right) = 2 \mid E \mid$$

Corollary 1 If G = (V, E) be a directed graph with vertices V = {V1, V2,⋯Vn}, then

$$\sum_{i=1}^{n} deg^{+}\left(V_i\right) = \mid E \mid = \sum_{i=1}^{n} deg^{-}\left(V_i\right)$$

Corollary 2 In any non-directed graph, the number of vertices with Odd degree is Even.
Corollary 3 In a non-directed graph, if the degree of each vertex is k, then

$$k \mid V \mid = 2 \mid E \mid$$

Corollary 4 In a non-directed graph, if the degree of each vertex is at least k, then

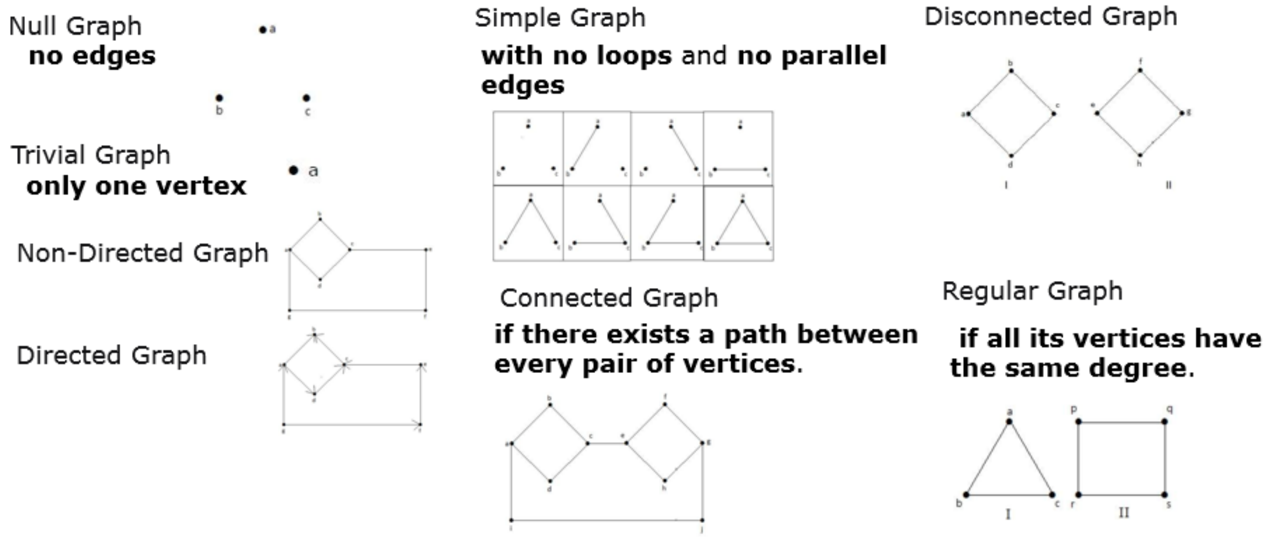
$$k \mid V \mid \leq 2 \mid E \mid$$

Corollary 5 In a non-directed graph, if the degree of each vertex is at most k, then

$$k \mid V \mid \geq 2 \mid E \mid$$

4 Types of Graphs

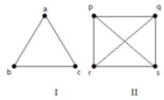
4-1. Null Graph
4-2. Trivial Graph
4-3. Non-Directed Graph
4-4. Directed Graph
4-5. Simple Graph
4-6. Connected Graph
4-7. Disconnected Graph
4-8. Regular Graph



4-9. Complete Graph
4-10. Cycle Graph
4-10. Wheel Graph
4-11. Cyclic Graph
4-12. Acyclic Graph

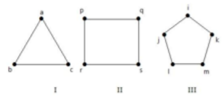
Complete Graph

A simple graph with 'n' mutual vertices is called a complete graph and it is **denoted by 'K_n'**.
In the graph, **a vertex should have edges with all other vertices,** with all the remaining vertices in the graph except by itself.



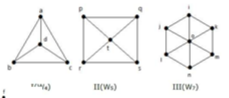
Cycle Graph– C_n

A simple graph with 'n' vertices (n >= 3) and 'n' edges is called a cycle graph if all its edges form a cycle of length 'n'.



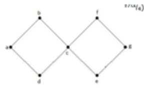
Wheel Graph– W_n

A wheel graph is obtained from a cycle graph C_{n-1} by adding a new vertex. That new vertex is called a **Hub** which is connected to all the vertices of C_n.



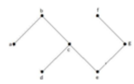
Cyclic Graph

A graph **with at least one** cycle is called a cyclic graph.



Acyclic Graph

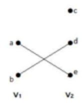
A graph **with no cycles** is called an acyclic graph.



4-13. Bipartite Graph
4-14. Complete Bipartite Graph
4-15. Star Graph
4-16. Complement of a Graph

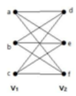
Bipartite Graph

A simple graph G = (V, E) with vertex partition V = {V_1, V_2} is called a bipartite graph **if every edge of E joins a vertex in V_1 to a vertex in V_2.**



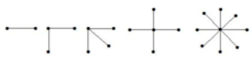
Complete Bipartite Graph

A bipartite graph 'G', G = (V, E) with partition V = {V_1, V_2} is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2.



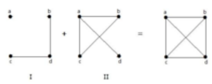
Star Graph

A complete bipartite graph of the form $K_{1,\, n-1}$ is a star graph with n-vertices.
In the above graphs, out of 'n' vertices, all the 'n−1' vertices are connected to a single vertex. Hence it is in the form of $K_{1,\, n-1}$ which are star graphs.



Complement of a Graph

If the edges that exist in graph I are absent in another graph II, and if both graph I and graph II are combined together to form a complete graph, then graph I and graph II are called complements of each other.



5 Trees

5-1. Tree

A connected acyclic graph is called a tree. In other words, a connected graph with no cycles is called a tree.
The edges of a tree are known as branches. Elements of trees are called their nodes. The nodes without child nodes are called leaf nodes.

5-2. Forest

A disconnected acyclic graph is called a forest. In other words, a disjoint collection of trees is called a forest.

5-3. Spanning Trees

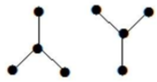
Let G be a connected graph, then the sub-graph H of G is called a spanning tree of G if − H is a tree and H contains all vertices of G.

Tree

It has four vertices and three edges, i.e., for 'n' vertices 'n-1' edges.
Every tree has at least two vertices of degree one.

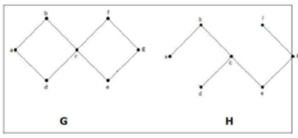
The vertices 'a' and 'd' has degree one. And the other two vertices 'b' and 'c' has degree two.

Forest The following graph looks like two sub-graphs; but it is a single disconnected graph. There are no cycles in this graph. Hence, clearly it is a forest.



Spanning Trees

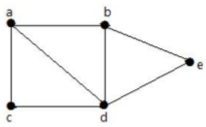
G is a connected graph and H is a sub-graph of G. Clearly, the graph H has no cycles, it is a tree with six edges which is one less than the total number of vertices. Hence H is the Spanning tree of G.



5-4. Circuit Rank

Let 'G' be a connected graph with 'n' vertices and 'm' edges. A spanning tree 'T' of G contains (n-1) edges.
Therefore, the number of edges you need to delete from 'G' in order to get a spanning tree = m-(n-1), which is called the circuit rank of G.

Circuit Rank You have m=7 edges and n=5 vertices.
Then the circuit rank is
G = m – (n – 1)
= 7 – (5 – 1)
= 3



Let 'G' be a connected graph with six vertices and the degree of each vertex is three. Find the circuit rank of 'G'.
By the sum of degree of vertices theorem,

$\sum_{i=1}^{n} \deg(V_i) = 2|E|$

$6 \times 3 = 2|E|$

$|E| = 9$

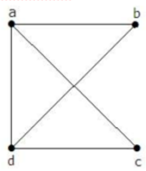
Circuit rank = $|E| - (|V| - 1)$

$= 9 - (6 - 1) = 4$

5-5. Kirchoff's Theorem

Kirchoff's theorem is useful in finding the number of spanning trees that can be formed from a connected graph.

Kirchoff's Theorem



The matrix 'A' be filled as, if there is an edge between two vertices, then it should be given as '1', else '0'.

$$A = \begin{vmatrix} 0 & a & b & c & d \\ a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

By using kirchoff's theorem, it should be changed as replacing the principle diagonal values with the degree of vertices and all other elements with -1.A

$$= \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix} = M$$

$$M = \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix}$$

$$\text{Cofactor of m1} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 8$$

Thus, the number of spanning trees = 8.

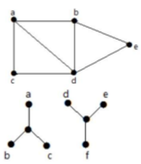
6 Connectivity

6-1. Connectivity

A graph is said to be connected if there is a path between every pair of vertex.

Connectivity In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.

Traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.



6-2. Cut Vertex

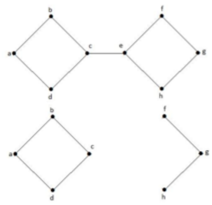
Let 'G' be a connected graph. A vertex V ∈ G is called a cut vertex of 'G', if 'G-V' (Delete 'V' from 'G') results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more graphs.

Cut Vertex

Vertices ‘e’ and ‘c’ are the cut vertices.

By removing ‘e’ or ‘c’, the graph will become a disconnected graph.

Without ‘g’, there is no path between vertex ‘c’ and vertex ‘h’ and many other. Hence it is a disconnected graph with cut vertex as ‘e’. Similarly, ‘c’ is also a cut vertex for the above graph.



Cut Edge (Bridge)

The cut edge is [(c, e)]

By removing the edge (c, e) from the graph, it becomes a disconnected graph.

6-3. Cut Edge (Bridge)

If removing an edge in a graph results in to two or more graphs, then that edge is called a Cut Edge.

6-4. Cut Set of a Graph

Let ‘G’ = (V, E) be a connected graph. A subset E’ of E is called a cut set of G if deletion of all the edges of E’ from G makes G disconnect.

Cut Set of a Graph

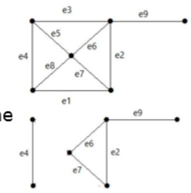
Its cut set is E1 = {e1, e3, e5, e8}.

After removing the cut set E1 from the graph, it would appear as follows –

Similarly there are other cut sets that can disconnect the graph –

•E3 = {e9} – Smallest cut set of the graph.

•E4 = {e3, e4, e5}



6-5. Edge Connectivity – λ (G)

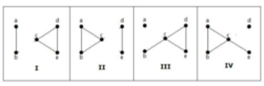
Let ‘G’ be a connected graph. The minimum number of edges whose removal makes ‘G’ disconnected is called edge connectivity of G. In other words, the number of edges in a smallest cut set of G is called the edge connectivity of G.

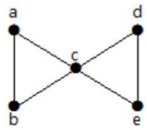
If ‘G’ has a cut edge, then λ (G) is 1. (edge connectivity of G.)

Edge Connectivity

By removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity (λ(G)) is 2.

Here are the four ways to disconnect the graph by removing two edges –



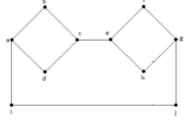


6-6. Vertex Connectivity – K(G)

Let ‘G’ be a connected graph. The minimum number of vertices whose removal makes ‘G’ either disconnected or reduces ‘G’ in to a trivial graph is called its vertex connectivity.

Vertex Connectivity

Removing the vertices ‘e’ and ‘i’ makes the graph disconnected. If G has a cut vertex, then K(G) = 1.



Notation – For any connected graph G,

K(G) ≤ λ(G) ≤ δ(G)

Vertex connectivity (K(G)), edge connectivity (λ(G)), minimum number of degrees of G(δ(G)).

Solution

From the graph,

δ(G) = 3

K(G) ≤ λ(G) ≤ δ(G) = 3 (1)

K(G) ≥ 2 (2)

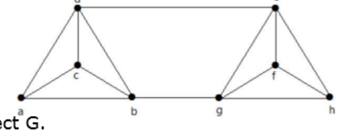
Deleting the edges {d, e} and {b, h}, we can disconnect G.

Therefore,

λ(G) = 2

2 ≤ λ(G) ≤ δ(G) = 2 (3)

From (2) and (3), vertex connectivity K(G) = 2



7 Coverings

7-1. Line Covering

Let G = (V, E) be a graph. A subset C(E) is called a line covering of G if every vertex of G is incident with at least one edge in C, i.e.,

$$deg\,(v) \leq n - 1 \forall \vee \in G$$

because each vertex is connected with another vertex by an edge. Hence it has a minimum degree of 1.

7-2. Minimal Line Covering

A line covering C of a graph G is said to be minimal if no edge can be deleted from C.

7-3. Minimum Line Covering

It is also known as Smallest Minimal Line Covering. A minimal line covering with minimum number of edges is called a minimum line covering of 'G'. The number of edges in a minimum line covering in 'G' is called the line covering number of 'G' (α 1).

7-4. Vertex Covering

Let 'G' = (V, E) be a graph. A subset K of V is called a vertex covering of 'G', if every edge of 'G' is incident with or covered by a vertex in 'K'.

7-5. Minimal Vertex Covering

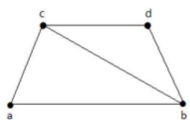
A vertex 'K' of graph 'G' is said to be minimal vertex covering if no vertex can be deleted from 'K'.

7-6. Minimum Vertex Covering

A vertex 'K' of graph 'G' is said to be minimal vertex covering if no vertex can be deleted from 'K'.

Line Covering

Its subgraphs having line covering are as follows –
C1 = {{a, b}, {c, d}}
C2 = {{a, d}, {b, c}}
C3 = {{a, b}, {b, c}, {b, d}}
C4 = {{a, b}, {b, c}, {c, d}}



Minimal Line Covering — Here, C1, C2, C3 are minimal line coverings, while C4 is not because we can delete {b, c}.

Minimum Line Covering — C_1 and C_2 are the minimum line covering of G and α_1 = 2.

Vertex Covering — K_1, K_2, and K_3 have vertex covering, whereas K_4 does not have any vertex covering as it does not cover the edge {bc}.

Minimal Vertex Covering — Here, K_1, K_2, and K_3 have vertex covering, whereas K_4 does not have any vertex covering as it does not cover the edge {bc}.
The subgraphs having vertex covering are as follows –
K_1 = {b, c}
K_2 = {a, b, c}
K_3 = {b, c, d}
Here, K_1 and K_2 are minimal vertex coverings, whereas in K_3, vertex 'd' can be deleted.

8 Matchings

8-1. Matching
8-2. Maximal Matching
8-3. Maximum Matching
8-4. Perfect Matching

9 Independent Sets

9-1. Independent Line Set
9-2. Maximal Independent Line Set
9-3. Maximum Independent Line Set
9-4. Independent Vertex Set
9-5. Maximal Independent Vertex Set
9-6. Maximum Independent Vertex Set

10 Coloring