vertices

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0.1 Merger

Consider a weighted complete graph $G(V, E)$ with a set of vertices V and a set of edges E formed by pairs of elements in V. The set V has cardinality $|V| = N$ and the set E has cardinality $|E| = N(N-1)/2$. The latter number will appear often, so we will denote it compactly as T_{N-1}, the $N-1$th triangular number. We will denote by $w_{ij} = w(v_i, v_j)$ the weight of the edge connecting the vertices i and j.

Choosing a subset $A \subset V$ of cardinality k induces a partition in the edges:

$$E = E_A \cup E_{AA^c} \cup E_{A^c} \tag{1}$$

Where E_A are the edges formed by the complete subgraph $G(A, E_A)$. The elements of this set are called the intra-edges of A. The set E_{AA^c} represents the edges with one vertex in A and another in its complement A^c, and its elements are called the inter-edges of A:

$$E_A := \{\{a, b\} \in E | a \in A \wedge b \in A\} \tag{2}$$
$$E_{AA^c} := \{\{a, b\} \in E | (a \in A \wedge b \in A^c) \vee (a \in A^c \wedge b \in A)\} \tag{3}$$

.

Since $G(A, E_A)$ is complete, the cardinality of E_A is T_{k-1} whereas the cardinality of E_{AA^c} is $(N-k)k$, since

$$|E| = |E_A \cup E_{AA^c} \cup E_{A^c}| = |E_A| + |E_{AA^c}| + |E_{A^c}| \tag{4}$$
$$T_{N-1} = T_{k-1} + |E_{AA^c}| + T_{N-k-1} \tag{5}$$
$$|E_{AA^c}| = (N-k)k \tag{6}$$

.

With this construction, our goal is to compute the expected value of the average of weights of E_A provided we choose the vertices of A with uniform probability, that is

$$\mathbb{E}\left[\frac{1}{|E_A|}\sum_{e \in E_A} w(e) \Big| |A| = k\right] \tag{7}$$

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From this point, it will be convenient to enumerate the set of vertices with a natural ordering $V = \{v\}_{i=1}^{N}$, so we can rewrite equation 7 as

$$\frac{1}{T_{k-1}}\mathbb{E}\left[\sum_{\substack{i>j \\ v_i,v_j\in A}} w((v_i,v_j))\Big||A|=k\right] \tag{8}$$

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Since the choice of vertices completely define the outcome, we define the set of events $\mathcal{U}$ to be a sigma algebra over the vertices as the power set of V:

$$\mathcal{U} = 2^V \tag{9}$$

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We are particularly interested in the subset $\mathcal{U}_k \coloneqq \{A \in \mathcal{U}||A| = k\}$ with probability

$$p_k \coloneqq \mathbb{P}(A|\mathcal{U}_k) = \frac{1}{\binom{N}{k}} \tag{10}$$

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Now, we can expand equation 8 as

$$\frac{1}{T_{k-1}}\mathbb{E}\left[\sum_{\substack{i>j \\ v_i,v_j\in A}} w((v_i,v_j))\Big||A|=k\right] = \frac{1}{T_{k-1}}\sum_{A\in\mathcal{U}_k} p_k \sum_{\substack{i>j \\ v_i,v_j\in A}} w((v_i,v_j)) \tag{11}$$

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The rightmost sum can be simplified with the use of indicator functions:

$$\chi_A(v) = \begin{cases} 1, & v \in A \\ 0 \end{cases} \tag{12}$$

Which allows to rewrite the sum as

$$\sum_{\substack{i>j \\ v_i,v_j\in A}} w((v_i,v_j)) = \sum_{\substack{i>j \\ v_i,v_j\in V}} \chi_A(v_i)\chi_A(v_j)w((v_i,v_j)) \tag{13}$$

Resulting in equation 11 to become

$$\frac{1}{T_{k-1}}\sum_{A\in\mathcal{U}_k} p_k \sum_{\substack{i>j \\ v_i,v_j\in A}} w((v_i,v_j)) = \frac{p_k}{T_{k-1}} \sum_{\substack{i>j \\ v_i,v_j\in V}} w((v_i,v_j)) \sum_{A\in\mathcal{U}_k} \chi_A(v_i)\chi_A(v_j) \tag{14}$$

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The rightmost sum is a count over how many times the edge $\{v_i, v_j\}$ appears in different sets of $\mathcal{U}_k$, more specifically, we look for the cardinality of the set $\mathcal{U}_k(v_i, v_j)$ defined as

$$\mathcal{U}_k(v_i, v_j) \coloneqq \{A \in \mathcal{U}_k | A \cap \{v_i, v_j\} \neq \emptyset\} \tag{15}$$

Which is isomorphic to

$$\mathcal{U}'_k(v_i, v_j) \coloneqq \{A - \{v_i, v_j\} | A \in \mathcal{U}_k\} - \{\emptyset\} \tag{16}$$

With cardinality $\binom{N-2}{k-2}$. Replacing this value with the sum in equation 14 yields

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{T_{k-1}} \sum_{e \in E_A} w(e) \middle| |A| = k\right] = \ldots &= \frac{p_k}{T_{k-1}} \sum_{\substack{i>j \\ v_i, v_j \in V}} w((v_i, v_j)) \binom{N-2}{k-2} &(17)\\
&= \frac{\cancel{p_k}}{T_{k-1}} \sum_{\substack{i>j \\ v_i, v_j \in V}} w((v_i, v_j)) \cancel{\binom{N}{k}} \frac{T_{k-1}}{T_{N-1}} &(18)\\
&= \frac{1}{T_{N-1}} \sum_{\substack{i>j \\ v_i, v_j \in V}} w((v_i, v_j)) &(19)
\end{aligned}$$

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