Covariance in processes with noisy readout

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We start by recalling the three main observations concerning our spine data:

1. spine signals are log-normally distributed
2. subsequent changes in signal are anticorrelated
3. this anticorrelation decays in time without crossing zero

We now want to investigate some theoretical setups and understand, in particular their covariance structure. Let's start with some definitions.
Let X_t be a discrete stationary stochastic process, so that $X_{t+1} = f(X_t, t),\ t \in \mathbb{N}$. Then

- the expected value of X_t is $\langle X \rangle$
- the **variance** of X_t is $\mathrm{Var}\,(X)$
- the **autocorrleation** of X_t is $G_X(j) = \sum_t X_t\, X_{t-j}$

These are just operative definitions so that later on we don't get confused with the nomenclature.

1 Problem statement

Consider a discrete stationary stochastic process X with an associated (additive) noisy readout R:

$$\begin{cases} \Delta X_t = f(X_t, t) \\ R_t = X_t + N_t = X_t + n(X_t, t) \end{cases} \tag{1}$$

where we define $\Delta X_t = \Delta X_t(1) = X_{t+1} - X_t$, with the generic variation being $\Delta X_t(d) = X_{t+1} - X_{t-d+1}$.

We are interested in understanding how the covariance structure of R and $\Delta R(d)$ depends on the underlying stochastic process X, the noise $n(X, t)$ and their possible correlation.

1.1 Some general properties

We can directly compute

$$G_R(j) = G_X(j) + G_N(j) + 2G_{X,N}(j) \tag{2}$$

and see that if noise and signals are not correlated, the covariance of the readout reduces to $G_R(j) = G_X(j) + G_N(j)$. Covariance should be linear under independence of arguments. With this property:

$$G_{\Delta R(d)}(j) = 2G_R(j) - G_R(d+j) - G_R(d-j) \tag{3}$$

$$\tag{4}$$

where we can then substitute $G_R(j)$.

1.2 Model 1: OU with uncorrelated normal noise

The system in question reads

$$\begin{cases} \Delta X_t = \theta(\mu - X) + \xi_t \\ R_t = X_t + \eta_t \end{cases} \tag{5}$$

where $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$ and $\xi_t \sim \mathcal{N}(0, \sigma_\xi^2)$ are two sets of IID normal random variables. A handy equation for X_t is

$$X_{t+1} = \mu\theta + (1-\theta)X_t + \xi_t \tag{6}$$

To compute the average and the variance of the OU process, we proceed in the usual way, taking the average of X_t or X_{t+1} and then imposing stationarity ($X_{t+1} = X_t$)

$$\langle X \rangle = \mu, \qquad \langle X^2 \rangle = \mu^2 + \frac{\sigma_\xi^2}{\theta(2-\theta)} \tag{7}$$

Observe that the variance is not finite for $\theta = 0$ (random walk) and $\theta = 2$ (convergence radius?); also, it is minimal for $\theta = 1$ (each X_t is an independent sample from a normal distribution).
To compute the autocorrelation, we observe that

$$X_{t+1} = \theta\mu \sum_{k=0}^{n} (1-\theta)^k + (1-\theta)^{n+1} X_{t-n} + \sum_{k=0}^{n} (1-\theta)^k \xi_{t-k} \tag{8}$$

Recalling that

$$\sum_{k=0}^{n} x^k = \sum_{k=0}^{\infty} x^k - \sum_{k=0}^{\infty} x^{k+n+1} = \frac{1-x^{n+1}}{1-x} \tag{9}$$

under the condition that $|1-\theta| < 1$, remembering that the sum of IID normal r.v. has variance equal the sum of the variances, we get that

$$\begin{align} X_{t+1} &= \left[1 - (1-\theta)^{j+1}\right]\mu + (1-\theta)^{j+1} X_{t-j} + \sqrt{\frac{1-(1-\theta)^{2(j+1)}}{1-(1-\theta)^2}}\,\xi_{t-j} \tag{10} \\ &= a(j) + b(j)\, X_{t-j} + c(j)\, \xi_{t-j} \tag{11} \end{align}$$

Now we can compute the quantity

$$
\begin{aligned}
& G_X(j) = \langle X_{t+1} X_{t-j+1} \rangle = && (12)\\
& = \langle [\mu\theta + (1-\theta)X_t + \xi_t][\mu\theta + (1-\theta)X_{t-j} + \xi_{t-j}] \rangle && (13)\\
& = [a(j) + b(j)X_{t-j} + c(j)\xi_{t-j}][\mu\theta + (1-\theta)X_{t-j} + \xi_{t-j}] && (14)\\
& = a(j)\mu\theta + \langle X \rangle \left[a(j)(1-\theta) + b(j)\mu\theta\right] + b(j)(1-\theta)\langle X^2 \rangle + c(j)\sigma_\xi^2 && (15)\\
& = [1 + (1-\theta)^{j+1}]\mu^2\theta + \mu^2 \left\{[1 + (1-\theta)^{j+1}](1-\theta) + \theta(1-\theta)^{j+1}\right\} + && (16)\\
& = \mu^2 \left[1 + (1-\theta)^{j+1} + \theta(1-\theta)^{j+1} + (1-\theta)^{j+2}\right] + \sigma_\xi^2 \left[\frac{(1-\theta)^{j+2}}{\theta(2-\theta)} + \sqrt{\frac{1 + (1-\theta)^{2(j+1)}}{1 + (1-\theta)^2}}\right] && (17)\\
& = \mu^2 \left[1 + 2(1-\theta)^{j+1}\right] + \sigma_\xi^2 \left[\frac{(1-\theta)^{j+2}}{\theta(2-\theta)} + \sqrt{\frac{1 + (1-\theta)^{2(j+1)}}{1 + (1-\theta)^2}}\right] && (18)
\end{aligned}
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