Some derivation of the axon problem

March 6, 2024

This is about Nestor's diffusion advection problem. We have a finite axon of length L, with some influx j_0 at the soma ($x = 0$). We also know that the dendrite is closed, so no net flux will be possible at $x = L$. Our chemical species $u : (x, t) \mapsto u(x, t)$ undergoes diffusion, advection, and decay. If I am not mistaken, the problem can be formulated as the initial boundary problem:

$$\begin{cases} u_t = D\, u_{xx} - v\, u_x - k\, u \\ [D\, u_x - v\, u]_{(0,t)} = j_0 \\ [D\, u_x - v\, u]_{(L,t)} = 0 \\ u(x, 0) = \varphi(x) \end{cases} \tag{1}$$

on the domain $x \in D = [0, L]$ and $t \in [0, +\infty]$. This is very classic stuff and I sure hope there exists a solution of some sort. Let us start first with the steady-state solution

Steady-state In order to find the steady-state solution $s(x) = u(x, t \to \infty)$, we impose $u_t = 0$. That leaves us with the Sturm-Liouville problem

$$\begin{cases} Ds'' - vs' - ks = 0 \\ Ds'(0) - vs(0) = j0 \\ Ds'(L) - vs(L) = 0 \end{cases} \tag{2}$$

which admits the general solution

$$s(x) = a\, e^{\lambda_1 x} + b\, e^{\lambda_2 x}, \quad a, b \in \mathbb{R} \tag{3}$$

Since $D, v, k > 0$ (things diffuse, move rightwards and degrade in time), also the eigenvalues λ_1 and λ_2 will be real. We can now proceed and find the exact values of a and b by direct substitution in the boundary conditions. Before going ahead, we rewrite the expressions for $s(x)$ and $s'(x)$ in the more convenient form

$$s(x) = e^{\frac{v}{2D}x} \left[a\, e^{\frac{\Delta}{2D}x} + b\, e^{-\frac{\Delta}{2D}x} \right] \tag{4}$$

$$s'(x) = \frac{v}{2D} s(x) + \frac{\Delta}{2D} e^{\frac{v}{2D}x} \left[a\, e^{\frac{\Delta}{2D}x} - b\, e^{-\frac{\Delta}{2D}x} \right] \tag{5}$$

where $\Delta = \sqrt{v^2 + 4Dk}$. By inserting these expressions in the boundary conditions, we obtain

$$\begin{cases} b = \dfrac{\Delta - v}{\Delta + v}\, a - \dfrac{2j_0}{\Delta - v} \\ b = \exp\left\{\dfrac{\Delta}{D} L\right\} \dfrac{\Delta - v}{\Delta + v}\, a \end{cases}$$

This yields the constant values

$$a = \frac{2j_0}{(\Delta - v)(1 - e^{\frac{\Delta}{D}L})}, \qquad b = \frac{2j_0\, e^{\frac{\Delta}{D}L}}{(\Delta + v)(1 - e^{\frac{\Delta}{D}L})} \tag{6}$$

and the final expression for $s(x)$ as

$$s(x) = \frac{2j_0}{1 - e^{\frac{\Delta}{D}L}}\, e^{\frac{v}{2D}x} \left[\frac{e^{\frac{\Delta}{2D}x}}{\Delta - v} + \frac{e^{-\frac{\Delta}{2D}(x-2L)}}{\Delta + v} \right] \tag{7}$$

Interestingly, the flux j_0 acts as a scaling parameter.

Full solution I don't really know how to do this for now but soon. Below are some failed attempts. We start with the substitution

$$u(x,t) = w(x,t)\, e^{\eta t - \mu x} \tag{8}$$

where

$$\eta = k + \frac{v^2}{4D}, \qquad \mu = \frac{v}{2D} \tag{9}$$

by direct substitution we get

$$\begin{cases} w_t = D\, w_{xx} \\ D\, w_x(0,t) - (D\mu + v) w(0,t) = j_0\, e^{-\eta t} \\ D\, w_x(L,t) - (D\mu + v) w(L,t) = 0 \\ w(x,0) = \varphi(x)\, e^{\mu x - \eta t} \end{cases} \tag{10}$$

We have, therefore, a initial boundary problem for the diffusion of $w(x,t)$, with Robin boundary conditions that depend on time. To try and tackle this problem, we will follow the recipe in [1] and integrating it with [2].
We start by separating our solution $w(x,t)$ in two parts, one denominated $S(x,t)$, representing the steady-state of (10), and the other $U(x,t)$ representing the transient perturbation of $w(x,t)$ from its asymptotic steady state

$$w(x,t) = S(x,t) + U(x,t) \tag{11}$$

By construction, $S(x,t)$ has to respect the boundary value problem (10) for $t \to \infty$. Following, without any pretentions of originality, the method illustrated in [1], we posit for $S(x,t)$ the following "quasi-steady-state" form

$$S(x,t) = A(t)\left(1 - \frac{x}{l}\right) + B(t)\frac{x}{l} \tag{12}$$

The two unknown functions can be promptly found by substituting $S(x,t)$ in the boundary conditions

$$\begin{cases} \dfrac{D}{l}(B-A) - \dfrac{3}{2}vA = j_0 e^{-\eta t} \\ \dfrac{D}{l}(B-A) - \dfrac{3}{2}vB = 0 \end{cases} \tag{13}$$

obtaining

$$A(t) = \left[\frac{2D}{3lv} - 1\right]\frac{2j_0}{3v}e^{-\eta t}, \quad B(t) = \frac{4Dj_0}{9lv^2}e^{-\eta t} \tag{14}$$

and therefore

$$S(x,t) = \left[\frac{x}{l} + \frac{2D}{3lv} - 1\right]\frac{2j_0}{3v}e^{-\eta t} \tag{15}$$

After finding $S(x,t)$, we can correctly define the problem for the transient solution $U(x,t)$, subject to Dirichlet boundary conditions:

$$\begin{cases} U_t = DU_{xx} \\ D\,U_x(0,t) - \dfrac{3}{2}U(0,t) = 0 \\ D\,U_x(L,t) - \dfrac{3}{2}U(L,t) = 0 \\ U(x,0) = \varphi(x)e^{\mu x - \eta t} - S(x,0) =: \tilde{\varphi}(x) \end{cases} \tag{16}$$

We can proceed with separation of variables, and posit that $U(x,t) = X(x)T(t)$. We first solve the spatial boundary value problem

$$\begin{cases} X'' + \lambda X = 0 \\ D\,X'(0) - \dfrac{3}{2}X(0) = 0 \\ D\,X'(L) - \dfrac{3}{2}X(L) = 0 \end{cases} \tag{17}$$

The only non-trivial solution of this BVP can be written, assuming $\lambda > 0$, as

$$X(x) = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x \tag{18}$$

and obtain the values for λ and the relationship between a and b by substituting in the boundary conditions

$$\begin{cases} Db\sqrt{\lambda} - \dfrac{3}{2}va = 0 \\ D\sqrt{\lambda}\left[b\cos\sqrt{\lambda}l - a\sin\sqrt{\lambda}l\right] - \dfrac{3}{2}v\left[a\cos\sqrt{\lambda}l + b\sin\sqrt{\lambda}l\right] = 0 \end{cases} \tag{19}$$

$$\begin{cases} Db\sqrt{\lambda} - \dfrac{3}{2}va = 0 \\ \cos\sqrt{\lambda}l\left[Db\sqrt{\lambda} - \dfrac{3}{2}va\right] + \sin\sqrt{\lambda}l\left[Da\sqrt{\lambda} + \dfrac{3}{2}vb\right] = 0 \end{cases} \tag{20}$$

$$\begin{cases} a = \dfrac{2Db}{3v}\sqrt{\lambda} \\ \dfrac{b}{6v}\sin\sqrt{\lambda}l\left[4D^2\lambda + 9v^2\right] = 0 \end{cases} \tag{21}$$

Since $\lambda > 0$, its only possible values are $\lambda = \frac{n^2\pi^2}{l}$, $n \in \mathbb{N}$, yielding the final form for $X(x)$ is

$$X(x) = C_n\left[\frac{2Dn\pi}{3vl}\cos\frac{n\pi}{l}x + \sin\frac{n\pi}{l}x\right] \tag{22}$$

References

1. Drábek, P. & Holubová, G. *Elements of Partial Differential Equations* in (2007). `https://api.semanticscholar.org/CorpusID:118449101`.
2. Salsa, S. Partial Differential Equations in Action. *UNITEXT*. `https://api.semanticscholar.org/CorpusID:124525275` (2022).