Označíme

$$C_\alpha(\theta) = \left(\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x\right)^{\frac{\alpha}{1+\alpha}} \tag{1}$$

pak Rényiho odhady po dosazení empirického rozdělení pravděpodobnosti P_n vypadají

$$\theta_{\alpha,n} = \begin{cases} \arg\max\limits_{\theta\in\Theta} C_\alpha(\theta)^{-1} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} p_\theta^\alpha(x_i) & \text{pro } 0 < \alpha \le \beta, \\ \arg\max\limits_{\theta\in\Theta} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \ln p_\theta(x_i) & \text{pro } \alpha = 0 \end{cases} \tag{2}$$

1 Exponenciální rozdělení

$$\begin{aligned} \int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \int_\mu^\infty \left(\frac{1}{\theta}\exp\left[-\frac{(x-\mu)}{\theta}\right]\right)^{1+\alpha} \mathrm{d}x \\ &= \int_\mu^\infty \frac{1}{\theta^{1+\alpha}} \exp\left[-\frac{(1+\alpha)(x-\mu)}{\theta}\right] \mathrm{d}x \end{aligned}$$

substituujeme $y = \frac{(1+\alpha)(x-\mu)}{\theta}$, tedy $\mathrm{d}y = \frac{1+\alpha}{\theta}\mathrm{d}x$, pak

$$\begin{aligned} \int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \frac{1}{\theta^{1+\alpha}} \int_0^\infty \frac{\theta}{1+\alpha} \exp[-y]\,\mathrm{d}x \\ &= \frac{\theta^{-\alpha}}{1+\alpha}. \end{aligned} \tag{3}$$

Následující postup nefunguje, protože v empirické distribuci stále ještě není zahrnut interval, na kterém je rozdělení definováno.

$$\begin{aligned} \theta_{\alpha,n} &= \arg\max_{\theta\in\Theta} \left(\frac{\theta^{-\alpha}}{1+\alpha}\right)^{-\frac{\alpha}{1+\alpha}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\theta^\alpha} \exp\left[-\alpha\frac{x_i-\mu}{\theta}\right] \\ &= \arg\max_{\theta\in\Theta} \theta^{-\frac{\alpha}{1+\alpha}} \frac{1}{n}\sum_{i=1}^{n} \exp\left[-\alpha\frac{x_i-\mu}{\theta}\right] \end{aligned} \tag{4}$$

2 Laplaceovo rozdělení

$$\begin{aligned} \int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \int_{-\infty}^\infty \left(\frac{1}{2\theta}\exp\left[-\frac{|x-\mu|}{\theta}\right]\right)^{1+\alpha} \mathrm{d}x \\ &= \int_{-\infty}^\mu \frac{1}{(2\theta)^{1+\alpha}} \exp\left[\frac{(1+\alpha)(x-\mu)}{\theta}\right] \mathrm{d}x \\ &+ \int_\mu^\infty \frac{1}{(2\theta)^{1+\alpha}} \exp\left[-\frac{(1+\alpha)(x-\mu)}{\theta}\right] \mathrm{d}x \end{aligned}$$

substituujeme $y = \frac{(1+\alpha)(x-\mu)}{\theta}$, tedy $\mathrm{d}y = \frac{1+\alpha}{\theta}\mathrm{d}x$, pak

$$
\begin{aligned}
\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \frac{1}{(2\theta)^{1+\alpha}}\frac{\theta}{1+\alpha}\left(\int_{-\infty}^{0}\exp\left[y\right]\mathrm{d}x + \int_{0}^{\infty}\exp\left[-y\right]\mathrm{d}x\right) \\
&= \frac{1}{(2\theta)^{1+\alpha}}\frac{\theta}{1+\alpha}\cdot 2 \\
&= \frac{(2\theta)^{-\alpha}}{(1+\alpha)}
\end{aligned}
\tag{5}
$$

tedy

$$
\begin{aligned}
\theta_{\alpha,n} &= \arg\max_{\theta\in\Theta}\left(\frac{(2\theta)^{-\alpha}}{(1+\alpha)}\right)^{-\frac{\alpha}{1+\alpha}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(2\theta)^{\alpha}}\exp\left[-\alpha\frac{|x_i-\mu|}{\theta}\right] \\
&= \arg\max_{\theta\in\Theta}(2\theta)^{-\frac{\alpha}{1+\alpha}}\frac{1}{n}\sum_{i=1}^{n}\exp\left[-\alpha\frac{|x_i-\mu|}{\theta}\right]
\end{aligned}
\tag{6}
$$

3 Rovnoměrné rozdělení

$$
\begin{aligned}
\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \int_a^b\left(\frac{1}{b-a}\right)^{1+\alpha}\mathrm{d}x \\
&= \frac{b-a}{(b-a)^{\alpha+1}}
\end{aligned}
$$

tedy

$$
\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x = (b-a)^{-\alpha} \tag{7}
$$

Následující postup nefunguje, protože v empirické distribuci stále ještě není zahrnut interval, na kterém je rozdělení definováno.

$$
\begin{aligned}
\theta_{\alpha,n} &= \arg\max_{\theta\in\Theta}\left((b-a)^{-\alpha}\right)^{-\frac{\alpha}{1+\alpha}}\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbf{I}_{(a,b)}}{(b-a)^{\alpha}} \\
&= \arg\max_{\theta\in\Theta}\left((b-a)^{-\alpha}\right)^{-\frac{\alpha}{1+\alpha}}\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbf{I}_{(a,b)}}{\left(2\sqrt{3\mathrm{V}x_i}\right)^{\alpha}} \\
&= \arg\max_{\theta\in\Theta}(b-a)^{\frac{\alpha^2}{1+\alpha}}\frac{1}{\left(2\sqrt{3\mathrm{V}X}\right)^{\alpha}}
\end{aligned}
\tag{8}
$$

4 Cauchyovo rozdělení

$$\begin{aligned}\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty}\left(\frac{1}{\pi\sigma}\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-1}\right)^{1+\alpha}\mathrm{d}x\\ &= \dots\end{aligned}$$

pak

$$\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x = \frac{1}{\pi^{\frac{1}{2}+\alpha}\sigma^\alpha}\frac{\Gamma(\frac{1}{2}+\alpha)}{\alpha\Gamma(\alpha)} \tag{9}$$

tedy

$$\begin{aligned}\theta_{\alpha,n} &= \arg\max_{\theta\in\Theta}\left(\frac{1}{\pi^{\frac{1}{2}+\alpha}\sigma^\alpha}\frac{\Gamma(\frac{1}{2}+\alpha)}{\Gamma(1+\alpha)}\right)^{-\frac{\alpha}{1+\alpha}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\pi^\alpha\sigma^\alpha}\left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)^{-\alpha}\\ &= \arg\max_{\theta\in\Theta}\sigma^{-\frac{\alpha}{1+\alpha}}\frac{1}{n}\sum_{i=1}^{n}\left(1+\left(\frac{x_i-\mu}{\sigma}\right)^2\right)^{-\alpha}\end{aligned}$$

5 Weibullovo rozdělení

$$\begin{aligned}\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x &= \int_{-\mu}^{\infty}\left(\frac{k}{\lambda}\left(\frac{x-\mu}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{x-\mu}{\lambda}\right)^k\right]\right)^{1+\alpha}\mathrm{d}x\\ &= \dots\end{aligned}$$

pak

$$\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x = \frac{k^\alpha}{\lambda^\alpha}(1+\alpha)^{-\frac{1+\alpha+k}{k}}\Gamma\left(\frac{1+\alpha+k}{k}\right) \tag{10}$$

Následující postup nefunguje, protože v empirické distribuci stále ještě není zahrnut interval, na kterém je rozdělení definováno.

$$\begin{aligned}\theta_{\alpha,n} &= \arg\max_{\theta\in\Theta}\left(\frac{k^\alpha}{\lambda^\alpha}(1+\alpha)^{-\frac{1+\alpha+k}{k}}\Gamma\left(\frac{1+\alpha+k}{k}\right)\right)^{-\frac{\alpha}{1+\alpha}}\\ &\quad \frac{1}{n}\sum_{i=1}^{n}\frac{k^\alpha}{\lambda^\alpha}\left(\frac{x_i-\mu}{\lambda}\right)^{\alpha(k-1)}\exp\left[-\alpha\left(\frac{x_i-\mu}{\lambda}\right)^k\right]\\ &= \arg\max_{\theta\in\Theta}\left(\frac{k}{\lambda}\right)^{\frac{\alpha}{1+\alpha}}(1+\alpha)^{\frac{\alpha}{1+\alpha}\frac{1+\alpha+k}{k}}\Gamma\left(\frac{1+\alpha+k}{k}\right)^{-\frac{\alpha}{1+\alpha}}\\ &\quad \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\lambda}\right)^{\alpha(k-1)}\exp\left[-\alpha\left(\frac{x_i-\mu}{\lambda}\right)^k\right]\end{aligned} \tag{11}$$