Rényi pseudo-distances

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Decomposable pseudo-distance

We say, that $\mathfrak{D} : \mathcal{P} \to \mathbb{R}$ is *pseudo-distance* if $\forall P, Q \in \mathcal{P}$

$$\mathfrak{D}(P, Q) \geq 0 \qquad \text{and} \qquad \mathfrak{D}(P, Q) = 0 \;\Leftrightarrow\; P = Q$$

and this pseudo-distance is *decomposable* if there exist functionals so that $\mathfrak{D}^0 : \mathcal{P} \to \mathbb{R}$, $\mathfrak{D}^1 : \tilde{\mathcal{P}} \to \mathbb{R}$ and measurable $\rho_\theta : \mathcal{X} \to \mathbb{R}$, $\theta \in \Theta$, so that $\forall \theta \in \Theta$ and $\forall Q \in \tilde{\mathcal{P}}$ there exists finite $\int \rho_\theta \mathrm{d}Q$ and

$$\mathfrak{D}(P_\theta, Q) = \mathfrak{D}^0(P_\theta) + \mathfrak{D}^1(Q) + \int \rho_\theta \mathrm{d}Q.$$

- we don't presume triangle inequality or symmetry

Minimal distance estimator

Functional $T_{\mathfrak{D}} : \tilde{\mathcal{P}} \to \Theta$ defines *minimal distance* estimator if $\mathfrak{D}(P_\theta, Q)$ is decomposable and $T_{\mathfrak{D}}(Q) \in \Theta$ minimizes

$$T_{\mathfrak{D}}(Q) = \arg\min_{\theta \in \Theta} \left[\mathfrak{D}^0(P_\theta) + \int \rho_\theta \mathrm{d}Q \right], \quad \forall Q \in \tilde{\mathcal{P}}$$

Rényi pseudo-distance

Let for some $\beta > 0$

$$p^{\beta}, q^{\beta}, \ln p \in \mathrm{L}_1(Q), \quad \forall P \in \mathcal{P}, Q \in \tilde{\mathcal{P}}.$$

holds. Then for some α, $0 < \alpha \leq \beta$, $P \in \mathcal{P}$, $Q \in \tilde{\mathcal{P}}$ is decomposable, that is

$$\mathcal{R}_{\alpha}(P, Q) = \mathcal{R}_{\alpha}^{0}(P) + \mathcal{R}_{\alpha}^{1}(Q) - \frac{1}{\alpha} \ln \left(\int p^{\alpha} \mathrm{d}Q \right),$$

where

$$\mathcal{R}_{\alpha}^{0}(P) = \frac{1}{1+\alpha} \ln \left(\int p^{\alpha} \mathrm{d}P \right), \quad \mathcal{R}_{\alpha}^{1}(Q) = \frac{1}{\alpha(1+\alpha)} \ln \left(\int q^{\alpha} \mathrm{d}Q \right).$$

Moreover for $\alpha \searrow 0$

$$\mathcal{R}_{0}(P, Q) = \int (\ln q - \ln p) \, \mathrm{d}Q$$

If we replace Q by the empirical distribution P_n, we get

$$\theta_{\mathfrak{R}_\alpha,n} = \begin{cases} \arg\max\limits_{\theta\in\Theta} \left(\int p_\theta^{1+\alpha}(x)\,\mathrm{d}x\right)^{-\frac{\alpha}{1+\alpha}} \frac{1}{n}\sum\limits_{i=1}^{n} p_\theta^{\alpha}(x_i), \\ \qquad \text{for } 0 < \alpha \leq \beta, \\ \arg\max\limits_{\theta\in\Theta} \frac{1}{n}\sum\limits_{i=1}^{n} \ln p_\theta(x_i), \\ \qquad \text{for } \alpha = 0. \end{cases}$$

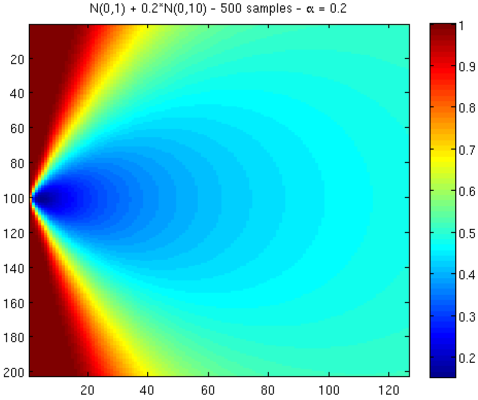
So for $\alpha = 0$ the $\theta_{\alpha,n} = \theta_{MLE}$

Normal distribution

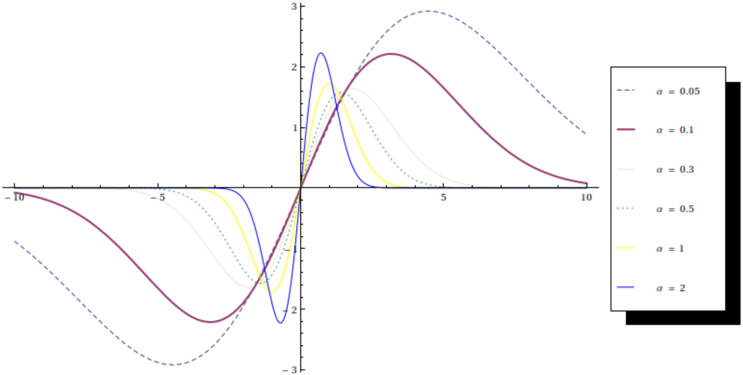
$$p_\theta = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma}\right]$$

$$\theta_{\mathfrak{R}_\alpha,n} = \arg\max_{\theta\in\Theta} \frac{1}{n\sigma^{\frac{\alpha}{1+\alpha}}} \sum_{i=1}^{n} \exp\left[-\alpha\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

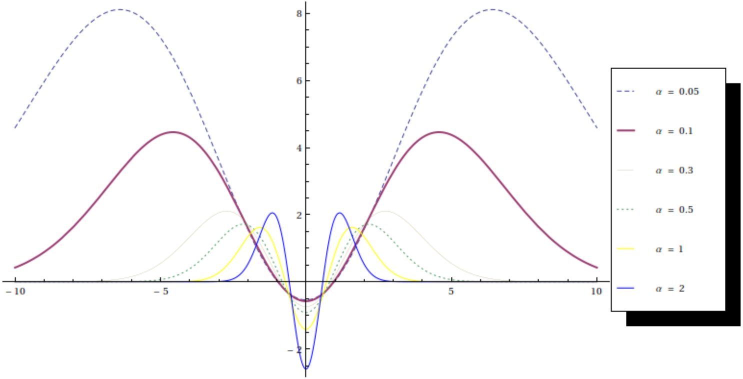
distances in the parametric space:



$$\mathrm{IF}(x; T_{\mathfrak{R}_\alpha}, \mu) = (1+\alpha)^{\frac{3}{2}}(x-\mu)e^{-\frac{\alpha}{2}(x-\mu)^2}$$



$$\mathrm{IF}(x; T_{\mathfrak{R}_\alpha}, \sigma) = \frac{(1+\alpha)^{\frac{5}{2}}\sigma}{2}\left(\left(\frac{x}{\sigma}\right)^2 - \frac{1}{1+\alpha}\right)e^{-\frac{\alpha x^2}{2\sigma^2}}$$



$\lim_{x\to\pm\infty} \mathrm{IF}(x; T_{\mathfrak{R}_\alpha}, \cdot) = 0$
$\to$ estimator is robust against outliers
$\sup_x |\mathrm{IF}(x; T_{\mathfrak{R}_\alpha}, \cdot)|$ is finite
$\to$ estimator is robust against point errors

$\alpha \backslash n$	500		
	$m(\mu)$ $m(\sigma)$	$s(\mu)$ $s(\sigma)$	$eref(\mu)$ $eref(\sigma)$
0.0	0.000 1.672	0.037 0.051	1.000 1.000
0.01	−0.000 1.652	0.037 0.050	1.032 1.031
0.05	0.001 1.572	0.033 0.046	1.248 1.244
0.1	−0.000 1.478	0.031 0.041	1.426 1.560
0.2	−0.000 1.333	0.029 0.035	1.650 2.097
0.5	−0.000 1.156	0.028 0.028	1.733 3.295

Table: Renyi: $p = \mathrm{N}(0,1)$, data: $(1-\varepsilon)\mathrm{N}(0,1) + \varepsilon\mathrm{N}(0,10)$, $\varepsilon = 0.2$, $K = 10000$

$\alpha \backslash n$	500		
	$m(\mu)$ $m(\sigma)$	$s(\mu)$ $s(\sigma)$	$eref(\mu)$ $eref(\sigma)$
0.0	-0.000 0.949	0.021 0.016	1.000 1.000
0.01	-0.000 0.949	0.021 0.016	1.002 0.969
0.05	0.001 0.945	0.021 0.016	0.991 0.967
0.1	0.000 0.942	0.021 0.017	0.990 0.889
0.2	-0.001 0.931	0.021 0.017	1.032 0.865
0.5	-0.000 0.905	0.021 0.020	0.998 0.641

Table: Renyi: $p = \mathrm{N}(0,1)$, data: $0.9\mathrm{N}(0,1) + 0.1\mathrm{N}_{0.1x}(0,1)$, $K = 1000$

$\alpha \backslash n$	500		
	$m(\mu)$ $m(\sigma)$	$s(\mu)$ $s(\sigma)$	$eref(\mu)$ $eref(\sigma)$
0.0	−0.002 3.294	0.075 0.231	1.000 1.000
0.01	−0.000 3.104	0.065 0.220	1.327 1.103
0.05	0.001 2.118	0.040 0.189	3.489 1.493
0.1	0.001 1.277	0.026 0.053	8.356 18.986
0.2	0.000 1.087	0.025 0.024	8.796 91.365
0.5	−0.000 1.029	0.026 0.021	8.390 123.291

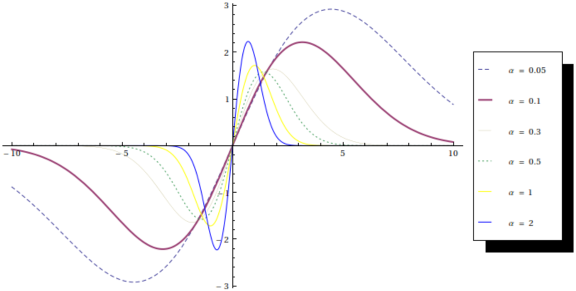
Table: Renyi: $p = \mathrm{N}(0,1)$, data: $0.9\mathrm{N}(0,1) + 0.1\mathrm{N}_{10.0x}(0,1)$, $K = 1000$

Laplace distribution

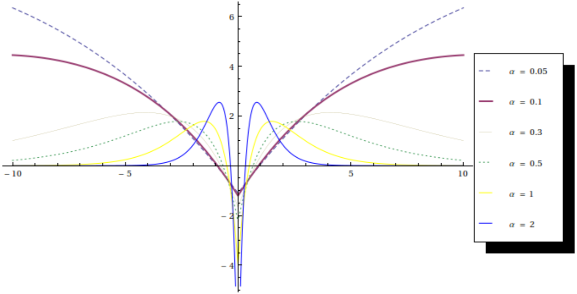
$$p_\theta = \frac{1}{2\lambda} e^{-\frac{|x-\mu|}{\lambda}}$$

$$\theta_{\mathfrak{R}_\alpha,n} = \arg\max_{\theta\in\Theta} (2\lambda)^{-\frac{\alpha}{1+\alpha}} \frac{1}{n} \sum_{i=1}^{n} \exp\left[-\alpha \frac{|x_i - \mu|}{\lambda}\right]$$

$$\mathrm{IF}(x;\, T_{\mathfrak{R}_\alpha}, \mu) = (1+\alpha)^{\frac{3}{2}}(x-\mu)e^{-\frac{\alpha}{2}(x-\mu)^2}$$



$$\mathrm{IF}(x;\, T_{\mathfrak{R}_\alpha}, \lambda) = (1+\alpha)^2\left(-\lambda + (1+\alpha)|x|\right) e^{-\frac{\alpha|x|}{\lambda}}$$



$\alpha \backslash n$	500		
	$m(\mu)$ $m(\lambda)$	$s(\mu)$ $s(\lambda)$	$eref(\mu)$ $eref(\lambda)$
0.0	0.000 3.723	0.065 0.355	1.000 1.000
0.01	0.003 3.593	0.063 0.338	1.090 1.102
0.05	0.002 3.239	0.063 0.306	1.063 1.348
0.1	−0.002 2.796	0.062 0.285	1.118 1.556
0.2	0.000 2.059	0.062 0.205	1.105 3.005
0.3	−0.003 1.626	0.060 0.154	1.198 5.326
0.5	0.000 1.300	0.058 0.108	1.267 10.756
1.0	−0.001 1.126	0.068 0.110	0.932 10.386

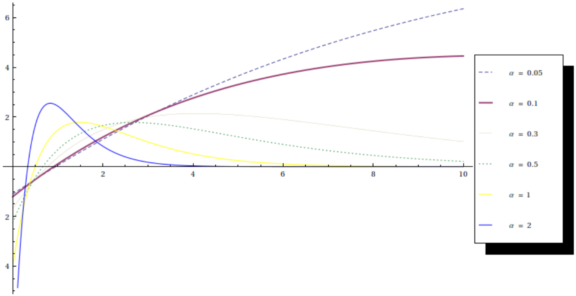
Table: Renyi: $p = \mathrm{L}(0,1)$, data: $(1-\varepsilon)\mathrm{L}(0,1) + \varepsilon\mathrm{L}(0,10)$, $\varepsilon = 0.3$, $K = 1000$

Exponential distribution

$$p_\theta = \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}$$

$$\theta_{\mathfrak{R}_\alpha,n} = \arg\max_{\theta\in\Theta} \lambda^{-\frac{\alpha}{1+\alpha}} \frac{1}{n} \sum_{i=1}^{n} \exp\left[-\alpha \frac{x_i-\mu}{\lambda}\right]$$

$$\mathrm{IF}(x; T_{\mathfrak{R}_\alpha}, \lambda) = (1+\alpha)^2(-\lambda + (1+\alpha)x)e^{-\frac{\alpha x}{\lambda}}$$



$\alpha \backslash n$	500		
	$m(\lambda)$	$s(\lambda)$	$eref(\lambda)$
0.0	3.901	0.432	1.000
0.01	3.614	0.356	1.475
0.05	3.241	0.317	1.862
0.1	2.796	0.274	2.495
0.15	2.389	0.239	3.258
0.2	2.049	0.209	4.281
0.3	1.622	0.153	7.999
0.5	1.310	0.110	15.330
1.0	1.139	0.109	15.806

Table: Renyi: $p = \mathrm{E}(0,1)$, data: $(1-\varepsilon)\mathrm{E}(0,1) + \varepsilon\mathrm{E}(0,10)$, $\varepsilon = 0.3$, $K = 1000$

Weibull distribution

$$p_\theta = \frac{k}{\lambda}\left(\frac{x-\mu}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x-\mu}{\lambda}\right)^{k}\right]$$

$$\begin{aligned}\theta_{\mathfrak{R}_\alpha,n} \quad = \quad & \arg\max_{\theta\in\Theta}\left(\frac{k}{\lambda}\right)^{\frac{\alpha}{1+\alpha}} (1+\alpha)^{\frac{\alpha}{1+\alpha}\frac{1+\alpha+k}{k}}\Gamma\left(\frac{1+\alpha+k}{k}\right)^{-\frac{\alpha}{1+\alpha}} \\ & \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\lambda}\right)^{\alpha(k-1)} \exp\left[-\alpha\left(\frac{x_i-\mu}{\lambda}\right)^{k}\right]\end{aligned}$$

$\alpha\backslash n$	500		
	$m(\lambda)$	$s(\lambda)$	$eref(\lambda)$
0.0	4.713	0.387	1.000
0.01	4.568	0.364	1.130
0.05	4.149	0.373	1.077
0.1	3.511	0.383	1.020
0.15	2.574	0.428	0.815
0.2	1.664	0.261	2.201
0.3	1.226	0.076	25.766
0.5	1.096	0.057	46.725
1.0	1.043	0.055	49.962

Table: Renyi: $p = \mathrm{W}(1.5, 1)$, data: $(1-\varepsilon)\mathrm{W}(1.5, 1) + \varepsilon\mathrm{W}(1.5, 10)$, $\varepsilon = 0.3$, $K = 1000$

Thank you for your attention.