Online Learning Setup and Stochastic Bandits

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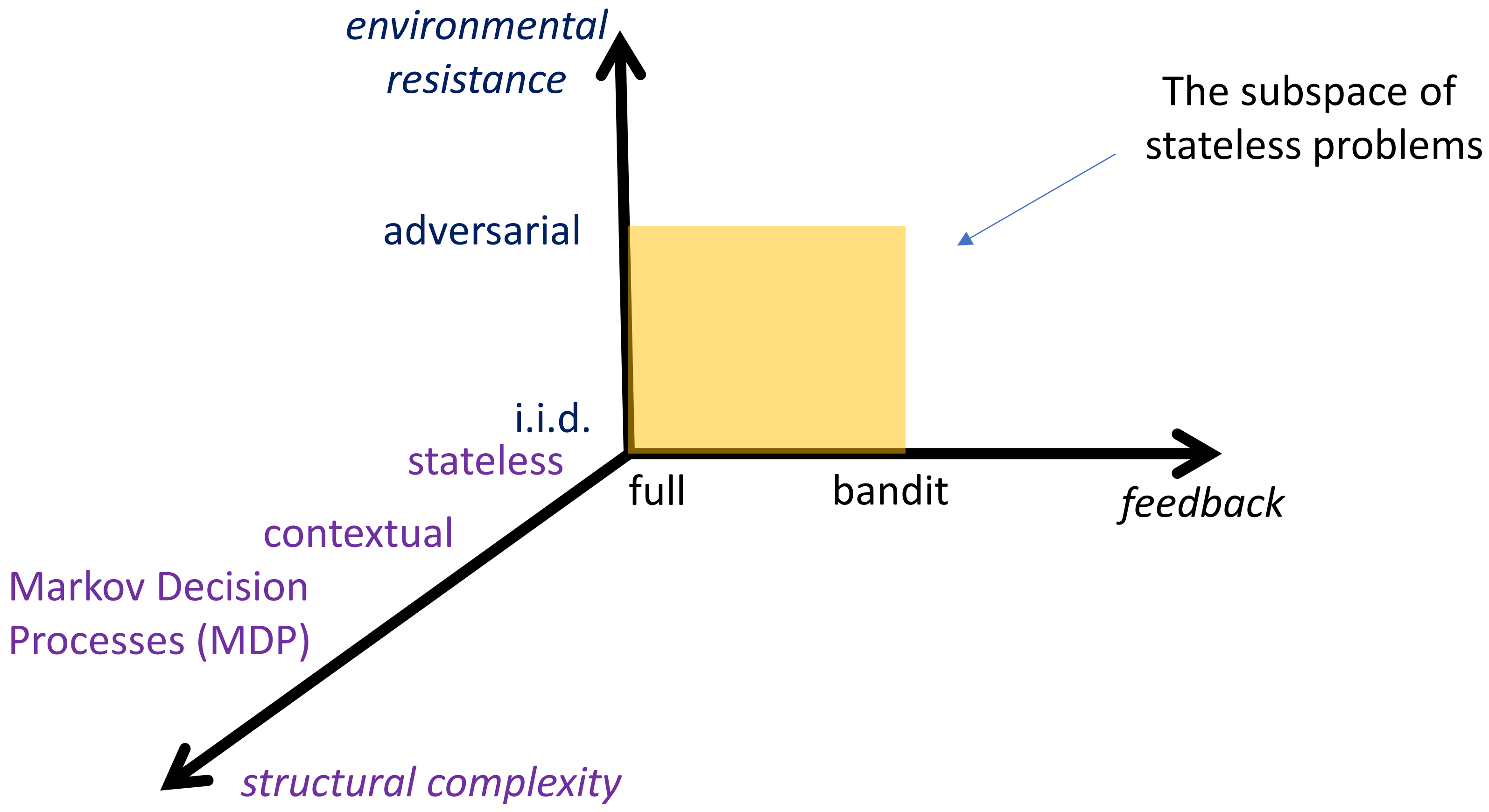
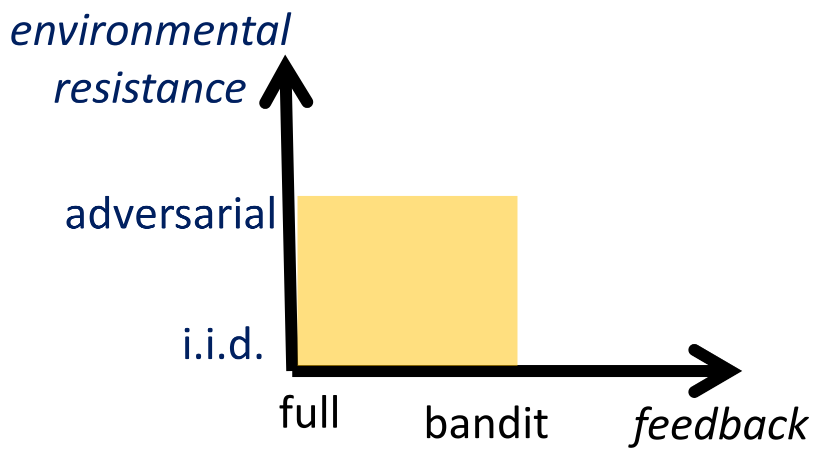
Online Learning Setup

The space of online learning problems



The stateless setting



$$K\text{ actions}\left\{\begin{array}{ccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array}\right.$$

time $\longrightarrow$

$\ell_{t,a} \in [0,1]$

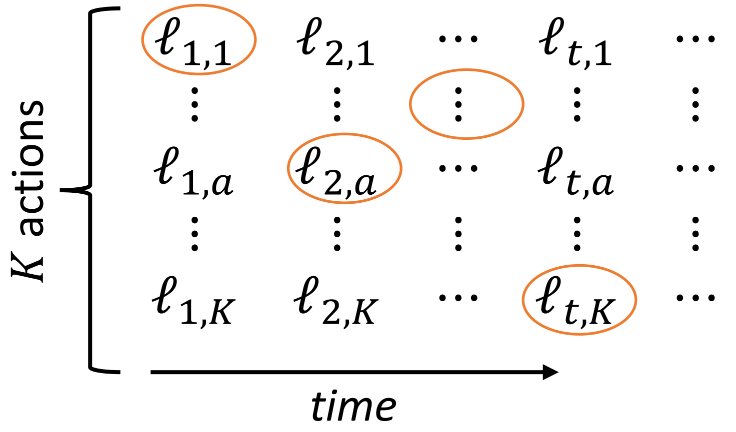
<u>Game protocol:</u>

For $t = 1,2,\ldots$:

1. Pick a row A_t
2. Suffer the loss ℓ_{t,A_t}
3. Observe …

Observations / Generation of $\ell_{t,a}$	Full: $\ell_{t,1}, \vdots, \ell_{t,K}$	Bandit: ℓ_{t,A_t}
Adversarial: $\ell_{t,a}$ arbitrary		
I.I.D.: $\ell_{t,a}$ sampled i.i.d., such that $\mathbb{E}[\ell_{t,a}] = \mu(a)$		

Performance measure



$\ell_{t,a} \in [0,1]$

- Regret: $R_T = \underbrace{\sum_{t=1}^{T} \ell_{t,A_t}}_{\text{Loss of the algorithm}} - \underbrace{\min_a \sum_{t=1}^{T} \ell_{t,a}}_{\text{Loss of the best action in hindsight}}$

- Regret of order T means no learning
  - The loss of A_t stays at the same distance from the loss of the optimal action as the game proceeds
- The aim is to achieve sublinear regret

- Why do we compare to the best fixed action in hindsight and not to the best path in hindsight?
  - "The best path in hindsight" is an overly strong competitor – we cannot guarantee sublinear regret
  - Show that the regret relative to the best path in hindsight can be as large as $\frac{K-1}{K}T$

Performance measures

$$K \text{ actions} \left\{ \begin{array}{ccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array} \right.$$

$$\xrightarrow[\textit{time}]{\qquad\qquad}$$

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- Regret: $R_T = \underbrace{\sum_{t=1}^{T} \ell_{t,A_t}}_{\text{Loss of the algorithm}} - \underbrace{\min_a \sum_{t=1}^{T} \ell_{t,a}}_{\text{Loss of the best action in hindsight}}$

- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$

$$\underbrace{=}_{\substack{\text{oblivious} \\ \text{adversary}}} \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \sum_{t=1}^{T} \ell_{t,a}$$

- Oblivious adversary:
  - $\ell_{t,a}$ is independent of $A_1, \dots, A_{t-1}$
  - The losses can be written down before the game starts
- Adaptive adversary:
  - $\ell_{t,a}$ may depend on $A_1, \dots, A_{t-1}$

Performance measures

$$K \text{ actions} \left\{ \begin{array}{ccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array} \right.$$

$\xrightarrow{\hspace{3em}}$ *time*

Regret: $R_T = \underbrace{\sum_{t=1}^{T} \ell_{t,A_t}}_{\text{Loss of the algorithm}} - \underbrace{\min_a \sum_{t=1}^{T} \ell_{t,a}}_{\text{Loss of the best action in hindsight}}$

Expected regret:

$$\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$$

- Pseudo-regret (stochastic setting):

$$\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - T \underbrace{\min_a \mu(a)}_{\mu^*}$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} (\ell_{t,A_t} - \mu^*)\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \Delta(A_t)\right]$$

$$= \mathbb{E}\left[\sum_{a=1}^{K} \Delta(a) N_T(a)\right]$$

$$= \sum_{a=1}^{K} \Delta(a) \mathbb{E}[N_T(a)]$$

- $\mathbb{E}[\ell_{t,a}] = \mu(a)$
- $\mu^* = \min_a \mu(a)$
- $a^* \in \arg\min_a \mu(a)$
  - An optimal arm (may be multiple optimal arms with the same μ^*)
- $\Delta(a) = \mu(a) - \mu^*$ suboptimality gap

- $\mathbb{E}[\ell_{t,A_t} - \mu^*] = \mathbb{E}\left[\mathbb{E}[\ell_{t,A_t} - \mu^* | A_1, \dots, A_t]\right] = \mathbb{E}[\mu(A_t) - \mu^*] = \mathbb{E}[\Delta(A_t)]$

Expected regret vs. Pseudo regret

$$K \text{ actions} \left\{ \begin{array}{ccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array} \right.$$

$$\xrightarrow[\textit{time}]{}$$

- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$
- Pseudo-regret: $\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - T\mu^*$

- $\mathbb{E}\left[\min_a f(a,B)\right] \leq \min_a \mathbb{E}[f(a,B)] \;\; \Rightarrow \;\; \bar{R}_T \leq \mathbb{E}[R_T]$

- Oblivious adversarial setting:
  - $\ell_{t,a}$ are deterministic and the two notions of regret coincide
  - $\mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right] = \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \min_a \sum_{t=1}^{T} \ell_{t,a}$

Expected regret vs. Pseudo regret

$$K\text{ actions}\left\{\begin{array}{ccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array}\right.$$

$$\xrightarrow{\quad\quad\quad\quad\quad\quad}$$
time

- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^T \ell_{t,a}\right]$
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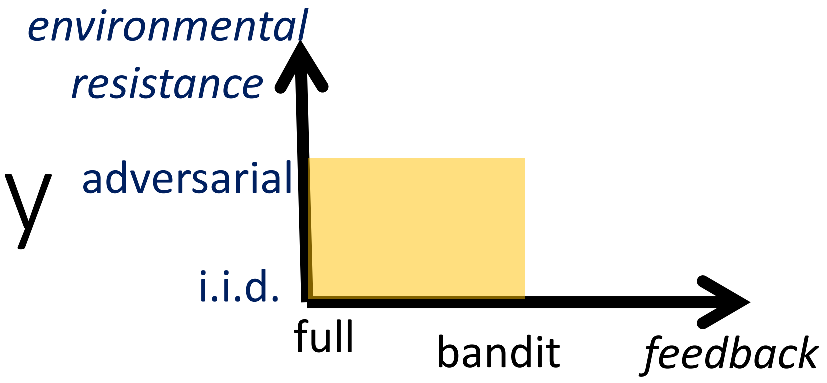
- $\mathbb{E}\left[\min_a f(a,B)\right] \leq \min_a \mathbb{E}[f(a,B)] \;\Rightarrow\; \bar{R}_T \leq \mathbb{E}[R_T]$

- Stochastic setting: imagine that $\mu(a) = \frac{1}{2}$ for all a. Then
  - $\mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] = \frac{1}{2}T$
  - $\mathbb{E}\left[\sum_{t=1}^T \ell_{t,a}\right] = \frac{1}{2}T$ for all a
  - $\bar{R}_T = 0$
  - $\mathbb{E}\left[\min_a \sum_{t=1}^T \ell_{t,a}\right] \approx T\mu^* - \sqrt{\frac{1}{2}\left(\frac{1}{2}T\right)\ln K}$
  - $\mathbb{E}[R_T] \approx \sqrt{\frac{1}{2}\left(\frac{1}{2}T\right)\ln K}$
  - Pseudo-regret is a more reasonable quantity to look at
  - Expected regret provides an artificial advantage to the competitor due to their ability to select out of K trials

Online Learning Setup - Summary



Observations / Generation of $\ell_{t,a}$	Full: $\ell_{t,1}, \vdots, \ell_{t,K}$	Bandit: ℓ_{t,A_t}
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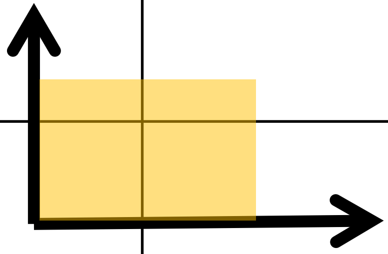
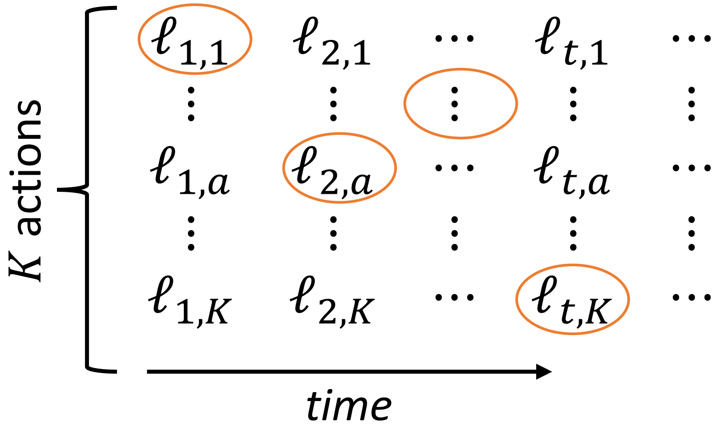
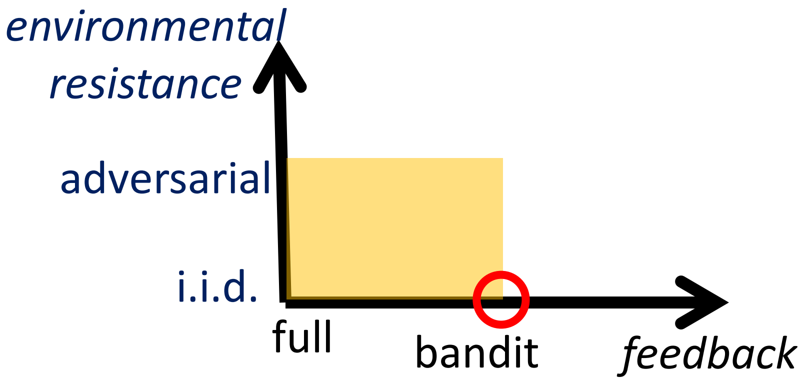




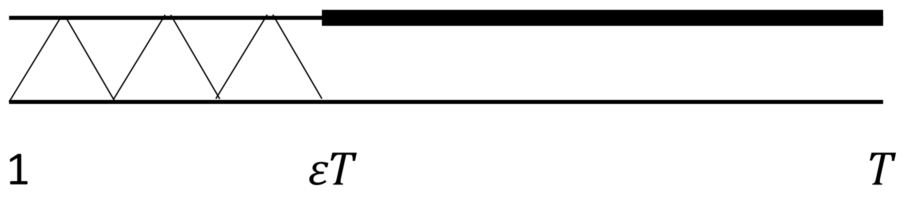
- Regret: $R_T = \sum_{t=1}^T \ell_{t,A_t} - \min_a \sum_{t=1}^T \ell_{t,a}$
- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^T \ell_{t,a}\right]$
- Pseudo-regret: $\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^T \ell_{t,a}\right] = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - T\mu^*$

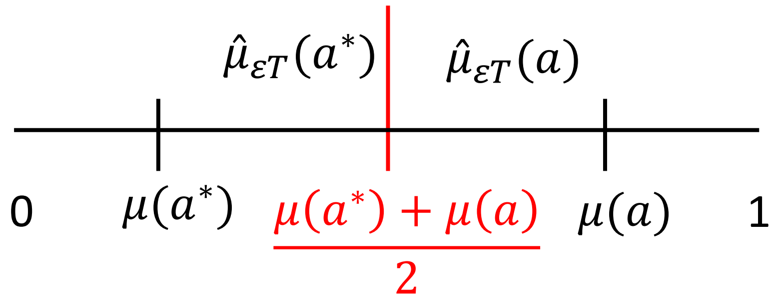


Stochastic (i.i.d.) bandits

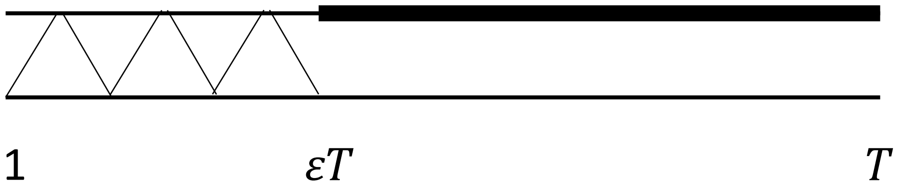
Exploration-Exploitation trade-off: a simple approach

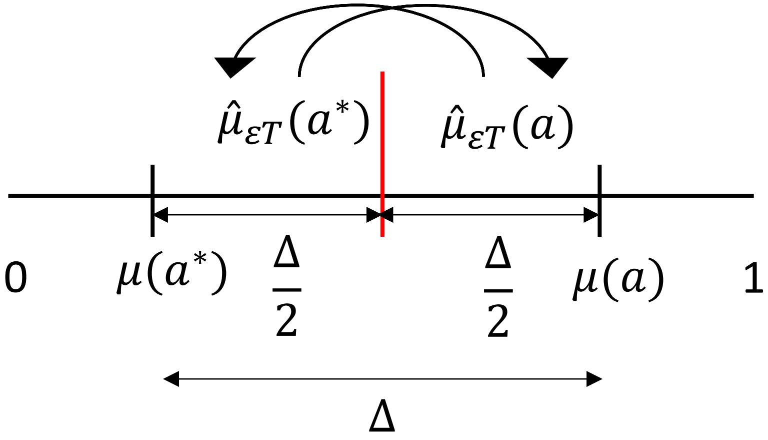
- Setting:
  - Two actions
  - Bandit feedback
  - T is known
  - Δ is known
- Approach:
  - Explore 50/50 for εT rounds
  - Exploit for the remaining rounds
- Analysis approach:
  - Take a separation line at $\frac{\mu(a^*)+\mu(a)}{2}$
  - If at time εT the empirical means are on the "correct" side of the separation line, the arm selection for exploitation will be correct
  - Bound the probability that at $\varepsilon \mathrm{T}$ the empirical means are estimated incorrectly





Analysis





- Let $\delta(\varepsilon) = \mathbb{P}\big(\hat{\mu}_{\varepsilon T}(a) \le \hat{\mu}_{\varepsilon T}(a^*)\big)$ be the prob. of confusion

- $\bar{R}_T = \sum_{t=1}^{T} \Delta(A_t) \le \underbrace{\frac{1}{2}\varepsilon T\Delta}_{\text{Exploration}} + \underbrace{\delta(\varepsilon)(1-\varepsilon)T\Delta}_{\text{Exploitation}} \le \left(\frac{\varepsilon}{2} + \delta(\varepsilon)\right)T\Delta$

- $$\begin{aligned}\delta(\varepsilon) &= \mathbb{P}\big(\hat{\mu}_{\varepsilon T}(a) \le \hat{\mu}_{\varepsilon T}(a^*)\big)\\ &\le \mathbb{P}\left(\hat{\mu}_{\varepsilon T}(a^*) \ge \mu(a^*) + \frac{1}{2}\Delta\right) + \mathbb{P}\left(\hat{\mu}_{\varepsilon T}(a) \le \mu(a) - \frac{1}{2}\Delta\right)\\ &\le 2e^{-2\frac{\varepsilon T}{2}\left(\frac{1}{2}\Delta\right)^2} = 2e^{-\varepsilon T\Delta^2/4}\end{aligned}$$

- Minimization of $\frac{\varepsilon}{2} + 2e^{-\varepsilon T\Delta^2/4}$ with respect to ε gives $\varepsilon^* = \frac{4\ln(T\Delta^2)}{T\Delta^2}$

- With exploration phase of length $\varepsilon^* T$, we get $\bar{R}_T \le \frac{2(\ln(T\Delta^2)+1)}{\Delta}$

Reflection

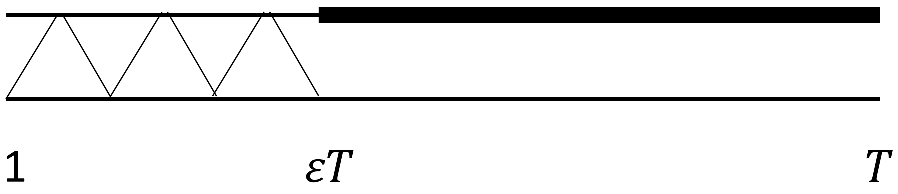
- $\varepsilon^* = \frac{4\ln(T\Delta^2)}{T\Delta^2}$
- Exploration phase: $\varepsilon^* T = \frac{4\ln(T\Delta^2)}{\Delta^2}$
- $\delta(\varepsilon^*) = 2e^{-\varepsilon^* T\Delta^2/4} = \frac{1}{T\Delta^2}$
- $\bar{R}_T \leq \frac{2\left(\ln(T\Delta^2)+1\right)}{\Delta}$

- It takes $\sim \frac{\ln T}{\Delta^2}$ rounds to identify the best arm with confidence $\frac{1}{T\Delta^2}$
- On each exploration round we pay Δ
- Total regret order:

$$\bar{\mathrm{R}}_T \approx \underbrace{\frac{\ln T}{\Delta^2}\Delta}_{\text{Exploration}} + \underbrace{\frac{1}{T\Delta^2}\mathrm{T}\Delta}_{\text{Exploitation}} \approx \frac{\ln T}{\Delta}$$

- Small $\Delta \Longrightarrow$ **Harder** problem (**Larger** $\bar{R}_T$)

Limitations



- Assumes knowledge of T
- Assumes knowledge of Δ
- Generalization to more than two actions is not straightforward

Lower Confidence Bound (LCB) algorithm for losses (Originally Upper Confidence Bound (UCB) for rewards) ("Optimism in the face of uncertainty" approach)

- Define $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{3 \ln t}{2N_{t-1}(a)}}$ lower confidence bound
  - (We will show that with high probability $L_t^{CB}(a) \leq \mu(a)$ for all t)

- LCB Algorithm:
  - Play each arm once
  - For $t = K + 1, K + 2, \ldots$:
    - Play $A_t = \arg\min_a L_t^{CB}(a)$

- Theorem:

$$\bar{R}_T \leq 6 \sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)} + \left(1 + \frac{\pi^2}{3}\right) \sum_a \Delta(a)$$

- Proof: next time

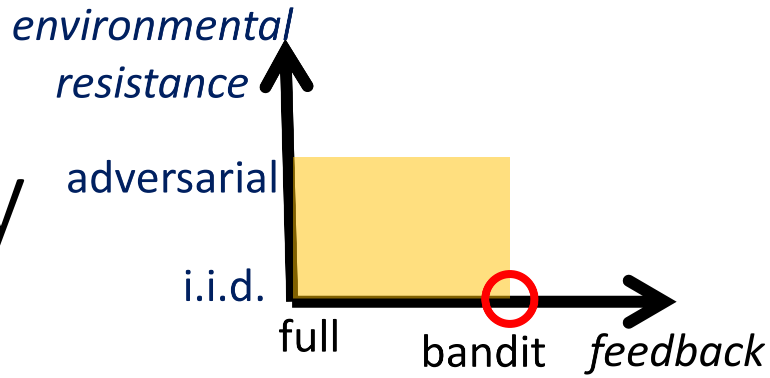
- No knowledge of T
- No knowledge of Δ
- Works for any K

Rewards ↔ Losses

$\ell_{t,a} = 1 - r_{t,a}$

$r_{t,a} = 1 - \ell_{t,a}$

Stochastic bandits – mid-summary

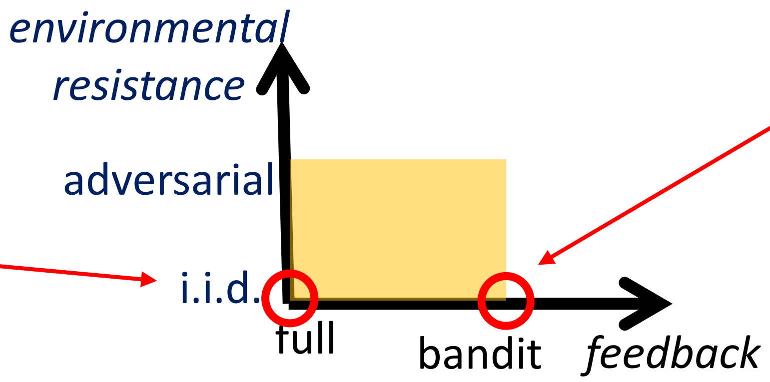


- It takes $\sim \frac{\ln T}{\Delta^2}$ rounds to identify the best arm with confidence $\frac{1}{T\Delta^2}$
- Each exploration round costs Δ, but their number grows as $\frac{1}{\Delta^2}$!
- $\overline{\mathrm{R}}_T \approx \underbrace{\frac{\ln T}{\Delta^2}\Delta}_{\text{Exploration}} + \underbrace{\frac{1}{T\Delta^2}\mathrm{T}\Delta}_{\text{Exploitation}} \approx \frac{\ln T}{\Delta}$
- Problems with small Δ are **harder** than problems with large Δ!

Home Assignment

$$\bar{R}_T = O\left(\sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)}\right)$$

$$\bar{R}_T = O\left(\sum_{a:\Delta(a)>0} \frac{1}{\Delta(a)}\right)$$



- In full information there is no need for exploration
- $\ln T$ factor is the cost of exploration (the cost of bandit feedback) in i.i.d.