Prediction with Expert Advice (Adversarial Full Info)

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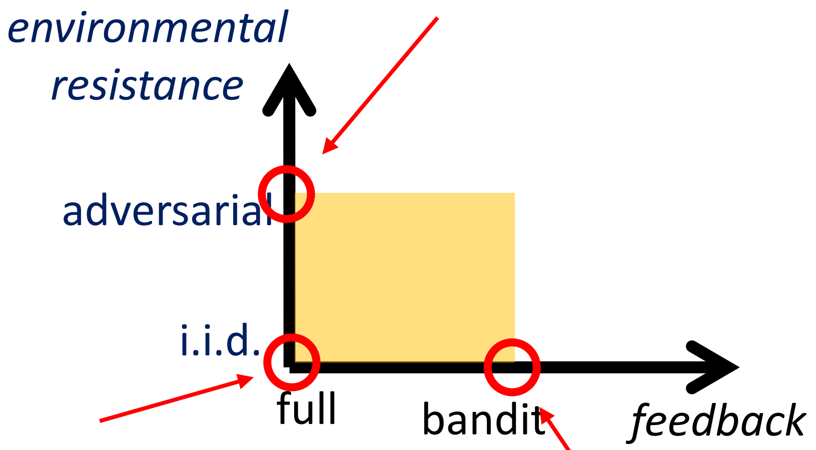
So far



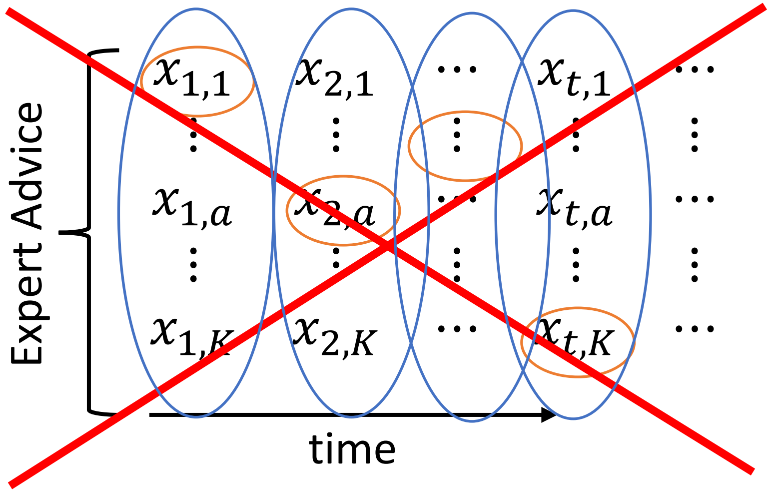
Now we look at this

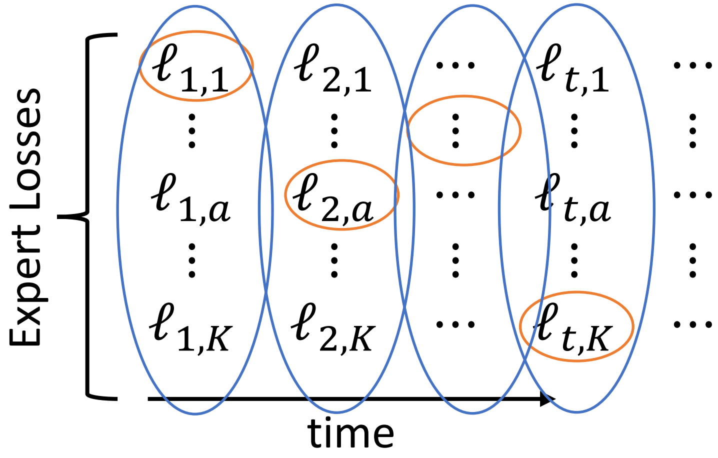
HA1:
Algorithm: FTL
$\bar{R}_T$ not growing with T

Algorithm: UCB / LCB

$$\bar{R}_T = O\left(\sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)}\right)$$

Prediction with Expert Advice (Adversarial full info game)





- Performance measures
- Regret:

$$R_T = \sum_{t=1}^{T} \ell_{t,A_t} - \min_a \sum_{t=1}^{T} \ell_{t,a}$$

- Expected regret (oblivious setting):

$$\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \sum_{t=1}^{T} \ell_{t,a}$$

$$\text{Expert Losses}\left\{\begin{array}{ccccc}
\ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots
\end{array}\right.$$

$$\xrightarrow[\text{time}]{\hspace{3cm}}$$

General observation

- Deterministic algorithms are not suitable for adversarial environments
  - Why?
  - What deterministic algorithms have we seen so far?
- We need to randomize
- Playing uniformly at random will not work
- We need to balance between randomizing and giving preference to better actions

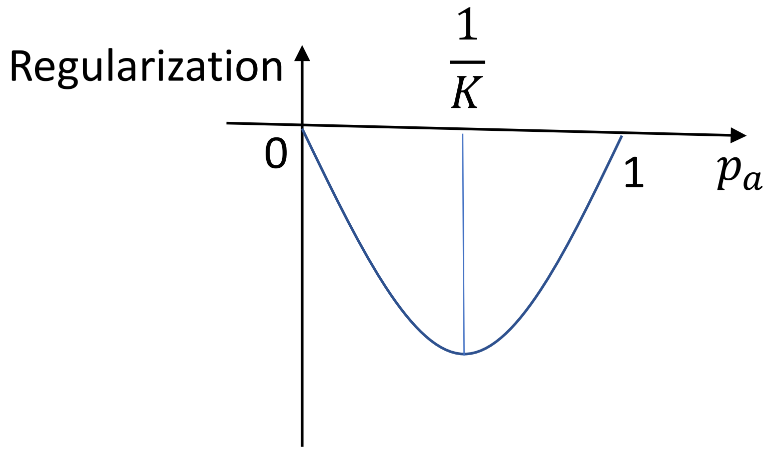
Algorithm for adversarial full info: Hedge / Exponential weights

$$\langle p, L_{t-1}\rangle = \sum_a p_a L_{t-1}(a)$$

- $\forall a: L_0(a) = 0$
- For $t = 1, 2, \ldots$
  - $\forall a$: $p_t(a) = \frac{e^{-\eta_t L_{t-1}(a)}}{\sum_{a'} e^{-\eta_t L_{t-1}(a')}}$
  - $A_t \sim p_t$
  - [Observe $\ell_{t,1}, \ldots, \ell_{t,K}$]
  - $\forall a: L_t(a) = L_{t-1}(a) + \ell_{t,a}$

$$p_t = \arg\min_p \left(\langle p, L_{t-1}\rangle + \frac{1}{\eta_t} \underbrace{\sum_a p_a \ln p_a}_{\text{Regularization}} \right)$$

$$= \arg\min_p \left(\sum_a p_a L_{t-1}(a) + \frac{1}{\eta_t} \underbrace{\sum_a p_a \ln p_a}_{\text{Regularization}} \right)$$



- In FTL: $p_t = \arg\min_p \langle p, L_{t-1}\rangle$

- Some intuition:
  - In the early versions $p_t(a) \propto p_{t-1}(a)(1-\varepsilon)^{\ell_{t,a}}$
    - $\ell_{t,a} \in \{0,1\}$
  - In Hedge: $p_t(a) \propto p_{t-1}(a) e^{-\eta_t \ell_{t,a}}$

Numerically stable calculation of p_t

- $p_t(a) = \frac{e^{-\eta_t L_{t-1}(a)}}{\sum_{a'} e^{-\eta_t L_{t-1}(a')}}$

- For large t, $L_{t-1}(a)$ can be large and $e^{-\eta_t L_{t-1}(a)}$ numerically become zero, leading to $\frac{0}{0}$ numerical instability

- Remedy:
  - $\frac{e^{-\eta_t L_{t-1}(a)}}{\sum_{a'} e^{-\eta_t L_{t-1}(a')}} = \frac{e^{-\eta_t \left(L_{t-1}(a) - \min_{a''} L_{t-1}(a'')\right)}}{\sum_{a'} e^{-\eta_t \left(L_{t-1}(a') - \min\limits_{a''} L_{t-1}(a'')\right)}}$
  - In the expression on the right the denominator is at least 1, resolving numerical instability

$$p_t(a) = \frac{e^{-\eta_t L_{t-1}(a)}}{\sum_{a'} e^{-\eta_t L_{t-1}(a')}}$$

Analysis

- Lemma: For any sequence of non-negative $\ell_{t,a}$ and $p_t(a)$ as in Hedge

$$\underbrace{\underbrace{\sum_{t=1}^{T} \underbrace{\sum_{a=1}^{K} p_t(a)\ell_{t,a}}_{\text{The expected loss of Hedge at round } t}}_{\text{The expected loss of Hedge}} - \underbrace{\min_a L_T(a)}_{\text{The best loss in hindsight}}}_{\text{The expected regret of Hedge } \mathbb{E}[R_T]} \le \frac{\ln K}{\eta} + \frac{\eta}{2} \underbrace{\sum_{t=1}^{T} \underbrace{\sum_{a=1}^{K} p_t(a) \underbrace{\left(\ell_{t,a}\right)^2}_{\le 1}}_{\le 1}}_{\le T}$$

- Corollary: $\mathbb{E}[R_T] \le \frac{\ln K}{\eta} + \frac{\eta}{2} T$

- Take $\eta = \sqrt{\frac{2 \ln K}{T}}$, then $\mathbb{E}[R_T] \le \sqrt{2T \ln K}$

Proof of the lemma

$$p_t(a) = \frac{e^{-\eta_t L_{t-1}(a)}}{\sum_{a'} e^{-\eta_t L_{t-1}(a')}}$$

$$\sum_{t=1}^{T}\sum_{a=1}^{K} p_t(a)\ell_{t,a} - \min_a L_T(a) \le \frac{\ln K}{\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\sum_{a=1}^{K} p_t(a)\left(\ell_{t,a}\right)^2$$

- Define $W_t = \sum_a e^{-\eta L_t(a)}$

$$\frac{W_t}{W_{t-1}} = \frac{\sum_a e^{-\eta L_t(a)}}{\sum_{a'} e^{-\eta L_{t-1}(a')}}$$

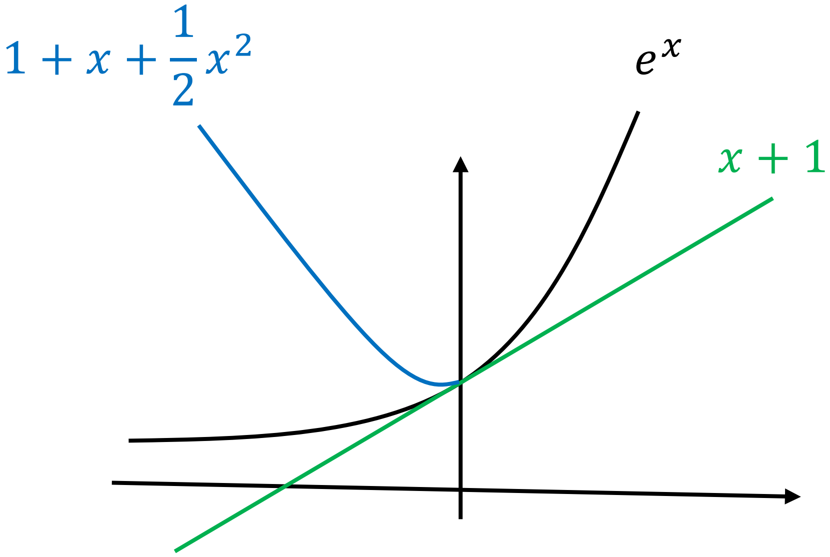
$$= \sum_a e^{-\eta \ell_{t,a}} \underbrace{\frac{e^{-\eta L_{t-1}(a)}}{\sum_{a'} e^{-\eta L_{t-1}(a')}}}_{p_t(a)}$$

$$= \sum_a e^{-\eta \ell_{t,a}} p_t(a)$$

$$\le \sum_a \left(1 - \eta \ell_{t,a} + \frac{1}{2}\eta^2 \left(\ell_{t,a}\right)^2\right) p_t(a)$$

$$= 1 - \eta \sum_a \ell_{t,a} p_t(a) + \frac{\eta^2}{2}\sum_a \left(\ell_{t,a}\right)^2 p_t(a)$$

$$\le e^{-\eta \sum_a \ell_{t,a} p_t(a) + \frac{\eta^2}{2}\sum_a \left(\ell_{t,a}\right)^2 p_t(a)}$$



- For $x \le 0$:

$$e^x \le 1 + x + \frac{1}{2}x^2$$

- For any x:

$$1 + x \le e^x$$

$$W_t = \sum_a e^{-\eta L_t(a)}$$

$$\frac{W_t}{W_{t-1}} \le e^{-\eta \sum_a \ell_{t,a} p_t(a) + \frac{\eta^2}{2} \sum_a (\ell_{t,a})^2 p_t(a)}$$

Proof continued

$$\frac{W_T}{W_0} = \frac{W_1}{W_0}\frac{W_2}{W_1}\frac{W_3}{W_2}\cdots\frac{W_T}{W_{T-1}} \le e^{-\eta \sum_{t=1}^T \sum_a \ell_{t,a} p_t(a) + \frac{\eta^2}{2} \sum_{t=1}^T \sum_a (\ell_{t,a})^2 p_t(a)}$$

$$\frac{W_T}{W_0} = \frac{\sum_a e^{-\eta L_T(a)}}{K} \ge \frac{\max_a e^{-\eta L_T(a)}}{K} = \frac{e^{-\eta \min_a L_T(a)}}{K}$$

Put the two sides together, take a logarithm and normalize by η:

$$\sum_{t=1}^{T} \sum_{a=1}^{K} p_t(a) \ell_{t,a} - \min_a L_T(a) \le \frac{\ln K}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{a=1}^{K} p_t(a) (\ell_{t,a})^2$$

Summary

- Hedge:
  - $p_t(a) = \frac{e^{-\eta_t L_{t-1}(a)}}{\sum_{a'} e^{-\eta_t L_{t-1}(a')}}$
- Analysis:
  - Evolution of the potential function $W_t = \sum_a e^{-\eta L_t(a)}$