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Master's thesis

Rationality problems and decompositions of the diagonal

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Contents

Introduction

Consider the polynomial ring $k[x_0, \dots, x_n]$ in n variables over some field k. An old question in mathematics asks to find the common zeroes of some given polynomials $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ in some field k (or in some field extension of k). For example, find the zeroes of the polynomial

$$x^2 + y^2 - z^2 \in \mathbb{R}[x, y, z]$$

in $\mathbb{R}$. For fixed z, the zeroes of the polynomial are the points on the circle centered at the origin with radius z in $\mathbb{R}^2$, i.e. are parametrized by $z \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ in the following way:

$$x = z \sin \phi, \ y = z \cos \phi, \ \text{and } z = z.$$

This answers the question for $\mathbb{R}$, but it does not generalize to other fields. So instead we want to find a parametrization by rational functions, i.e. by fractions of polynomials. It is well-known that for any field k of characteristic different from 2 the zeroes of the polynomial

$$x^2 + y^2 - z^2 \in k[x, y, z]$$

are of the form

$$x = 2st, \ y = s^2 - t^2, \ \text{and } z = s^2 + t^2$$

with $s, t \in k$.

From the viewpoint of algebraic geometry this observation can be rephrased as follows: The (smooth) projective curve

$$Q := \{x^2 + y^2 - z^2 = 0\} \subset \mathbb{P}^2_k$$

is rational which means that Q is isomorphic to the one-dimensional projective space $\mathbb{P}^1_k$. More generally, the question

"Parametrize the common zeroes of a given system of polynomials by rational functions."

can be rephrased to

"Is a given (smooth) projective variety rational?".

In dimension one and two there are explicit criteria answering this question: An irreducible curve is rational if and only if its geometric genus is zero. A smooth, projective surface is rational if and only if its irregularity and its second plurigenus vanish (over the complex numbers this is known as Castelnuovo's Rationality theorem). In higher dimensions there is currently no complete answer to the above question, even in the case of (Fano) hypersurfaces or more generally complete intersections in projective space. For Fano hypersurfaces some progress was made over the years, mostly over the complex numbers in [IM71; CG72; Puh87; Kol95; Puh98; deF13; deF16].

Recently, a new strategy to tackle the problem for Fano hypersurfaces and complete intersection was developed using the observation that (retract) rational varieties admit a decomposition of the diagonal, i.e. the diagonal point $\delta_X \in X_{k(X)}$ is rationally equivalent to the base change $z_{k(X)}$ of some k-rational point $z \in X$. Voisin ([Voi15]) introduced a cycle-theoretic degeneration technique: For a variety X which degenerates to a mildly singular variety Y, she disproved (retract) rationality for X by showing that Y does not admit a decomposition of the diagonal. Voisin's approach was generalized and refined in [CTP16; Sch19a]. Totaro used this technique in [Tot16] to improve Kollár's bound on very general hypersurfaces in all dimensions ([Kol95]). Moreover, this approach was successfully used to construct new examples by [HPT18] and to find a logarithmic bound for very general hypersurfaces in arbitrary dimension [Sch19b].

Nicaise and Shinder ([NS19]) introduced a motivic obstruction to stable rationality in characteristic 0 by using the weak factorization theorem. They consider the free abelian group $\mathbb{Z}[\mathrm{SB}_k]$ generated by stably birational equivalence classes of smooth projective k-varieties and showed stably irrationality for varieties admitting a degeneration to a simple normal crossing variety $Y = \bigcup_{i\in I} Y_i$ with

$$\left[\mathbb{P}_k^{\dim Y}\right] + \sum_{\emptyset \neq J \subset I} (-1)^{|J|} \left[\left(\bigcap_{j\in J} Y_j\right) \times \mathbb{P}_k^{|J|-1}\right] \neq 0 \in \mathbb{Z}[\mathrm{SB}_k].$$

Kontsevich and Tschinkel [KT19] proved the same statement for irrationality instead of stably irrationality after replacing $\mathbb{Z}[\mathrm{SB}_k]$ by the free abelian group $\mathbb{Z}[\mathrm{Bir}(k)]$, which is generated by birational equivalence classes of smooth projective k-varieties. Using this obstruction Nicaise and Ottem [NO22] found a new example of a stably irrational Fano hypersurface, namely very general quartic fivefolds, and some complete intersections, e.g. very general $(3,3)$ complete intersections in $\mathbb{P}^7$.

Recently, Pavic and Schreieder [PS21] introduced a cycle-theoretic obstruction which uses also degenerations to simple normal crossing varieties. Their obstruction works also in positive characteristic and can be seen as a generalization of the motivic obstruction. They were able to disprove retract rationality for a very general quartic fivefold by using a similar decomposition as [NO22].

In this thesis we will give another application of the method of [PS21] by showing that a very general $(3,3)$ complete intersection is not retract rational.

Theorem 1.1. *Let k be an uncountable field of characteristic different from 2. A very general $(3,3)$ complete intersection in $\mathbb{P}_k^7$ does not admit a decomposition of the diagonal, in particular is not retract rational.*

Even the rationality of $(3,3)$ complete intersections was previously open in positive characteristic. Similar to the quartic fivefold example by [PS21], we will use a degeneration which is inspired by the degeneration of [NO22]. The key input for the obstruction to rationality is the example of a quadric surface bundle by [HPT18].

Preliminaries

Conventions. *A variety is a separated, integral scheme of finite type over a field k. We denote the function field of a k-variety X by $k(X)$. For a separated scheme X over a ring R and a ring extension A/R we write $X_A := X \times_R A := X \times_{\operatorname{Spec} R} \operatorname{Spec} A$ for the base change.*

Let X be a variety or more generally an algebraic scheme over a field k, i.e. a separated scheme of finite type over k. Then we denote the free abelian group of algebraic l-cycles by $Z_l(X)$ and the Chow group of l-cycles by $\mathrm{CH}_l(X)$, i.e. $Z_l(X)$ modulo rational equivalence.

A very general point of an irreducible separated scheme is a closed point outside a countable union of proper closed subsets.

Let X be a variety over an algebraically closed field k. An alteration of X is a proper, generically finite, and surjective morphism $\tau\colon X' \to X$ such that X' is smooth over k. The existence of alterations in any characteristic was shown by de Jong [deJ96]. By work of Gabber, see e.g. [IT14], the degree of the alteration can be choosen to be coprime to any prime not dividing the characteristic of the field.

2.1 Retract rationality and decomposition of the diagonal

Recall that two varieties over some field k are birational if they contain isomorphic Zariski-open, dense subsets. A variety is called rational if it is birational to the projective space $\mathbb{P}^n_k$ where n is the dimension of the variety. We call a variety X stably rational if $X \times \mathbb{P}^m_k$ is rational for some $m \geq 0$. Morover, a variety X is retract rational if there is an integer $N \in \mathbb{Z}_{\geq 0}$ and rational maps $f\colon X \dashrightarrow \mathbb{P}^N_k$ and $g\colon \mathbb{P}^N_k \dashrightarrow X$ such that $g \circ f$ is defined and coincides with the identity id_X as a rational map. We call a variety X unirational if there exists a dominant rational map $\mathbb{P}^N_k \dashrightarrow X$ for some $N \geq 0$. The following relations between these notions are well-known.

Lemma 2.1. *Let X be a variety over some field k. Then*

$$X \text{ is rational} \Longrightarrow X \text{ is stably rational} \Longrightarrow X \text{ is retract rational} \Longrightarrow X \text{ is unirational.}$$

Proof. The first implication is obvious. Let X be stably rational, i.e. there exists $n, N \in \mathbb{Z}_{\geq 0}$ such that

$$\varphi\colon X \times_k \mathbb{P}^n_k \overset{\sim}{\dashrightarrow} \mathbb{P}^N_k$$

is a birational map. Let $\iota\colon X = X \times_k \{\mathrm{pt}\} \hookrightarrow X \times_k \mathbb{P}^n_k$ be a section of the projection

$$\mathrm{pr}_1\colon X \times_k \mathbb{P}^n_k \longrightarrow X$$

onto the first factor and define $f := \varphi \circ \iota$ and $g := \mathrm{pr}_1 \circ \varphi^{-1}$. Then f and g are rational maps such that the composition $g \circ f$ is defined and coincides with the identity, i.e. X is retract rational. The last implication follows immediately from the definitions. □

The first and last implication are known to be strict by [BCTSSD85] and [AM72], respectively. Whereas it is currently unknown whether the second implication is strict.

Definition 2.2. *Let X be a variety over a field k and let $\Delta_X \subset X \times_k X$ be the diagonal. Pulling back Δ_X via the natural morphism $X_{k(X)} \to X \times_k X$ yields a zero-cycle $\delta_X \in \mathrm{CH}_0(X_{k(X)})$. We say that X admits a decomposition of the diagonal if*

$$\delta_X = z_{k(X)} \in \mathrm{CH}_0(X_{k(X)})$$

for some zero-cycle $z \in \mathrm{CH}_0(X)$.

There is an equivalent definition of the decomposition of the diagonal which also works for algebraic schemes and is therefore sometimes preferred.

Lemma 2.3 ([Sch21, Lemma 7.3])**.** *A variety X over some field k admits a decomposition of the diagonal if and only if there exists a zero-cycle $z \in Z_0(X)$ and a cycle $Z_X \in Z_0(X \times_k X)$ which does not dominate the first factor such that*

$$[\Delta_X] = [X \times_k z] + [Z_X] \in \mathrm{CH}_n(X \times_k X), \tag{2.1}$$

where $\Delta_X \subset X \times_k X$ denotes the diagonal and $n = \dim X$.

Proof. We recall first the well-known result

$$\mathrm{CH}_0(X_{k(X)}) \cong \varinjlim_{U \neq \emptyset} \mathrm{CH}_n(U \times_k X). \tag{2.2}$$

For a closed point $z \in X_{k(X)}$ the closure of the image under the natural morphism $X_{k(X)} \to U \times_k X$ defines an element in $\mathrm{CH}_n(U \times_k X)$ by taking the associated cycle of the n-dimensional subvariety. Extending this map $\mathbb{Z}$-linearly, we obtain a map

$$Z_0(X_{k(X)}) \longrightarrow \mathrm{CH}_n(U \times_k X). \tag{2.3}$$

Any one-dimensional subvariety L in $X_{k(X)}$ give rise to an $(n+1)$-dimensional subvariety in $U \times_k X$ by taking the closure $\overline{L}$ of the image under the natural morphism $X_{k(X)} \to U \times_k X$. Moreover, any rational function on L can be viewed also as a rational function on $\overline{L}$. Thus we see that the map (2.3) factors through rational equivalence, i.e. for every open subset $U \subset X$ there exists a map

$$\rho_U \colon \mathrm{CH}_0(X_{k(X)}) \longrightarrow \mathrm{CH}_n(U \times_k X).$$

Since the map is given by taking the closure of the image under the natural map and since the pullback along flat morphism of the form $V \times_k X \to U \times_k X$ defines a natural restriction morphism $\mathrm{CH}_n(V \times_k X) \to \mathrm{CH}_n(U \times_k X)$, the homomorphisms ρ_U give rise to a homomorphism

$$\rho \colon \mathrm{CH}_0(X_{k(X)}) \longrightarrow \varinjlim_{U \neq \emptyset} \mathrm{CH}_n(U \times_k X).$$

To show (2.2) it suffices to prove that ρ is an isomorphism. Since any n-dimensional subvariety S of $U \times_k X$, which represents a non-trivial cycle in

$$\varinjlim_{U \neq \emptyset} \mathrm{CH}_n(U \times_k X),$$

maps dominantly via the projection onto the first factor, the variety S can be restricted to the generic fibre of the projection $U \times_k X \to U$. In other words the preimage under the natural map $X_{k(X)} \to U \times_k X$ exists and is zero-dimensional. Hence, ρ is surjective.

The injectivity of ρ follows from the observation that any $(n+1)$-dimensional variety of some $U \times_k X$, which maps dominantly via the first projection, can also be restricted to the generic fibre, i.e. give rise to a one-dimensional variety in $X_{k(X)}$.

Let us come back to the proof of the lemma. Assume there exists a decomposition as in (2.1). Since ρ is an isomorphism,

$$\delta_X = \rho^{-1}([\Delta_X]) = \rho^{-1}([X \times_k z]) + \rho^{-1}([Z_X]) = [z_{k(X)}] \in \mathrm{CH}_0(X_{k(X)}).$$

as $Z_X \in X \times_k X$ does not dominate the first factor. Hence, X admits a decomposition of the diagonal as defined in Definition 2.2.

Assume now that X admits a decomposition of the diagonal as in Definition 2.2. By (2.2) there exists some open subset $U \subset X$ such that

$$i^*[\Delta_X] = \rho_U(\delta_X) = \rho_U([z_{k(X)}]) = [U \times_k z] \in \mathrm{CH}_n(U \times_k X),$$

where $i\colon U \times_k X \hookrightarrow X \times_k X$ is the base change of the open embedding $U \hookrightarrow X$. Using the localization exact sequence [Ful98, Proposition 1.8] we get that

$$[\Delta_X] = [X \times_k z] + [Z_X] \in \mathrm{CH}_n(X \times_k X),$$

where Z_X is a cycle which does not dominate the first factor. □

As already mentioned in the introduction, the degeneration methods use the observation that (retract) rational varieties admit a decomposition of the diagonal, see e.g. [Sch21, Lemma 7.5]:

Lemma 2.4. *A retract rational variety over a field k admits a decomposition of the diagonal.*

Proof. Let $f\colon X \dashrightarrow \mathbb{P}^N_k$ and $g\colon \mathbb{P}^N_k \dashrightarrow X$ be rational maps as in the definition of retract rationality. Let $\Gamma_f \subset X \times \mathbb{P}^N_k$ and $\Gamma_g \subset \mathbb{P}^N_k \times X$ denote the closure of the graphs of f and g, respectively. After replacing X by a projective model, we may assume that X is proper. Up to replacing X by Γ_f, we may also assume that f is a morphism which is automatically proper as X is proper. Let $K = k(X)$ be the function field of X. Then, we obtain a well-defined morphism

$$f_*\colon \mathrm{CH}_0(X_K) \longrightarrow \mathrm{CH}_0(\mathbb{P}^N_K).$$

The projection $\mathrm{pr}_2\colon X_K \times_K \mathbb{P}^n_K \to \mathbb{P}^n_K$ onto the second factor is a flat morphism, i.e. there is a well-defined flat pull-back map

$$\mathrm{pr}_2^*\colon \mathrm{CH}_0(\mathbb{P}^N_K) \longrightarrow \mathrm{CH}_n(X_K \times_K \mathbb{P}^N_K).$$

Since $\mathbb{P}^N_K$ is smooth, the closed embedding $\Gamma_{f,K} \hookrightarrow X_K \times_K \mathbb{P}^N_K$ is a regular embedding and we can define the refined Gysin homomorphism (see [Ful98, Definition 8.1.2])

$$\mathrm{CH}_n(X_K \times_K \mathbb{P}^n_K) \longrightarrow \mathrm{CH}_0(\Gamma_{f,K}).$$

Combining these two morphisms with the pushforward via the natural morphism $\mathrm{pr}_1\colon \Gamma_{f,K} \to X_K$ we obtain a morphism

$$g^\star\colon \mathrm{CH}_0(\mathbb{P}^N_K) \longrightarrow \mathrm{CH}_0(X_K).$$

We claim that $g^\star \circ f_* = \mathrm{id}_{\mathrm{CH}_0(X_K)}$. It suffices to check this equality for every closed point $z \in X_K$. By definition of the Gysin homomorphism

$$g^\star f_*[z] = g^\star[f(z)] = \mathrm{pr}_{1,*}\left([X_K \times_K f(z)] \cdot [\Gamma_{f,K}]\right) = [z],$$

because $g \circ f = \mathrm{id}$. Furthermore, $\mathrm{CH}_0(\mathbb{P}^N_K) \cong \mathbb{Z}$, i.e. $f_*\delta_X = [z_K]$ for some closed point $z \in \mathbb{P}^n_k$ because δ_X has degree 1. We can choose even z such that g is defined in z. Combining all the results yields

$$\delta_X = g^\star(f_*\delta_X) = g^\star[z_K] = [(g_*z)_K] \in \mathrm{CH}_0(X_K).$$

That concludes the proof of the lemma. □

We also introduce the related notion of universally trivial Chow group of zero-cycles.

Definition 2.5. *Let X be a proper variety over a field k. We say that X has universally trivial Chow group of zero-cycles if for any field extension F/k, the degree map*

$$\deg\colon \mathrm{CH}_0(X_F) \longrightarrow \mathbb{Z}$$

is an isomorphism.

From the definition it is obvious that if X has universally trivial Chow group of zero-cycles, then X admits a decomposition of the diagonal. The converse is also true for geometrically integral and smooth varieties.

Proposition 2.6 ([CTP16, Proposition 1.4])**.** *Let X be a geometrically integral and smooth variety over a field k. Then X has universally trivial Chow group of zero-cycles if and only if X admits a decomposition of the diagonal.*

2.2 Chow-theoretic obstruction to (retract) rationality

In this section we recall the constructions in [PS21, Section 3]. To this end let R be a discrete valuation ring with residue field k and fraction field K.

Definition 2.7. *(1) A proper flat R-scheme $\mathcal{X} \to \operatorname{Spec} R$ is called strictly semi-stable, if the special fiber $Y := \mathcal{X} \times_R k$ is a geometrically reduced simple normal crossing divisor on $\mathcal{X}$, i.e. the components of Y are smooth Cartier divisors on $\mathcal{X}$ and the intersections of r different components is either empty or smooth and of codimension r.*

(2) Let $Y = \bigcup_{i=1}^{m} Y_i$ be the irreducible components of the special fiber Y. The variety Y is called a chain of Cartier divisors if additionally $Y_{i-1} \cap Y_i$ and $Y_i \cap Y_{i+1}$ are irreducible and disjoint in Y_i for $1 < i < m$ and all other intersection are empty.

Remark 2.8. *If $m = 2$, the last condition says that the intersection $Y_1 \cap Y_2$ is irreducible.*

Definition 2.9. *We use the same notations as above. Assume that $\mathcal{X} \to \operatorname{Spec} R$ is a strictly semi-stable R-scheme with special fiber Y. (In particular we do not assume that Y is a chain of Cartier divisors.) Denote the natural inclusion by $\iota\colon Y \to \mathcal{X}$ and $\iota_i\colon Y_i \to \mathcal{X}$ for $1 \le i \le m$. Then we define for every $i \in \{1, \dots, m\}$*

$$\Phi_{\mathcal{X},Y_i} := \iota_i^* \circ \iota_*\colon \mathrm{CH}_1(Y) \longrightarrow \mathrm{CH}_0(Y_i).$$

The obstruction map is the sum of all the homomorphisms

$$\Phi_{\mathcal{X}} := \sum_{i=1}^{m} \Phi_{\mathcal{X},Y_i} = \sum_{i=1}^{m} \iota_i^* \circ \iota_*\colon \mathrm{CH}_1(Y) \longrightarrow \bigoplus_{i=1}^{m} \mathrm{CH}_0(Y_i). \tag{2.4}$$

Remark 2.10. *Notet that the pull-back maps ι_i^* exist and are well-defined by [Ful98, Section 2.6] because each $Y_i \subset \mathcal{X}$ is Cartier.*

Although the involved Chow groups depend only on Y, the obstruction map might a priori depend on the choice of the strictly semi-stable model. We recall the observation made by [PS21] that the obstruction map does only depend on the special fiber Y, and not on the total space $\mathcal{X}$. To this end let $Y = \bigcup_{i=1}^{m} Y_i$ be the irreducible components of the special fiber. Denote the natural inclusions by

$$\iota_i \colon Y_i \hookrightarrow Y, \quad \iota_{i,j} \colon Y_{i,j} \hookrightarrow Y_j,$$

where $Y_{i,j} := Y_i \cap Y_j$ is the scheme-theoretic intersection of Y_i and Y_j for $i, j \in \{1, \dots, m\}$ different. As Y is a simple normal crossing divisor, $\iota_{i,j}$ are regular embedding of codimension 1. Thus, there exist well-defined homomorphisms $\iota_{i,j}^* \colon \mathrm{CH}_1(Y_j) \to \mathrm{CH}_0(Y_{i,j})$, see e.g. [Ful98, Example 5.2.1]. For $\gamma_j \in \mathrm{CH}_1(Y_j)$ we write $\gamma_j|_{Y_{i,j}} := \iota_{i,j}^* \gamma_j$.

Lemma 2.11. *Using the observations and notations made above, any $\gamma_j \in \mathrm{CH}_1(Y_j)$ satisfies*

$$\Phi_{\mathcal{X},Y_i}((\iota_j)_*\gamma_j) = \begin{cases} (\iota_{j,i})_*(\gamma_j|_{Y_{i,j}}) \in \mathrm{CH}_0(Y_i) & \text{for } j \neq i, \\ -\sum_{k \neq j} ((\iota_{k,i})_*(\gamma_j|_{Y_{k,j}})) \in \mathrm{CH}_0(Y_i) & \text{for } j = i. \end{cases}$$

In particular, for $\gamma = \sum_{k=1}^{m} (\iota_k)_* \gamma_k \in \mathrm{CH}_1(Y)$ and $i \in \{1, \dots, m\}$:

$$\Phi_{\mathcal{X},Y_i}(\gamma) = \sum_{j \neq i} (\iota_{j,i})_* \, \gamma_j|_{Y_{i,j}} - \sum_{j \neq i} (\iota_{j,i})_* \, \gamma_i|_{Y_{j,i}} . \tag{2.5}$$

Proof. The case $i \neq j$ follows from [Ful98, Theorem 6.2 (a)] applied to the fiber squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\mathrm{id}} & Y_i \\ \downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \iota_i} \\ Y & \xrightarrow{\iota} & \mathcal{X} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_{i,j} & \xrightarrow{\iota_{j,i}} & Y_j \\ \downarrow{\scriptstyle \iota_{i,j}} & & \downarrow{\scriptstyle \iota_j} \\ Y_i & \xrightarrow{\iota_i} & Y, \end{array}$$

i.e. for $\gamma_j \in \mathrm{CH}_1(Y_j)$:

$$\iota_i^* \iota_* (\iota_j)_* \gamma_j = \mathrm{id}_* \, \iota_i^* (\iota_j)_* \gamma_j = \iota_i^* (\iota_j)_* \gamma_j = (\iota_{i,j})_* \iota_{j,i}^* \gamma_j .$$

The second case follows directly from this by noting that $[Y] = \sum_{k=1}^{n} [Y_k] = \mathrm{div}(t) \subset \mathcal{X}$, i.e.

$$[Y_i] = -\sum_{k \neq i} [Y_k] \in \mathrm{Pic}\, \mathcal{X}.$$

This finishes the proof of the lemma. $\square$

Remark 2.12. *The homomorphism*

$$\sum_{i=1}^{m} (\iota_i)_* \colon \bigoplus_{i=1}^{m} \mathrm{CH}_1(Y_i) \twoheadrightarrow \mathrm{CH}_1(Y)$$

is obviously surjective as any codimension one subvariety of Y is contained in at least one irreducible component. The map is in general not injective as one-cycles in the intersection of two irreducible components have (at least) two different preimages. But we see from (2.4) that the obstruction map does not depend on the choice of the "decomposition" $\gamma = \sum_{k=1}^{m} (\iota_k)_* \gamma_k \in \mathrm{CH}_1(Y)$.

The main theorem in [PS21] uses two observations made by Pavic and Schreieder. These are written down in the following lemma.

Lemma 2.13 ([PS21, Section 3.2])**.** *Let* $\mathcal{X} \to \operatorname{Spec} R$ *be a strictly semi-stable model and let* $\Phi_{\mathcal{X}}$ *be defined as in* (2.4)*.*

(a) If $\gamma \in \mathrm{CH}_1(Y)$ *is a one-cycle on* Y*, then* $\deg(\Phi_{\mathcal{X}}(\gamma)) = 0$*.*

(b) Let A/R *be an unramified extension of discrete valuation rings, i.e.* $R \to A$ *is injective and local with* $m_R \cdot A = m_A$*, then* $\mathcal{X}_A := \mathcal{X} \times_R A$ *is a strictly semi-stable* A*-scheme.*

Proof. We start proving the first item. Let $Y = \bigcup_{i=1}^{m} Y_i$ be the irreducible components of the special fiber Y of $\mathcal{X} \to \operatorname{Spec} R$ and let $\gamma = \sum_{k=1}^{m} \gamma_k \in \mathrm{CH}_1(Y)$ be a "decomposition" with $\gamma_k \in \mathrm{CH}_1(Y_k)$ for $1 \le k \le m$. Using the concrete description of the obstruction map in (2.5), we find by rearranging the summation that

$$\begin{aligned}
\Phi_{\mathcal{X}}(\gamma) &= \sum_{i=1}^{m} \left(\sum_{j \neq i} (\iota_{j,i})_* \, \gamma_j|_{Y_{i,j}} - \sum_{j \neq i} (\iota_{j,i})_* \, \gamma_i|_{Y_{j,i}} \right) \\
&= \left(\sum_{j=1}^{m} \sum_{i \neq j} (\iota_{j,i})_* \, \gamma_j|_{Y_{i,j}} \right) - \left(\sum_{i=1}^{m} \sum_{j \neq i} (\iota_{j,i})_* \, \gamma_i|_{Y_{j,i}} \right) \\
&= \sum_{i=1}^{m} \sum_{i \neq j} \Big((\iota_{i,j})_* \, \gamma_i|_{Y_{j,i}} - (\iota_{j,i})_* \, \gamma_i|_{Y_{j,i}} \Big) \in \bigoplus_{k=1}^{m} \mathrm{CH}_0(Y_k).
\end{aligned}$$

Hence,

$$\begin{aligned}
\deg \Phi_{\mathcal{X}}(\gamma) &= \deg \left(\sum_{i=1}^{m} \sum_{i \neq j} \Big((\iota_{i,j})_* \, \gamma_i|_{Y_{j,i}} - (\iota_{j,i})_* \, \gamma_i|_{Y_{j,i}} \Big) \right) \\
&= \sum_{i=1}^{m} \sum_{i \neq j} \Big(\deg\big(\gamma_i|_{Y_{j,i}}\big) - \deg\big(\gamma_i|_{Y_{j,i}}\big) \Big) \\
&= 0.
\end{aligned}$$

This proves item (a). Let us prove (b) now. Let $\mathcal{X} \to \operatorname{Spec} R$ be strictly semi-stable, i.e. the morphism $\mathcal{X} \to \operatorname{Spec} R$ is proper and flat and the components of the special fiber are smooth Cartier divisors such that the intersection of r different components is either empty or smooth of codimension r. As properness and flatness are preserved under base change, $\mathcal{X}_A \to \operatorname{Spec} A$ is proper and flat. Let L denote the residue field of A. Since A/R is unramified,

$$L = {}^{A}\!/_{m_A} = {}^{A}\!/_{m_R A} = k \otimes_R A,$$

where m_R and m_A denote the unique maximal ideals of R and A, respectively. Hence, the special fiber of $\mathcal{X}_A \to \operatorname{Spec} A$ is the base change of the special fibre Y of $\mathcal{X} \to \operatorname{Spec} R$, i.e.

$$\mathcal{X}_A \times_A L = Y \times_R A = Y_L.$$

Thus, we immediately conclude that $\mathcal{X}_A$ is again strictly semi-stable as wanted which proves item (b). $\square$

Rephrasing these two observations we conclude that the image of the obstruction map is contained in the kernel of the degree map, i.e.

$$\Phi_{\mathcal{X}}\colon \mathrm{CH}_1(Y) \longrightarrow \mathrm{Ker}\left(\deg\colon \bigoplus_{i=1}^{m} \mathrm{CH}_0(Y_i) \longrightarrow \mathbb{Z}\right),$$

and that for any unramified extension A/R of discrete valuation rings there exists a homomorphism

$$\Phi_{\mathcal{X}_A}\colon \mathrm{CH}_1(Y_L) \longrightarrow \mathrm{Ker}\left(\deg\colon \bigoplus_{i=1}^{m} \mathrm{CH}_0(Y_{i,L}) \longrightarrow \mathbb{Z}\right)$$

where L is the residue field of A. Studying these maps can give an obstruction to the decomposition of the diagonal of the geometric generic fiber.

Theorem 2.14 ([PS21, Theorem 4.1])**.** *Let R be a discrete valuation ring with algebraically closed residue field and let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective R-scheme whose special fiber $Y = \bigcup_{i\in I} Y_i$ is a chain of Cartier divisors. Assume that the geometric generic fibre of $\mathcal{X} \to \operatorname{Spec} R$ has a decomposition of the diagonal. Then for any unramified extension A/R of discrete valuation rings, with induced extension L/k of residue fields, the natural map*

$$\Phi_{\mathcal{X}_A}\colon \mathrm{CH}_1(Y_L)/2 \longrightarrow \mathrm{Ker}\left(\bigoplus_{i\in I} \mathrm{CH}_0(Y_{i,L})/2 \xrightarrow{\deg} \mathbb{Z}/2\right)$$

is surjective.

2.3 Degenerations and Specializations

We will often use degenerations of varieties, or more generally reduced algebraic schemes, e.g. to show that certain varieties are smooth (see Remark 3.5). Therefore we introduce this notion here by following [Sch19a, Section 2.2]. Let X and Y be reduced algebraic schemes over a field L and an algebraically closed field k, respectively. We say that X degenerates (or specializes) to Y if there exists a discrete valuation ring R with residue field k and fraction field K together with an injection of field $K \to L$ such that the following holds: There exists a proper, flat morphism

$$\mathcal{X} \longrightarrow \operatorname{Spec} R$$

of finite type such that Y is isomorphic its special fibre $\mathcal{X}_k$ and X is isomorphic to the base change $\mathcal{X}_L = \mathcal{X}_K \times L$ of the generic fibre.

The following lemma shows that in "nice" families $\mathcal{X} \to B$ a very general fibre specialize to the fibre over some fixed closed point $b \in B$, see e.g. [Sch19a, Lemma 8].

Lemma 2.15. *Let $f\colon \mathcal{X} \to B$ be a surjective, proper, and flat morphism of reduced, quasi-projective algebraic schemes over an algebraically closed field k and assume further that B is integral. Let $0 \in B$ be a closed point. Then a very general fibre specialize to the fibre X_0 over the point 0 in the above sense.*

Proof. A very general fibre of f is abstractly isomorphic to the geometric generic fibre of f, see e.g. [Via13, Lemma 2.1]. Hence, it suffices to prove that one fibre which is very general specializes to X_0, i.e. we can reduce to the case where B is an (integral) curve. Since normalization commutes with localization, we can assume furthermore that B is smooth by passing to the normalization. Since the local ring $\mathcal{O}_{B,0}$ is an integrally closed Noetherian local ring, it is a discrete valuation ring and the lemma follows. $\square$

Let us also mention the following result by Colliot-Thélène and Pirutka. This statement is not correct:

Theorem 2.16 ([CTP16, Theorem 1.12])**.** *Let A be a discrete valuation ring with residue field k and fraction field K. Let $\mathcal{X}$ be a proper and flat A-scheme with geometrically integral fibers. Assume furthermore that the special fibre $\mathcal{X}_k$ and generic fibre $\mathcal{X}_K$ of $\mathcal{X} \to \operatorname{Spec} A$ are smooth. Then each one implies the next one:*

(i) *$\mathcal{X}_K$ is retract rational.*

(ii) *$\mathcal{X}_K$ admits a decomposition of the diagonal.*

(iii) *$\mathcal{X}_k$ admits a decomposition of the diagonal.*

Remark 2.17. *If k is algebraically closed, then the same statement holds for the geometric generic fibre instead of the generic fibre, see [CTP16, Theorem 1.14].*

2.4 Unramified cohomology and Merkurjev pairing

We follow mainly [Sch21] for this quick overview. For a scheme X and a sheaf $\mathcal{F}$ in the étale topology, we denote by $H^i(X, \mathcal{F})$ the i-th étale cohomology group of $\mathcal{F}$. If $X = \operatorname{Spec} A$ is the spectrum of a ring A, we write $H^i(A, \mathcal{F}) := H^i(\operatorname{Spec} A, \mathcal{F})$. We solely use the constant (étale) sheaf $\mathbb{Z}/2$. The theory of unramified cohomology and Merkurjev pairing works also in more generality, e.g. where one considers the sheaf of m-th roots of unity μ_m and its tensor powers. As we work over fields of characteristic different from 2, the sheaf μ_2 is isomorphic to the constant sheaf $\mathbb{Z}/2$ and thus also its tensor products, i.e. we work in a special case of this more general theory.

For the rest of this section k denotes always a field of characteristic different from 2 and K/k denotes a finitely generated field extension. We start with some preliminary results on étale cohomology:

By Hilbert 90, $H^1(K, \mathbb{G}_m) = \operatorname{Pic}(\operatorname{Spec} K) = 0$. Thus, the long exact sequence in étale cohomology associated to the Kummer exact sequence

$$0 \longrightarrow \mu_2 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \to 0$$

yields that

$$H^1(K, \mathbb{Z}/2) = H^1(K, \mu_2) = {}^{K^*}/_{(K^*)^2},$$

where the first equality comes from the fact that $\mu_2 \cong \mathbb{Z}/2$ as étale sheaves. Hence, for every element $a \in K^*$ its residue class $\overline{a} \in {}^{K^*}/_{(K^*)^2}$ defines an étale cohomology class

$$(a) \in H^1(K, \mathbb{Z}/2).$$

Using the cup product in étale cohomology (see e.g. [Sch21, Section 2.4]) any $a_1, \dots, a_n \in K^*$ give rise to a class

$$(a_1, \dots, a_n) := (a_1) \cup \cdots \cup (a_n) \in H^n(K, \mathbb{Z}/2).$$

These classes are closely related to quadrics as the following discussion (see e.g. [Sch19b, Section 2.4]) shows: For $c_0, \dots, c_r \in K^*$ we denote by $\langle c_0, c_1, \dots, c_r \rangle$ the quadratic form $q = \sum_{i=0}^{r} c_i z_i^2$ over K. The tensor product of two quadratic form q and q' is denoted by $q \otimes q'$. A quadratic form over K is called a Pfister form if it is isomorphic to

$$\langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$$

where $a_i \in K^*$ for $i = 1, \ldots, n$. We denote this Pfister form by $\langle\langle a_1, \ldots, a_n\rangle\rangle$. The following result due to the work of many people, including Arason, Elman, Lam, Knebusch and Voevodsky, shows the above mentioned connection to étale cohomology of fields, cf. [Sch19b, Theorem 2.2]

Theorem 2.18 ([EL72, Main Theorem 3.2] and [Voe03])**.** *Let K be a field of characteric different from 2 and let $a_1, \ldots, a_n \in K^*$. The Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is isotropic if and only if $(a_1, \ldots, a_n) = 0 \in H^n(K, \mathbb{Z}/2)$.*

The long exact sequence of pairs [SGA4.2, p. V.6.5.4] together with Gabber's proof of Grothendieck's purity conjecture implies the existence of a Gysin sequence, cf. [Sch21, Theorem 2.3].

Theorem 2.19. *Let V be a regular Noetherian scheme over k with a regular closed subscheme $Z \subset V$ of pure codimension c and complement $U := V \setminus Z$. Then there exists a long exact sequence*

$$\cdots \longrightarrow H^i(V, \mathbb{Z}/2) \longrightarrow H^i(U, \mathbb{Z}/2) \xrightarrow{\partial} H^{i+1-2c}(Z, \mathbb{Z}/2) \longrightarrow H^{i+1}(V, \mathbb{Z}/2) \longrightarrow \ldots .$$

Next, we turn to the definition of unramified cohomology groups which are certain subgroups in étale cohomology. Their definition, cf. [Sch21, Definition 4.1], requires so called geometric valuation.

Definition 2.20. *Let K/k be a finitely generated field extension. A geometric valuation ν on K over k is a discrete valuation on K, which is trivial on k, such that the transcendence degree of the residue field κ_ν over k is given by*

$$\operatorname{trdeg}_k(\kappa_\nu) = \operatorname{trdeg}_k(K) - 1.$$

We denote the discrete valuation ring associated to the discrete valuation ν by A_ν.

Remark 2.21. *Every geometric valuation ν is given by the order of a prime divisor E on some normal k-variety Y with $k(Y) \cong K$, i.e. $\nu(\phi) = \operatorname{ord}_E(\phi)$ for every $\phi \in K^*$ (see e.g. [Mer08, Proposition 1.7]).*

Definition 2.22 ([Mer08])**.** *The i-th unramified cohomology group of K over k with coefficients in $\mathbb{Z}/2$ is the subgroup*

$$H^i_{nr}(K/k, \mathbb{Z}/2) \subset H^i(K, \mathbb{Z}/2),$$

that consists of all elements $\alpha \in H^i(K, \mathbb{Z}/2)$ such that for any geometric valuation ν on K over k, we have $\partial_\nu \alpha = 0$ where

$$-\partial_\nu \colon H^i(K, \mathbb{Z}/2) \longrightarrow H^{i-1}(k, \mathbb{Z}/2)$$

is the boundary map of the Gysin sequence (Theorem 2.19) for $\operatorname{Spec} K = \operatorname{Spec} A_\nu \setminus \operatorname{Spec} k$.

Remark 2.23. *We use the definition given by Merkurjev [Mer08, Section 2.2] which differs slightly from the original definition by Colliot-Thélène and Ojanguren [CTO89, Definition 1.1.1]. The latter requires that $\partial_\nu \alpha$ vanishes for any discrete valuation of K which is trivial on k. The two definitions agree when $K = k(X)$ is the function field of a smooth projective variety X over k, see e.g. [Sch21, Proposition 4.10 and Remark 4.4].*

Let us collect some results about unramified cohomology. We start with the functioriality which is induced by the functoriality of étale cohomology.

Proposition 2.24 ([Sch21, Proposition 4.7]). *Let $K'/K/k$ be finitely generated field extensions and let $f\colon \operatorname{Spec} K' \to \operatorname{Spec} K$ be the natural morphism.*

(a) Then $f^*\colon H^i(K, \mathbb{Z}/2) \to H^i(K', \mathbb{Z}/2)$ induces a pullback map

$$f^*\colon H^i_{nr}(K/k, \mathbb{Z}/2) \longrightarrow H^i_{nr}(K'/k, \mathbb{Z}/2).$$

(b) If f is finite, then $f_*\colon H^i(K', \mathbb{Z}/2) \to H^i(K, \mathbb{Z}/2)$ induces a pushforward map

$$f_*\colon H^i_{nr}(K'/k, \mathbb{Z}/2) \longrightarrow H^i_{nr}(K/k, \mathbb{Z}/2)$$

with $f_* \circ f^* = \deg(f) \cdot \mathrm{id}$.

The functioriality of unramified cohomology allows us to define the base extension of some unramified cohomology class.

Definition 2.25. *Let K/k be a finitely generated field extension and X be a variety over k. Let $\alpha \in H^i_{nr}(k(X)/k, \mathbb{Z}/2)$ be an unramified cohomology class. Moreover, let $\psi\colon \operatorname{Spec} K(X_K) \to \operatorname{Spec} k(X)$ be the natural morphism corresponding to the inclusion $k(X) \hookrightarrow K(X_K)$ of fields. Then, we define the class*

$$\alpha_K := \psi^*\alpha \in H^i(K(X_K)/k, \mathbb{Z}/2) \subset H^i(K(X_K)/K, \mathbb{Z}/2).$$

Bloch-Ogus' proof [BO74] of the Gersten conjecture for étale cohomology yields the following theorem, known as injectivity and codimension 1 purity for étale cohomology [CT95, Theorem 3.8.1 and Theorem 3.8.2].

Theorem 2.26 ([Sch21, Theorem 3.6]). *Let X be a variety over a field k of characteristic different from 2 and let x be a point in the smooth locus of X. Then the following holds:*

(a) The natural morphism

$$H^i(\mathcal{O}_{X,x}, \mathbb{Z}/2) \longrightarrow H^i(k(X), \mathbb{Z}/2) \tag{2.6}$$

is injective.

(b) A class $\alpha \in H^i(k(X), \mathbb{Z}/2)$ lies in the image of (2.6) if and only if α has trivial residue along each prime divisor on X that passes through x.

Using these results we are able to define a well-defined restriction map for unramified cohomology classes. Let X be a proper and smooth k-variety and let $\alpha \in H^i_{\mathrm{nr}}(k(X)/k, \mathbb{Z}/2)$ be an unramified cohomology class. Moreover, let $x \in X$ be a point. By part (b) of Theorem 2.26 we know that α admits a lift $\tilde{\alpha} \in H^i(\mathcal{O}_{X,x}, \mathbb{Z}/2)$ and this lift is unique by part (a) of Theorem 2.26. Thus, we may define the restriction $\alpha|_x$ of α to x as the image of $\tilde{\alpha}$ via the natural morphism

$$H^i(\mathcal{O}_{X,x}, \mathbb{Z}/2) \longrightarrow H^i(\kappa(x), \mathbb{Z}/2).$$

One can show that this element is in fact again unramified.

Proposition 2.27 ([Sch21, Proposition 4.8]). *Let X be a proper, smooth variety over k and let $\alpha \in H^i_{nr}(k(X)/k, \mathbb{Z}/2)$. Then for any $x \in X$ there exists a well-defined restriction*

$$\alpha|_x \in H^i_{nr}(\kappa(x), \mathbb{Z}/2),$$

where $\kappa(x)$ is the residue field of the (not necessarily closed) point x.

Corollary 2.28 ([Sch21, Corollary 4.9]). *Let $f\colon X \to Y$ be a morphism between proper varieties and assume additionally that Y is smooth. Then, there is a well-defined pullback map*

$$f^*\colon H^i_{nr}(k(Y)/k, \mathbb{Z}/2) \longrightarrow H^i_{nr}(k(X)/k, \mathbb{Z}/2)$$

which is given by restricting an unramified class $\alpha \in H^i_{nr}(k(Y)/k, \mathbb{Z}/2)$ to the generic point of the image of f and pulling it back to $k(X)$.

Now we are able to introduce the Merkurjev pairing which allows us to find a cohomological obstruction to rationality, see Proposition 4.7.

Definition 2.29 (Merkurjev pairing). *Let X be a smooth proper variety over a field k of characteristic different from 2 and K/k be a finitely generated field extension. Let $z \in X_K$ be a closed point and let $f_z\colon \operatorname{Spec} \kappa(z) \to \operatorname{Spec} K$ be the structure morphism. For any unramified class $\alpha \in H^i_{nr}(k(X)/k, \mathbb{Z}/2)$ we define*

$$\langle z, \alpha \rangle := (f_z)_* \left(\alpha_K|_z\right) \in H^i(K, \mathbb{Z}/2).$$

Extending this definition linearly to arbitrary zero-cycles $z \in Z_0(X)$ yields a bilinear pairing

$$Z_0(X_K) \times H^i_{nr}(k(X)/k, \mathbb{Z}/2) \longrightarrow H^i(K, \mathbb{Z}/2), \quad (z, \alpha) \mapsto \langle z, \alpha \rangle .$$

Remark 2.30. *This pairing is a slight variant of the Merkurjev pairing defined in [Mer08, Section 2.4]. More precisely our definition of the Merkurjev pairing is the composition*

$$Z_0(X_K) \times H^i_{nr}(k(X)/k, \mathbb{Z}/2) \xrightarrow{(\mathrm{id}, \psi^*)} Z_0(X_K) \times H^i_{nr}(K(X)/K, \mathbb{Z}/2) \longrightarrow H^i(K, \mathbb{Z}/2)$$

where $\psi\colon \operatorname{Spec} K(X_K) \to \operatorname{Spec} k(X)$ is the natural morphism and the latter map is the Merkurjev pairing as defined in [Mer08, Section 2.4]. Since the Merkurjev pairing descends to the level of Chow groups due to Merkurjev [Mer08, Corollary 2.9], there is a well-defined pairing

$$\mathrm{CH}_0(X_K) \times H^i_{nr}(k(X)/k, \mathbb{Z}/2) \longrightarrow H^i(K, \mathbb{Z}/2), \quad (z, \alpha) \mapsto \langle z, \alpha \rangle . \tag{2.7}$$

The following lemma is an easy consequence of the various definitions made so far and will be used later on.

Lemma 2.31. *Let K/k be a finitely generated field extension and let $\psi\colon \operatorname{Spec} K \to \operatorname{Spec} k$ be the natural morphism. Moreover, let X be a variety over k and let $\alpha \in H^i_{nr}(k(X)/k, \mathbb{Z}/2)$ be an unramified cohomology class.*

(a) Let $z \in Z_0(X)$ be a zero-cycle on X. Then

$$\langle z_K, \alpha \rangle = \psi^* \langle z, \alpha \rangle \in H^i(K, \mathbb{Z}/2).$$

(b) Let $\tau_0\colon : Y \to X$ be a generically finite morphism of varieties over k and $\tau\colon Y_K \to X_K$ the corresponding morphism after the base change with ψ. Then for any zero-cycle $w \in Z_0(Y_K)$ on Y_K

$$\langle w, \tau_0^* \alpha \rangle = \langle \tau(w), \alpha \rangle \in H^i(K, \mathbb{Z}/2).$$

Proof. We first prove item (a). Recall the definition of the Merkurjev pairing $\langle z_K, \alpha \rangle$: We start with $\alpha \in H^i_{\mathrm{nr}}\left(k(X)/k, \mathbb{Z}/2\right)$. Pulling back this class via the morphism $X_K \to X$ we obtain an unramified class $\alpha_K \in H^i_{\mathrm{nr}}\left(K(X_K)/K, \mathbb{Z}/2\right)$. Since α_K is unramified, it has trivial residue for every geometric valuation, in particular by Theorem 2.26 (b) α_K is contained in

$$\operatorname{Im}\left(H^i(\mathcal{O}_{X_K, z_K}, \mathbb{Z}/2) \hookrightarrow H^i(K(Y_K), \mathbb{Z}/2)\right).$$

Hence, we find a (unique) element in $H^i(\mathcal{O}_{X_K,z_K},\mathbb{Z}/2)$ mapping to α_K. Pulling back this element via the natural map $\operatorname{Spec}\kappa(z_K) \to \operatorname{Spec}\mathcal{O}_{X_K,z_K}$ and pushing it forward via the natural map $\operatorname{Spec}\kappa(z_K) \to \operatorname{Spec} K$ gives the pairing $\langle z_K,\alpha\rangle$. The diagram below summarizes this discussion:

$$\begin{tikzcd}
\alpha_K \arrow[r, phantom, "\in"] & H^i(K(X_K),\mathbb{Z}/2) & H^i(\mathcal{O}_{X_K,z_K},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] & H^i(\kappa(z_K),\mathbb{Z}/2) \arrow[r, "(f_{z_K})_*"] & H^i(K,\mathbb{Z}/2) \ni \langle z_K,\alpha\rangle \\
\alpha \arrow[u, mapsto] \arrow[r, phantom, "\in"] & H^i(k(X),\mathbb{Z}/2). \arrow[u] & & &
\end{tikzcd}$$

Recall that all unnamed maps in this diagram are given by pull-backs of a morphism between spectra of rings. We can complete this diagram in the following way:

$$\begin{tikzcd}
\alpha_K \arrow[r, phantom, "\in"] & H^i(K(X_K),\mathbb{Z}/2) & H^i(\mathcal{O}_{X_K,z_K},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] & H^i(\kappa(z_K),\mathbb{Z}/2) \arrow[r, "(f_{z_K})_*"] & H^i(K,\mathbb{Z}/2) \ni \langle z_K,\alpha\rangle \\
\alpha \arrow[u, mapsto] \arrow[r, phantom, "\in"] & H^i(k(X),\mathbb{Z}/2) \arrow[u] & H^i(\mathcal{O}_{X,z},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] \arrow[u] & H^i(\kappa(z),\mathbb{Z}/2) \arrow[r, "(f_z)_*"] \arrow[u] & H^i(k,\mathbb{Z}/2) \ni \langle z,\alpha\rangle\,. \arrow[u, "\psi^*"]
\end{tikzcd}$$

The commutativity of this diagram follows directly from the commutativity of the corresponding diagram of k-algebras together with the functoriality of étale cohomology. In particular, we find that

$$\langle z_K,\alpha\rangle = \psi^*\langle z,\alpha\rangle\,,$$

which proves item (a). We prove item (b) in a similar fashion: Let $z=\tau(w)$ be the image of w under τ. The following diagram describes the pairing $\langle w,\tau_0^*\alpha\rangle$:

$$\begin{tikzcd}
H^i(k(Y),\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H^i(K(Y_K),\mathbb{Z}/2) & H^i(\mathcal{O}_{Y_K,w},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] & H^i(\kappa(w),\mathbb{Z}/2) \arrow[r, "(f_w)_*"] & H^i(K,\mathbb{Z}/2) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\tau_0^*\alpha \arrow[rrrr, mapsto] & & & & \langle w,\tau_0^*\alpha\rangle\,.
\end{tikzcd}$$

Similarly, the pairing $\langle z,\alpha\rangle = \langle \tau(w),\alpha\rangle$ is given by

$$\begin{tikzcd}
H^i(k(X),\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H^i(K(X_K),\mathbb{Z}/2) & H^i(\mathcal{O}_{X_K,z},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] & H^i(\kappa(z),\mathbb{Z}/2) \arrow[r, "(f_z)_*"] & H^i(K,\mathbb{Z}/2) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\alpha \arrow[rrrr, mapsto] & & & & \langle z,\alpha\rangle\,.
\end{tikzcd}$$

Since $\tau_0^*\alpha$ is a pull-back of α we can connect the two diagrams as follows:

$$\begin{tikzcd}
\tau_0^*\alpha \arrow[rrrr, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & & & \langle w,\tau_0^*\alpha\rangle \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
H^i(k(Y),\mathbb{Z}/2) \arrow[r] & H^i(K(Y_K),\mathbb{Z}/2) & H^i(\mathcal{O}_{Y_K,w},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] & H^i(\kappa(w),\mathbb{Z}/2) \arrow[r, "(f_w)_*"] & H^i(K,\mathbb{Z}/2) \\
H^i(k(X),\mathbb{Z}/2) \arrow[r] \arrow[u, "\tau_0^*"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H^i(K(X_K),\mathbb{Z}/2) \arrow[u, "\tau^*"] & H^i(\mathcal{O}_{X_K,z},\mathbb{Z}/2) \arrow[l, hook'] \arrow[r] \arrow[u] & H^i(\kappa(z),\mathbb{Z}/2) \arrow[r, "(f_z)_*"] \arrow[u] & H^i(K,\mathbb{Z}/2) \arrow[u, equal] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\alpha \arrow[rrrr, mapsto] & & & & \langle z,\alpha\rangle\,.
\end{tikzcd}$$

The commutativity follows from the commutativity of the corresponding diagrams of k-algebras together with the functoriality of étale cohomology. Thus we get

$$\langle w,\tau_0^*\alpha\rangle = \langle z,\alpha\rangle\,,$$

which finishes the proof of item (b) and thus also concludes the proof of the lemma. $\square$

Lastly, let us mention a vanishing result by Schreieder.

Theorem 2.32 ([Sch19b, Theorem 9.2]). *Let $f\colon Y \to S$ be a surjective morphism of proper varieties over an algebraically closed field k of characteristic $\operatorname{char} k \neq 2$ whose generic fibre is birational to a smooth quadric over $k(S)$. Let $n = \dim S$ and assume that there exists a class $\beta \in H^n(k(S), \mathbb{Z}/2)$ with $f^*\beta \in H^n_{nr}(k(Y)/k, \mathbb{Z}/2)$.*

Then for any dominant and generically finite morphism $\tau\colon Y' \to Y$ of varieties and for any subvariety $E \subset Y'$ which meets the smooth locus of Y' and which does not dominate S via $f \circ \tau$, we have $(\tau^* f^* \beta)|_E = 0 \in H^n(k(E), \mathbb{Z}/2)$.

A strictly semi-stable model

Now we turn to our example regarding a $(3,3)$ complete intersection in $\mathbb{P}^7$. We will use a degeneration similar to the one by Nicaise and Ottem in [NO22, Theorem 7.2]. Their model is not strictly semi-stable, so we construct such a model by blowing-up one component of the special fiber. The obstruction found by Nicaise and Ottem lies in the intersection of the two components of the special fiber. In order to use the argument from [PS21] we need to blow-up the intersection. To end up with a strictly semi-stable model after the blow-up we also need to perform a $2:1$ base change.

Let k_0 be an algebraically closed field of characteristic different from 2. Let

$$k := \overline{k_0(u,v,w)}$$

be the algebraic closure of a purely transcendental, degree 3 extension of k_0. We assume first that $\operatorname{char} k \neq 3$ and give later on a strictly semi-stable model in characteristic 3 for which we can use basically the same arguments in Chapter 4.

Definition 3.1. *Let $f^w, g^w, h^w \in k_0[x_0, \ldots, x_6]$ be homogeneous polynomials with $\deg f^w = 2$, $\deg g^w = \deg h^w = 3$ such that the hypersurfaces $F := \{f^w = 0\}$, $G := \{g^w = 0\}$, $H := \{h^w = 0\} \subset \mathbb{P}^6_{k_0}$ and all the complete intersections $F \cap G$, $F \cap H$, $G \cap H$ and $F \cap G \cap H \subset \mathbb{P}^6_{k_0}$ are smooth. (Bertini's theorem implies the existence of such polynomials, in fact this works for general f^w, g^w and h^w.) Furthermore, we define the following three polynomials in $k[x_0, \ldots, x_5]$:*

$$\begin{aligned} f_2 &:= \sqrt[3]{4}(x_1x_2 + x_4x_5) + x_3^2, \\ f_3 &:= x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3, \\ \hat{f}_3 &:= x_0^3 - x_1^3 + \rho x_2^3 - \rho x_3^3 + \rho^2 x_4^3 - \rho^2 x_5^3, \end{aligned}$$

where $\rho \in k_0$ is a primitive third root of unity. (Note that such ρ exists as k_0 is algebraically closed.) Consider the following polynomials in $k[x_0, \ldots, x_7]$:

$$\begin{aligned} c_{u,v,w} &:= c_0 + u(x_6 f_2 + f_3) + v(f_3 + x_6^3) + w(g^w + x_7^3), \\ q_{v,w} &:= x_3x_6 - x_4x_5 + v(x_3x_7 + f_2 + x_6^2) + w f^w, \\ \hat{c}_{v,w} &:= x_6^3 + v\hat{f}_3 + w h^w, \end{aligned}$$

where

$$c_0 := x_0^2x_5 + x_1^2x_4 + x_2^2x_6 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5 + x_6)) \in k_0[x_0, \ldots, x_6].$$

Remark 3.2. *Before starting the actual argument, let us make some short comments on the chosen polynomials. At the end we specialize u, v, w all to 0. Thus we are mainly interested in c_0 and the term $x_3x_6 - x_4x_5$ which are also used in [Ska22]. The other terms ensure that our model is strictly semi-stable and enable us to simplify the contribution of the Chow groups in the obstruction map (2.4). In particular our choice of involving a cubic root of 4 and a third root of unity is made solely to be able to work also in characteristic 5 and 7.*

There are many instances in which certain varieties must be smooth. This will be shown by using a degeneration argument, i.e. we degenerate to another proper variety and check that the latter is smooth. We will give now a list of varieties to which we will later degenerate in order to show smoothness.

Lemma 3.3. *With the same notation as above:*

$$\begin{aligned}
A_{(i)} &:= \{f^w = g^w = h^w = 0\} \subset \mathbb{P}^6_k, \\
A_{(ii)} &:= \{g^w + x_7^3 = 0\} \subset \mathbb{P}^7_k, \\
A_{(iii)} &:= \{g^w + x_7^3 = f^w = 0\} \subset \mathbb{P}^7_k, \\
A_{(iv)} &:= \{g^w + x_7^3 = h^w = 0\} \subset \mathbb{P}^7_k, \\
A_{(v)} &:= \{f_3 = 0\} \subset \mathbb{P}^5_k, \\
A_{(vi)} &:= \{f_3 + x_6^3 = 0\} \subset \mathbb{P}^6_k, \\
A_{(vii)} &:= \{f_3 + x_6^3 = \hat{f}_3 = 0\} \subset \mathbb{P}^6_k, \\
A_{(viii)} &:= \{f_2 + x_6^2 = f_3 + x_6^3 = 0\} \subset \mathbb{P}^6_k, \\
A_{(ix)} &:= \{f_2 = f_3 = 0\} \subset \mathbb{P}^5_k, \\
A_{(x)} &:= \{f_2 + x_6^2 = f_3 + x_6^3 = x_3 = 0\} \subset \mathbb{P}^6_k
\end{aligned}$$

are smooth complete intersections.

Proof. Recall that we use the notation $F := \{f^w = 0\}$, $G := \{g^w = 0\}$, $H := \{h^w = 0\} \subset \mathbb{P}^6_k$. Moreover, we chose $f^w,\ g^w,\ h^w \in k_0[x_0, \dots, x_6]$ such that F, G, H, $F \cap G$, $F \cap H$, $G \cap H$ and $F \cap G \cap H \subset \mathbb{P}^6_k$ are smooth, in particular $A_{(i)} = F \cap G \cap H$ is smooth. We will prove the smoothness of the remaining varieties via the Jacobian criterion.

Consider next $A_{(ii)}$. The vanishing of its Jacobian immediately implies $x_7 = 0$. Thus the smoothness of $A_{(ii)}$ follows directly from G being smooth. The Jacobian of $A_{(iii)}$ reads

$$\begin{pmatrix} \partial_0 g^w & \partial_1 g^w & \dots & \partial_6 g^w & 3x_7^2 \\ \partial_0 f^w & \partial_1 f^w & \dots & \partial_6 f^w & 0 \end{pmatrix}.$$

As F is smooth, we immediately obtain that $x_7 = 0$. Therefore $A_{(iii)}$ is smooth, because $F \cap G$ is smooth. The smoothness of $A_{(iv)}$ follows by the same argument from H and $G \cap H$ being smooth. The varieties $A_{(v)}$ and $A_{(vi)}$ are Fermat cubics and thus smooth as the characteristic of k is different from 3. Next we turn to $A_{(vii)}$. Its Jacobian reads

$$\begin{pmatrix} 3x_0^2 & 3x_1^2 & 3x_2^2 & 3x_3^2 & 3x_4^2 & 3x_5^2 & 3x_6^2 \\ 3x_0^2 & -3x_1^2 & 3\rho x_2^2 & -3\rho x_3^2 & 3\rho^2 x_4^2 & -3\rho^2 x_5^2 & 0 \end{pmatrix}.$$

Recall that ρ is a fixed primitive third root of unity. Let $P = [x_0 : \dots : x_6]$ be a singular point of $A_{(vii)}$. The vanishing of all 2×2 minors of the Jacobian yields

$$a_i x_i^2 x_j^2 = a_j x_i^2 x_j^2, \quad i, j \in \{0, 1, \dots, 5\} \tag{3.1}$$

where a_i is the coefficient of the monomial x_i^3 in $\hat{f}_3$, i.e. $a_i \in \{1, -1, \rho, -\rho, \rho^2, -\rho^2\}$. Since $a_i \neq a_j$ for $i \neq j$, (3.1) implies that at most one coordinate of $x_0, \dots, x_5$ is non-zero. Then all coordinates $x_0, \dots, x_5$ have to be zero because $\hat{f}_3$ vanishes at P. The point $[0 : \dots : 0 : 1] \in \mathbb{P}^6_k$ is not contained in $A_{(vii)}$ because $f_3 + x_6^3 = x_6^3 \neq 0$ at that point. Thus we find that $A_{(vii)}$ is smooth.

The arguments for the smoothness of the last three varieties $A_{(viii)}$, $A_{(ix)}$ and $A_{(x)}$ are similar, so we will mainly consider $A_{(viii)}$ and only indicate the changes for the other two varieties. The Jacobian of $A_{(viii)}$ reads

$$\begin{pmatrix} 3x_0^2 & 3x_1^2 & 3x_2^2 & 3x_3^2 & 3x_4^2 & 3x_5^2 & 3x_6^2 \\ 0 & \sqrt[3]{4}x_2 & \sqrt[3]{4}x_1 & 2x_3 & \sqrt[3]{4}x_5 & \sqrt[3]{4}x_4 & 2x_6 \end{pmatrix}.$$

The Jacobian of $A_{(ix)}$ and $A_{(x)}$ are obtained by removing the last and forth column, respectively. Let $[x_0 : \cdots : x_6] \in \mathbb{P}_k^6$ be a singular point of $A_{(viii)}$, i.e. all 2×2 minors vanish. Since $A_{(viii)}$ should be smooth, we aim to find a contradiction: If $x_0 \neq 0$, then

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0,$$

which contradicts $0 = f_3 + x_6^3 = x_0^3$. Hence, we can assume that $x_0 = 0$. Moreover, we obtain the following conditions:

$$m_{1,2}: \quad 0 = x_1^3 - x_2^3, \tag{3.2}$$
$$m_{4,5}: \quad 0 = x_4^3 - x_5^3, \tag{3.3}$$
$$m_{1,4}: \quad 0 = x_1^2 x_5 - x_4^2 x_2, \tag{3.4}$$
$$m_{i,3}: \quad 0 = 2x_i^2 x_3 - \sqrt[3]{4} x_j x_3^2, \quad (i,j) \in \{(1,2),(2,1),(4,5),(5,4)\}, \tag{3.5}$$
$$m_{3,6}: \quad 0 = x_3 x_6 (x_3 - x_6). \tag{3.6}$$

Note that by considering the minor $m_{i,6}$ instead of $m_{i,3}$ we can replace x_3 in (3.4) by x_6. Moreover, the last condition exists and is needed only for $A_{(viii)}$. (It would be also trivial for $A_{(ix)}$ and $A_{(x)}$.) We will distinguish the following two cases. Note that for the varieties $A_{(ix)}$ and $A_{(x)}$ the cases reduce to $x_6 = 0$, $x_6 \neq 0$ and $x_3 = 0$, $x_3 \neq 0$, respectively.

Case 1. Assume $x_3 = x_6 = 0$. We may asssume without loss of generality $x_1 = 1$. The equation (3.2) implies that $x_2^3 = 1$. Moreover,

$$x_5^3 = x_1^6 x_5^3 \overset{(3.4)}{=} x_4^6 x_2^3 = x_4^6 \overset{(3.3)}{=} x_5^6,$$

i.e. $x_4^3 \overset{(3.3)}{=} x_5^3 \in \{0, 1\}$. Then,

$$0 \overset{!}{=} f_3 + x_6^3 = 2 + 2x_4^3 \in \{2, 4\}$$

yields a contradiction.

Case 2. Assume now $x_3 \neq 0$ or $x_6 \neq 0$. Without loss of generality we may assume $x_3 = 1$. The equation (3.6) implies $x_6^3 \in \{0, 1\}$. Moreover, for $(i,j) \in \{(1,2),(2,1),(4,5),(5,4)\}$

$$8x_i^6 = (2x_i^2)^3 \overset{(3.5)}{=} (\sqrt[3]{4}x_j)^3 = 4x_j^3 \overset{(3.2),(3.3)}{=} 4x_i^3.$$

Thus, $x_i^3 \in \left\{0, \frac{1}{2}\right\}$ for $i \in \{1,2,4,5\}$. Together with (3.2) and (3.3) we find that

$$0 \overset{!}{=} f_3 + x_6^3 = 2x_1^3 + 2x_3^3 + 1 + x_6^3 \in \{1, 2, 3, 4\},$$

which yields the desired contradiction. □

Let us construct now the model inspired by Nicaise and Ottem [NO22, Theorem 7.2]. Let $R := k[[t]]$ and consider the R-scheme

$$\mathcal{X} := \{c_{u,v,w} = t\hat{c}_{v,w} + q_{v,w} x_7 = 0\} \subset \mathbb{P}_R^7.$$

The special fibre Y of $\mathcal{X} \to \operatorname{Spec} R$ has two irreducible components, namely

$$Y = Y_1 \cup Y_2,$$

where

$$Y_1 = \{c_{u,v,w} = x_7 = 0\}, \quad Y_2 = \{c_{u,v,w} = q_{v,w} = 0\} \subset \mathbb{P}^7_k.$$

We denote their scheme-theoretic intersection by $Z := Y_1 \cap Y_2$.

Lemma 3.4. *The varieties Y_1, Y_2 and $Z \subset \mathbb{P}^7$ are smooth. The geometric generic fibre*

$$X_{\overline{K}} := \{c_{u,v,w} = \hat{c}_{v,w} + t^{-1}x_7 q_{v,w} = 0\} \subset \mathbb{P}^7_{\overline{K}}$$

of $\mathcal{X} \to \operatorname{Spec} R$ is a smooth $(3,3)$ complete intersection.

Proof. Since the models in which we specialize are proper over some discrete valuation ring, any singular point will specialize via $u \to \infty$, $v \to \infty$ or $w \to \infty$ to a singular point. Thus, it suffices to prove that some specialization is smooth. This in turn follows from Lemma 3.3 as follows

$$\begin{array}{rlll} Y_1 & \text{specializes via} & v \to \infty & \text{to } A_{(vi)} \text{ from Lemma 3.3,} \\ Y_2 & \text{specializes via} & w \to \infty & \text{to } A_{(iii)} \text{ from Lemma 3.3,} \\ Z & \text{specializes via} & v \to \infty & \text{to } A_{(vii)} \text{ from Lemma 3.3,} \\ X_{\overline{K}} & \text{specializes via} & t \to \infty \text{ and } w \to \infty & \text{to } A_{(iv)} \text{ from Lemma 3.3.} \end{array}$$

This concludes the proof. □

Remark 3.5. *Let us write down one such specialization concretely: We consider the specialization $v \to \infty$ for Y_1. Recall that*

$$Y_1 = \{c_{u,v,w} = x_7 = 0\} = \left\{c_0 + u(x_6 f_2 + f_3) + v(f_3 + x_6^3) + w(g^w + x_7^3) = x_7 = 0\right\} \subset \mathbb{P}^7_k,$$

see also Definition 3.1 for more details. Consider $\lambda := \frac{1}{v}$ and

$$\psi \colon \mathcal{Y}_1 := \{\lambda\,(c_0 + u(x_6 f_2 + f_3) + w g^w) + f_3 + x_6^3 = 0\} \subset \mathbb{P}^6_{\overline{k_0(u,w)}[[\lambda]]} \longrightarrow \operatorname{Spec} \overline{k_0(u,w)}[[\lambda]].$$

Then, the geometric generic fiber of ψ is isomorphic to Y_1 and the special fiber is isomorphic to $A_{(vi)}$ of Lemma 3.3, i.e. Y_1 specializes to $A_{(vi)}$ in the sense of Section 2.3. The smoothness of Y_1 can be seen now as follows: Assume that the generic fiber of ψ is singular. Any singular point of the generic fiber yields a point in $\operatorname{Sing}\psi$. Since $\operatorname{Sing}\psi \subset \mathcal{Y}_1$ is closed and ψ is proper, the special fiber is also singular which contradicts Lemma 3.3, i.e. the generic fibre of ψ is smooth. Since Y_1 is isomorphic to the base change of the generic fibre with $\operatorname{Spec} k \to \operatorname{Spec} \overline{k_0(u,w)}[[\lambda]]$, the variety Y_1 is smooth. Thus, we convinced ourselves that it suffices to check smoothness after some specialization in proper families.

Our current model $\mathcal{X} \to \operatorname{Spec} R$ is flat and proper. Moreover the irreducible components of the special fiber and their intersections are smooth. However the components of the special fiber of $\mathcal{X} \to \operatorname{Spec} R$ are not Cartier divisors in $\mathcal{X}$, i.e. $\mathcal{X} \to \operatorname{Spec} R$ is not strictly semi-stable. The problem is that the total space is singular.

Lemma 3.6. *The singular locus of the total space $\mathcal{X}$ is given by*

$$S := \{c_{u,v,w} = \hat{c}_{v,w} = q_{v,w} = x_7 = t = 0\} \subset \mathcal{X}.$$

Moreover, S is smooth and $\mathcal{X}$ has ordinary quadratic singularities along S.

Proof. Let us first show that S is smooth. Using the definitions in Definition 3.1, S is isomorphic to the variety

$$\left\{ \begin{matrix} c_0+u(x_6f_2+f_3)+v(f_3+x_6^3)+wg^w=x_6^3+v\hat{f}_3+wh^w=0, \\ x_3x_6-x_4x_5+v(f_2+x_6^2)+wf^w=0 \end{matrix} \right\} \subset \mathbb{P}_k^6.$$

By the same argument as in Remark 3.5 it suffices to check smoothness after some specialization. The variety S specializes via $w \to \infty$ to the smooth variety $A_{(i)}$ from Lemma 3.3, i.e. S is smooth.

Next we check that S is indeed the singular locus of $\mathcal{X}$. Recall that the singular locus $\operatorname{Sing}\mathcal{X}$ of $\mathcal{X}$ is given by the vanishing of the defining equations of $\mathcal{X}$ as well as all minors of the Jacobian. The Jacobian of $\mathcal{X}$ is given by

$$\begin{pmatrix} \partial_0 c_{u,v,w} & \dots & \partial_6 c_{u,v,w} & \partial_7 c_{u,v,w} & 0 \\ t\partial_0\hat{c}_{v,w}+x_7\partial_0 q_{v,w} & \dots & t\partial_6\hat{c}_{v,w}+x_7\partial_6 q_{v,w} & t\partial_7\hat{c}_{v,w}+q_{v,w}+x_7\partial_7 q_{v,w} & \hat{c}_{v,w} \end{pmatrix}. \tag{3.7}$$

Obviously, the variety S is contained in $\operatorname{Sing}\mathcal{X}$ because the defining equation and the second row of (3.7) vanish. We show the opposite inclusion: Let $([x_0:\dots:x_7],t) \in \mathcal{X}$ be a singular point. Note that $\{c_{u,v,w}=0\} \subset \mathbb{P}_k^7$ is smooth because it specializes via $w \to \infty$ to $A_{(ii)}$ from Lemma 3.3. Hence $\hat{c}_{v,w}=0$, because otherwise the Jacobian would have full rank. This implies furthermore, by definition of $\mathcal{X}$,

$$x_7 q_{v,w}=0.$$

Since $\{c_{u,v,w}=\hat{c}_{v,w}=0\} \subset \mathbb{P}_k^7$ specializes via $w \to \infty$ to $A_{(iv)}$ from Lemma 3.3,

$$\{c_{u,v,w}=\hat{c}_{v,w}=0\} \subset \mathbb{P}_k^7$$

is smooth. Hence, the singular locus of $\mathcal{X}$ is contained in the special fiber. We have thus shown that the singular locus of $\mathcal{X}$ is contained in

$$\{c_{u,v,w}=\hat{c}_{v,w}=t=x_7q_{v,w}=0\} \subset \mathcal{X}.$$

Hence it suffices to show under the assumption $c_{u,v,w}=\hat{c}_{v,w}=t=0$ that

$$x_7=0 \iff q_{v,w}=0.$$

We start by showing "$\Rightarrow$": Note that $\{c_0+u(x_6f_2+f_3)+v(f_3+x_6^3)+wg^w=0\} \subset \mathbb{P}_k^6$ is smooth because it specializes via $v \to \infty$ to $A_{(vi)}$ from Lemma 3.3. Thus, we conclude that $q_{v,w}=0$ as wanted because otherwise (3.7) has rank 2. For "$\Leftarrow$", we note that

$$\{c_{u,v,w}=q_{v,w}=0\} \subset \mathbb{P}_k^7$$

is smooth because it specializes via $w \to \infty$ to $A_{(iii)}$ from Lemma 3.3. Thus, x_7 has to be equal to 0 as otherwise the Jacobian would have full rank. This shows $\operatorname{Sing}\mathcal{X} \subset S$ and thus $S=\operatorname{Sing}\mathcal{X}$.

Lastly, we describe the type of the singularities of $\mathcal{X}$. Let $P \in S$ be any point, i.e. P is a singular point of $\mathcal{X}$. Note that $\{\hat{c}_{v,w}=0\}$, $\{q_{v,w}=0\} \subset \mathbb{P}_k^7$ are smooth because they specialize to F and H from Definition 3.1 which are smooth by definition, respectively. Thus the tangent cone of $\{t\hat{c}_{v,w}+x_7q_{v,w}=0\} \subset \mathbb{P}_R^7$ at P is Zariski locally isomorphic to the tangent cone of the ordinary quadratic singularity $\{tx+yz=0\}$. Moreover, the tangent cone of $\{t\hat{c}_{v,w}+x_7q_{v,w}=0\} \subset \mathbb{P}_R^7$ at P intersects the tangent space of $\{c_{u,v,w}=0\} \subset \mathbb{P}_R^7$ at P transversely because $\{c_{u,v,w}=0\} \subset \mathbb{P}_k^7$ is smooth (as it specializes to $A_{(ii)}$ from Lemma 3.3 via $w \to \infty$). This concludes the proof of the lemma. □

Remark 3.7. *In the proof that the singularities of $\mathcal{X}$ are ordinary quadratic, we used that $\{\hat{c}_{v,w}=0\}$ and $\{q_{v,w}=0\} \subset \mathbb{P}_k^7$ are smooth. But the argument shows that it suffices that the two varieties are smooth at every point of S. This slightly weaker assumption will be used for the strictly semi-stable model in characteristic 3.*

Lemma 3.8. *The blow-up $\mathcal{X}' := \mathrm{Bl}_{Y_1}\mathcal{X} \to \operatorname{Spec} R$ is strictly semi-stable with special fibre $\tilde{Y}_1 \cup Y_2$ where $\tilde{Y}_1 = \mathrm{Bl}_S Y_1$. Moreover,*

$$\tilde{Y}_1 \cap Y_2 = \mathrm{Bl}_S Z = Z = Y_1 \cap Y_2.$$

Proof. The family $\mathcal{X}' \to \operatorname{Spec} R$ is clearly proper, as $\mathcal{X}$ is projective over R. Since $R = k[[t]]$, the affine scheme $\operatorname{Spec} R$ is an integral, regular scheme of dimension 1. By [Har77, III. Proposition 9.7] the family $\mathcal{X}' \to \operatorname{Spec} R$ is flat, because $\mathcal{X}'$ is integral and the morphism $\mathcal{X}' \to \operatorname{Spec} R$ is dominant.

By Lemmata 3.4 and 3.6, we know that Y_1, Y_2, and S are smooth. Locally at a point of S, $\mathcal{X}$ has ordinary quadratic singularities (see Lemma 3.6) and a local computation shows that the special fibre of $\mathcal{X}'$ is given by $\tilde{Y}_1 \cup Y_2$ where $\tilde{Y}_1 = \mathrm{Bl}_S Y_1$: Recall that

$$\mathcal{X} = \{c_{u,v,w} = t\hat{c}_{v,w} + x_7 q_{v,w} = 0\} \subset \mathbb{P}^7_R$$

and $Y_1 = \{t = x_7 = 0\} \subset \mathcal{X}$. Let $U_i = \{x_i \neq 0\} \subset \mathbb{P}^7_R$ be the standard affine charts for $0 \leq i \leq 7$. Then,

$$\mathrm{Bl}_{Y_1 \cap U_i}(\mathcal{X} \cap U_i) = \left\{ \begin{matrix} a_i \frac{x_7}{x_i} - t b_i = c_{u,v,w}\left(\frac{x_0}{x_i}, \ldots, \frac{x_7}{x_i}\right) = 0, \\ a_i \hat{c}_{v,w}\left(\frac{x_0}{x_i}, \ldots, \frac{x_7}{x_i}\right) + b_i q_{v,w}\left(\frac{x_0}{x_i}, \ldots, \frac{x_7}{x_i}\right) = 0 \end{matrix} \right\} \subset \mathbb{A}^7_k \times \operatorname{Spec} k[[t]] \times \mathbb{P}^1_k \tag{3.8}$$

where $[a_i : b_i]$ are the projective coordinates of $\mathbb{P}^1_k$ for $i \in \{0, \ldots, 7\}$. In particular, we find that the special fiber is the union of two subvarieties, namely in the local description of the blow-up

$$\left\{ t = \frac{x_7}{x_i} = 0 \right\} \cup \{t = a_i = 0\} \subset \mathrm{Bl}_{Y_1 \cap U_i}(\mathcal{X} \cap U_i).$$

The latter one, i.e. $\{t = a_i = 0\}$ is obviously isomorphic to $Y_2 \cap U_i$. Let us compute now the blow-up of Y_1 in S to see that it is the other irreducible component of the special fibre of $\mathcal{X}' \to \operatorname{Spec} R$: Consider

$$Y_1 = \{c_{u,v,w}(y_0, \ldots, y_6, 0) = 0\} \subset \mathbb{P}^6_k$$

and let $V_j = \{y_j \neq 0\} \subset \mathbb{P}^6_k$ be the standard affine charts for $0 \leq j \leq 6$. Recall that

$$S = \{\hat{c}_{v,w}(y_0, \ldots, y_6, 0) = q_{v,w}(y_0, \ldots, y_6) = 0\} \subset Y_1.$$

Then we find that

$$\mathrm{Bl}_{S \cap V_j}(Y_1 \cap V_j) = \left\{ \begin{matrix} \alpha_j \hat{c}_{v,w}\left(\frac{y_0}{y_j}, \ldots, \frac{y_6}{y_j}, 0\right) - \beta_j q_{v,w}\left(\frac{y_0}{y_j}, \ldots, \frac{y_6}{y_j}, 0\right) = 0, \\ c_{u,v,w}\left(\frac{y_0}{y_j}, \ldots, \frac{y_6}{y_j}, 0\right) = 0 \end{matrix} \right\} \subset \mathbb{A}^6_k \times \mathbb{P}^1_k, \tag{3.9}$$

where $[\alpha_j : \beta_j]$ are the projective coordinates of $\mathbb{P}^1_k$ for $j \in \{0, \ldots, 6\}$. Comparing now (3.8) and (3.9), we see that the first component of the special fiber is precisely the blow-up of Y_1 in S as claimed. We can use this local description to check that the irreducible components are smooth and Cartier in $\mathcal{X}$, but we prefer the general theory of blow-ups here: Since Y_1 and S are smooth, the blow-up $\tilde{Y}_1 = \mathrm{Bl}_S Y_1$ is also smooth. Moreover, $\tilde{Y}_1 \cap Y_2 = \mathrm{Bl}_S Z = Z$ where the latter equation comes from the fact that S is Cartier in Z. Thus, the irreducible components of the special fiber of $\mathcal{X}'$ and their intersection are smooth. By construction, $\tilde{Y}_1 \subset \mathcal{X}$ is Cartier and (3.8) shows that Y_2 is Cartier, i.e. $\mathcal{X}'$ is strictly semi-stable. $\square$

Lemma 3.9. *Let $\mathcal{X}'' = \mathcal{X}' \times_{\substack{R\to R\\ t\to t^2}} R$ be the $2:1$ base change of $\mathcal{X}'$. The blow-up*

$$\tilde{\mathcal{X}} := \mathrm{Bl}_Z \mathcal{X}'' \longrightarrow \operatorname{Spec} R \tag{3.10}$$

is a strictly semi-stable R-scheme with special fibre $\tilde{X}_k = \tilde{Y}_1 \cup P_Z \cup Y_2$ where P_Z is a $\mathbb{P}^1_k$-bundle over Z. The intersections $\tilde{Y}_1 \cap P_Z$ and $Y_2 \cap P_Z$ are disjoint sections of $P_Z \to Z$. The geometric generic fibre

$$\tilde{X}_{\overline{K}} := \{c_{u,v,w} = t^2 \hat{c}_{v,w} + x_7 q_{v,w} = 0\} \subset \mathbb{P}^7_{\overline{K}}$$

is a smooth $(3,3)$ complete intersection.

Proof. Recall that $\mathcal{X}' \to \operatorname{Spec} R$ is strictly semi-stable by Lemma 3.8. A local computation as in Lemma 3.6 shows that the singular locus of $\mathcal{X}''$ is the intersection Z of the two components of the special fibre. Hence, the $2:1$ base change is regular away from the singular locus Z of the special fiber. A computation in affine charts shows that the special fibre of the blow-up $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ is $\tilde{Y}_1 \cup P_Z \cup Y_2$ where P_Z is a $\mathbb{P}^1$-bundle, because it is a smooth conic bundle with a section, e.g. $\tilde{Y}_1 \cap P_Z$. As the singularities of $\mathcal{X}''$ are ordinary quadratic, the blow-up resolves them, i.e. $\tilde{\mathcal{X}}$ is regular and thus in particular strictly semi-stable. The smoothness of the geometric generic fibre follows from Lemma 3.4. □

Remark 3.10. *The $2:1$ base change is necessary to ensure that the components of the special fiber are reduced.*

3.1 A strictly semi-stable model in characteristic 3

As mentioned in the beginning of the previous section, we construct also a strictly semi-stable model in characteristic 3. Let k_0 be an algebraically closed field of characteristic 3 and let $k := \overline{k_0(u,v,w)}$ be the algebraic closure of a purely transcendental extension of k_0 of transcendental degree 3.

Definition 3.11. *We define the following three polynomials:*

$$\begin{aligned}
f_2^{(3)} &:= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \in k[x_0, \dots, x_5],\\
f_3^{(3)} &:= x_0^3 + x_0 x_1^2 + x_1 x_2^2 + x_2 x_4^2 + x_4 x_5^2 + x_5 x_3^2 \in k[x_0, \dots, x_5],\\
\hat{f}_3^{(3)} &:= x_1^2 x_2 + x_2^2 x_4 + x_4^2 x_5 + x_5^2 x_3 + x_3^2 x_6 + x_6^3 \in k[x_0, \dots, x_6].
\end{aligned}$$

Moreover, we consider the following three polynomials in $k[x_0, \dots, x_7]$:

$$\begin{aligned}
c_{u,v,w}^{(3)} &:= c_0 + u\left(x_6 f_2^{(3)} + f_3^{(3)}\right) + v\left(f_3^{(3)} + x_3 x_6^2\right) + w\left(f_3^{(3)} + x_3 x_6^2 + x_6 x_7^2\right),\\
q_{v,w}^{(3)} &:= x_3 x_6 - x_4 x_5 + v\left(x_3 x_7 + f_2^{(3)} - x_6^2\right) + w\left(f_2^{(3)} - x_6^2 + x_7^2\right),\\
\hat{c}_{v,w}^{(3)} &:= x_6^3 + v \hat{f}_3^{(3)} + w\left(\hat{f}_3^{(3)} + x_6^2 x_7 + x_7^3\right),
\end{aligned}$$

where

$$c_0 := x_0^2 x_5 + x_1^2 x_4 + x_2^2 x_6 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5 + x_6)) \in k_0[x_0, \dots, x_6].$$

Remark 3.12. *The polynomials are chosen such that the arguments and constructions which we made in characteristic different from 2 and 3 can easily adapted to this model. In particular we need that certain varieties are smooth, see Lemma 3.3. The analogue statement is written down in the lemma below. Note that the varieties are labelled such that the labeling agrees with the labeling from Lemma 3.3.*

Lemma 3.13. *With the same notation as in Definition 3.11:*

$$\begin{aligned}
A'_{(i)} &:= \{f_3^{(3)} + x_3x_6^2 = \hat{f}_3^{(3)} = f_2^{(3)} + x_6^2 = 0\} \subset \mathbb{P}^6_k,\\
A'_{(ii)} &:= \{f_3^{(3)} + x_3x_6^2 + x_6x_7^2 = 0\} \subset \mathbb{P}^7_k,\\
A'_{(iii)} &:= \{f_3^{(3)} + x_3x_6^2 + x_6x_7^2 = f_2^{(3)} - x_6^2 + x_7^2 = 0\} \subset \mathbb{P}^7_k,\\
A'_{(iv)} &:= \{f_3^{(3)} + x_3x_6^2 + x_6x_7^2 = \hat{f}_3^{(3)} + x_6^2x_7 + x_7^3 = 0\} \subset \mathbb{P}^7_k,\\
A'_{(v)} &:= \{f_3^{(3)} = 0\} \subset \mathbb{P}^5_k,\\
A'_{(vi)} &:= \{f_3^{(3)} + x_3x_6^2 = 0\} \subset \mathbb{P}^6_k,\\
A'_{(vii)} &:= \{f_3^{(3)} + x_3x_6^2 = \hat{f}_3^{(3)} = 0\} \subset \mathbb{P}^6_k,\\
A'_{(viii)} &:= \{f_2^{(3)} - x_6^2 = f_3^{(3)} + x_3x_6^2 = 0\} \subset \mathbb{P}^6_k,\\
A'_{(ix)} &:= \{f_2^{(3)} = f_3^{(3)} = 0\} \subset \mathbb{P}^5_k,\\
A'_{(x)} &:= \{f_2^{(3)} - x_6^2 = f_3^{(3)} + x_3x_6^2 = x_3 = 0\} \subset \mathbb{P}^6_k
\end{aligned}$$

are smooth complete intersections.

Proof. Recall that k is a field of characteristic 3. We use the Jacobian criterion as in the proof of Lemma 3.3 to show smoothness. Let us start with $A'_{(ii)}$. Its Jacobian is given by

$$\begin{pmatrix} x_1^2 & 2x_0x_1 + x_2^2 & 2x_1x_2 + x_4^2 & 2x_3x_5 + x_6^2 & 2x_2x_4 + x_5^2 & 2x_4x_5 + x_3^2 & 2x_3x_6 + x_7^2 & 2x_6x_7 \end{pmatrix}.$$

Thus we immediately see that the Jacobian vanishes if and only if

$$x_1 = x_2 = x_4 = x_5 = x_3 = x_6 = x_7 = 0.$$

Hence, the only singular point of $A'_{(ii)}$ might be the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^7_k$. Since that point does not lie on $A'_{(ii)}$, the variety $A'_{(ii)}$ is smooth. The same argument shows that $A'_{(v)}$ and $A'_{(vi)}$ are smooth.

Next we consider $A'_{(iii)}$. Its Jacobian reads

$$\begin{pmatrix} x_1^2 & 2x_0x_1 + x_2^2 & 2x_1x_2 + x_4^2 & 2x_3x_5 + x_6^2 & 2x_2x_4 + x_5^2 & 2x_4x_5 + x_3^2 & 2x_3x_6 + x_7^2 & 2x_6x_7 \\ 0 & 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 & -2x_6 & 2x_7 \end{pmatrix}.$$

We denote the 2×2 minor given by the i-th and j-th column by $m^{(iii)}_{i,j}$. Using the vanishing of

$$m^{(iii)}_{0,1},\ m^{(iii)}_{1,2},\ m^{(iii)}_{2,4},\ m^{(iii)}_{4,5},\ m^{(iii)}_{5,3},\ m^{(iii)}_{3,6} \text{ and } m^{(iii)}_{6,7}$$

in this order we find that

$$x_1 = x_2 = x_4 = x_5 = x_3 = x_6 = x_7 = 0.$$

More precisely, the vanishing of $m^{(iii)}_{0,1}$ implies $x_1 = 0$. Then, the vanishing of $m^{(iii)}_{1,2}$ yields $x_2 = 0$ and so on. We note that the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^7_k$ does not lie on $A'_{(iii)}$. Hence the previous argument shows that $A'_{(iii)}$ is smooth. The same argument shows the smoothness of $A'_{(viii)}$ and $A'_{(ix)}$ because we used the sixth and seventh row of the Jacobian only to conclude $x_6 = 0$ and $x_7 = 0$, respectively.

The Jacobian of $A'_{(x)}$ is given by

$$\begin{pmatrix} x_1^2 & 2x_0x_1+x_2^2 & 2x_1x_2+x_4^2 & 2x_3x_5+x_6^2 & 2x_2x_4+x_5^2 & 2x_4x_5+x_3^2 & 2x_3x_6 \\ 0 & 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 & -2x_6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The same approach as for $A'_{(iii)}$ shows that

$$x_1 = x_2 = x_4 = x_5 = 0.$$

Since the line

$$\{x_1 = x_2 = x_4 = x_5 = x_3 = 0\} \subset \mathbb{P}^6_k$$

does not intersect $A'_{(x)}$, it follows that $A'_{(x)}$ is smooth.

Consider $A'_{(iv)}$ and look at its Jacobian

$$\begin{pmatrix} x_1^2 & 2x_0x_1+x_2^2 & 2x_1x_2+x_4^2 & 2x_3x_5+x_6^2 & 2x_2x_4+x_5^2 & 2x_4x_5+x_3^2 & 2x_3x_6+x_7^2 & 2x_6x_7 \\ 0 & 2x_1x_2 & 2x_2x_4+x_1^2 & 2x_3x_6+x_5^2 & 2x_4x_5+x_2^2 & 2x_3x_5+x_4^2 & 2x_6x_7+x_3^2 & x_6^2 \end{pmatrix}.$$

We denote the determinant of the 2×2 minor involving the i-th and j-the column by $m^{(iv)}_{i,j}$. Then the equations

$$0 = m^{(iv)}_{0,1} = x_1^3x_2,$$
$$0 = m^{(iv)}_{0,2} = x_1^2x_2x_4 + x_1^4$$

imply $x_1 = 0$. Similarly, the vanishing of

$$\left(m^{(iv)}_{1,2}, m^{(iv)}_{1,4}\right), \ \left(m^{(iv)}_{2,4}, m^{(iv)}_{2,5}\right), \ \left(m^{(iv)}_{4,5}, m^{(iv)}_{4,3}\right), \ \left(m^{(iv)}_{5,3}, m^{(iv)}_{5,6}\right) \text{ and } \left(m^{(iv)}_{3,7}\right)$$

implies that

$$x_2 = x_4 = x_5 = x_3 = x_6 = 0.$$

Thus $A'_{(iv)}$ is smooth, because it does not intersect the line

$$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0\} \subset \mathbb{P}^7_k.$$

The smoothness of $A'_{(vii)}$ is shown by the same argument. Note that it is crucial that the polynomial $\hat{f}^{(3)}_3$ contains the monomial x_6^3.

It is left to prove that $A'_{(i)}$ is smooth. Its Jacobian reads

$$\begin{pmatrix} x_1^2 & 2x_0x_1+x_2^2 & 2x_1x_2+x_4^2 & 2x_3x_5+x_6^2 & 2x_2x_4+x_5^2 & 2x_4x_5+x_3^2 & 2x_3x_6 \\ 0 & 2x_1x_2 & 2x_2x_4+x_1^2 & 2x_3x_6+x_5^2 & 2x_4x_5+x_2^2 & 2x_3x_5+x_4^2 & x_3^2 \\ 0 & 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 & -2x_6 \end{pmatrix}. \tag{3.11}$$

We claim that any singular point satisfies

$$x_1 = x_2 = x_4 = x_5 = 0. \tag{3.12}$$

Assume that this claim holds. Then it follows immediately that $A'_{(i)}$ is smooth: Let $[x_0 : \cdots : x_6]$ be a singular point of $A'_{(i)}$. The vanishing of $\hat{f}^{(3)}_3$ and $f^{(3)}_2 - x_6^2$ reads

$$0 = x_3^2x_6 + x_6^3 = x_3^2 - x_6^2,$$

i.e. implies $x_3 = x_6 = 0$. Since $[1 : 0 \cdots : 0] \in \mathbb{P}^6_k$ is not contained in $A'_{(i)}$, we conclude the variety $A'_{(i)}$ is smooth if the claim that any singular point satisfies (3.12) holds. Thus it is left to show the claim. We will prove it by contradiction and assume that one of the four variables is non-zero. Let us order the four variables as follows

$$x_1, x_2, x_4, \text{ and } x_5. \tag{3.13}$$

Note that if the first k of these variables are zero, then the Jacobian looks similar to (3.11) in the sense that up to removing some zero columns and rearranging the columns it has the form

$$\begin{pmatrix} x^2 & \dots & \dots & \dots \\ 0 & 2xy & x^2 + 2yz & \dots \\ 0 & 2x & 2y & \dots \end{pmatrix}.$$

If $x \neq 0$, the Jacobian of $A'_{(i)}$ has not the full rank if and only if the 2×2 minors of the last two rows vanish, i.e. we obtain equation of the form

$$0 = 2xy(\dots) - 2x(\dots) \Longleftrightarrow 0 = y(\dots) - (\dots).$$

We use this observation to find a contradiction in each of the following four cases which we distinguish by the first non-zero variable in (3.13):

Case 1. Assume $x_1 \neq 0$: Using the approach we describe above, we obtain the following equations:

$$\begin{aligned} 0 &= 2x_2^2 - x_1^2 - 2x_2x_4, \\ 0 &= 2x_2x_4 - x_2^2 - 2x_4x_5, \\ 0 &= -2x_2x_6 - x_3^2, \\ 0 &= 2x_2x_3 - x_5^2 - 2x_3x_6, \\ 0 &= 2x_2x_5 - x_4^2 - 2x_3x_5. \end{aligned}$$

We see from the first equation that $x_2 \neq 0$ (otherwise $x_1 = 0$). Thus we can assume without loss of generality $x_2 = 1$. Rearranging these equations yields

$$x_1^2 = x_4 - 1 \tag{3.14}$$
$$x_4x_5 = x_4 + 1 \tag{3.15}$$
$$x_6 = x_3^2, \tag{3.16}$$
$$x_5^2 = x_3x_6 - x_3 = x_3^3 - x_3, \tag{3.17}$$
$$x_4^2 = x_5(x_3 - 1). \tag{3.18}$$

The vanishing of $\hat{f}_3^{(3)}$ yields

$$\begin{aligned} 0 &= x_1^2 + x_4 + x_4^2x_5 + x_5^2x_3 + x_3^2x_6 + x_6^3 \\ &\overset{(3.14)}{=} x_4 - 1 + x_4 + x_4^2x_5 + x_5^2x_3 + x_3^2x_6 + x_6^3 \\ &\overset{(3.15)}{=} 2x_4 - 1 + x_4(x_4 + 1) + x_5^2x_3 + x_3^2x_6 + x_6^3 \\ &\overset{(3.16)}{=} 2x_4 - 1 + x_4(x_4 + 1) + x_3^4 - x_3^2 + x_3^2x_6 + x_6^3 \\ &\overset{(3.17)}{=} x_4^2 - 1 + x_3^4 - x_3^2 + x_3^4 + x_3^6 \end{aligned}$$

$$= x_4^2 - 1 - x_3^2 - x_3^4 + x_3^6, \tag{3.19}$$

and also

$$\begin{aligned}
0 &= x_1^2 + x_4 + x_4^2 x_5 + x_5^2 x_3 + x_3^2 x_6 + x_6^3 \\
&\overset{(3.14)}{=} x_4 - 1 + x_4 + x_4^2 x_5 + x_5^2 x_3 + x_3^2 x_6 + x_6^3 \\
&\overset{(3.18)}{=} -x_4 - 1 + x_5^2(x_3 - 1) + x_5^2 x_3 + x_3^2 x_6 + x_6^3 \\
&\overset{(3.16)}{=} -x_4 - 1 + (x_3^3 - x_3)(-x_3 - 1) + x_3^2 x_6 + x_6^3 \\
&\overset{(3.17)}{=} -x_4 - 1 + (x_3^3 - x_3)(-x_3 - 1) + x_3^4 + x_3^6 \\
&= -x_4 - 1 + x_3 + x_3^2 - x_3^3 + x_3^6.
\end{aligned} \tag{3.20}$$

Plugging (3.20) in (3.19) yields

$$0 = (-1 + x_3 + x_3^2 - x_3^3 + x_3^6)^2 - 1 - x_3^2 - x_3^4 + x_3^6 = x_3^{12} + x_3^9 - x_3^8 - x_3^7 + x_3^5 + x_3^4 + x_3^3 + x_3^2 + x_3 =: p_1.$$

Moreover, squaring (3.18) and using (3.19) yields

$$0 = (1 + x_3^2 + x_3^4 - x_3^6)^2 - (x_3^3 - x_3)(x_3 - 1)^2 = x_3^{12} + x_3^{10} - x_3^8 - x_3^5 - x_3^4 + x_3 + 1 =: p_2.$$

If $A'_{(i)}$ has a singular point with $x_1 \neq 0$, then two polynomial p_1 and p_2 in $k[x_3]$ must have a non-trivial greatest common divisor. Thus, we find a contradiction by showing that the greatest common divisor of these two polynomials is a unit. Note that

$$p_1 - p_2 = -x_3^{10} + x_3^9 - x_3^7 - x_3^5 - x_3^4 + x_3^3 + x_3^2 - 1.$$

Thus, we compute the greatest common divisor of p_2 and $p_1 - p_2$:

$$\begin{aligned}
p_2 &= (-x_3^2 - x_3 + 1) \cdot (p_1 - p_2) + x_3^9 + x_3^8 + x_3^6 - x_3^4 + x_3^2 - 1, \\
p_1 - p_2 &= (-x_3 - 1) \cdot (x_3^9 + x_3^8 + x_3^6 - x_3^4 + x_3^2 - 1) + p_3, \\
x_3^9 + x_3^8 + x_3^6 - x_3^4 + x_3^2 - 1 &= (x_3 + 1) \cdot p_3 - x_3^7 - x_3^6 + x_3^5 - x_3^4 - x_3^3 + 1, \\
p_3 &= (-x_3 + 1) \cdot (-x_3^7 - x_3^6 + x_3^5 - x_3^4 - x_3^3 + 1) - x_3^5 + x_3^4 - x_3^2, \\
-x_3^7 - x_3^6 + x_3^5 - x_3^4 - x_3^3 + 1 &= (x_3^4 - x_3^3 + x_3^2 + x_3) \cdot (-x_3^3 + x_3^2 - 1) + x_3^2 + x_3 + 1, \\
-x_3^3 + x_3^2 - 1 &= (-x_3 - 1) \cdot (x_3^2 + x_3 + 1) - x_3, \\
x_3^2 + x_3 + 1 &= (x_3 + 1) \cdot x_3 + 1,
\end{aligned}$$

where $p_3 = x_3^8 + x_3^6 + x_3^5 + x_3^4 - x_3^3 - x_3^2 - x_3 + 1$.

Case 2. Assume $x_1 = 0$, $x_2 \neq 0$: By the same argument as in Case 1, we can assume without loss of generality $x_4 = 1$ and obtain the following equations:

$$x_2^2 = x_5 - 1, \tag{3.21}$$

$$x_3 x_5 = x_5 + 1, \tag{3.22}$$

$$x_6 = x_3^2, \tag{3.23}$$

$$x_5^2 = x_3(x_6 - 1) = x_3^3 - x_3. \tag{3.24}$$

Since $\hat{f}_3^{(3)}$ and $f_2^{(3)} - x_6^2$ vanish, we obtain the condition

$$0 = (x_2^2 + x_5 + x_5^2 x_3 + x_3^2 x_6 + x_6^3) + (x_2^2 + x_3^2 + 1 + x_5^2 - x_6^2)$$

$$
\begin{aligned}
&\overset{(3.21)}{=} 3x_5 - 2 + x_5^2(x_3+1) + x_3^2 x_6 + x_6^3 + x_3^2 + 1 - x_6^2 \\
&\overset{(3.23)}{=} -1 + x_5^2(x_3+1) + x_3^4 + x_3^6 + x_3^2 - x_3^4 \\
&\overset{(3.24)}{=} -1 + (x_3^3 - x_3)(x_3+1) + x_3^2 + x_3^4 + x_3^6 - x_3^4 \\
&= x_3^6 + x_3^4 + x_3^3 - x_3 - 1.
\end{aligned}
$$

On the other hand,

$$1 \overset{(3.22)}{=} x_5^2(x_3-1)^2 \overset{(3.24)}{=} (x_3^3 - x_3)(x_3-1)^2,$$

i.e.

$$0 = x_3^5 + x_3^4 - x_3^2 - x_3 - 1.$$

We check again that the greatest common divisors of the two polynomials is a unit and conclude that no singular point of $A'_{(i)}$ satisfies $x_1 = 0, x_2 \neq 0$:

$$
\begin{aligned}
x_3^6 + x_3^4 + x_3^3 - x_3 - 1 &= (x_3 - 1)\cdot(x_3^5 + x_3^4 - x_3^2 - x_3 - 1) - x_3^4 - x_3^3 - x_3 + 1, \\
x_3^5 + x_3^4 - x_3^2 - x_3 - 1 &= x_3 \cdot (x_3^4 + x_3^3 + x_3 - 1) + x_3^2 - 1, \\
x_3^4 + x_3^3 + x_3 - 1 &= (x_3^2 + x_3 + 1)\cdot(x_3^2 - 1) - x_3, \\
x_3^2 - 1 &= x_3 \cdot x_3 - 1.
\end{aligned}
$$

Case 3. Assume $x_1 = x_2 = 0,\ x_4 \neq 0$: By the same argument as in Case 1 and Case 2 we can assume without loss of generality $x_5 = 1$ and obtain the following equations:

$$
\begin{aligned}
&x_4^2 = x_3 - 1, \\
&x_6 = x_3^2, \\
&\ \ 0 = -x_3 x_6 + x_3 + 1 = -x_3^3 + x_3 + 1.
\end{aligned}
$$

Moreover, the vanishing of $\hat{f}_3^{(3)}$ implies

$$0 = x_4^2 + x_3 + x_3^2 x_6 + x_6^3 = x_3 - 1 + x_3 + x_3^4 + x_3^6 = x_3^6 + x_3^4 - x_3 - 1.$$

The greatest common divisor of these two polynomials is a unit:

$$
\begin{aligned}
x_3^6 + x_3^4 - x_3 - 1 &= (x_3^3 - x_3 + 1)(x_3^3 - x_3 - 1) - x_3^2 - x_3 \\
x_3^3 - x_3 - 1 &= (-x_3 + 1)(-x_3^2 - x_3) - 1.
\end{aligned}
$$

Hence, no singular point of $A'_{(i)}$ satisfies $x_1 = 0, x_2 = 0, x_4 \neq 0$.

Case 4. Assume $x_1 = x_2 = x_4 = 0,\ x_5 \neq 0$: As in the previous cases we can assume without loss of generality $x_3 = 1$. Moreover, any singular point of $A'_{(i)}$ satisfies

$$x_6 = 1,\ x_5^2 = x_6 - 1 = 0$$

which contradicts our assumption that $x_5 \neq 0$.

This concludes the proof of the claim that any singular point of $A'_{(i)}$ satisfies (3.12) and thus finishes the proof of the lemma. □

Recall that $k = \overline{k_0(u, v, w)}$ where k_0 is an algebraically closed field of characteristic 3. Let $R = k[[t]]$ be the formal power series in one variable over k and consider the R-scheme

$$\mathcal{X} := \left\{ c_{u,v,w}^{(3)} = t\hat{c}_{v,w}^{(3)} + x_7 q_{v,w}^{(3)} = 0 \right\} \subset \mathbb{P}_R^7$$

where the polynomials are defined as in Definition 3.11. The special fiber $\mathcal{X}_k$ of $\mathcal{X} \to \operatorname{Spec} R$ has the two irreducible components

$$Y_1 := \left\{ c^{(3)}_{u,v,w} = x_7 = 0 \right\}, \quad Y_2 := \left\{ c^{(3)}_{u,v,w} = q^{(3)}_{v,w} = 0 \right\} \subset \mathbb{P}^7_k.$$

Let $Z := Y_1 \cap Y_2$ denote their scheme-theoretic intersection. The model $\mathcal{X} \to \operatorname{Spec} R$ is again not strictly semi-stable because $\mathcal{X}$ is not smooth. More precisely, the singular locus of $\mathcal{X}$ is given by

$$S := \left\{ c^{(3)}_{u,v,w} = \hat{c}^{(3)}_{v,w} = q^{(3)}_{v,w} = x_7 = t = 0 \right\} \subset \mathcal{X}.$$

This can be seen by following the argument in Lemma 3.6 and replacing the $A_{(\dots)}$ from Lemma 3.3 with $A'_{(\dots)}$ from Lemma 3.13. Applying the same construction to $\mathcal{X} \to \operatorname{Spec} R$ as in characteristic different from 3, which was done in the previous section, we obtain a strictly semi-stable model.

Lemma 3.14. *The blow-up $\mathcal{X}' := \operatorname{Bl}_{Y_1} \mathcal{X}$ is a strictly semi-stable R-scheme with special fibre $\tilde{Y}_1 \cup Y_2$ where $\tilde{Y}_1 = \operatorname{Bl}_S Y_1$. Furthermore, consider the $2:1$ base change $\mathcal{X}'' := \mathcal{X}' \times_{\substack{R \to R \\ t \to t^2}} R$. Then, the blow-up*

$$\tilde{\mathcal{X}} := \operatorname{Bl}_Z \left(\mathcal{X}'' \right) \longrightarrow \operatorname{Spec} R \tag{3.25}$$

of $\mathcal{X}''$ in the subvariety Z is a strictly semi-stable R-scheme with special fibre $\tilde{Y}_1 \cup P_Z \cup Y_2$ where P_Z is a $\mathbb{P}^1_k$-bundle over Z. The intersections $\tilde{Y}_1 \cap P_Z$ and $Y_2 \cap P_Z$ are disjoint sections of $P_Z \to Z$ and the geometric generic fibre

$$\tilde{X}_{\overline{K}} := \left\{ c^{(3)}_{u,v,w} = t^2 \hat{c}^{(3)}_{v,w} + x_7 q^{(3)}_{v,w} = 0 \right\} \subset \mathbb{P}^7_{\overline{K}}$$

of $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ is a smooth $(3,3)$ complete intersection.

Remark 3.15. *We use the same letters for this strictly semi-stable model as for the model constructed before in characteristic different from 3 to simplify the statements in the next chapter. But we will repeatedly mention that in characteristic 3 the model looks different.*

The lemma is proved by adapting the proofs of Lemmata 3.4, 3.6, 3.8 and 3.9. Let us make this more precise: In Lemmata 3.4, 3.6, 3.8 and 3.9 we often degenerate to a variety $A_{(\dots)}$ from Lemma 3.3 to show smoothness. For the proof of Lemma 3.14 we instead degenerate to the variety $A'_{(\dots)}$ (with the same index) from Lemma 3.13. Moreover in the proof of Lemma 3.6 we used that

$$\{\hat{c}_{v,w} = 0\}, \ \{q_{v,w} = 0\} \subset \mathbb{P}^7_k$$

are smooth to describe the singularities of $\mathcal{X} \to \operatorname{Spec} R$ (in characteristic different from 2 and 3). In characteristic 3 the varieties

$$\{\hat{c}_{v,w} = 0\}, \ \{q^{(3)}_{v,w} = 0\} \subset \mathbb{P}^7_k$$

are singular. But as already mentioned in Remark 3.7 it suffices that the two latter varieties are smooth along S which is proved in the lemma below. Besides this two adjustments the arguments from Lemmata 3.4, 3.6, 3.8 and 3.9 can be copied to prove Lemma 3.14.

Lemma 3.16. *The varieties*

$$A'_{(xi)} := \left\{ q^{(3)}_{v,w} = 0 \right\} \subset \mathbb{P}^7_k,$$
$$A'_{(xii)} := \left\{ \hat{c}^{(3)}_{v,w} = 0 \right\} \subset \mathbb{P}^7_k$$

are smooth along $S = \left\{ x_7 = q^{(3)}_{v,w} = \hat{c}^{(3)}_{v,w} = c^{(3)}_{u,v,w} = 0 \right\} \subset \mathbb{P}^7_k$.

Proof. The Jacobian of $A'_{(xi)}$ reads (up to some factors of the form $2(v+w)$)

$$\begin{pmatrix} 0 & x_1 & x_2 & 2(v+w)x_3 + vx_7 + x_6 & x_4 - \frac{1}{2(v+w)}x_5 & x_5 - \frac{1}{2(v+w)}x_4 & -x_6 + \frac{1}{2(v+w)}x_3 & vx_3 + 2wx_7 \end{pmatrix}.$$

Using that $S \subset \{x_7 = 0\}$, we immediately see that the only singular point of $A'_{(xi)}$ which might be contained in S is $[1:0:\cdots:0]$. Since $c^{(3)}_{u,v,w}$ does not vanish at this point, we find that $A'_{(xi)}$ is smooth along S. The Jacobian of $A'_{(xii)}$ is (up to some factors of the form $(v+w)$) given by

$$\begin{pmatrix} 0 & 2x_1x_2 & 2x_2x_4 + x_1^2 & 2x_3x_6 + x_5^2 & 2x_4x_5 + x_2^2 & 2x_3x_5 + x_4^2 & x_3^2 + 2wx_6x_7 & wx_6^2 \end{pmatrix}.$$

Thus $A'_{(xii)}$ is smooth away from the line $l = \{x_1 = x_2 = \cdots = x_6 = 0\} \subset \mathbb{P}^7_k$ which does not intersects S, i.e. $A'_{(xii)}$ is smooth along S. $\square$

Very general (3,3) fivefolds are irrational

In this section we prove that the geometric generic fibre $\tilde{X}_{\overline{K}}$ of the strictly semi-stable family $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ does not admit a decomposition of the diagonal which implies Theorem 1.1. Recall that we defined the strictly semi-stable family $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ in characteristic 3 (see Lemma 3.14) slightly different from the one in characteristic $\operatorname{char} k \neq 3$ (see Lemma 3.9).

To this end let $A := \mathcal{O}_{\tilde{\mathcal{X}},\eta_{P_Z}}$ be the local ring of $\tilde{\mathcal{X}}$ at the generic point η_{P_Z} of P_Z with residue field $\kappa(P_Z)$. Then $\tilde{\mathcal{X}}_A \to \operatorname{Spec} A$ is strictly semi-stable and we obtain a homomorphism

$$\Phi_{\tilde{\mathcal{X}}_A,P_Z} \colon \mathrm{CH}_1(\tilde{X}_k \times \kappa(P_Z)) \longrightarrow \mathrm{CH}_0(P_Z \times \kappa(P_Z)),$$

where $\tilde{\mathcal{X}}_k$ denotes the special fiber of the strictly semi-stable model $\tilde{\mathcal{X}} \to \operatorname{Spec} R$. The main result of this section is the following proposition which implies that $\tilde{X}_{\overline{K}}$ does not admit a decomposition of the diagonal by Theorem 2.14.

Proposition 4.1. *Let $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ be the strictly semi-stable model with special fibre $\tilde{Y}_1 \cup P_Z \cup Y_2$ from* (3.10)*, or from* (3.25) *if $\operatorname{char} k = 3$. Let $A = \mathcal{O}_{\tilde{\mathcal{X}},P_Z}$ be the local ring at the generic point of P_Z with residue field $\kappa(P_Z)$. Then for any zero-cycle $z \in \mathrm{CH}_0(P_Z)$, the element*

$$\delta_{P_Z} - z_{\kappa(P_Z)} \in \mathrm{CH}_0(P_Z \times \kappa(P_Z))$$

does not lie in the image of $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ modulo 2, where δ_{P_Z} denotes the diagonal point of $P_Z \times \kappa(P_Z)$.

Let us lay down the strategy of the proof first: We will assume that the element is contained in the image of $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ modulo 2. Similar to [PS21] the strategy is to simplify the contribution from $\mathrm{CH}_1(\tilde{\mathcal{X}}_k \times \kappa(P_Z))$ by repeatedly applying Fulton's specialization map. A new ingredient is the idea of degenerations to cones which have trivial Chow groups of zero-cycles. We will arrive at the conclusion that the diagonal point $\delta_{Z_0} \in \mathrm{CH}_0(Z_0 \times \kappa(Z_0))$ satisfies

$$\delta_{Z_0} \in \operatorname{Im}\left(\mathrm{CH}_0(Z_0) \longrightarrow \mathrm{CH}_0(Z_0 \times \kappa(Z_0))\right) \mod 2.$$

This contradicts some property of the non-trivial unramified cohomology with $\mathbb{Z}/2$-coefficients on Z_0 which is also studied in [Ska22].

Before we give the details of the proof we start with a couple of observations regarding certain Chow groups and an cohomological obstruction which gives the contradiction in the proof of Proposition 4.1. These are laid down in the following two section. In the last section we give the actual argument as well as some consequences of Proposition 4.1.

4.1 Chow groups of certain varieties

We describe some Chow groups of certain varieties which we consider later in the proof of Proposition 4.1. The following lemma works quite generally and describes the Chow group of one-cycles for a variety which is the intersection of a cone with a hypersurface that intersects the vertex of the cone with high multiplicity.

Lemma 4.2. *Let $W := \{F_1 = \cdots = F_r = 0\} \subset \mathbb{P}^n_\kappa$ be a smooth variety over some field κ where $F_1, \ldots, F_r \in \kappa[x_0, \ldots, x_n]$ are some homogeneous polynomials and $r \geq 0$. Let*

$$V := \{F_1 = \cdots = F_r = g_1 x_{n+1} + g_0 = 0\} \subset \mathbb{P}^{n+1}_\kappa$$

where $g_i \in k[x_0, \ldots, x_n]$ are homogeneous polynomials of degree $d - i$. Assume further that

$$D := \{F_1 = \cdots = F_r = g_1 = g_0 = 0\} \subset \mathbb{P}^n_\kappa$$

is smooth. Then for any field extension κ'/κ there is a surjective homomorphism

$$\mathrm{CH}_0(D \times_\kappa \kappa') \oplus \mathrm{CH}_1(W \times_\kappa \kappa') \twoheadrightarrow \mathrm{CH}_1(V \times_\kappa \kappa').$$

Remark 4.3. *Our assumption on D and W implies that V is smooth away from Q.*

Proof. We first note that V is the intersection of the cone C_W over $W \subset \mathbb{P}^n_\kappa \cong \{x_{n+1} = 0\} \subset \mathbb{P}^{n+1}_\kappa$ with vertex $Q := [0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}_\kappa$ and the degree d hypersurface $H := \{g_1 x_{n+1} + g_0 = 0\} \subset \mathbb{P}^{n+1}_\kappa$. As C_W is a cone with vertex Q, the projection from Q induces a rational map

$$\varphi \colon C_W \dashrightarrow W.$$

Away from the point Q, φ is a morphism $C_W \setminus Q \to W$ whose fiber over some κ-rational point $P \in W$ is the line l_P through Q and $P \in W$ where we view W as a subvariety of the hyperplane $\{x_{n+1} = 0\} \subset \mathbb{P}^{n+1}_\kappa$. The hypersurface H intersects the line l_P at the point Q with multiplicity $d - 1$. Hence, there is a unique other point on that line in the intersection $V = C_W \cap H$ which is mapped to P under φ. Thus, $\varphi|_V$ yields a birational map

$$V \dashrightarrow W.$$

This map is resolved by the isomorphism

$$\mathrm{Bl}_Q V \xrightarrow{\cong} \mathrm{Bl}_D W$$

which can be checked by an explicit computation: We consider the blow-up of H at the point Q because $\mathrm{Bl}_Q V$ is the strict transform of V under the blow-up $\mathrm{Bl}_Q H \to H$. Recall that the blow-up $\mathrm{Bl}_Q H$ is given by the closure of the graph of the rational map $[x_0 : \cdots : x_{n+1}] \mapsto [x_0 : \cdots : x_n]$. Let $x_0, \ldots, x_{n+1}$ and $y_0, \ldots, y_n$ denote the coordinates of $\mathbb{P}^{n+1}_\kappa \times \mathbb{P}^n_\kappa$. The blow-up $\mathrm{Bl}_Q H$ is given in the chart $U'_i := \{y_i \neq 0\} \subset \mathbb{P}^{n+1}_\kappa \times \mathbb{P}^n_\kappa$ by

$$\mathrm{Bl}_Q H|_{U'_i} = \left\{ \begin{matrix} x_{n+1} g_1\left(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}\right) + x_i g_0\left(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}\right) = 0, \\ x_j = x_i \frac{y_j}{y_i} \quad \text{for } j \in \{0, \ldots, n\} \setminus \{i\} \end{matrix} \right\} \subset \mathbb{P}^{n+1}_\kappa \times \mathbb{A}^n_\kappa.$$

The equations $x_j = x_i \frac{y_j}{y_i}$ describe hyperplanes in the first factor, i.e. we can also describe the blow-up as

$$\mathrm{Bl}_Q H|_{U'_i} \cong \left\{ x_{n+1} g_1\left(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}\right) + x_i g_0\left(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}\right) = 0 \right\} \subset \mathbb{P}^1_\kappa \times \mathbb{A}^n_\kappa$$

where x_i, x_{n+1} and $\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}$ are the coordinates of $\mathbb{P}^1_\kappa$ and $\mathbb{A}^n_\kappa$, respectively. Hence,

$$\mathrm{Bl}_Q V|_{U_i'} = \left\{ \begin{matrix} F_1\left(\frac{y_0}{y_i},\ldots,\frac{y_n}{y_i}\right)=\cdots=F_r\left(\frac{y_0}{y_i},\ldots,\frac{y_n}{y_i}\right)=0, \\ x_{n+1}g_1\left(\frac{y_0}{y_i},\ldots,\frac{y_n}{y_i}\right)+x_i g_0\left(\frac{y_0}{y_i},\ldots,\frac{y_n}{y_i}\right)=0 \end{matrix} \right\} \subset \mathbb{P}^1_\kappa \times \mathbb{A}^n_\kappa. \tag{4.1}$$

Similarly, the blow-up of W in D is the strict transform of the blow-up of $\mathbb{P}^n_\kappa$ in $\{g_0 = g_1 = 0\} \subset \mathbb{P}^n_\kappa$. Hence, in the affine charts $U_l := \{x_l \neq 0\}$:

$$\mathrm{Bl}_{D\cap U_l}(W \cap U_l) = \left\{ \begin{matrix} F_1\left(\frac{x_0}{x_l},\ldots,\frac{x_n}{x_l}\right)=\cdots=F_r\left(\frac{x_0}{x_l},\ldots,\frac{x_n}{x_l}\right)=0, \\ sg_1\left(\frac{x_0}{x_l},\ldots,\frac{x_n}{x_l}\right)-tg_0\left(\frac{x_0}{x_l},\ldots,\frac{x_n}{x_l}\right)=0 \end{matrix} \right\} \subset \mathbb{A}^n_\kappa \times \mathbb{P}^1_\kappa \tag{4.2}$$

where $\frac{x_0}{x_l}, \ldots, \frac{x_n}{x_l}$ and s, t are the coordinates of $\mathbb{A}^n_\kappa$ and $\mathbb{P}^1_\kappa$, respectively. Moreover, the birational map $\varphi|_V$ extends to a morphism

$$\mathrm{Bl}_Q V \longrightarrow \mathrm{Bl}_D W.$$

The morphism is given in the charts which are described in (4.1) and (4.2) for $i = l \in \{0, \ldots, n\}$ by

$$\left([x_i : x_{n+1}], \left(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}\right)\right) \mapsto \left(\left(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}\right), [x_i : -x_{n+1}]\right).$$

Thus, we see that $\mathrm{Bl}_Q V \to \mathrm{Bl}_D W$ is a birational morphism which is locally an isomorphism, i.e. it is an isomorphism which extends $\varphi|_V$.

Hence, there is an isomorphism on the level of Chow groups

$$\mathrm{CH}_1(\mathrm{Bl}_D W) \xrightarrow{\cong} \mathrm{CH}_1(\mathrm{Bl}_Q V).$$

Since W and D are smooth, we can apply the blow-up formula for Chow groups (see e.g. [Voi03, Theorem 9.27]), i.e. there exists an isomorphism

$$\mathrm{CH}_0(D) \oplus \mathrm{CH}_1(W) \cong \mathrm{CH}_1(\mathrm{Bl}_D W).$$

Consider the natural pushforward

$$\mathrm{CH}_1(\mathrm{Bl}_Q V) \longrightarrow \mathrm{CH}_1(V)$$

of the proper morphism $\pi\colon \mathrm{Bl}_Q V \to V$. Since π is an isomorphism away from Q, which is a (singular) point, the pushforward on the Chow groups of one-cycles is surjective. Indeed the strict transform of a one-cycle on V is mapped to that one-cycle under the pushforward map.

As blow-ups commute with extension of the base field, the above construction also holds after some base extension. Thus, we obtain a surjective homomorphism

$$\mathrm{CH}_0(D \times_\kappa \kappa') \oplus \mathrm{CH}_1(W \times_\kappa \kappa') \twoheadrightarrow \mathrm{CH}_1(V \times_\kappa \kappa').$$

This conclude the proof of the lemma. □

Next we observe that cones in $\mathbb{P}^n_k$ with a k-rational point as vertex have universally trivial Chow group of zero-cycles.

Definition 4.4. *Let $V = \{F_1 = \cdots = F_r = 0\} \subset \mathbb{P}^n_k$ be a variety with a k-rational point P where $F_1, \ldots, F_r \in k[x_0, \ldots, x_n]$ are some homogeneous polynomials. We say that V is a cone over P if after a suitable coordinate transformation $x_0, \ldots, x_n \mapsto y_0, \ldots, y_n$ the point P is the point $[0 : \cdots : 0 : 1]$ and the homogeneous polynomials $F_1, \ldots, F_r$ are contained in $k[y_0, \ldots, y_{n-1}]$.*

Lemma 4.5. *Let $V \subset \mathbb{P}^n_k$ be a cone with a k-rational point P as vertex. Then V has universally trivial Chow group of zero-cycles.*

Proof. We need to show that for any field extension F/k the degree map $\deg\colon \mathrm{CH}_0(V_F) \to \mathbb{Z}$ is an isomorphism. Since P is k-rational point, it suffices to show that any closed point $Q \in V_F$ is rationally equivalent to $\deg(Q) \cdot P_F$.

Let $Q \in V_F$ be a closed point and let $r := \deg(Q) = [F(Q) : F]$. Consider the base change of V_F to the algebraic closure $\overline{F}$ of F. Then there are exactly r closed point $Q_1, \dots, Q_r \in V_{\overline{F}}$ which map to $Q \in V_F$ under the natural map $V_{\overline{F}} \to V_F$. Since V is cone over P, the base-change $V_{\overline{F}}$ is also a cone with vertex $P_{\overline{F}}$. Hence, there exist (unique) lines $l_1, \dots, l_r$ such that l_i passes through $P_{\overline{F}}$ and Q_i. Let $L = l_1 \cup \dots \cup l_r$ be the union of these lines. As $P_{\overline{F}}$ is the base-change of the F-rational point P_F and the points Q_i are mapped to Q under the map $V_{\overline{F}} \to V_F$, we see that the point $P_{\overline{F}}$ and the union of points $Q_1 \cup \dots \cup Q_r$ are invariant under the action of $\mathrm{Gal}(\overline{F}/F)$. Thus, L is invariant under the action of $\mathrm{Gal}(\overline{F}/F)$. In particular we find that L is the base change of some subvariety $L' \subset \mathbb{P}^n_F$.

Since L_i is a line for every $i \in \{1, 2, \dots, r\}$, there exist some $f_i \in \overline{F}(L_i)$ such that

$$P_{\overline{F}} - Q_i = \mathrm{div}(f_i).$$

In particular

$$rP_{\overline{F}} - \sum_{i=1}^{r} Q_i = \mathrm{div}(f)$$

where $f := \prod_{i=1}^{r} f_i \in \overline{F}(L)$. Since L is invariant under $\mathrm{Gal}(\overline{F}/F)$, the rational function f is contained in $F(L')$, i.e.

$$rP_F - Q = \mathrm{div}\,(f)\,.$$

Hence we showed that any closed point $Q \in V_F$ is rational equivalent to a multiple of P_F for any field extension F/k. Thus, V has universally trivial Chow group of zero-cycles. □

Using this observation we prove that certain algebraic schemes have universally trivial Chow group of zero-cycles. The considered algebraic schemes will come up in the proof of Proposition 4.1 when using Lemma 4.2.

Lemma 4.6. *Let k_0 be an algebraically closed field of characteristic different from 2. We define the following reduced algebraic schemes*

$$\begin{aligned}
D_1 &:= \{x_2^2 - 2x_3^2 = x_0^2 x_5 + x_1^2 x_4 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5)) = 0\} \subset \mathbb{P}^5_{k_0},\\
D_2 &:= \{c_0 = x_3 = x_4 x_5 = 0\} \subset \mathbb{P}^6_{k_0},\\
S_0^{red} &:= \{c_0 = x_6 = x_4 x_5 = 0\} \subset \mathbb{P}^6_{k_0},
\end{aligned}$$

where $c_0 \in k_0[x_0, \dots, x_6]$ is defined as in Definition 3.1 or Definition 3.11. Then $D_1,\ D_2,$ and S_0^{red} have universally trivial Chow group of zero-cycles.

Note that the polynomials $c_0 \in k_0[x_0, \dots, x_6]$ in Definition 3.1 and Definition 3.11 are the same, i.e. a distinction between characteristic 3 and characteristic $\mathrm{char}\, k_0 \neq 3$ is not necessary.

Proof. As k_0 is algebraically closed, there exists a square root $\sqrt{-1}$ of -1. The transformation

$$x_5 \mapsto \sqrt{-1} y_5, \quad x_4 \mapsto y_4 + y_5, \quad x_i \mapsto y_i \text{ for } i \in \{0, 1, 2, 3\}$$

induces an isomorphism $D_1 \cong D_1'$ where D_1' is the algebraic subscheme of $\mathbb{P}^5_{k_0}$ given by

$$\left\{\sqrt{-1}y_0^2y_5 + y_1^2y_4 + y_1^2y_5 + y_3(y_3^2 + y_4^2 + 2y_4y_5 - 2y_3(y_4 + y_5 + \sqrt{-1}y_5)) = y_2^2 - 2y_3^2 = 0\right\}.$$

The projection from the point $P = [0 : \cdots : 0 : 1] \in \mathbb{P}^5_{k_0}$ induces a birational map

$$D_1' \overset{\sim}{\dashrightarrow} D_1'' := \{y_2^2 - 2y_3^2 = 0\} \subset \mathbb{P}^4_{k_0}.$$

Let $W := \left\{\sqrt{-1}y_0^2 + y_1^2 + 2y_3y_4 - 2(1+\sqrt{-1})y_3^2 = 0\right\} \subset D_1'$ and let $U := D_1' \setminus W$ be the complement. Let $z \in Z_0(D_1')$ be any zero-cycle. Obviously, we can write

$$z = z_1 + z_2 \in Z_0(D_1')$$

for some $z_1 \in Z_0(W)$ and $z_2 \in Z_0(U)$. Note that W is a cone over the k_0-rational point $P = [0 : \cdots : 0 : 1] \in \mathbb{P}^5_{k_0}$. Thus W has universally trivial Chow group of zero-cycles by Lemma 4.5, in particular $z_1 = k \cdot P \in CH_0(W)$ for some $k \in \mathbb{Z}$. Similarly, D_1'' is a cone over the k_0-rational point $[1 : 0 : 0 : 0 : 0] \in \mathbb{P}^4_{k_0}$ and thus has universally trivial Chow group of zero-cycles by Lemma 4.5. Since U is isomorphic to an open subvariety of D_1'', U has also universally trivial Chow group of zero-cycles, by the localization exact sequence. Thus we can write

$$z_2 = l \cdot P' \in CH_0(U)$$

for some $l \in \mathbb{Z}$ where $P' = [1 : 0 : \cdots : 0] \in \mathbb{P}^5_{k_0}$. As the lines

$$[s : t] \mapsto [s : 0 : 0 : 0 : t : 0], \quad [u : v] \mapsto [0 : 0 : 0 : 0 : u : v] \tag{4.3}$$

are contained in D_1', the k_0-rational points P and P' are linearly equivalent in D_1' and the previous discussion shows that

$$z = (k + l) \cdot P \in \mathrm{CH}_0(D_1').$$

Since the lines in (4.3) are defined over k_0, the argument works also after some field extension. Thus, D_1' and therefore also $D_1 \cong D_1'$ have universally trivial Chow groups of zero-cycles.

Consider next

$$D_2 = \{x_3 = x_4x_5 = x_0^2x_5 + x_1^2x_4 + x_2^2x_6\} = D_2^1 \cup D_2^2$$

where $D_2^1 = \{x_3 = x_5 = x_1^2x_4 + x_2^2x_6\}$, $D_2^2 = \{x_3 = x_4 = x_0^2x_5 + x_2^2x_6\} \subset \mathbb{P}^6_{k_0}$ are two subvarieties which are isomorphic to each other. The varieties $D_2^1,\ D_2^2 \subset \mathbb{P}^6_{k_0}$, and their intersection

$$D_2^1 \cap D_2^2 = \{x_3 = x_4 = x_5 = x_2^2x_6\} \subset \mathbb{P}^6_{k_0}$$

are cones with a k_0-rational point as vertex. Thus, we see from Lemma 4.5 and the Mayer-Vietoris exact sequence for Chow groups [Ful98, Example 1.3.1 (c)]

$$\mathrm{CH}_0(D_2^1 \cap D_2^2) \longrightarrow \mathrm{CH}_0(D_2^1) \oplus \mathrm{CH}_0(D_2^2) \longrightarrow \mathrm{CH}_0(D_2) \longrightarrow 0$$

that D_2 has universally trivial Chow group of zero-cycles.

Lastly, $S_0^{\mathrm{red}} = \{x_0^2x_5 + x_1^2x_4 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5)) = x_4x_5 = 0\} \subset \mathbb{P}^5_{k_0}$ is a cone over the k_0-rational point $[0 : 0 : 1 : 0 : 0 : 0]$ and thus has universally trivial Chow group of zero-cycles by Lemma 4.5. $\square$

4.2 A cohomological obstruction

Proposition 4.7. *Let k_0 be an algebraically closed field of characteristic different from 2 and let*

$$Z_0 = \{x_0^2x_5 + x_1^2x_4 + x_2^2x_6 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5 + x_6)) = x_3x_6 - x_4x_5 = 0\} \subset \mathbb{P}^6_{k_0}.$$

Then the class $\delta_{Z_0} \in \mathrm{CH}_0(Z_{0,k_0(Z_0)})$ is non-zero in the quotient

$$0 \neq [\delta_{Z_0}] \in \frac{\mathrm{CH}_0\left(Z_{0,k_0(Z_0)}\right)/2}{\mathrm{CH}_0(Z_0)/2}.$$

The proof of this proposition follows mainly the argument by Skauli [Ska22], which uses ideas of Schreieder in [Sch19b]. The main idea is to compute the Merkurjev pairing of δ_{Z_0} with some non-zero unramified class α. This pairing will give a non-zero element in some étale cohomology. On the other hand, if $[\delta_{Z_0}] = 0$ in the quotient, then it turns out that the pairing has to be also zero which gives us a contradiction. This approach has several technical difficulties, mostly caused by the the singularities of Z_0. These issues have been resolved in [Sch19b].

We recall the computations from [Ska22]. Let Z_0 be defined as in Proposition 4.7 and let

$$Q := \{x_3x_6 - x_4x_5 = 0\} \subset \mathbb{P}^6_{k_0}$$

be the cone over $\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$ embedded in $\mathbb{P}^3_{k_0} \cong \{x_0 = x_1 = x_2 = 0\} \subset \mathbb{P}^6_{k_0}$ with vertex plane $\{x_3 = x_4 = x_5 = x_6 = 0\} \subset \mathbb{P}^6_{k_0}$. Recall that the Segre embedding of $\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$ is given by

$$\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0} \longrightarrow \mathbb{P}^3_{k_0}, \quad ([z_0 : z_1], [z_2 : z_3]) \mapsto [z_0z_2 : z_0z_3 : z_1z_2 : z_1z_3].$$

Thus, we get a rational map

$$\varphi\colon Q \dashrightarrow \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}, \quad [x_0 : \cdots : x_6] \mapsto ([x_3 : x_5], [x_3 : x_4]).$$

Lemma 4.8. *The rational map φ induces a morphism*

$$Q' := \mathrm{Bl}_D Q \longrightarrow \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$$

where $D := \{x_3 = x_4 = x_5 = x_6 = 0\} \subset Q$ is the vertex plane of the cone Q over $\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$. Restricting this morphism to the strict transform of Z_0 via the morphism $\mathrm{Bl}_D Q \to Q$ yields a surjective morphism

$$f_0\colon Z_0' := \mathrm{Bl}_{D\cap Z_0} Z_0 \longrightarrow \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}.$$

Moreover, the generic fiber of f_0 is smooth and the singular locus does not dominate $\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$.

Proof. Define

$$Q' := \mathrm{Bl}_D Q := \{y_3y_6 - y_4y_5 = x_iy_j - x_jy_i = 0, \quad \text{for } i, j \in \{3, 4, 5, 6\}\} \subset \mathbb{P}^6_{k_0} \times \mathbb{P}^3_{k_0},$$

where $x_0, x_1, \dots, x_6$ are the coordinates of $\mathbb{P}^6_{k_0}$ and $y_3, \dots, y_6$ are the coordinates of $\mathbb{P}^3_{k_0}$. Then it is immediate to check that the projection onto the first factor

$$\mathrm{pr}_1\colon \mathrm{Bl}_D Q \longrightarrow Q$$

is the blow-up of Q in D. Moreover, the projection onto the second factor

$$\mathrm{pr}_2\colon Q' \longrightarrow \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$$

is induced by φ. The strict transform Z_0' of Z_0 is given in the chart $\{y_i \neq 0\} \subset Q'$ ($i \in \{3,4,5,6\}$) by

$$\left\{x_0^2\frac{y_5}{y_i} + x_1^2\frac{y_4}{y_i} + x_2^2\frac{y_6}{y_i} + x_i^2\left(\left(\frac{y_3}{y_i}\right)^2 + \left(\frac{y_4}{y_i}\right)^2 + \left(\frac{y_5}{y_i}\right)^2 - 2\frac{y_3}{y_i}\left(\frac{y_4}{y_i} + \frac{y_5}{y_i} + \frac{y_6}{y_i}\right)\right) = 0\right\}, \tag{4.4}$$

and the exceptional divisor is given by

$$E := \left\{x_0^2\frac{y_5}{y_i} + x_1^2\frac{y_4}{y_i} + x_2^2\frac{y_6}{y_i} = x_i = 0\right\}. \tag{4.5}$$

The projection onto the second factor pr_2 induces a morphism

$$f_0 \colon Z_0' \longrightarrow \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}.$$

Using (4.4), the variety Z_0' is given in the chart $\{y_3 \neq 0\}$ by

$$\left\{x_0^2\frac{y_5}{y_3} + x_1^2\frac{y_4}{y_3} + x_2^2\frac{y_4y_5}{y_3^2} + x_3^2\left(1 + \left(\frac{y_4}{y_3}\right)^2 + \left(\frac{y_5}{y_3}\right)^2 - 2\left(\frac{y_4}{y_3} + \frac{y_5}{y_3} + \frac{y_4y_5}{y_3^2}\right)\right) = 0\right\} \subset \mathbb{P}^3_{k_0}\times\mathbb{A}^2_{k_0}, \tag{4.6}$$

and the restriction of the morphism f_0 to that chart is

$$Z_0' \longrightarrow \mathbb{A}^1_{k_0} \times \mathbb{A}^1_{k_0} \subset \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}, \quad \left([x_0 : x_1 : x_2 : x_3], \left(\frac{y_4}{y_3}, \frac{y_5}{y_3}\right)\right) \mapsto \left(\left[1 : \frac{y_5}{y_3}\right], \left[1 : \frac{y_4}{y_3}\right]\right). \tag{4.7}$$

Similarly, the variety Z_0' restricted to the chart $\{y_4 \neq 0\}$ is isomorphic to

$$\left\{x_0^2\frac{y_3y_6}{y_4^2} + x_1^2 + x_2^2\frac{y_6}{y_4} + x_4^2\left(\left(\frac{y_3}{y_4}\right)^2 + 1 + \left(\frac{y_3y_6}{y_4^2}\right)^2 - 2\frac{y_3}{y_4}\left(1 + \frac{y_3y_6}{y_4^2} + \frac{y_6}{y_4}\right)\right) = 0\right\} \subset \mathbb{P}^3_{k_0}\times\mathbb{A}^2_{k_0},$$

and the morphism f_0 is in that chart given by

$$Z_0' \longrightarrow \mathbb{A}^1_{k_0} \times \mathbb{A}^1_{k_0} \subset \mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}, \quad \left([x_0 : x_1 : x_2 : x_4], \left(\frac{y_3}{y_4}, \frac{y_6}{y_4}\right)\right) \mapsto \left(\left[\frac{y_6}{y_4} : 1\right], \left[\frac{y_3}{y_4} : 1\right]\right). \tag{4.8}$$

Hence, we directly see that f_0 is surjective and the generic fiber of f_0 is smooth because it is a Fermat quadric over the function field of the generic point of $\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$. The statement about the singular locus follows from [Ska22, Remark 3.6]. For the convenience of the reader we give an alternative proof: We claim that every singular point of Z_0' satisfies

$$y_3y_6 = y_4y_5 = 0.$$

Then, by the local description of f_0 in (4.7) and (4.8) the image of the singular locus under the morphism f_0 is contained in the union of the coordinate axis, i.e. a proper closed subset of $\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}$.

To prove the claim it clearly suffices to show that the singular locus of Z_0' in the chart $\{y_3 \neq 0\}$ is contained in $y_4y_5 = 0$ because we can assume $y_3 = 0$ outside of this chart and then $y_4y_5 = 0$ follows from the definition of Q'. Assume without loss of generality $y_3 = 1$. Then (4.6) reads

$$Z_0' \cong \left\{x_0^2y_5 + x_1^2y_4 + x_2^2y_4y_5 + x_3^2\left(1 + y_4^2 + y_5^2 - 2\left(y_4 + y_5 + y_4y_5\right)\right) = 0\right\} \subset \mathbb{P}^3_{k_0} \times \mathbb{A}^2_{k_0}.$$

For any singular point the Jacobian

$$\left(2x_0y_5 \quad 2x_1y_4 \quad 2x_2y_4y_5 \quad 2x_3\,(\dots) \quad x_1^2+y_5x_2^2+x_3^2(2y_4-2y_5-2) \quad x_0^2+y_4x_2^2+x_3^2(2y_5-2y_4-2)\right)$$

has to vanish. Assume that $x_0=x_1=x_2=0$ and thus $x_3\neq 0$ because $x_0,\ x_1,\ x_2$, and x_3 are projective coordinates. Then the vanishing of the last two components of the Jacobian yields a contradiction

$$0=(2y_4-2y_5-2)+(2y_5-2y_4-2)=-4\neq 0.$$

Hence, one of the coordinates $x_0,\ x_1$, and x_2 is non-zero which immediately implies $y_4y_5=0$. $\square$

The non-zero unramified cohomology class comes from a quadric surface bundle studied by Hassett, Pirutka and Tschinkel [HPT18, Example 8]:

Proposition 4.9. *The variety $Z_0'\subset \mathrm{Bl}_D Q$ is birational to the following quadric surface bundle*

$$\mathcal{Q}:=\{yzs^2+xzt^2+xyu^2+(x^2+y^2+z^2-2(xy+xz+yz))v^2=0\}\subset \mathbb{P}^2_{k_0}\times\mathbb{P}^3_{k_0}.$$

Hence, there exists a non-trivial class

$$0\neq\alpha=f_0^*\left(\frac{z_1}{z_0},\frac{y_1}{y_0}\right)\in H^2_{nr}\left({}^{k_0(Z_0')}/_{k_0},\mathbb{Z}/2\right).$$

Proof. The ambient spaces $\mathrm{Bl}_D Q$ and $\mathbb{P}^2_{k_0}\times\mathbb{P}^3_{k_0}$ are birational, e.g.

$$\mathbb{P}^2_{k_0}\times\mathbb{P}^3_{k_0}\dashrightarrow \mathrm{Bl}_D Q,[x:y:z],[s:t:u:v]\mapsto\left[s:t:u:v:v\frac{x}{z}:v\frac{y}{z}:v\frac{xy}{z}\right],\left[z:x:y:\frac{xy}{z}\right]$$

defines a birational map. Moreover, the birational map is an isomorphism in the charts $z=1$ and $y_3=1$. Setting now $z=1$ in the defining equation of $\mathcal{Q}$ and comparing the result with (4.6), we immediately see that $\mathcal{Q}$ and Z_0' are birational. Thus the unramified cohomology groups are isomorphic. Schreieder observed in [Sch19b, Proposition 9.6] that the class

$$\mathrm{pr}_1^*\left(\frac{x}{z},\frac{y}{z}\right)\neq 0\in H^2_{\mathrm{nr}}\left(k_0(\mathcal{Q})/k_0,\mathbb{Z}/2)\right)$$

is non-trivial which was proven in [HPT18, Proposition 11]. Following the birational maps we thus see that

$$\alpha=f^*\left(\frac{z_1}{z_0},\frac{y_1}{y_0}\right)\neq 0\in H^2_{\mathrm{nr}}\left(k_0(Z_0)/k_0,\mathbb{Z}/2\right).$$

This finishes the proof of the proposition. $\square$

The explicit description of the unramified cohomology class α allows us to obtain the following vanishing result.

Lemma 4.10. *Let $E\subset Z_0'$ be the exceptional divisor and let α be the unramified class from Proposition 4.9. Let F/k_0 be some finitely generated field extension. Then for any closed point e in the smooth locus of E_F*

$$\alpha_F|_e=0\in H^2(\kappa(e),\mathbb{Z}/2).$$

Proof. Let $\eta_{E_F}\in E_F\subset Z'_{0,F}$ denote the generic point of E_F. Recall that $\alpha=f_0^*\left(\frac{z_1}{z_0},\frac{y_1}{y_0}\right)$ where $f_0\colon Z_0'\to\mathbb{P}^1_{k_0}\times\mathbb{P}^1_{k_0}$ is defined as in Lemma 4.8. Since $\left(\frac{z_1}{z_0},\frac{y_1}{y_0}\right)$ corresponds to the quadratic form

$$\left\langle 1,\frac{z_1}{z_0},\frac{y_1}{y_0},\frac{z_1}{z_0}\frac{y_1}{y_0}\right\rangle\cong\left\langle 1,-\frac{z_1}{z_0},-\frac{y_1}{y_0},\frac{z_1}{z_0}\frac{y_1}{y_0}\right\rangle$$

over $k_0(\mathbb{P}^1_{k_0} \times \mathbb{P}^1_{k_0}) = k_0\left(\frac{y_1}{y_0}, \frac{z_1}{z_0}\right)$, the restriction of α_F to the generic point of E_F corresponds to the same quadratic form over $F(E_F)$. To be more precise: As $f_{E_F} \colon E_F \to \mathbb{P}^1_F \times \mathbb{P}^1_F$ is dominant and $\frac{y_4}{y_3}$, $\frac{y_5}{y_3}$ are mapped to $\frac{z_1}{z_0}$ and $\frac{y_1}{y_0}$ respectively, $\alpha_F|_{\eta_{E_F}}$ corresponds to the quadratic form

$$\left\langle 1, \frac{y_5}{y_3}, \frac{y_4}{y_3}, \frac{y_5}{y_3}\frac{y_4}{y_3} \right\rangle$$

on $F(E_F)$. This quadratic form is isotropic over $F(E_F)$ because the equation for $i = 3$ in (4.5) is a subform of $\left\langle 1, \frac{y_5}{y_3}, \frac{y_4}{y_3}, \frac{y_5}{y_3}\frac{y_4}{y_3} \right\rangle$. Thus, the restriction $\alpha_F|_{\eta_{E_F}}$ vanishes in $H^2(F(E_F), \mathbb{Z}/2)$ by Theorem 2.18. Since $H^2(\mathcal{O}_{E_F,e}, \mathbb{Z}/2) \to H^2(F(E_F), \mathbb{Z}/2)$ is injective by Theorem 2.26 (a), the restriction $\alpha_F|_e \in H^2(\kappa(e), \mathbb{Z}/2)$ of α to the regular point e of E_F vanishes. □

Proof of Proposition 4.7. Let

$$Z_0 = \{x_0^2x_5 + x_1^2x_4 + x_2^2x_6 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5 + x_6)) = x_3x_6 - x_4x_5 = 0\} \subset \mathbb{P}^6_{k_0}$$

and let $\delta_{Z_0} \in \mathrm{CH}_0(Z_{0,k_0(Z_0)})$ be the class of the diagonal. We aim to show that

$$0 \neq [\delta_{Z_0}] \in \frac{\mathrm{CH}_0\left(Z_{0,k_0(Z_0)}\right)/2}{\mathrm{CH}_0(Z_0)/2}.$$

To abbreviate some notation let us define $K := k_0(Z_0)$ and $k := k_0$. For a contradiction assume that

$$\delta_{Z_0} = 2z_1 + z_{2,K} \in \mathrm{CH}_0(Z_{0,K}) \tag{4.9}$$

where $z_1 \in \mathrm{CH}_0(Z_{0,K})$ and $z_2 \in \mathrm{CH}_0(Z_0)$ are some zero-cycles. Recall that there exists a non-trivial element $\alpha \in H^2_{\mathrm{nr}}(k(Z_0)/k, \mathbb{Z}/2)$ which was constructed by [HPT18] (see also Proposition 4.9). Since Z_0' is the blow-up of Z_0 in some subvariety $D \cap Z_0$ (see Lemma 4.8), the varieties Z_0' and Z_0 are birational, i.e.

$$\alpha \in H^2_{\mathrm{nr}}(k(Z_0)/k, \mathbb{Z}/2) = H^2_{\mathrm{nr}}(k(Z_0')/k, \mathbb{Z}/2).$$

We denote the blow-down morphism by $\rho_0 \colon Z_0' \to Z_0$ and the exceptional divisor by E. As already mentioned, the main idea for this proof is to compute the Merkurjev pairing $\langle \delta_{Z_0}, \alpha \rangle$ and use (4.9) to find a contradiction. Since Z_0 is not smooth, the Merkurjev pairing might not factor through rational equivalence. Therefore we need a smooth "model" for Z_0 which we construct by using an alteration. (If resolution of singularities would be known also in positive characteristic, the argument becomes slightly easier.) Let $\tau_0 \colon Z_0'' \to Z_0'$ be an alteration of odd degree which exists by work of de Jong [deJ96] and Gabber, see e.g. [IT14]. Since k_0 is perfect, Z_0'' is smooth. Then, we obtain the following diagram

$$\begin{tikzcd} Z_0'' \arrow[r, "\tau_0"] & Z_0' \arrow[r, "\rho_0"] \arrow[d, "f_0"] & Z_0 \\ & \mathbb{P}^1_k \times \mathbb{P}^1_k. & \end{tikzcd}$$

Extending the base field to K/k, the diagram reads

$$Z_{0,K}'' \xrightarrow{\tau} Z_{0,K}' \xrightarrow{\rho} Z_{0,K},$$

where τ and ρ denote the base-change of the morphism τ_0 and ρ_0, respectively. Note that τ is again an alteration of odd degree.

Next we need to pull-back the zero-cycle (4.9) and the unramified cohomology class to $Z''_{0,K}$ and Z''_0, respectively. Let us start with the latter one: Since, τ_0 is surjective it maps the generic point of Z''_0 to the generic point of Z'_0. Thus τ_0 induces a morphism

$$\tau_0\colon \operatorname{Spec} k(Z''_0) \longrightarrow \operatorname{Spec} k(Z'_0), \tag{4.10}$$

which we also denote by τ_0. We apply Proposition 2.24 (a) to obtain an element

$$\tau_0^*\alpha \in H^2_{\mathrm{nr}}\left(k(Z''_0)/k, \mathbb{Z}/2\right).$$

Note that (4.10) is a finite morphism, as τ_0 is generically finite. Thus there exists also a well-defined push-forward morphism

$$(\tau_0)_*\colon H^2_{\mathrm{nr}}(k(Z''_0)/k, \mathbb{Z}/2) \longrightarrow H^2_{\mathrm{nr}}(k(Z'_0)/k, \mathbb{Z}/2)$$

by Proposition 2.24 (b). Next we pull-back the zero-cycle δ_{Z_0} from $\mathrm{CH}_0(Z_{0,K})$ to $\mathrm{CH}_0(Z''_{0,K})$: Since $\rho\colon Z'_{0,K} \to Z_{0,K}$ is proper, there exists a well-defined push-forward map on the level of Chow groups. Restricting this push-forward map to the open subvariety $Z'_{0,K} \setminus E_K$ gives an isomorphism

$$\mathrm{CH}_0\left(Z'_{0,K} \setminus E_K\right) \xrightarrow{\cong} \mathrm{CH}_0\left(Z_{0,K} \setminus (Z_{0,K} \cap D_K)\right),$$

as the map on varieties is an isomorphism. Using the localization exact sequence for Chow groups (see e.g. [Ful98, Proposition 1.8]), we find the following commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_0(E_K) \arrow[r] & \mathrm{CH}_0(Z'_{0,K}) \arrow[r, two heads] \arrow[d, "\rho_*"] & \mathrm{CH}_0(Z'_{0,K} \setminus E_K) \arrow[d, "\cong"] \\
& \mathrm{CH}_0(Z_{0,K}) \arrow[r, two heads] & \mathrm{CH}_0(Z_{0,K} \setminus (Z_{0,K} \cap D_K))).
\end{tikzcd}$$

Thus, restricting the class (4.9) to the open subvariety $Z_{0,K} \setminus (Z_{0,K} \cap D_K)$ and pulling it back via ρ, we find by the above diagram that

$$\delta_{Z'_0} = 2z'_1 + z'_{2,K} + z'_3 \in \mathrm{CH}_0(Z'_{0,K}) \tag{4.11}$$

for some zero-cycles $z'_1 \in \mathrm{CH}_0(Z'_{0,K})$, $z'_2 \in \mathrm{CH}_0(Z'_0)$ and a zero-cycle z'_3 supported on E_K.

Consider the pull-back of (4.11) along τ. More precisely, if we restrict τ to the preimage of the smooth locus, there exists a well-defined pull-back map on the level of Chow groups (see [Ful98, Section 8.1]). Thus, by the localization exact sequence ([Ful98, Proposition 1.8]) we obtain the following diagram

$$\begin{tikzcd}
\mathrm{CH}_0\left(\tau^{-1}\left(\left(Z'_{0,K}\right)^{\mathrm{sing}}\right)\right) \arrow[r] & \mathrm{CH}_0\left(Z''_{0,K}\right) \arrow[r, two heads] & \mathrm{CH}_0\left(\tau^{-1}\left(\left(Z'_{0,K}\right)^{\mathrm{sm}}\right)\right) \\
& \mathrm{CH}_0\left(Z'_{0,K}\right) \arrow[r, two heads] & \mathrm{CH}_0\left(\left(Z'_{0,K}\right)^{\mathrm{sm}}\right). \arrow[u, "\tau^*"]
\end{tikzcd}$$

By restricting (4.11) to the smooth locus and pulling back the zero-cycle along τ we find that

$$\delta_\tau = 2z''_1 + z''_{2,K} + z''_3 + z''_4 \in \mathrm{CH}_0(Z''_{0,K}) \tag{4.12}$$

where δ_τ is the zero-cycle on $Z''_{0,K}$ induced by the graph of τ, z''_4 is supported on $\tau^{-1}\left(\left(Z'_{0,K}\right)^{\mathrm{sing}}\right)$, z''_3 is supported on $\tau^{-1}\left(E_K \cap \left(Z'_{0,K}\right)^{\mathrm{sm}}\right)$, $z''_2 \in \mathrm{CH}_0(Z''_0)$, and $z''_1 \in \mathrm{CH}_0(Z''_{0,K})$.

Lastly, we compute the Merkurjev pairing of the cycle (4.12) with $\tau_0^*\alpha$. Let us compute first

$$\langle \delta_\tau, \tau_0^*\alpha \rangle \in H^2(K, \mathbb{Z}/2).$$

Recall that the graph Γ_τ is isomorphic to Z_0'', so the generic point η_{Γ_τ} of the graph Γ_τ which represents the cycles δ_τ has function field isomorphic to Z_0'' and the natural morphism

$$\operatorname{Spec} k(Z_0'') \longrightarrow \operatorname{Spec} K = \operatorname{Spec} k(Z_0')$$

is the same morphism as (4.10). By construction of the Merkurjev pairing (2.7),

$$\langle \delta_\tau, \tau_0^*\alpha \rangle = (\tau_0)_* \left((\tau_0^*\alpha)_K|_{\eta_{\Gamma_\tau}} \right).$$

Since, the composition of the base extension by K and restriction to the generic point is the identity, we obtain by Proposition 2.24 (b)

$$\langle \delta_\tau, \tau_0^*\alpha \rangle = (\tau_0)_*(\tau_0)^*\alpha = \deg \tau_0 \cdot \alpha \in H^2(k(Z_0'), \mathbb{Z}/2) = H^2(K, \mathbb{Z}/2).$$

As $\deg \tau_0$ is odd and $\alpha \neq 0 \in H^2(K, \mathbb{Z}/2)$ by Proposition 4.9, the pairing $\langle \delta_\tau, \tau_0^*\alpha \rangle$ is non-zero. We claim now that

$$\left\langle 2z_1'' + z_{2,K}'' + z_3'' + z_4'', \tau_0^*\alpha \right\rangle = \left\langle z_{2,K}'' + z_3'' + z_4'', \tau_0^*\alpha \right\rangle = 0 \in H^2(K, \mathbb{Z}/2)$$

which yields the contradiction

$$0 \neq \langle \delta_\tau, \tau_0^*\alpha \rangle = 0 \in H^2(K, \mathbb{Z}/2),$$

i.e. our initial assumption (4.9) is false and this proves Proposition 4.7.

It is left to show the claim: Since the Merkurjev pairing is linear in the first argument, it suffices to prove that each summand of the zero-cycle pairs zero with $\tau_0^*\alpha$. The pairing $\left\langle z_{2,K}'', \tau_0^*\alpha \right\rangle$ is the pull-back of the pairing $\langle z_2'', \tau_0^*\alpha \rangle$ via the natural morphism $\psi\colon \operatorname{Spec} K \to \operatorname{Spec} k$, see Lemma 2.31 (a). Since k is algebraically closed, all higher étale cohomology of $\operatorname{Spec} k$ vanish, in particular $H^2(k, \mathbb{Z}/2) = 0$. Hence,

$$\left\langle z_{2,K}'', \tau_0^*\alpha \right\rangle = \psi^* \left\langle z_2'', \tau_0^*\alpha \right\rangle = \psi^* 0 = 0.$$

Next, we show that $\langle z_3'', \tau_0^*\alpha \rangle = 0$. Recall that the zero-cycle z_3'' is supported on $\tau^{-1}((Z_{0,K}')^{\mathrm{sm}} \cap E_K)$. Using the local descriptions of E and Z_0', see (4.4) and (4.5), we see that $(Z_{0,K}')^{\mathrm{sm}} \cap E_K \subset E_K^{\mathrm{sm}}$. Hence it suffices to check that the pairing $\langle z_3'', \tau_0^*\alpha \rangle$ vanishes for a single closed point z_3'' which is mapped via τ to a regular point z_3' of E_K. Lemma 2.31 (b) and Lemma 4.10 imply

$$\left\langle z_3'', \tau_0^*\alpha \right\rangle = \left\langle z_3', \alpha \right\rangle = 0 \in H^2(K, \mathbb{Z}/2).$$

Lastly we check that $\langle z_4'', \tau_0^*\alpha \rangle = 0$. Recall that z_4'' is supported on $\tau^{-1}\left((Z_0')^{\mathrm{sing}}\right)$. By Lemma 4.8, the singular locus does not dominate $\mathbb{P}^1_K \times \mathbb{P}^1_K$. Thus, we can apply Theorem 2.32 and conclude that

$$\left\langle z_4'', \tau_0^*\alpha \right\rangle = 0.$$

This proves the claim and thus Proposition 4.7. ☐

4.3 Proof of the main result

Proof of Proposition 4.1: Let us recall what we want to prove: Let k_0 be an algebraically closed field of characteristic different from 2 and let $k := \overline{k_0(u,v,w)}$ be the algebraic closure of a purely transcendental extension of k_0. Moreover, let $R = k[[t]]$ be the formal power series in one variable over k. We consider the strictly semi-stable model

$$\tilde{\mathcal{X}} \longrightarrow \operatorname{Spec} R$$

from (3.10), or (3.25) if $\operatorname{char} k = 3$. Recall that the special fibre $\tilde{\mathcal{X}}_k$ of $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ has three irreducible components

$$\tilde{Y}_1 \cup P_Z \cup Y_2$$

where $\tilde{Y}_1 = \operatorname{Bl}_S Y_1$ is the blow-up of Y_1 in the singular locus S of $\mathcal{X}$ and P_Z is a $\mathbb{P}^1_k$-bundle over $Z = Y_1 \cap Y_2$, see also (3.10) and (3.25). Moreover, let $A := \mathcal{O}_{\tilde{\mathcal{X}},P_Z}$ be the local ring at the generic point of P_Z with residue field $k(P_Z)$. We want to show that for any zero-cycle $z \in \mathrm{CH}_0(P_Z)$ the element

$$\delta_{P_Z} - z_{k(P_Z)} \in \mathrm{CH}_0(P_Z \times k(P_Z)) \tag{4.13}$$

does not lie in the image of the obstruction map

$$\Phi_{\tilde{\mathcal{X}}_A,P_Z} \colon \mathrm{CH}_1(\tilde{\mathcal{X}}_k \times_k k(P_Z)) \longrightarrow \mathrm{CH}_0(P_Z \times_k k(P_Z))$$

modulo 2 where $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ is defined as in (2.4).

Assume that (4.13) is contained in the image of $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ modulo 2. The idea of the proof is to simplify the Chow groups in the obstruction map as much as possible and conclude that

$$\delta_{Z_0} \in \operatorname{Im}\left(\mathrm{CH}_0(Z_0) \longrightarrow \mathrm{CH}_0(Z_0 \times k_0(Z_0))\right) \quad \text{mod } 2. \tag{4.14}$$

which contradicts Proposition 4.7.

Step 0. Clearly,

$$\mathrm{CH}_1(\tilde{Y}_1 \times k(P_Z)) \oplus \mathrm{CH}_1(P_Z \times k(P_Z)) \oplus \mathrm{CH}_1(Y_2 \times k(P_Z)) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{CH}_1(\tilde{X}_k \times k(P_Z)),$$

where the map is given by the push-forward along the corresponding inclusions of varieties. Let us consider the contribution of $\mathrm{CH}_1(P_Z \times k(P_Z))$. We know that $P_Z \to Z$ is a $\mathbb{P}^1_k$-bundle (see Lemma 3.9 and Lemma 3.14 respectively) and Z is smooth, so there is a canonical isomorphism

$$\mathrm{CH}_0(Z \times k(P_Z)) \oplus \mathrm{CH}_1(Z \times k(P_Z)) \xrightarrow{\cong} \mathrm{CH}_1(P_Z \times k(P_Z)). \tag{4.15}$$

For the convenience of the reader we quickly describe this map: The map from the first summand is given by pulling back a zero-cycle via base extension of the flat morphism $P_Z \to Z$ with $k(P_Z)$. Geometrically, the class of a closed point in $Z \times k(P_Z)$ is mapped to the class of the line over that point. The map from the second summand is given by pulling back a one-cycle via base extension of the flat morphism $P_Z \to Z$ with $k(P_Z)$ and intersecting it with a section of the $\mathbb{P}^1$-bundle $P_Z \times k(P_Z) \to Z \times k(P_Z)$. Combining this description with the concrete description of the obstruction morphism (2.5) we find that the contribution of $\mathrm{CH}_0(Z \times k(P_Z))$ to the obstruction morphism vanishes modulo 2. Indeed, take a closed point z in $Z \times k(P_Z)$, the above description of the isomorphism (4.15) tells us that we take the fiber over this point which is a $\mathbb{P}^1$. By the concrete description of the obstruction morphism (see (2.5)), we then intersect this line in $P_Z \times k(P_Z)$ with $\tilde{Y}_1 \times k(P_Z)$ and $Y_2 \times k(P_Z)$. These intersection points are equal to the point z we started with by viewing z as a point in $P_Z \times k(P_Z)$ via the sections $Z \times k(P_Z) \to (\tilde{Y}_1 \cap P_Z) \times k(P_Z)$ and $Z \times k(P_Z) \to (Y_2 \cap P_Z) \times k(P_Z)$ respectively. Since these two intersection points lie in the fiber over z, which is a $\mathbb{P}^1$, they are rationally equivalent. Hence, the image of the $\mathrm{CH}_0(Z \times k(P_Z))$ via the obstruction morphism is zero modulo 2.

Next we discuss the contribution of $\mathrm{CH}_1(Z \times k(P_Z))$. As $Z \to Y_2 \cap P_Z$ is a section of the projective bundle $P_Z \to Z$ and the isomorphism (4.15) is given by pulling back the one-cycle and intersecting it with a section, we find that any one-cycle of $Z \times k(P_Z)$ is mapped under (4.15) to a one-cycle supported on $Y_2 \cap P_Z$. Hence, the contribution of $\mathrm{CH}_1(Z \times k(P_Z))$ to the obstruction map is absorbed by $\mathrm{CH}_1(Y_2 \times k(P_Z))$. After this discussion we thus find that the image of the obstruction map $\Phi_{\tilde{\mathcal{X}}_A, P_Z}$ modulo 2 is contained in

$$\operatorname{Im}\left(\mathrm{CH}_1(\tilde{Y}_1 \times k(P_Z)) \oplus \mathrm{CH}_1(Y_2 \times k(P_Z)) \longrightarrow \mathrm{CH}_0(P_Z \times k(P_Z))\right) \mod 2. \tag{4.16}$$

Next we take a look at $\mathrm{CH}_1(\tilde{Y}_1 \times k(P_Z))$. As blow-ups commute with extension of the base field, the blow-up formula for Chow groups (see e.g. [Voi03, Theorem 9.27]) yields a canonical isomorphism

$$\mathrm{CH}_1(Y_1 \times k(P_Z)) \oplus \mathrm{CH}_0(S \times k(P_Z)) \cong \mathrm{CH}_1(\tilde{Y}_1 \times k(P_Z)).$$

Thus we conclude that (4.16) is contained in

$$\operatorname{Im}\left(\mathrm{CH}_1(Y_1 \times k(P_Z)) \oplus \mathrm{CH}_1(Y_2 \times k(P_Z)) \oplus \mathrm{CH}_0(S \times k(P_Z)) \longrightarrow \mathrm{CH}_0(P_Z \times k(P_Z))\right) \mod 2. \tag{4.17}$$

Next we will use Fulton's specialization map to make the Chow groups more accessible. For this we recall an observation made by Pavic and Schreieder.

Lemma 4.11 ([PS21, Lemma 5.7])**.** *Let B be a discrete valuation ring with fraction field F and residue field L. Let $p\colon \mathcal{X} \to \operatorname{Spec} B$ and $q\colon \mathcal{Y} \to \operatorname{Spec} B$ be proper, flat B-schemes with connected fibers. Denote by X_η, Y_η and X_0, Y_0 the generic and the special fibers of p, q respectively. Assume Y_0 is integral, i.e. $A = \mathcal{O}_{\mathcal{Y}, Y_0}$ is a discrete valuation ring, and consider the flat proper A-scheme $\mathcal{X}_A \to \operatorname{Spec} A$, given by base change of p. Then Fulton's specialization map induces a specialization map*

$$\mathrm{sp}\colon\ \mathrm{CH}_i(X_\eta \times_F \overline{F}(Y_\eta)) \longrightarrow \mathrm{CH}_i(X_0 \times_L \overline{L}(Y_0)),$$

where $\overline{F}$ and $\overline{L}$ denote the algebraic closures of F and L, respectively, such that the following holds:

(1) sp *commutes with pushforwards along proper maps and pullbacks along regular embeddings;*

(2) If $\mathcal{X} = \mathcal{Y}$, then $\mathrm{sp}(\delta_{X_\eta}) = \delta_{X_0}$, where $\delta_{X_\eta} \in \mathrm{CH}_0(X_\eta \times_F \overline{F}(X_\eta))$ and $\delta_{X_0} \in CH_0(X_0 \times_L \overline{L}(X_0))$ denote the diagonal points.

Recall that we aim to conclude (4.14) from the assumption that (4.13) is contained in the image of $\Phi_{\tilde{\mathcal{X}}_A, P_Z}$. This is done in the following three steps. In each step we apply Fulton's specialization map, see Lemma 4.11, by specializing one parameter u, v, w to 0 in order to control the Chow groups in (4.17): The first specialization allows us to write the first Chow group of Y_2 as the first Chow group of Y_1 and some Chow group of zero-cycles. In the second step the first Chow group of Y_1 will simplify to the first Chow group of some projective space, which is isomorphic to $\mathbb{Z}$, and again some Chow group of zero-cycles. Lastly the remaining Chow groups of zero-cycles, namely the two added in the previous steps and the Chow group of S, specialize in the third and final step to Chow groups of the reduced algebraic schemes in Lemma 4.6, i.e. are also isomorphic to $\mathbb{Z}$. Thus we will be able to conclude (4.14). The following diagram summarizes the strategy in a very informal way:

$$\begin{aligned} Y_2 &\overset{w\to 0}{\rightsquigarrow} Y_1 \;+\; \text{something},\\ Y_1 &\overset{v\to 0}{\rightsquigarrow} \mathbb{P}^5 \;+\; \text{something},\\ S \;+\; \text{something} &\overset{u\to 0}{\rightsquigarrow} S_0^{\mathrm{red}},\; D_1,\; \text{and } D_2 \text{ from Lemma 4.6.} \end{aligned}$$

We start with some general remarks and lay down the details afterwards. Since the specialization map in Lemma 4.11 commutes with pushforwards along proper maps and pullbacks along regular embeddings [Ful98, Proposition 20.3], we may compute the specialization of (4.17) by specializing the involved varieties. The ground field $k = \overline{k_0(u,v,w)}$ will change in each step, but remains algebraically closed by the construction in Lemma 4.11. To simplify the notation we denote the ground field in each step by κ which we specify at the beginning of each step. Note that the varieties in (4.17) depend on u, v, and w. To distinguish the specialized varieties from each other we denote the parameters as indices, e.g. $Z = Z_{u,v,w}$. We will omit the parameters after the specialization to 0. If there might be confusion after specializing all parameters we will write a 0 as index, e.g. Z_0. As a last remark, we try to explicitly write down the specialization, i.e. write down $\mathcal{X}$ and $\mathcal{Y}$ from Lemma 4.11. In order to keep the text readable, we omit the explicit specialization for characteristic 3 and we think of P_Z as a trivial $\mathbb{P}^1$-bundle over Z which is technically not correct because P_Z is only a locally trivial $\mathbb{P}^1$-bundle (see Lemma 3.9 and Lemma 3.14).

Recall that we defined the following polynomials in $k[x_0,\dots,x_7]$:

$$\begin{aligned} c_0 &= x_0^2x_5 + x_1^2x_4 + x_2^2x_6 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4+x_5+x_6)),\\ f_2 &= \sqrt[3]{4}(x_1x_2 + x_4x_5) + x_3^2,\\ f_3 &= x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3,\\ \hat{f}_3 &= x_0^3 - x_1^3 + \rho x_2^3 - \rho x_3^3 + \rho^2 x_4^3 - \rho^2 x_5^3,\\ c_{u,v,w} &= c_0 + u(x_6 f_2 + f_3) + v(f_3 + x_6^3) + w(g^w + x_7^3),\\ q_{v,w} &= x_3x_6 - x_4x_5 + v(x_3x_7 + f_2 + x_6^2) + wf^w,\\ \hat{c}_{v,w} &= x_6^3 + v\hat{f}_3 + wh^w, \end{aligned}$$

where $\rho \in k_0$ is a primitive third root of unity, see also Definition 3.1.

Step 1. We simplify the contribution of Y_2: The ground field κ is $\overline{k_0(u,v)}$ and we want to specialize $w \to 0$. More precisely, we consider the proper and flat family

$$\mathcal{Y} = \left\{c_{u,v,0} + wg^w = x_3x_6 - x_4x_5 + v(f_2 + x_6^2) + wf^w = 0\right\} \subset \mathbb{P}^6_{\kappa[[w]]} \times \mathbb{P}^1_{\kappa[[w]]} \longrightarrow \operatorname{Spec} \kappa[[w]].$$

Its geometric generic fibre is $P_{Z_{u,v,w}}$ and its special fibre is $P_{Z_{u,v}}$. (Recall again that we assume for simplicity that P_Z is a trivial $\mathbb{P}^1$-bundle in order to write down the family $\mathcal{Y} \to \operatorname{Spec} k[[w]]$ more nicely.) The special fibre $P_{Z_{u,v}}$ of the family $\mathcal{Y} \to \operatorname{Spec} \kappa[[w]]$ is integral because $P_{Z_{u,v}}$ is a $\mathbb{P}^1$-bundle of $Z_{u,v}$ and the latter variety degenerates via $v \to \infty$ to $A_{(viii)}$ from Lemma 3.3 and is thus smooth and integral. Hence, we can apply Lemma 4.11 to

$$\begin{aligned} &\mathcal{X} = \{c_{u,v,0} + wg^w = 0\} \subset \mathbb{P}^6_{\kappa[[w]]} \longrightarrow \operatorname{Spec} \kappa[[w]] && \text{for } Y_1,\\ &\mathcal{X} = \left\{c_{u,v,0} + w(g^w + x_7^3) = q_{v,0} + wf^w = 0\right\} \subset \mathbb{P}^7_{\kappa[[w]]} \longrightarrow \operatorname{Spec} \kappa[[w]] && \text{for } Y_2,\\ &\mathcal{X} = \{c_{u,v,0} + wg^w = q_{v,0} + wf^w = \hat{c}_{v,0} + wh^w = x_7 = 0\} \subset \mathbb{P}^7_{\kappa[[w]]} \longrightarrow \operatorname{Spec} \kappa[[w]] && \text{for } S, \text{ and}\\ &\mathcal{X} = \mathcal{Y} \longrightarrow \operatorname{Spec} \kappa[[w]] && \text{for } P_Z, \end{aligned}$$

respectively. We are mostly interested in Y_2 as we want to simplify its contribution in the obstruction, i.e. in (4.17). Note that $Y_{2,u,v}$ is a cone over $Y_{1,u,v} \subset \mathbb{P}^6_\kappa$ with vertex $[0 : \cdots : 0 : 1] \in \mathbb{P}^7_\kappa$ which is precisely the situation described in Lemma 4.2. The varieties

$$\begin{aligned} W &= Y_{1,u,v} \subset \mathbb{P}^6_\kappa, \text{ and} \\ D &= D_{2,u,v} := \{c_{u,v,0} = x_3 = x_3x_6 - x_4x_5 + v(f_2 + x_6^2) = 0\} \subset \mathbb{P}^6_\kappa \end{aligned}$$

are smooth as they degenerate via $v \to \infty$ to $A_{(vi)}$ and $A_{(x)}$ from Lemma 3.3 respectively. Note that in characteristic 3 the varieties

$$\begin{aligned} W &= Y_{1,u,v} \subset \mathbb{P}^6_\kappa, \text{ and} \\ D &= D_{2,u,v} := \left\{c^{(3)}_{u,v,0} = x_3 = x_3x_6 - x_4x_5 + v(f_2^{(3)} - x_6^2) = 0\right\} \subset \mathbb{P}^6_\kappa \end{aligned}$$

are also smooth as they degenerate to $A'_{(vi)}$ and $A'_{(x)}$ from Lemma 3.13 respectively. Thus, we can apply Lemma 4.2 with $V = Y_{2,u,v}$ and $d = 2$ and obtain a surjective homomorphism

$$\mathrm{CH}_0(D_{2,u,v} \times \kappa(P_{Z_{u,v}})) \oplus \mathrm{CH}_1(Y_{1,u,v} \times \kappa(P_{Z_{u,v}})) \twoheadrightarrow \mathrm{CH}_1(Y_{2,u,v} \times \kappa(P_{Z_{u,v}})),$$

i.e. the specialization of (4.17) is contained in

$$\mathrm{Im}\Big(\mathrm{CH}_1\left(Y_{1,u,v,L_1}\right) \oplus \mathrm{CH}_0\left(D_{2,u,v,L_1}\right) \oplus \mathrm{CH}_0\left(S_{u,v,L_1}\right) \longrightarrow \mathrm{CH}_0\left(P_{Z_{u,v,L_1}}\right)\Big) \mod 2 \tag{4.18}$$

where $L_1 := \kappa(P_{Z_{u,v}})$.

Step 2. We simplify the contribution of Y_1: The ground field κ is $\overline{k_0(u)}$ and we specialize $v \to 0$. Consider the proper and flat family

$$\mathcal{Y} = \left\{c_{u,0,0} + v(f_3 + x_6^3) = x_3x_6 - x_4x_5 + v(f_2 + x_6^2) = 0\right\} \subset \mathbb{P}^6_{\kappa[[v]]} \times \mathbb{P}^1_{\kappa[[v]]} \longrightarrow \operatorname{Spec} \kappa[[v]].$$

The variety $P_{Z_{u,v}}$ is the geometric generic fibre of $\mathcal{Y} \to \operatorname{Spec} R$ and the variety P_{Z_u} is the special fibre. The latter variety is a $\mathbb{P}^1_\kappa$-bundle over Z_u which is integral because its defining equations are linear in the variable x_6. Thus P_{Z_u} is integral and we can apply Lemma 4.11 to the flat and proper families

$$\begin{aligned} &\mathcal{X} = \left\{c_0 + u(x_6f_2 + f_3) + v(f_3 + x_6^3) = 0\right\} \subset \mathbb{P}^6_{\kappa[[v]]} \longrightarrow \operatorname{Spec} \kappa[[v]] && \text{for } Y_1, \\ &\mathcal{X} = \left\{c_{u,0,0} + v(f_3 + x_6^3) = x_3 = x_4x_5 - v(f_2 + x_6^2) = 0\right\} \subset \mathbb{P}^6_{\kappa[[v]]} \longrightarrow \operatorname{Spec} \kappa[[v]] && \text{for } D_2, \\ &\mathcal{X} = \left\{\begin{smallmatrix} c_{u,0,0}+v(f_3+x_6^3)=x_6^3+v\hat{f}_3=0, \\ x_3x_6-x_4x_5+v(f_2+x_6^2)=0 \end{smallmatrix}\right\} \subset \mathbb{P}^6_{\kappa[[v]]} \longrightarrow \operatorname{Spec} \kappa[[v]] && \text{for } S, \text{ and} \\ &\mathcal{X} = \mathcal{Y} \longrightarrow \operatorname{Spec} \kappa[[v]] && \text{for } P_Z, \end{aligned}$$

respectively. The only interesting specialization is the specialization of Y_1: We note that $Y_{1,u}$ is a cone in $\mathbb{P}^6_\kappa$ with vertex $[0 : \cdots : 0 : 1] \in \mathbb{P}^6_\kappa$. Similar to Step 1 we want to apply Lemma 4.2. Therefore we need to check that $W = \mathbb{P}^5_\kappa$ and $D = D_{1,u} \subset \mathbb{P}^6_\kappa$ are smooth where $D_{1,u}$ is given by

$$\left\{x_2^2 - 2x_3^2 + uf_2 = x_0^2x_5 + x_1^2x_4 + x_3(x_3^2 + x_4^2 + x_5^2 - 2x_3(x_4 + x_5)) + uf_3 = 0\right\} \subset \mathbb{P}^6_\kappa.$$

The projective space W is obviously smooth and D specializes via $u \to \infty$ to $A_{(ix)}$ from Lemma 3.3. In characteristic 3 we need to replace f_2 and f_3 in the above definition of D by $f_2^{(3)}$ and $f_3^{(3)}$, respectively. The variety D degenerates via $v \to \infty$ then to $A'_{(ix)}$ from Lemma 3.13, i.e. D is also smooth in characteristic 3. Applying Lemma 4.2 with $V = Y_{1,u}$ and $d = 3$ yields a surjective homomorphism

$$\mathrm{CH}_0(D_{1,u} \times \kappa(P_{Z_u})) \oplus \mathrm{CH}_1(\mathbb{P}_{\kappa(P_{Z_u})}) \twoheadrightarrow \mathrm{CH}_1(Y_{1,u} \times \kappa(P_{Z_u})).$$

Moreover, $\mathrm{CH}_1(\mathbb{P}^5_{\kappa(P_{Z_u})}) \cong \mathbb{Z}$ is generated by a line. Without loss of generality we can assume that the line l is defined over κ. By Lemma 4.2 with $\kappa' = \kappa$ (and the same W, D, and V as above) the line l give rise to a one-cycle on $Y_{1,u}$. The image of that one-cycle under the obstruction yields a zero-cycle in $P_{Z_u} \times \kappa(P_{Z_u})$ which is defined over κ. Hence, the specialization of (4.18) is contained in

$$\mathrm{Im}\left(\mathrm{CH}_0\left(P_{Z_u}\right) \oplus \mathrm{CH}_0\left(D_{1,u,L_2}\right) \oplus \mathrm{CH}_0\left(D_{2,u,L_2}\right) \oplus \mathrm{CH}_0\left(S_{u,L_2}\right) \longrightarrow \mathrm{CH}_0\left(P_{Z_{u,L_2}}\right)\right) \mod 2 \tag{4.19}$$

where $L_2 = \kappa(P_{Z_u})$.

Step 3. We simplify the remaining Chow groups: The ground field is $\kappa = k_0$ and we specialize $u \to 0$. We consider the proper and flat family

$$\mathcal{Y} = \{c_0 + u(f_3 + x_6 f_2) = x_3x_6 - x_4x_5 = 0\} \subset \mathbb{P}^6_{\kappa[[u]]} \times \mathbb{P}^1_{\kappa[[u]]} \longrightarrow \operatorname{Spec} \kappa[[u]].$$

The variety P_{Z_u} is the geometric generic fibre of $\mathcal{Y} \to \operatorname{Spec} \kappa[[u]]$. The special fibre P_{Z_0} of the family $\mathcal{Y} \to \operatorname{Spec} \kappa[[u]]$ is a $\mathbb{P}^1$-bundle over Z_0 which is integral because its defining equation is linear in x_6, i.e. P_{Z_0} is integral and we can apply Lemma 4.11 to

$$\begin{aligned}
\mathcal{X} &= \left\{\begin{matrix} x_0^2x_5+x_1^2x_4+x_3(x_3^2+x_4^2+x_5^2-2x_3(x_4+x_5))+uf_3=0, \\ x_2^2-2x_3^2+uf_2=0 \end{matrix}\right\} \subset \mathbb{P}^6_{\kappa[[u]]} \longrightarrow \operatorname{Spec} \kappa[[u]] && \text{for } D_1,\\
\mathcal{X} &= \{c_0 + u(x_6 f_2 + f_3) = x_3 = -x_4x_5 = 0\} \subset \mathbb{P}^6_{\kappa[[u]]} \longrightarrow \operatorname{Spec} \kappa[[u]] && \text{for } D_2,\\
\mathcal{X} &= \left\{c_0 + u(x_6 f_2 + f_3) = x_3x_6 - x_4x_5 = x_6^3 = 0\right\} \subset \mathbb{P}^6_{\kappa[[u]]} \longrightarrow \operatorname{Spec} \kappa[[u] && \text{for } S, \text{ and}\\
\mathcal{X} &= \mathcal{Y} \longrightarrow \operatorname{Spec} \kappa[[u]] && \text{for } P_Z,
\end{aligned}$$

respectively. Note that $D_{1,u}$ and $D_{2,u}$ specialize via $u \to 0$ to D_1 and D_2 from Lemma 4.6, respectively. Moreover, the algebraic scheme S_u specializes via $u \to 0$ to

$$S_0 = \{c_0 = x_3x_6 - x_4x_5 = x_6^3 = 0\} \subset \mathbb{P}^6_\kappa.$$

Its reduced scheme is S_0^{red} from Lemma 4.6. By Lemma 4.6,

$$D_1, \ D_2, \ \text{and } S_0^{\mathrm{red}}$$

have universally trivial Chow group of zero-cycles. Thus, the specialization of (4.19) is contained in

$$\mathrm{Im}\left(\mathrm{CH}_0(P_{Z_0}) \longrightarrow \mathrm{CH}_0(P_{Z_0} \times \kappa(P_{Z_0}))\right) \mod 2. \tag{4.20}$$

Moreover, applying Lemma 4.11 (2) repeatedly to our initial assumption that (4.13) is contained in (4.17) implies that $\delta_{P_{Z_0}}$ is contained in (4.20).

Since P_{Z_0} is a $\mathbb{P}^1_{k_0}$-bundle over Z_0, the pushforward of zero-cycles yields a canonical isomorphism $\mathrm{CH}_0(P_{Z_0}) \cong \mathrm{CH}_0(Z_0)$. As Chow groups do not change under purely transcendental field extensions, there is an isomorphism $\mathrm{CH}_0(Z_0 \times k_0(Z_0)) \cong \mathrm{CH}_0(Z_0 \times k_0(P_{Z_0}))$. Moreover, the diagonal point $\delta_{P_{Z_0}}$ is mapped to the diagonal point δ_{Z_0} under the composition

$$\mathrm{CH}_0(P_{Z_0} \times k_0(P_{Z_0})) \xrightarrow{\cong} \mathrm{CH}_0(Z_0 \times k_0(P_{Z_0})) \xrightarrow{\cong} \mathrm{CH}_0(Z_0 \times k_0(Z_0)).$$

Hence, we conclude that

$$\delta_{Z_0} \in \mathrm{Im}\left(\mathrm{CH}_0(Z_0) \longrightarrow \mathrm{CH}_0(Z_0 \times k_0(Z_0))\right) \mod 2$$

which contradicts Proposition 4.7. Thus our initial assumption is wrong and we conclude that the element $\delta_{P_Z} - z_{k(P_Z)} \in \mathrm{CH}_0(P_Z \times k(P_Z))$ is not contained in the image of $\Phi_{\tilde{\mathcal{X}}_A, P_Z}$ modulo 2 which proves Proposition 4.1. $\square$

Corollary 4.12. *The smooth* $(3,3)$ *complete intersection*

$$\tilde{X}_{\overline{K}} = \left\{c_{u,v,w} = t^2\hat{c}_{v,w} + x_7 q_{v,w}\right\} \subset \mathbb{P}^7_{\overline{K}}$$

from Lemma 3.9, or from Lemma 3.14 in characteristic 3, *does not admit a decomposition of the diagonal, in particular* $\tilde{X}_{\overline{K}}$ *is not retract rational.*

Proof. Assume that $\tilde{X}_{\overline{K}}$ admits a decomposition of the diagonal. Consider the strictly semi-stable family $\tilde{\mathcal{X}} \to \operatorname{Spec} R$ from Lemma 3.9, or from Lemma 3.14 in characteristic 3. Recall that $\tilde{X}_{\overline{K}}$ is its geometric generic fibre and the special fibre $\tilde{\mathcal{X}}_k$ of this family is given by

$$\tilde{Y}_1 \cup P_Z \cup Y_2,$$

where the varieties are defined as in the above mentioned lemmata. Let $A = \mathcal{O}_{\tilde{X},P_Z}$ be the local ring at the generic point of P_Z. Then A/R is an unramified extension of discrete valuation rings. Thus it follows from Theorem 2.14 (i.e. [PS21, Theorem 4.1]) that the natural morphism

$$\Phi_{\tilde{\mathcal{X}}_A} \colon \mathrm{CH}_1(\tilde{\mathcal{X}}_k \times_k k(P_Z))/2 \longrightarrow \operatorname{Ker}\left(\mathrm{CH}_0(\tilde{Y}_{1,k(P_Z)})/2 \oplus \mathrm{CH}_0(P_{Z,k(P_Z)})/2 \oplus \mathrm{CH}_0(Y_{2,k(P_Z)})/2 \xrightarrow{\deg} \mathbb{Z}/2\right) \tag{4.21}$$

is surjective. Note that for any zero-cycle $z \in \mathrm{CH}_0(P_Z)$ of degree 1, the element

$$\delta_{P_Z} - z_{k(P_Z)} \in \mathrm{CH}_0(P_Z \times k(P_Z))$$

has degree 0. Thus the image of (4.21) contains this zero-cycle. More precisely, since the zero-cycle is supported on $P_Z \times k(P_Z)$ it is contained in the image of $\Phi_{\tilde{\mathcal{X}}_A,P_Z}$ modulo 2 which contradicts Proposition 4.1. Hence, $\tilde{\mathcal{X}}_{\overline{K}}$ does not admit a decomposition of the diagonal and is thus not retract rational by Lemma 2.4. □

Hence we constructed an example of a smooth $(3,3)$ complete intersection in $\mathbb{P}^7$ which is not retract rational which implies that a very general $(3,3)$ complete intersection in $\mathbb{P}^7$ is not retract rational and thus proves Theorem 1.1.

Corollary 4.13. *Let k be an uncountable field of characteristic different from* 2. *A very general* $(3,3)$ *complete intersection in* $\mathbb{P}^7_k$ *does not admit a decomposition of the diagonal.*

Proof. Let B be the variety parametrizing smooth $(3,3)$ complete intersection of $\mathbb{P}^7_k$, i.e. B is an open subvariety of $\mathbb{P}^{119}_k \times \mathbb{P}^{119}_k$. (Note that $119 = \binom{10}{3} - 1$.) Let $f \colon \mathcal{Y} \to B$ be the family of smooth $(3,3)$ complete intersections, i.e. $\mathcal{Y}$ is a closed subvariety of $B \times \mathbb{P}^7_k$. We claim that it suffices to prove the argument for (uncountable) algebraically closed fields k. Indeed, assume that the statement holds over algebraically closed fields. If a $(3,3)$ complete intersection over a (not necessarily algebraically closed) field k admits a decomposition of the diagonal then its base change with $\overline{k}$ also admits a decomposition of the diagonal. Thus the closed point in B whose fibre admits a decomposition of the diagonal lie in the image of a countable union of proper closed subsets under the natural morphism

$$B_{\overline{k}} \to B,$$

i.e. is a countable union of proper closed subsets. (Recall that B is a projective variety.) Thus we can assume without loss of generality that k is algebraically closed. By Corollary 4.12 there exists a smooth $(3,3)$ complete intersection in $\mathbb{P}^7_k$ which does not admit a decomposition of the diagonal. Hence the statement follows directly from Lemma 2.15 and Theorem 2.16. □

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Selbständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst und keine anderen Hilfsmittel als angegeben verwendet habe. Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Werken entnommen sind, habe ich als solche kenntlich gemacht.

Diese Arbeit wurde in gleicher oder ähnlicher Form noch bei keinem anderen Prüfer als Prüfungsleistung eingereicht und ist auch noch nicht veröffentlicht.

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