ON ZERO-CYCLES OF VARIETIES OVER LAURENT FIELDS

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Abstract. We generalize a recent result of Pavic–Schreieder regarding the surjectivity of the obstruction morphism defined in [PS23]. As a consequence of this result, we show that geometrically (retract) rational varieties over a Laurent field of characteristic 0, which admit a strictly semi-stable model, have trivial Chow group of zero-cycles. Our key new ingredient comes from toric geometry.

1. Introduction

Kontsevich–Tschinkel [KT19] and Nicaise–Shinder [NS19] study the behaviour of rationality and stable rationality in families in characteristic 0. In particular for degenerations of a smooth projective variety, they construct a motivic obstruction to (stable) rationality which depends only on the special fibre of the degeneration. This approach was successfully applied by Nicaise–Ottem [NO22] to show the stable irrationality of quartic fivefolds and several complete intersections by reducing to previously known irrationality results [HPT18; Sch19b]. Building on [Sch19b], [Moe23] uses the approach in [NO22] to improve Schreieder's logarithmic bound for stably irrational hypersurfaces in characteristic 0.

Motivated by the cycle-theoretic approaches to stable rationality in [Voi15; CTP16; Sch19a], Pavic–Schreieder [PS23] introduce a Chow-theoretic analogue of the motivic approach for strictly semi-stable schemes $\mathfrak{X}$ over a dvr R. These are integral separated regular flat R-schemes such that the generic fibre $X = \mathfrak{X}_\eta$ is a smooth variety and the special fibre is a geometrically reduced simple normal crossing divisor $Y = \bigcup_{i\in I} Y_i$ with irreducible components Y_i. For each strictly semi-stable and proper $\mathfrak{X} \to \operatorname{Spec} R$ they consider the complex

$$\bigoplus_{j\in I} \mathrm{CH}_1(Y_j) \xrightarrow{\Phi_{\mathfrak{X}}} \bigoplus_{i\in I} \mathrm{CH}_0(Y_i) \xrightarrow{\sum_i \deg} \mathbb{Z} \longrightarrow 0, \tag{1.1}$$

where $\Phi_{\mathfrak{X}} = \sum\limits_{i\in I}\sum\limits_{j\in I} \iota_i^*\iota_{j\,*}$ with $\iota_i\colon Y_i \hookrightarrow \mathfrak{X}$ the natural inclusion for $i \in I$.

Theorem 1.1. *Let R be a discrete valuation ring with algebraically closed residue field k, and let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective R-scheme. Assume that the geometric generic fibre admits a decomposition of the diagonal (e.g. is retract rational). Then the complex (1.1) is exact after base-change to any field extension L/k, i.e. the complex (1.1) is exact for the strictly semi-stable family $\mathfrak{X} \times_R A \to \operatorname{Spec} A$, where A/R is any unramified extension of dvr's with induced extension L/k of residue fields.*

The theorem implies that the complex (1.1) is exact modulo m for all $m \in \mathbb{N}$, as the tensor product is right exact, see also Corollary 6.4 for a more general version. While the residue field of A does not need to be algebraically closed, we would like to emphasize

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that the assumption on the residue field of R being algebraically closed is crucial, see Remark 6.5. Theorem 1.1 improves [PS23, Theorem 1.2 (2)] which shows the exactness of the complex tensored with $\mathbb{Z}/2$ if the special fibre is a chain of Cartier divisors. We can remove the restriction to $\mathbb{Z}/2$-coefficients and allow an arbitrary configuration of the special fibre in the strictly semi-stable family. The theorem of Pavic–Schreieder implies the (retract) irrationality of very general quartic fivefolds [PS23, Theorem 1.1] and of very general complete intersections of two cubics in $\mathbb{P}^7$ [LS23, Theorem 1.1]. One might hope to approach the other complete intersections in [NO22] and the hypersurfaces in [Moe23] with Theorem 1.1.

As a consequence of Theorem 1.1, we study in this paper the Chow group of zero-cycles of geometrically retract rational varieties over Laurent fields $k((t))$ where k is algebraically closed. Colliot-Thélène shows in [CT83, Theorem A (iv)] that the Chow group of degree 0 zero-cycles is trivial for geometrically rational surfaces over fields of characteristic 0 and cohomological dimension at most 1. In particular [CT83] applies to Laurent fields in characteristic 0. More generally, Tian considers rationally connected varieties over Laurent fields in characteristic 0 and shows the triviality of the Chow group of zero-cycles unconditionally in dimension at most 3 and in all dimensions if the Tate conjecture for surfaces holds, see [Tia20, Theorem 1.1]. We obtain a partial, but unconditional result in this direction and a similar result in positive characteristic.

Corollary 1.2. *Let X be a smooth, projective variety over a Laurent field $k((t))$ with k algebraically closed. Assume that X admits a strictly semi-stable projective model $\mathfrak{X} \to \operatorname{Spec} k[[t]]$. If X admits geometrically an integral decomposition of the diagonal, e.g. X is geometrically retract rational, then the following holds:*

- *(i) The degree map* $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ *is an isomorphism if* $\operatorname{char} k = 0$.
- *(ii) If* $p = \operatorname{char} k > 0$*, then* $\deg\colon \mathrm{CH}_0(X)/l \to \mathbb{Z}/l$ *is an isomorphism for every l coprime to p.*

For smooth families $\mathfrak{X} \to \operatorname{Spec} k[[t]]$, Corollary 1.2 (i) follows from [Kol04, Theorem 2 (2)]. Over $\mathbb{C}((t))$, the result is known for rationally simply connected varieties by [Pir12, Theorem 1.5]. Note that there are smooth projective varieties over $\mathbb{C}$ admitting an integral decomposition of diagonal which are not rationally connected and thus in particular not rationally simply connected, e.g. Barlow surfaces, see [ACTP17, Proposition 1.9] and [Voi17, Corollary 2.2].

Remark 1.3. *We prove a more general version of Corollary 1.2 for a Λ-decomposition of the diagonal (see Section 2.2) and over fraction fields of excellent, henselian discrete valuation rings with algebraically closed residue field k in Corollary 6.6. The latter corollary is applicable to varieties, which are geometrically Enriques surfaces ($\Lambda = \mathbb{Z}[1/2]$) or geometrically Godeaux surfaces ($\Lambda = \mathbb{Z}[1/5]$ for a general Godeaux surface).*

A variant of the homomorphism Φ in (1.1) is used to determine the kernel of the degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ for certain geometrically rational surfaces over local fields in [Dal05a; Dal05b] and for a cubic threefold over $\mathbb{C}((u))((t))$ in [Mad08].

We briefly sketch the strategy for the two results: The assumption of Theorem 1.1 ensures that the generic fibre of $\mathfrak{X} \to \operatorname{Spec} R$ admits a decomposition of the diagonal after a finite field extension. This implies by [PS23, Theorem 1.2 (1)] that the complex (1.1) is exact for a suitable resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ after a finite base change $\tilde{R}/R$. From this we aim to deduce the exactness over R. The issue is that the suitable resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ has many more components in the special fibre, so the associated complexes (1.1) are quite different. We describe the suitable resolution $\tilde{\mathfrak{X}}$ étale locally by toric geometry and obtain a global description via simplicial complexes. This way we can reduce Theorem 1.1 to a combinatorial problem which we solve in Proposition 5.9.

Corollary 1.2 follows from Theorem 1.1 via Fulton's localization exact sequence

$$\mathrm{CH}_1(Y) \longrightarrow \mathrm{CH}_1(\mathfrak{X}) \longrightarrow \mathrm{CH}_0(X) \longrightarrow 0$$

by using Saito-Sato's [SS10] bijectivity result for the étale cycle class map.

Some preliminaries are given in Section 2. In Section 3, we briefly recall some toric intersection theory and provide a description of a particular affine toric singularity and its resolution. The main technical aspects of this paper are contained in Section 4, where we use a subdivision of the dual complex of an snc scheme to describe the components of the resolution and one-cycles on the special fibre. In Section 5, we use the results from Section 4 to compare the complex (1.1) with the complex of a suitable resolution after a finite base change by constructing two auxiliary functions. This enables us to prove Theorem 1.1 in Section 6, where we also explain the argument for Corollary 1.2. In Appendix A, we provide two concrete examples, for which we illustrate the key proposition - Proposition 5.9 - by a direct computation. We achieve this by explicitly spelling out the constructions and arguments from Sections 4 and 5 for each example.

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2. Preliminaries

2.1. **Notations.** A ring is always a commutative ring with 1. For an abelian group G and an integer $l \in \mathbb{Z}_{\geq 0}$, we denote $G \otimes_{\mathbb{Z}} \mathbb{Z}/l\mathbb{Z}$ by G/l.

Let k be a field. A *k-variety* (or *variety*) X is an integral, separated scheme of finite type over k. The base-change of a separated scheme X over a ring R to some ring extension A/R is denoted by

$$X_A := X \times_R A := X \times_{\operatorname{Spec} R} \operatorname{Spec} A.$$

Let Y be a scheme of finite type over a field k. The *group of l-cycles* of Y is the free abelian group generated by subvarieties of dimension l. The *Chow group of l-cycles* $\mathrm{CH}_l(Y)$ of Y is the quotient of the group of l-cycles by rational equivalence. For a ring Λ, we denote the tensor product $\mathrm{CH}_l(Y) \otimes_{\mathbb{Z}} \Lambda$ by $\mathrm{CH}_l(Y, \Lambda)$.

2.2. **Decomposition of the diagonal.** Let Λ be a ring. We say that a k-variety X of dimension n admits a Λ-decomposition of the diagonal, if the diagonal $\Delta_X \subset X \times_k X$ satisfies

$$[\Delta_X] = [X \times z] + [Z] \in \mathrm{CH}_n(X \times_k X, \Lambda),$$

where z is a zero-cycle in X of degree 1 and Z is an n-cycle on $X \times_k X$ which does not dominate the first factor.

If $\Lambda = \mathbb{Z}$, we say that X admits an (integral) decomposition of the diagonal. For example, retract rational varieties admit a decomposition of the diagonal, see e.g. [Sch21, Lemma 7.5].

2.3. **Strictly semi-stable families.** Let R be a discrete valuation ring with fraction field K and residue field k. An integral regular separated flat R-scheme $\mathfrak{X} \to \operatorname{Spec} R$ is called *strictly semi-stable*, if the generic fibre $X = \mathfrak{X}_K$ is smooth over K and the special fibre $\mathfrak{X} \times_R k$ is a geometrically reduced simple normal crossing divisor, i.e. every component of the special fibre is a smooth Cartier divisor in $\mathfrak{X}$ and the scheme-theoretic intersection of the components is either empty or smooth of the expected codimension.

Let R be a discrete valuation ring and let $R \subset \tilde{R}$ be an unramified extension of discrete valuation rings, i.e. an extension $R \to \tilde{R}$ such that $m_{\tilde{R}} = m_R \tilde{R}$ where m_R and $m_{\tilde{R}}$ are the maximal ideal of R and $\tilde{R}$, respectively. Then for every strictly semi-stable R-scheme $\mathfrak{X}$, the base change $\mathfrak{X}_{\tilde{R}} = \mathfrak{X} \times_R \tilde{R}$ is a strictly semi-stable $\tilde{R}$-scheme by [Har01, Proposition 1.3]. For finite ramified extension of dvr's the base-change $\mathfrak{X}_{\tilde{R}}$ becomes a strictly semi-stable $\tilde{R}$-scheme after a finite sequence of blow-ups.

Proposition 2.1 ([Har01, Proposition 2.2])**.** *Let $R \subset \tilde{R}$ be a finite ramified extension of discrete valuation rings. Let $\mathfrak{X}$ be a strictly semi-stable R-scheme with special fibre Y. Assume that the irreducible components of Y are geometrically integral. Then there exists a finite sequence of blow-ups $\tilde{\mathfrak{X}} := V^m \to V^{m-1} \to \cdots \to V^0 =: \mathfrak{X}_{\tilde{R}}$ such that $\tilde{\mathfrak{X}}$ is a strictly semi-stable $\tilde{R}$-scheme and the center of each blow-up $V_{i+1} \to V_i$ is an irreducible component of the special fibre of V_i.*

2.4. **Basic constructions in toric geometry.** Throughout this section, let k be a field. A (split) toric variety is a normal k-variety X which contains a torus $T = (k^*)^{\dim X}$ as an open dense subset. Toric varieties are also constructed from collections of cones, called fans, and many properties of the toric variety relate to combinatorical data of the fan. We recall the construction of toric varieties from fans, introduce some notation and state a few standard facts from toric geometry.

Let $N \cong \mathbb{Z}^n$ be a lattice and denote by $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ its dual lattice. Consider the natural duality pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Morover, we denote the tensor products $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively.

A *(strongly convex, rational, polyhedral or short: scrp) cone* in $N_{\mathbb{R}}$ is a set

$$\sigma := \operatorname{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \geq 0 \right\} \subset N_{\mathbb{R}},$$

where $S \subset N$ is a finite subset such that $\sigma \cap (-\sigma) = \{0\}$. The elements in S are called *generators* of σ. The *dimension* of σ is the dimension of its linear span in $N_{\mathbb{R}}$. The *dual cone* of σ is the subset

$$\sigma^\vee := \{ m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \} \subset M_{\mathbb{R}},$$

which is again a (rational, polyhedral, not necessarily strongly convex) cone. Gordan's lemma says that $\sigma^\vee \cap M$ is a finitely generated semi-group such that $k[\sigma^\vee \cap M]$ is an integral domain, see e.g. [CLS11, Proposition 1.2.17]. Then the *affine toric variety U_σ* associated to a cone σ is $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$.

A *face* τ of an (scrp) cone σ is a subset of the form

$$\tau = \{ u \in \sigma : \langle m, u \rangle = 0 \} \subset \sigma$$

for some $m \in \sigma^\vee$ and it is again an (scrp) cone. A collection of (scrp) cones Σ is called a *fan* if each face of a cone is also in Σ and if the intersection of two cones in Σ is a common face of both. These two conditions ensure that we can glue the affine toric varieties associated to each cone $\sigma \in \Sigma$ along the intersection of the cones to obtain a toric variety, which we denote by X_Σ. We denote the set of n-dimensional cones in the fan Σ by $\Sigma(n)$ and we usually call 1-dimensional cones *rays*. Recall also the following facts in toric geometry:

(i) A $\mathbb{Z}$-linear map of lattices $N \to N'$ is *compatible* with a pair of fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$ if the $\mathbb{R}$-linear extension maps every cone $\sigma \in \Sigma$ into a cone $\sigma' \in \Sigma'$. Such a compatible map of lattices gives rise to a toric morphism of the associated toric varieties, i.e. a morphism of the varieties such that the restriction to the tori is a group homomorphism, see [Oda78, Theorem 4.1].

(ii) The orbit-cone correspondence yields an inclusion-reversing bijection between cones in a fan and orbits of the torus action of its associated toric variety, see e.g. [Oda78, Theorem 4.2]. If $\sigma \in \Sigma$ is a cone in the fan, then we denote the corresponding orbit in X_Σ by $O(\sigma)$ and its Zariski closure by $V(\sigma)$.

(iii) The generators v_i of the 1-dimensional faces ρ_i of an (scrp) cone $\sigma \subset N_\mathbb{R}$ yield a set of generators for σ. Up to scaling v_i, we can assume that $v_i \in (N \setminus \{0\}) \cap \rho_i$ is of minimal length. Then the v_i's are called the *minimal generators* of σ.

(iv) We say that a cone is *simplicial* if its minimal generators are $\mathbb{R}$-linear independent. A cone is called *regular* if its minimal generators form part of a $\mathbb{Z}$-basis of the lattice N. We say a fan is *simplicial* or *regular*, if all its cones are so. Note that regularity of the fan is equivalent to the regularity of the associated toric variety, see [Oda78, Theorem 4.3].

For a more detailed account of the constructions and the facts in toric geometry over arbitrary fields, we refer the reader to [Oda78] and the survey [Dan78]. An earlier description of toric varieties over algebraically closed fields can be found in [KKMS]. We refer the reader to [CLS11] for a modern treatment of toric geometry (over $\mathbb{C}$).

2.5. **The general setup.** We state here the general setup in this paper and refer to it later on.

Setup 2.2. *Let R be a discrete valuation ring with residue field k. Let $\tilde{R}/R$ be a finite extension of discrete valuation rings of ramification index r with induced extension L/k of residue fields. Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre $Y = \mathfrak{X} \times_R k$. Assume that the irreducible components of Y are geometrically integral. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution of $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ as in Proposition 2.1 and denote the natural morphism by $q \colon \tilde{\mathfrak{X}} \to \mathfrak{X}$. Let $\tilde{Y} = \tilde{\mathfrak{X}} \times_{\tilde{R}} L$ be the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ and denote the restriction of q to $\tilde{Y}$ also by $q \colon \tilde{Y} \to Y$.*

3. Resolution of a toric singularity

The proof of Theorem 1.1 relies on the analysis of a certain type of toric singularity together with an iterative description of its resolution as well as a combinatorial description of the intersection of toric curves with toric divisors. We recall the necessary intersection theory from [CLS11] in the first part of this section. The second part consists of a description of the toric singularity together with a resolution process.

3.1. **Some intersection theory.** We recall the required intersection theory which is described by combinatorical data of the fan. Here, we are interested in the intersection of toric divisors with one-cycles, which is described in [CLS11, Section 6.4].

Let Σ be a simplicial fan in $N_\mathbb{R} \cong \mathbb{R}^n$. Recall that we denote by $\Sigma(k)$ the set of k-dimensional cones in Σ and rays are one-dimensional cones $\rho \in \Sigma(1)$. Let $\tau \in \Sigma(n-1)$ be the intersection of two cones σ, σ' in $\Sigma(n)$. Since Σ is simplicial, we can label the minimal generators u_i of σ, σ', and τ such that

$$\sigma = \mathrm{Cone}(u_1, \dots, u_n), \quad \sigma' = \mathrm{Cone}(u_2, \dots, u_{n+1}), \quad \tau = \mathrm{Cone}(u_2, \dots, u_n). \tag{3.1}$$

Moreover, since the generators $u_1, \dots, u_{n+1}$ are linearly dependent, they satisfies a (up to multiplication with a constant unique) linear relation, called the *wall relation*

$$\sum_{i=1}^{n+1} b_i u_i = 0. \tag{3.2}$$

Since $u_2, \dots, u_n$ are linearly independent, we see that u_1, u_{n+1} are both non-zero.

Proposition 3.1 ([CLS11, Proposition 6.4.4])**.** *Let $\tau = \sigma \cap \sigma' \in \Sigma(n-1)$ be as in (3.1) and let $V(\tau)$ be the Zariski closure of the orbit corresponding to τ. Moreover, let $\rho_i = \mathbb{R}_{\geq 0} u_i$ be the rays in $N_{\mathbb{R}}$ generated by u_i. Then for every ray $\rho \in \Sigma(1)$ with associated divisor $D_\rho = V(\rho)$ in the toric variety X_Σ, the intersection number $D_\rho \cdot V(\tau)$ is given by*

$$D_\rho \cdot V(\tau) = \begin{cases} \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)} & \text{if } \rho = \rho_1, \\ \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma')} = \frac{b_{n+1}\,\mathrm{mult}(\tau)}{b_1\,\mathrm{mult}(\sigma)} & \text{if } \rho = \rho_{n+1}, \\ \frac{b_i\,\mathrm{mult}(\tau)}{b_1\,\mathrm{mult}(\sigma)} & \text{if } \rho = \rho_i, i \neq 1, n+1, \\ 0 & \text{otherwise,} \end{cases}$$

where the multiplicity of a cone $\gamma \subset N_{\mathbb{R}}$ with minimal generators $v_1, \dots, v_k$ is the index of the sublattice $\mathbb{Z}v_1 + \dots + \mathbb{Z}v_k \subset N_\gamma = (\mathbb{R}v_1 + \dots + \mathbb{R}v_k) \cap N$.

Sketch of the proof. Using the uniqueness up to scalar of the wall relation together with the observation that $\sum_{\rho \in \Sigma(1)} (D_\rho \cdot V(\tau))\, u_\rho = 0$ in $N_{\mathbb{R}}$, where u_ρ is the minimal generator of the ray $\rho \in \Sigma(1)$, it suffices to compute $D_{\rho_1} \cdot V(\tau)$ which can be done explicitly, see e.g. [CLS11, Lemma 6.4.2]. □

Remark 3.2. *If the fan Σ is regular, the multiplicity of all cones is 1. Moreover, we can assume without loss of generality that $b_1 = 1$ and thus also $b_{n+1} = 1$ by the above proposition. Hence the formula in Proposition 3.1 reduces to*

$$D_\rho \cdot V(\tau) = \begin{cases} 0 & \rho \notin \{\rho_1, \dots, \rho_{n+1}\}, \\ 1 & \rho = \rho_1, \ \rho_{n+1}, \\ b_i & \rho = \rho_i, \quad i \neq 1, n+1. \end{cases} \tag{3.3}$$

3.2. **A particular toric singularity.** We discuss the toric description of a particular singularity. This type of singularity appears naturally in our problem: We consider a strictly semi-stable R-scheme $\mathfrak{X} \to \operatorname{Spec} R$ for some discrete valuation ring R. Then étale locally at a point of the special fibre the scheme $\mathfrak{X}$ looks like

$$\{t - x_1 \cdots x_n = 0\} \subset \mathbb{A}^n_R, \tag{3.4}$$

where $t \in m_R \subset R$ is a uniformizer and $n \geq 1$. We are interested in Hartl's resolution after some finite, ramified base change $\tilde{R}/R$. The base-change $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ admits a resolution to a strictly semi-stable $\tilde{R}$-scheme by multiple blow-ups of the irreducible components, see Proposition 2.1. During the blow-ups the local equation looks like (3.5) below, see [Har01, proof of Proposition 2.2]. Thus the following lemma is a toroidal description of the behaviour under these blow-ups.

Lemma 3.3. *For any $m, n \geq 1$ and $r_1, \dots, r_m \geq 1$, the affine variety*

$$Z = \{x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m} - y_0 y_1 \cdots y_n = 0\} \subset \mathbb{A}_k^{n+m+1} \tag{3.5}$$

is the affine toric variety corresponding to the fan spanned by the cone

$$\sigma = \sigma_{n,r_1,\dots,r_m} = \mathrm{Cone}\left(\{e_i : i = 1, \dots, m\} \cup \{e_i + r_i f_j : i = 1, \dots, m,\ j = 1, \dots, n\}\right)$$

in $N_{\mathbb{R}} = \mathbb{R}^{m+n}$, where $e_1, \dots, e_m, f_1, \dots, f_n$ is a basis of $N = \mathbb{Z}^m \oplus \mathbb{Z}^n$. Moreover,

(a) The rays $\rho_{i,j} = \mathbb{R}_{\geq 0}(e_i + r_i f_j)$ and $\rho_{i,0} = \mathbb{R}_{\geq 0} e_i$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ correspond to the irreducible subvarieties $D_{\rho_{i,j}} = V(x_i, y_j)$ for $i = 1, \dots, m$ and $j = 0, 1, \dots, n$.

(b) For $i = 1, \dots, m$ the natural projection $\overline{\pi} = \mathrm{pr}_i \oplus 0 \colon \mathbb{Z}^m \oplus \mathbb{Z}^n \to \mathbb{Z}$ onto the i-th coordinate is compatible with the fan given by σ and $\mathbb{R}_{\geq 0} \cdot 1 \subset N'_{\mathbb{R}} = \mathbb{R}$ and the corresponding toric morphism is the projection onto the i-th coordinate x_i: $\pi = \mathrm{pr}_i \colon Z \to \mathbb{A}_k^1$.

(c) For $r_1 \geq 1$, the blow-up of Z along $V(x_1, y_1)$ is given by the fan spanned by the two cones

$$\sigma' = \mathrm{Cone}(\{e_i : i = 1, \dots, m+1\} \cup \{e_i + r_i f_j : i = 1, \dots, m+1,\ j = 2, \dots, n\})$$
$$\sigma'' = \mathrm{Cone}(\{e_i : i = 2, \dots, m+1\} \cup \{e_i + r_i f_j : i = 2, \dots, m+1,\ j = 1, \dots, n\}),$$

where $e_{m+1} := e_1 + f_1$ and $r_{m+1} := r_1 - 1$.

Remark 3.4. *For $m = 2$ and $r_1 = r_2 = 1$, [Shi22, Lemma 2.2] provides a similar computation.*

Proof. Let σ be the cone as defined in the statement. We claim that the dual cone $\sigma^\vee$ is given by

$$\sigma^\vee = \mathrm{Cone}(e_1^*, \dots, e_m^*, f_1^*, \dots, f_n^*, r_1 e_1^* + \dots + r_m e_m^* - f_1^* - \dots - f_n^*).$$

If this claim is true, the first statement follows immediately, because

$$k[\sigma^\vee \cap M] = k[x_1, \dots, x_m, y_0, \dots, y_n] / (x_1^{r_1} \cdots x_m^{r_m} - y_0 y_1 \cdots y_n).$$

To prove the claim, note that "$\supseteq$" is obvious. We prove the other inclusion: Let

$$\sum_{i=1}^m a_i e_i^* + \sum_{j=1}^n b_j f_j^* \in \sigma^\vee.$$

By definition, the coefficients a_i, b_j satisfy

$$a_i \geq 0, \quad a_i + r_i b_j \geq 0, \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, n.$$

Let $-\lambda = \min\{b_1, \dots, b_n, 0\} \leq 0$, then

$$\sum_{i=1}^m a_i e_i^* + \sum_{j=1}^n b_j f_j^* = \underbrace{\lambda}_{\geq 0}(r_1 e_1^* + \dots + r_m e_m^* - f_1^* - \dots - f_n^*) + \sum_{i=1}^m \underbrace{(a_i - \lambda r_i)}_{a_i + r_i b_j \text{ or } a_i \geq 0} e_i^* + \sum_{j=1}^n \underbrace{(\lambda + b_j)}_{\geq 0} f_j^*.$$

This shows "$\subseteq$" and thus the claim.

We prove item (a): Note that the orthogonal part of the cones $\rho_{i,j}$ for $i = 1, \dots, m$ and $j = 0, \dots, n$ are given by

$$\rho_{i,j}^\perp = \begin{cases} \mathrm{Cone}\left(e_1^*, \dots, \widehat{e_i^*}, \dots, e_m^*, f_1^*, \dots, f_n^*\right) & \text{if } j = 0, \\ \mathrm{Cone}\left(e_1^*, \dots, \widehat{e_i^*}, \dots, e_m^*, f_1^*, \dots, \widehat{f_j^*}, \dots, f_n^*, r_1 e_1^* + \dots r_m e_m^* - f_1^* - \dots - f_n^*\right) & \text{if } j \neq 0. \end{cases}$$

Hence, the distinguished points of the cones $\rho_{i,j}$ are the points $(x_1, \dots, x_m, y_0, y_1, \dots, y_n) \in \mathbb{A}_k^{n+m+1}$ with $x_a = \delta_{a,i}$ and $y_b = \delta_{b,j}$ where $\delta_{\cdot,\cdot}$ is the Kronecker delta. Thus statement

(a) follows from the orbit-cone correspondence. Statement (b) follows directly from the construction.

For statement (c), note that the blow-up of (3.5) along $V(x_1, y_1)$ is given by

$$x_1 = x_1' y_1, \quad (x_1')^{r_1} x_2^{r_2} \cdots x_m^{r_m} y_1^{r_1 - 1} - y_0 y_2 \cdots y_n = 0,$$
$$y_1 = y_1' x_1, \quad (x_1)^{r_1 - 1} x_2^{r_2} \cdots x_m^{r_m} - y_0 y_1' y_2 \cdots y_n = 0.$$

Thus we find that the corresponding dual vectors are given by

$$(e_1')^* = e_1^* - f_1^*,\ (e_i')^* = e_i^*,\ (e_{m+1}')^* = f_1^*,\ (f_j')^* = f_j^*, \quad i = 2, \ldots, m,\ j = 2, \ldots, n,$$
$$(f_1'')^* = f_1^* - e_1^*,\ (e_i'')^* = e_i^*,\ (f_j'')^* = f_j^*, \quad i = 1, \ldots, m,\ j = 2, \ldots, n,$$

i.e.

$$e_{m+1}' = e_1 + f_1,\ e_i' = e_i,\ f_j' = f_j, \quad i = 1, \ldots, m,\ j = 2, \ldots, n,$$
$$e_1'' = e_1 + f_1,\ e_i'' = e_i,\ f_j'' = f_j, \quad i = 2, \ldots, m,\ j = 1, \ldots, n.$$

Thus the description for σ' and σ'' follows from the description of Z in (3.5). □

Remark 3.5. *Up to relabeling the coordinates the blow-up description in (c) also holds for the blow-up along $V(x_i, y_j)$ where we think of f_0 as $f_0 = 0$ and do the calculation formally. More precisely, the blow-up of $V(x_1, y_0)$ corresponds to the subdivision of σ into the two cones*

$$\sigma' = \mathrm{Cone}\left(\{e_i + r_i f_j : i = 1, \ldots, m+1,\ j = 1, \ldots, n\}\right),$$
$$\sigma'' = \mathrm{Cone}\left(\{e_2, \ldots, e_{m+1}\} \cup \{e_i + r_i f_j : i = 2, \ldots, m+1,\ j = 1, \ldots, n\}\right),$$

where $e_{m+1} = e_1$ and $r_{m+1} = r_1 - 1$.

In concrete examples one can work with an explicit (log) resolution. For our purposes it suffices to work with some (log) resolution and use its properties. The first such property is the following.

Proposition 3.6. *Let $\sigma = \sigma_{n,r}$ be a cone from Lemma 3.3 with $m = 1$. Then any sequence of subdivisions of the same form as in Lemma 3.3(c) terminates with a regular fan $\tilde{\Sigma}$. Moreover, every minimal generator of a ray in $\tilde{\Sigma}$ lies in the hyperplane $\{e_1 = 1\}$ and is a lattice point of the lattice generated by $e_1, f_1, f_2, \ldots, f_n$ as defined in Lemma 3.3.*

Remark 3.7. *Note that the subdivision from Lemma 3.3(c) corresponds to the blow-up of the irreducible components of the special fibre which is the procedure in Proposition 2.1.*

Proof. The termination of this process is precisely [Har01, Proposition 2.2] which yields a resolution after some finite ramified base change, see also Proposition 2.1. Since the total space $\tilde{\mathfrak{X}}$ of a strictly semi-stable model is regular, the local model of the toric variety is regular. Alternatively, the explicit local descriptions and the termination condition of the algorithm in [Har01, Proposition 2.2] imply also the regularity of the refined fan.

The last statement follows from Lemma 3.3(c) by induction. Indeed, starting with $\sigma = \sigma_{n,r}$ for some $n, r \in \mathbb{N}$ we see that the minimal generators of each ray is of the form e_1 or $e_1 + r f_j$ for some $j = 1, \ldots, n$, i.e. they lie in the hyperplane $\{e_1 = 1\}$ and are lattice points. In the j-th step, we add a new vector $e_{j+1} := e_{j'} + f_{k'}$ for some $j' = 1, \ldots, j$. $k' = 1, \ldots, n$. The new minimal generators after the blow-up are of the form

$$e_{j+1} + r_{j+1} f_k$$

for some $k = 1, \ldots, n$ and $r_{j+1} \in \mathbb{Z}_{\geq 0}$. Thus, we see by induction that all minimal generators lie in the hyperplane $\{e_1 = 1\}$ and are lattice points. □

4. Chow group of the resolution

We recall the statement of Theorem 1.1 from the introduction and the strategy laid out in the introduction. Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable scheme over a dvr R with algebraically closed residue field. Assume that the geometric generic fibre of $\mathfrak{X} \to \operatorname{Spec} R$ admits a decomposition of the diagonal. Then the complex

$$\bigoplus_{j\in I} \mathrm{CH}_1\left(Y_{j,L'}\right) \xrightarrow{\Phi_{\mathfrak{X}_A}} \mathrm{CH}_0\left(Y_{i,L'}\right) \xrightarrow{\sum_i \deg} \mathbb{Z} \longrightarrow 0 \tag{4.1}$$

is exact for every unramified extension A/R with induced extension L'/k of residue fields. Recall that $\Phi_{\mathfrak{X}_A} = \sum_{i\in I}\sum_{j\in I} \iota_i^* \iota_{j*}$ with $\iota_i \colon Y_{i,L'} \hookrightarrow \mathfrak{X}_A$ the natural inclusions. The assumption on the geometric generic fibre implies the exactness of (4.1) for a strictly semi-stable family $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ which is obtained by a finite (possibly ramified) base change and a resolution as in Proposition 2.1. We want to relate $\Phi_{\tilde{\mathfrak{X}}}$ with $\Phi_{\mathfrak{X}}$ such that we can deduce the exactness of the complex (4.1) for $\mathfrak{X}$ from the exactness of the complex (4.1) for $\tilde{\mathfrak{X}}$, which then proves Theorem 1.1. In this section, we describe the special fibre $\tilde{Y}$ of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ using subdivisions of the dual complex of an snc scheme and express its Chow group of one-cycles in terms of cycles supported Y. Before that we recall dual complexes and subdivisions and introduce some necessary notation. The description of $\mathrm{CH}_1(\tilde{Y})$ is then used in Section 5 to relate $\Phi_{\mathfrak{X}}$ and $\Phi_{\tilde{\mathfrak{X}}}$.

We follow [Hat02, Appendix] and [dFKX, Definition 7] for the definition of Δ-complexes.

Definition 4.1 (Δ-complex)**.**
(1) The notion of an Δ-complex is defined by induction on the dimension. A 0-*skeleton* $\mathcal{C}^0$ is a collection of points, called *vertices*. Inductively, define an n-*skeleton* $\mathcal{C}^n$ by attaching a collection $\{\sigma_i : i \in I_n\}$ of simplices of dimension n to an $(n-1)$-skeleton $\mathcal{C}^{n-1}$ via *characteristic maps* $\varphi_i \colon \partial\sigma_i \to \mathcal{C}^{n-1}$ and identifying points on the boundary $\partial\sigma_i$ of σ_i with their image in $\mathcal{C}^{n-1}$. A (regular finite-dimensional unordered) Δ-*complex* $\mathcal{C}$ is an n-skeleton $\mathcal{C}^n$ for some $n \in \mathbb{Z}_{\geq 1}$ such that all characteristic maps are embeddings. Then n is called the *dimension* of $\mathcal{C}$.
(2) Let σ be a simplex of $\mathcal{C}$. A (proper) *face* of σ is a simplex σ' of $\mathcal{C}$ such that $\sigma' \subset \partial\sigma$.
(3) We call a simplex of dimension n an n-*simplex*. The set of n-simplices of a Δ-complex $\mathcal{C}$ is denoted by $\mathcal{C}(n)$.

Recall from graph theory: A vertex v' in a graph is adjacent to a vertex v if there exists an edge with endpoints v and v'. We extend this definition to Δ-complexes by considering the 1-skeleton of the Δ-complex which is a graph.

Definition 4.2 (Adjacent vertices)**.** Let $\mathcal{C}$ be a Δ-complex and let $v \in \mathcal{C}(0)$ be a vertex. A vertex $v' \in \mathcal{C}(0)$ is called *adjacent to* v if there exists an 1-simplex σ such that $\partial\sigma = \{v, v'\}$. We denote the set of adjacent vertices of v in $\mathcal{C}$ by $A_{\mathcal{C}}(v)$.

Moreover, we define walls of a Δ-complex inspired by the definition of walls for toric varieties, see e.g. [Dan78, Section 10.5].

Definition 4.3 (Walls)**.** Let $\mathcal{C}$ be a Δ-complex of dimension n. A simplex $\tau \in \mathcal{C}(n-1)$ is a *wall* of $\mathcal{C}$ if τ is a common face of two simplices $\sigma_1, \sigma_2 \in \mathcal{C}(n)$. Equivalently if τ is the (set-theoretic) intersection of the simplices σ_1, σ_2.

We recall the definition of a subdivision, see e.g. [Mun84, §15].

Definition 4.4 (Subdivision of Δ-complex)**.** Let $\mathcal{C}$ be a Δ-complex. A *subdivision* of $\mathcal{C}$ is a Δ-complex $\mathcal{C}'$ such that

(i) every simplex σ' of $\mathcal{C}'$ is contained in a simplex of $\mathcal{C}$.
(ii) every simplex σ of $\mathcal{C}$ is a non-empty union of finitely many simplices of $\mathcal{C}'$.

We formally denote the subdivision by a map $\psi\colon \mathcal{C}' \to \mathcal{C}$. For a simplex σ' of $\mathcal{C}'$, we denote the smallest simplex in $\mathcal{C}$ containing the simplex σ' by $\psi_*\sigma'$. The non-empty union of simplices of $\mathcal{C}'$ in (ii) is denoted by $\psi^*\sigma$ for a simplex σ of $\mathcal{C}$.

Remark 4.5. *(i) It follows from item (ii) in the above definition that ψ^*v consist of a single vertex for a vertex $v \in \mathcal{C}(0)$.*
(ii) Note that $\psi^*\sigma$ is a subcomplex of $\mathcal{C}'$ for every simplex σ of $\mathcal{C}$, i.e. a subset of $\mathcal{C}'$ which becomes a complex by restricting the characteristic maps of $\mathcal{C}'$ to that subset.

For such a subdivision we define relative versions of adjacent vertices and walls which become useful later on in Section 5.

Definition 4.6. Let $\mathcal{C}'$ be a subdivision of a Δ-complex $\mathcal{C}$ in the above sense.

(i) For any vertex $v' \in \mathcal{C}'(0)$ we define the *set of relative adjacent vertices* as
$$A_\psi(v') := \begin{cases} A_{\mathcal{C}}(\psi_*v') & \text{if } \psi_*v' \in \mathcal{C}(0), \\ \emptyset & \text{otherwise,} \end{cases}$$
where $A_{\mathcal{C}}$ is defined in Definition 4.2.
(ii) A *relative wall* τ of $\mathcal{C}'$ is a wall of the subcomplex $\psi^*\sigma$ for some simplex σ of $\mathcal{C}$. We denote the set of relative walls of the subdivision $\mathcal{C}'$ by $\mathrm{RelWall}(\psi)$.
(iii) Let $v' \in \mathcal{C}'$ be a vertex. We define the *set of lines through v' with adjacent vertex in $\mathcal{C}$* as
$$\mathcal{L}_\psi(v') := \left\{(\sigma, v) \in \mathcal{C}(1) \times \mathcal{C}(0) : v, v' \in \sigma \text{ and } \psi^*v \in A_{\mathcal{C}'}(v')\right\}.$$

In other words, $\mathcal{L}_\psi(v')$ is the set of 1-simplices σ in $\mathcal{C}$ which contain v' (viewed as an element in the Δ-complex $\mathcal{C}$) such that one of the faces of σ is an adjacent vertex of v'.

We recall the well-known definition of the dual complex of an snc-scheme, see e.g. [dFKX, Definition 8].

Definition 4.7 (Dual complex)**.** Let $Y = \bigcup_{i\in I} Y_i$ be a pure dimensional scheme with irreducible components Y_i such that for every non-empty $J \subset I$
$$Y_J := \bigcap_{i\in J} Y_j$$
is either empty or smooth and equidimensional of codimension $|J| - 1$. The *dual complex $\mathcal{D}(Y)$ of Y* is the (regular finite-dimensional unordered) Δ-complex defined as follows. The vertices are the irreducible components of Y. For each non-empty $J \subset I$, we associate a $(|J|-1)$-simplex σ_Z to each irreducible component $Z \subset Y_J$ with the obvious characteristic maps. We denote the irreducible subscheme associated to a simplex σ of $\mathcal{D}(Y)$ by Y_σ and call them the *strata* of Y.

Remark 4.8. *Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme for some discrete valuation ring R with residue field k, see Section 2.3. The special fibre $Y = \mathfrak{X} \times_R k$ is an snc scheme. Thus we obtain a dual complex $\mathcal{D}(Y)$ of Y.*

Lemma 4.9. *Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre Y and let $\tilde{R}/R$ be a finite ramified extension of dvr's with ramification index r. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution of the base-change $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ as in Proposition 2.1. We denote the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ by $\tilde{Y}$. Then the dual complex $\mathcal{D}(\tilde{Y})$ of $\tilde{Y}$ is a subdivision of the dual complex $\mathcal{D}(Y)$ of Y, which we denote formally by a map $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$.*

The proof of Lemma 4.9 uses the following observation which allows us to translate between the étale local toric picture and the global picture via the dual complex.

Remark 4.10. *Let $\sigma_Z \in \mathcal{D}(Y)$ be a simplex associated to a stratum Z of Y. Then Z is étale locally the (affine) toric variety associated to the cone $\sigma_{n,1}$ from Lemma 3.3 with $m = 1$, where $n = \dim \sigma_Z$. Moreover, Lemma 3.3(a) and the orbit-cone correspondence imply that this identification respects the strata of Y in the sense that a stratum in the dual complex is mapped to the same stratum in the étale local toric picture.*

Proof of Lemma 4.9. Let $\sigma_Z \in \mathcal{D}(Y)$ be a simplex associated to a stratum Z of Y and let $\sigma_{n,1}$ be the corresponding cone under the correspondence in Remark 4.10. Then the base-change $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ corresponds étale locally to the cone $\sigma_{n,r}$ from Lemma 3.3 with the obvious morphism of lattices, namely scaling each generator of N by $\frac{1}{r}$ except the first one. (Note that this morphism is compatible with the cones $\sigma_{n,r}$ and $\sigma_{n,1}$). The resolution $\tilde{\mathfrak{X}} \to \mathfrak{X}$ corresponds étale locally by Proposition 3.6 to a subdivision of the cone $\sigma_{n,r}$. Moreover, this subdivision of cone is induced by a subdivision of the n-simplex and gives a simplicial fan, see Proposition 3.6. Since this construction is the étale local analogue of $\tilde{\mathfrak{X}} \to \mathfrak{X}$ and the correspondence in Remark 4.10 respects the strata of Y and $\tilde{Y}$, we see that the constructed subdivision is a subdivision in the sense of Definition 4.4 and is equal to $\mathcal{D}(\tilde{Y})$, as claimed. □

Remark 4.11. *Note that the correspondence between the étale local toric picture and the global picture via dual complexes does not hold in each step of sequence of subdivisions in Proposition 3.6 as the cones obtained during this sequence of subdivision are in general not simplicial. Hence, we use the (global) description with dual complexes only for the strictly semi-stable models $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ and $\mathfrak{X} \to \operatorname{Spec} R$.*

We describe the new components appearing in the resolution process of Proposition 2.1, which we described étale locally in Section 3.2. This enables us to explicitly describe the Chow group of one-cycles of the special fibres of the resolution, see Proposition 4.14.

Proposition 4.12. *With the same notation as in Setup 2.2, let $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ be the subdivision of dual complexes from Lemma 4.9. Recall that each component in Y and $\tilde{Y}$ corresponds to a vertex in $\mathcal{D}(Y)$ and $\mathcal{D}(\tilde{Y})$, respectively. For every vertex $\tilde{v} \in \mathcal{D}(\tilde{Y})(0)$, the irreducible component $\tilde{Y}_{\tilde{v}}$ is obtained by a finite sequence of blow-ups in smooth centers and Zariski locally trivial $\mathbb{P}^1$-bundles of the base-change $Y_{\psi_*\tilde{v},L}$ where $Y_{\psi_*\tilde{v}}$ is the irreducible subscheme of Y associated to the simplex $\psi_*\tilde{v}$, see Definition 4.7.*

Proof. This follows from the description of the resolution process in [Har01, Proposition 2.2] together with the construction of the dual complex $\mathcal{D}(Y)$ and the toric construction of its subdivision. Indeed, Hartl's resolution process is iterative and in each step every component will be blown-up in a smooth (possibly empty) center and the new components in that step are étale locally trivial $\mathbb{P}^1$-bundles over the intersection of two components in the previous step, see Lemma 3.3 or [Har01, Proof of Proposition 2.2]. Since we blow-up in that step one of the two components, we see that the étale locally trivial $\mathbb{P}^1$-bundle admits a section. Thus it is a Zariski-locally trivial $\mathbb{P}^1$-bundle over the intersection of two components in the previous step. Hence each component $\tilde{Y}_{\tilde{v}}$ for $\tilde{v} \in \mathcal{D}(\tilde{Y})(0)$ is of the form described in the statement. $\square$

Definition 4.13. Let the notation be as in Proposition 4.12. Let $\tau \in \operatorname{RelWall}(\psi)$ be a relative wall as defined in Definition 4.6(ii). Let $\psi_*\tau$ be the smallest simplex of $\mathcal{D}(Y)$ containing τ. Then τ corresponds via the toric description to a (Zariski) locally trivial $\mathbb{P}^1$-bundle P_τ over $Y_{\psi_*\tau,L}$. We denote the bundle projection by $\tilde{q}_\tau \colon P_\tau \to Y_{\psi(\tau),L}$.

Additionally, we fix for any $\tau \in \operatorname{RelWall}(\psi)$ a vertex $v(\tau) \in \mathcal{D}(\tilde{Y})(0)$ such that $v(\tau)$ is a face of τ. This implies via the toric description that the projective bundle P_τ lies inside $\tilde{Y}_{v(\tau)}$ and we denote the natural inclusion by $\tilde{\iota}'_\tau \colon \tilde{Y}_{v(\tau)} \hookrightarrow \tilde{Y}$.

Proposition 4.14. *Let the notation be as in Proposition 4.12 and Definition 4.13.*

(a) *The Chow group $\mathrm{CH}_1(Y,\Lambda)$ of one-cycles on Y is generated by cycles of the form*

$$\gamma = \sum_{v\in\mathcal{D}(Y)(0)} \iota'_{v*}\gamma_v \in \mathrm{CH}_1(Y,\Lambda), \tag{4.2}$$

where $\gamma_v \in \mathrm{CH}_1(Y_v,\Lambda)$ is a one-cycle and $\iota'_v \colon Y_v \hookrightarrow Y$ is the natural inclusion.

(b) *The Chow group $\mathrm{CH}_1(\tilde{Y},\Lambda)$ of one-cycles on $\tilde{Y}$ is generated by cycles of the form*

$$\tilde{\gamma} = \sum_{v\in\mathcal{D}(Y)(0)} \tilde{\iota}'_{v*}\tilde{\gamma}_v + \sum_{\tau\in\operatorname{RelWall}(\psi)} \tilde{\iota}'_{\tau*}\tilde{q}^*_\tau\tilde{\alpha}_\tau \in \mathrm{CH}_1(\tilde{Y},\Lambda), \tag{4.3}$$

where $\tilde{\gamma}_v \in \mathrm{CH}_1(Y_{v,L},\Lambda)$ is a one-cycle, $\tilde{\iota}'_v \colon \tilde{Y}_v \hookrightarrow \tilde{Y}$ is the natural inclusion, $\tilde{\alpha}_\tau \in \mathrm{CH}_0\left(Y_{\psi_*\tau,L},\Lambda\right)$ is a zero-cycle, and $\tilde{\iota}'_\tau \colon \tilde{Y}_{v(\tau)} \hookrightarrow \tilde{Y}$ and $\tilde{q}_\tau \colon P_\tau \to Y_{\psi(\tau),L}$ are the natural morphisms, see also Definition 4.13.

Remark 4.15. (a) *This says that the Chow group of one-cycles on $\tilde{Y}$ is governed by $\mathrm{CH}_1(Y,\Lambda)$ and by a toric part. Indeed, we can see (4.3) as*

$$\tilde{\gamma} = \underbrace{\sum_{v\in\mathcal{D}(Y)(0)} \tilde{\iota}'_{v*}\tilde{\gamma}_v}_{\text{from } \mathrm{CH}_1(Y,\Lambda)} + \underbrace{\sum_{\tau\in\operatorname{RelWall}(\psi)} \tilde{\iota}'_{\tau*}\tilde{q}^*_\tau\tilde{\alpha}_\tau}_{\text{“toric part”}}.$$

Morally speaking, the difference between (4.2) and (4.3) lies in this toric part.

(b) *Two examples of such a description can be found in Appendix A.*

Proof. Since any prime one-cycle, i.e. every irreducible, 1-dimensional subvariety, is contained in an irreducible component, statement (a) follows immediately. The same argument shows that $\mathrm{CH}_1(\tilde{Y},\Lambda)$ is generated by one-cycles on the components of $\tilde{Y}$. We show statement (b) by using the structure of the components from Proposition 4.12: Recall the following two standard facts about Chow groups, see e.g. [Ful98, Theorem 3.3 (b) and Proposition 6.7 (e)]:

(i) Let $P = \mathbb{P}(E) \to W$ be the projectivization of a vector bundle E on a smooth variety W of relative dimension ≥ 2. Then

$$\mathrm{CH}_1(P,\Lambda) \cong \mathrm{CH}_0(W,\Lambda) \oplus \mathrm{CH}_1(W,\Lambda), \qquad \mathrm{CH}_0(P,\Lambda) \cong \mathrm{CH}_0(W,\Lambda),$$

where the isomorphisms are given by pulling-back a cycle on W along the flat map $P \to W$ and intersecting with a suitable power of the canonical line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ which is dual to the pull-back of the vector bundle E on W to $P = \mathbb{P}(E)$.

(ii) Let $\tilde{W}$ be the blow-up of a smooth variety W along a smooth subvariety Z. Then
$$\mathrm{CH}_1(\tilde{W},\Lambda) \cong \mathrm{CH}_0(Z,\Lambda) \oplus \mathrm{CH}_1(W,\Lambda),$$
where the isomorphism is given by applying the isomorphism from (i) to the projective bundle $\mathbb{P}(\mathcal{N}_{Z/W}) \to Z$ and pushing the one-cycle forward to $\tilde{W}$ as well as pushing a one-cycle on $W \setminus Z$ forward along the natural morphism $W \setminus Z \to \tilde{W}$.

Let $\tilde{v} \in \mathcal{D}(\tilde{Y})(0)$ be a vertex. Assume first $\tilde{v} = \psi^* v$ for a vertex $v \in \mathcal{D}(Y)(0)$. (Note that $\psi^* v$ consists of a single vertex by Remark 4.5(i).) By Proposition 4.12, the component $\tilde{Y}_{\tilde{v}}$ of $\tilde{Y}$ is a sequence of smooth blow-ups over $Y_{v,L}$, i.e.
$$\tilde{Y}_{\tilde{v}} = \mathrm{Bl}_{Z_r}\mathrm{Bl}_{Z_{r-1}}\cdots \mathrm{Bl}_{Z_1} Y_{v,L}$$
where $Z_1,\dots,Z_r$ are the smooth centers of the blow-ups. Then by fact (ii) we find
$$\mathrm{CH}_1(\tilde{Y}_{\tilde{v}},\Lambda) \cong \bigoplus_{i=1}^{r} \mathrm{CH}_0(Z_i,\Lambda) \oplus \mathrm{CH}_1\left(Y_{v,L},\Lambda\right). \tag{4.4}$$
The $\mathrm{CH}_0(Z_i,\Lambda)$ parts correspond under the orbit-cone correspondence to some relative walls. Indeed, every zero-cycle is a (formal) linear combination of closed points. For each closed point the image under the isomorphism (4.4) is the one-cycle given by pulling back the point to the corresponding $\mathbb{P}^1$-bundle over Z_i and pushing the one-cycle forward to $\tilde{Y}_{\tilde{v}}$ via the natural inclusion. This one-cycle is an irreducible rational curve which corresponds via the orbit-cone correspondence to a relative wall. Hence, we see that $\mathrm{CH}_1(\tilde{Y}_{\tilde{v}},\Lambda)$ is described by one-cycles corresponding to relative walls and $\mathrm{CH}_1\left(Y_{v,L},\Lambda\right)$.

We consider now the case $\tilde{v} \notin \psi^*\mathcal{D}(Y)(0)$, i.e. the simplex $\sigma = \psi_*\tilde{v}$ of $\mathcal{D}(Y)$ has $\dim(\sigma) \geq 1$. By Proposition 4.12, $\tilde{Y}_{\tilde{v}}$ is a sequence of smooth blow-ups and projective bundles over some $Y_{\sigma,L}$, i.e.
$$\tilde{Y}_{\tilde{v}} = \mathrm{Bl}_{Z_r}\mathrm{Bl}_{Z_{r-1}}\cdots \mathrm{Bl}_{Z_1} P_{\tilde{v}}$$
where $Z_1,\dots,Z_r$ are the smooth centers of blow-ups and $P_{\tilde{v}}$ is a Zariski locally trivial $\mathbb{P}^1$-bundle over $W_{\tilde{v}}$ such that the intersection with a different component $\tilde{Y}_{\tilde{v}'}$ yields a section. Then the facts (i) and (ii) above imply
$$\mathrm{CH}_1(\tilde{Y}_{\tilde{v}},\Lambda) \cong \bigoplus_{i=1}^{r} \mathrm{CH}_0(Z_i,\Lambda) \oplus \mathrm{CH}_0(W_{\tilde{v}},\Lambda) \oplus \mathrm{CH}_1(W_{\tilde{v}},\Lambda).$$
The latter part $\mathrm{CH}_1(W_{\tilde{v}},\Lambda)$ is contained in $\mathrm{CH}_1(Y_{\tilde{v}'},\Lambda)$ and by the same argument as in the case $\tilde{v} \in \psi^*\mathcal{D}(Y)(0)$ we find that the $\mathrm{CH}_0(Z_i,\Lambda)$ parts and the $\mathrm{CH}_0(W_{\tilde{v}},\Lambda)$ part correspond via the orbit-cone correspondence to some relative walls (in the above sense). Hence, we see that $\mathrm{CH}_1(\tilde{Y}_{\tilde{v}},\Lambda)$ is described by one-cycles corresponding to relative walls and the CH_1 which appear in a previous step of Proposition 2.1. Inductively we get that the CH_1 of the "new" components (corresponding to vertices in $\mathcal{D}(\tilde{Y})(0) \setminus \psi^*\mathcal{D}(Y)(0)$) is described by one-cycles corresponding to the relative walls and $\mathrm{CH}_1(Y_{v,L},\Lambda)$ for $v \in \mathcal{D}(Y)$. The one-cycles corresponding to the relative walls and the one-cycles supported on $Y_{v,L}$ might appear in multiple components, but they need to be counted once, as for an irreducible subvariety Z of $\tilde{Y}_v$ and $\tilde{Y}_w$ the diagram with the natural inclusions
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & \tilde{Y}_v \arrow[d, hook, "\iota'_v"] \\ \tilde{Y}_w \arrow[r, hook, "\iota'_w"] & \tilde{Y}, \end{tikzcd}$$

commutes. This proves part (b) of the proposition. □

5. Analysis of the base-change

We recall our setup (Setup 2.2), which we fix throughout this entire section. Let R be a discrete valuation ring with residue field k (not necessarily algebraically closed) and let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre Y whose irreducible components are geometrically integral. Let $\tilde{R}/R$ be a finite extension of dvr's with induced extension L/k of residue fields. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a strictly semi-stable $\tilde{R}$-scheme with special fibre $\tilde{Y}$ which is a resolution of the base change $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.1 and let $q\colon \tilde{\mathfrak{X}} \to \mathfrak{X}$ denote the natural morphism and also the restriction to the special fibre $q\colon \tilde{Y} \to Y$. Recall that the dual complex $\mathcal{D}(\tilde{Y})$ of $\tilde{Y}$ is a subdivision of the dual complex $\mathcal{D}(Y)$ of Y which we denote formally by $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$, see Lemma 4.9. Consider the homomorphism

$$\bigoplus_{j\in I} \mathrm{CH}_1(Y_j, \Lambda) \xrightarrow{\Phi^{\Lambda}_{\mathfrak{X}}} \bigoplus_{i\in I} \mathrm{CH}_0(Y_i, \Lambda), \tag{5.1}$$

where $Y_i \subset Y$ are the irreducible components of Y with $\iota_i\colon Y_i \hookrightarrow \mathfrak{X}$ the natural inclusions and $\Phi^{\Lambda}_{\mathfrak{X}} = \sum_{i\in I}\sum_{j\in I} \iota_i^* \iota_{j*}$. In this section we relate (5.1) with $\Phi^{\Lambda}_{\tilde{\mathfrak{X}}}$. This relays on the following two auxiliary functions.

Construction 5.1. We construct a function

$$d\colon \mathcal{D}(\tilde{Y})(0) \times \mathcal{D}(Y)(0) \longrightarrow \mathbb{Z}_{\geq 0}, \tag{5.2}$$

which measures a "distance" between the vertices of the simplicial complex $\mathcal{D}(\tilde{Y})$ to the vertices of $\mathcal{D}(Y)$.

Let $v' \in \mathcal{D}(\tilde{Y})(0)$ be a vertex and let $\sigma = \psi_* v'$ be the smallest simplex in $\mathcal{D}(Y)$ containing v'. Then σ corresponds by Remark 4.10 to a cone $\sigma_{n,1}$ from Lemma 3.3 with $m = 1$ and $n = \dim \sigma$. Let $w_0, w_1, \ldots, w_n \in \sigma_{n,1}$ be the minimal generators of the cone $\sigma_{n,1}$, i.e. $w_i = e_1 + f_i$ in the notation of Lemma 3.3 (with $f_0 = 0$). The vertex v' corresponds to a ray in the fan obtained by the resolution associated to ψ, see also the proof of Lemma 4.9. Let w' denote the minimal generator of this ray. Since this fan is a toric subdivision of the cone $\sigma_{n,1}$ in the sense of [CLS11, before Example 3.3.12], we can write

$$w' = \sum_{i=0}^{n} a_i w_i$$

for some $a_i \geq 0$. As $\sigma_{n,1}$ (or equivalently σ) is simplicial, these a_i's are unique. The a_i's additionally satisfy

$$\sum_{i=0}^{n} a_i = 1, \tag{5.3}$$

by Proposition 3.6. We define for any vertex $v \in \mathcal{D}(Y)(0)$

$$d(v', v) = \begin{cases} r(1 - a_i) & \text{if } v = v_i, \\ r & \text{otherwise,} \end{cases} \tag{5.4}$$

where $v_i \in \mathcal{D}(Y)(0)$ is the vertex associated to the minimal generator w_i in $\sigma_{n,1}$. It remains to check that $d(v', v) \in \mathbb{Z}_{\geq 0}$ for all $v \in \mathcal{D}(Y)(0)$. If $v \neq v_i$ for all $i = 0, 1, \ldots, n$, this is clear. For $i = 0, 1, \ldots, n$, we note that $0 \leq a_i \leq 1$, by (5.3). Hence $d(v', v_i) \geq 0$ for all $i = 0, 1, \ldots, n$. Moreover, the point w' is a lattice point in the sublattice generated by $e_1, \frac{1}{r}f_1, \frac{1}{r}f_2, \ldots, \frac{1}{r}f_n$, see Proposition 3.6. Since $w_j = e_1 + f_j$, this immediately implies $a_i \in \frac{1}{r}\mathbb{Z}$. These two observations yield $d(v', v_i) \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n$, i.e. we have a well-defined map of the form (5.2).

Construction 5.2. We construct a function

$$I\colon \operatorname{RelWall}(\psi) \times \mathcal{D}(\tilde{Y})(0) \longrightarrow \mathbb{Z}, \tag{5.5}$$

which encodes the intersection numbers of one-cycles corresponding to the relative walls with the divisors corresponding to the vertices of the simplicial complex.

Let σ be a simplex in $\mathcal{D}(Y)$ and let $\tau' = \sigma_1' \cap \sigma_2'$ be a wall of the subcomplex $\psi^*\sigma$ of $\mathcal{D}(\tilde{Y})$, i.e. τ' is a relative wall of ψ. Let $d-1 = \dim \tau'$ be the dimension of the simplex τ'. Let $v_0', \ldots, v_d'$ and $v_1', \ldots, v_{d+1}'$ be the vertices of σ_1' and σ_2', respectively. Recall once more that we associate to σ the simplicial cone $\sigma_{d,1}$ from Lemma 3.3 with $m = 1$, see Remark 4.10. Then $\psi^*\sigma$ corresponds to a fan Σ' which is a toric subdivision of the cone $\sigma_{\dim\sigma,1}$. The simplices σ_1', σ_2', and τ' correspond to cones of Σ' of dimensions $d+1, d+1$, and d respectively. We denote the cones by S_1', S_2', and T' respectively. Note that $T' = S_1' \cap S_2'$. Recall also that the vertices $v_0', \ldots, v_{d+1}'$ correspond to rays $\rho_0', \ldots, \rho_{d+1}'$ of S_1' or S_2'. We define for any $v' \in \mathcal{D}(\tilde{Y})(0)$

$$I_{\tau'}(v') := I(\tau', v') := \begin{cases} D_{\rho_i'} \cdot V(T') & \text{if } v' = v_i' \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

where $D_{\rho_i'}$ is the divisor associated to the ray ρ_i' and $V(T')$ is the Zariski closure of the orbit corresponding to T' via the orbit-cone correspondence, see also Proposition 3.1. It is obvious that $I_{\tau'}(v')$ is an integer, i.e. the function I is well-defined as claimed in (5.5).

These two auxiliary functions satisfy the following crucial relation, which is deduced from the wall relation, see (3.2).

Lemma 5.3. *The functions d defined in Construction 5.1 and I defined in Construction 5.2 satisfy for every $v \in \mathcal{D}(Y)(0)$ and $\tau' \in \operatorname{RelWall}(\psi)$*

$$\sum_{v' \in \mathcal{D}(\tilde{Y})(0)} (r - d(v', v))\, I_{\tau'}(v') = 0. \tag{5.7}$$

Proof. With the same notation as in Construction 5.2, let $w_0', \ldots, w_{d+1}'$ be the minimal generators of the rays $\rho_0', \ldots, \rho_{d+1}'$. The fan Σ' is a toric subdivision of the cone $\sigma_{d,1}$ associated to σ and let $w_1, \ldots, w_{d+1}$ be the minimal generators of the cone. We write for $i = 0, \ldots, d+1$ the minimal generators w_i' as a linear combination

$$w_i' = \sum_{j=1}^{d+1} a_{ij} w_j, \tag{5.8}$$

which uniquely exists, since σ_1' and σ_2' are contained in the simplex σ. Moreover, by the wall relation (3.2) there exist (up to scaling) unique $b_0, \ldots, b_{d+1}$ such that

$$\sum_{i=0}^{d+1} b_i w_i' = 0.$$

Together with (5.8) we find

$$0 = \sum_{i=0}^{d+1} b_i w_i' = \sum_{i=0}^{d+1} b_i \left(\sum_{j=1}^{d+1} a_{ij} w_j \right) = \sum_{j=1}^{d+1} \left(\sum_{i=0}^{d+1} a_{ij} b_i \right) w_j$$

$$= \lambda \sum_{j=1}^{d+1} \left(\sum_{i=0}^{d+1} (r - d(v_j, v_i')) I_{\tau'}(v_i') \right) w_j,$$

where $\lambda = \frac{b_0}{r} \neq 0$ is the necessary scaling due to the choice with regard to the b_j's. Note that we used in the last step the definition of d and I together with (3.3). Since $I_{\tau'}(v') = 0$ for every v' different to all (v_i')'s, this shows (5.7) as the cone σ is simplicial and $\lambda \neq 0$. $\square$

In order to relate the homomorphism (5.1) with the homomorphism for the resolution $\tilde{X}$ after a finite base change, we replace the term $\bigoplus_j \mathrm{CH}_1(Y_j, \Lambda)$ in (5.1) with $\mathrm{CH}_1(Y, \Lambda)$ which does not change the image of the map, see Remark 5.5.

Definition 5.4 ([PS23, Definition 3.1])**.** Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre Y. Denote the irreducible components of Y by Y_i with $i \in I$. Then we define

$$\Phi^{\Lambda}_{\mathcal{X},Y_i} \colon \mathrm{CH}_1(Y, \Lambda) \xrightarrow{\iota_*} \mathrm{CH}_1(\mathcal{X}, \Lambda) \xrightarrow{\iota_i^*} \mathrm{CH}_0(Y_i, \Lambda),$$

where $\iota \colon Y \to \mathcal{X}$ and $\iota_i \colon Y_i \to \mathcal{X}$ are the natural inclusions. Moreover, we define by abuse of notation (see Remark 5.5 below)

$$\Phi^{\Lambda}_{\mathcal{X}} := \sum_{i \in I} \Phi^{\Lambda}_{\mathcal{X},Y_i} \colon \mathrm{CH}_1(Y, \Lambda) \longrightarrow \bigoplus_{i \in I} \mathrm{CH}_0(Y_i, \Lambda).$$

Remark 5.5. *The map in (5.1) is the composition of the natural surjection*

$$\bigoplus_{i \in I} \mathrm{CH}_1(Y_i, \Lambda) \longrightarrow \mathrm{CH}_1(Y, \Lambda),$$

given by the pushforward along the inclusions $Y_i \hookrightarrow Y$, with the map $\Phi^{\Lambda}_{\mathfrak{X}}$ (defined in Definition 5.4), see also [PS23, Lemma 3.2]. In particular, the images of both maps named $\Phi^{\Lambda}_{\mathfrak{X}}$ are equal, which justifies the notation. Moreover, in the following we will mostly write $\Phi^{\Lambda}_{\mathfrak{X}}$ for the map in Definition 5.4.

Remark 5.6. *Let the notation be as in Definition 5.4. Let $\gamma_i \in \mathrm{CH}_1(Y_i, \Lambda)$ be a one-cycle for some $i \in I$ and let $\iota_i' \colon Y_i \hookrightarrow Y$ be the natural inclusion. Then for every $j \in I \setminus \{i\}$*

$$\Phi^{\Lambda}_{\mathcal{X},Y_j}(\iota'_{i*}\gamma_i) = \begin{cases} \iota_{ij,j*}\, \gamma_i|_{Y_i \cap Y_j} \in \mathrm{CH}_0(Y_j, \Lambda), & \text{if } Y_i \cap Y_j \neq \emptyset, \\ 0 \in \mathrm{CH}_0(Y_j, \Lambda), & \text{otherwise}, \end{cases}$$

and

$$\Phi^{\Lambda}_{\mathcal{X},Y_i}(\iota'_{i*}\gamma_i) = -\sum_{\substack{l \neq i \\ Y_i \cap Y_l \neq \emptyset}} \iota_{il,i*}\, \gamma_i|_{Y_i \cap Y_l} \in \mathrm{CH}_0(Y_i, \Lambda),$$

where $\iota_{il,i} \colon Y_i \cap Y_l \hookrightarrow Y_i$ is the natural inclusion and we set $\gamma_i|_{Y_i \cap Y_j} := \iota^*_{ij,i}\gamma_i$, see [PS23, Lemma 3.2].

Recall that we fixed throughout this section a strictly semi-stable family $\mathfrak{X} \to \operatorname{Spec} R$ over a dvr R and a resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ after a finite extension of dvr's $\tilde{R}/R$ with induced extension L/k of residue fields. We denote the special fibres by Y and $\tilde{Y}$ and their dual complexes by $\mathcal{D}(Y)$ and $\mathcal{D}(\tilde{Y})$, respectively. Recall that $\psi \colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ is a subdivision, see Lemma 4.9. Recall that we denote the stratum associated to a simplex σ in $\mathcal{D}(Y)$ by Y_σ.

Lemma 5.7. *The homomorphism $\Phi_{\mathcal{X}}$ from Definition 5.4 is given for the strictly semi-stable families $\mathfrak{X} \to \operatorname{Spec} R$ and $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ as follows:*

(a) For any $\gamma \in \mathrm{CH}_1(Y,\Lambda)$ of the form (4.2) and for any $v \in \mathcal{D}(Y)(0)$,

$$\Phi^{\Lambda}_{\mathfrak{X},Y_v}(\gamma) = \sum_{w\in A_{\mathcal{D}(Y)}(v)} \iota_{vw,v\,*}\left(\gamma_w|_{Y_v\cap Y_w} - \gamma_v|_{Y_v\cap Y_w}\right) \in \mathrm{CH}_0(Y_v,\Lambda),$$

where $\iota_{vw,v}\colon Y_v\cap Y_w \hookrightarrow Y_v$ is the natural inclusion and $A_{\mathcal{D}(Y)}(v)$ is defined in Definition 4.2.

(b) For any $\tilde\gamma \in \mathrm{CH}_1(\tilde Y,\Lambda)$ of the form (4.3) and for any $\tilde v \in \mathcal{D}(\tilde Y)(0)$,

$$\begin{aligned}\Phi^{\Lambda}_{\tilde{\mathfrak{X}},\tilde Y_{\tilde v}}(\tilde\gamma) = &\sum_{v\in A_\psi(\tilde v)} -\tilde\iota_{v\tilde v,\tilde v\,*}\,\tilde\gamma_{\psi_*\tilde v}|_{(Y_v\cap Y_{\psi_*\tilde v})_L} + \sum_{(\sigma,v)\in\mathcal{L}_\psi(\tilde v)} \tilde\iota_{\sigma,\tilde v\,*}\,\tilde\gamma_v|_{Y_{\sigma,L}}\\ &+ \sum_{\tau\in\mathrm{RelWall}(\psi)} \tilde\iota_{\tau,\tilde v\,*} I_\tau(\tilde v)\tilde\alpha_\tau\end{aligned} \tag{5.9}$$

in $\mathrm{CH}_0(\tilde Y_{\tilde v},\Lambda)$, where $A_{\mathcal{D}(\tilde Y)}(v)$ is defined in Definition 4.2, ψ_* in Definition 4.4, $A_\psi(\tilde v)$, $\mathrm{RelWall}(\psi)$, and $\mathcal{L}_\psi(\tilde v)$ in Definition 4.6, and $I_\tau(\tilde v)$ in (5.6). Moreover, $\tilde\iota_{v\tilde v;\tilde v}$, $\tilde\iota_{\sigma,\tilde v}$, and $\tilde\iota_{\tau,\tilde v}$ are the natural inclusions, see also the remark below.

Remark 5.8. *(a) The zero-cycles appearing on the right hand side of (5.9) lie in $(Y_v\cap Y_{\psi_*\tilde v})_L$ for $v \in A_\psi(\tilde v)$, in $Y_{\sigma,L}$ for $(\sigma,v)\in\mathcal{L}_\psi(\tilde v)$, or in $Y_{\psi_*\tau,L}$ for $\tau \in$* $\mathrm{RelWall}(\psi)$*. If $I_\tau(\tilde v)\neq 0$, then we know by the construction in (5.6) that $\tilde v$ is a vertex of the simplex τ. Otherwise we disregard the term, i.e. those α_τ with $I_\tau(\tilde v)=0$.*

We know that a finite sequence of blow-ups in smooth centers and (Zariski) locally trivial $\mathbb{P}^1$-bundles of $(Y_v\cap Y_{\psi(\tilde v)})_L$, $Y_{\sigma,L}$, or $Y_{\psi(\tau),L}$ (for $I_\tau(\tilde v)\neq 0$) lies in $\tilde Y_{\tilde v}$, see Proposition 4.12. These are the natural inclusions $\tilde\iota_{v\tilde v,\tilde v}$, $\tilde\iota_{\sigma,\tilde v}$, and $\tilde\iota_{\tau,\tilde v}$, respectively. Note that this makes sense as the Chow group of zero-cycles is invariant under blow-ups in smooth centers and (Zariski) locally trivial $\mathbb{P}^1$-bundles by [Ful98, Theorem 3.3 (b) and Proposition 6.7 (e)].

(b) In accordance with the previous remark, we restrict the one-cycle $\tilde\gamma_{\psi_*\tilde v}$ in (5.9) not to $(Y_v\cap Y_{\psi_*\tilde v})_L$, but rather the strict transform of $(Y_v\cap Y_{\psi_*\tilde v})_L$ under the sequence of blow-ups described in the previous remark. Note that $(Y_v\cap Y_{\psi_*\tilde v})_L$ is already a Cartier divisor in $Y_{\psi_*\tilde Y,L}$. The same remark holds for $\tilde\gamma_v|_{Y_{\sigma,L}}$.

(c) Two explicit examples of the map $\Phi_{\tilde{\mathfrak{X}}}$ are provided in Appendix A.

Proof of Lemma 5.7. Item (a) follows immediately from the explicit description of $\Phi^{\Lambda}_{\mathfrak{X}}$ in Remark 5.6 together with (4.2). For item (b), we can use the same argument, i.e. apply Remark 5.6 to the explicit description of the one-cycles in (4.3). The only non-trivial part is the intersection

$$\tilde Y_{\tilde v}\cdot \tilde\iota_*\tilde\iota'_{\tau\,*}\tilde q^*_\tau\tilde\alpha_\tau = \tilde\iota_{\tau,\tilde v\,*} I_\tau(\tilde v)\alpha_\tau, \tag{5.10}$$

where $\tilde\iota\colon \tilde Y \hookrightarrow \tilde{\mathfrak{X}}$ is the natural inclusion and $\tilde Y_{\tilde v}\cdot$ is the intersection with the Cartier divisor $\tilde Y_{\tilde v}\subset\tilde{\mathfrak{X}}$. Since α_τ is a linear combination of rational equivalent classes of points, it suffices to show (5.10) for every point in $Y_{\psi(\tau),L}$. For each such point we can consider the étale local toric description of $\tilde{\mathfrak{X}}$ and find that $I_\tau(\tilde v)$ is by (5.6) the intersection multiplicity at that point. Since this is independent of the point chosen, we obtain (5.10) and thus item (b). ∎

Proposition 5.9 (Key formula)**.** *Let $\mathfrak{X}\to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre Y, let $\tilde R/R$ be a finite extension of dvr's of ramification index r with induced extension L/k of residue fields, and let $\tilde{\mathfrak{X}}\to\operatorname{Spec}\tilde R$ be a resolution of the base-change $\mathfrak{X}_{\tilde R}$ from Proposition 2.1 and let $\tilde Y$ be its special fibre. Let $q\colon \tilde Y\to Y$ be the natural morphism* *and let* $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ *be the subdivision in Lemma 4.9. Then for any* $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ *and* $v \in \mathcal{D}(Y)(0)$ *the following holds:*

$$\Phi^{\Lambda}_{\mathfrak{X},Y_v}(q_*\tilde{\gamma}) = \sum_{\tilde{v}\in\mathcal{D}(\tilde{Y})(0)} \iota_{\psi_*\tilde{v},v\,*}(r - d(\tilde{v}, v))q_*\Phi^{\Lambda}_{\tilde{\mathfrak{X}},\tilde{Y}_{\tilde{v}}}(\tilde{\gamma}) \in \mathrm{CH}_0(Y_v, \Lambda), \tag{5.11}$$

where $d(\tilde{v}, v)$ *is defined in* (5.4) *and* $\iota_{\psi_*\tilde{v},v}\colon Y_{\psi_*\tilde{v}} \hookrightarrow Y_v$ *is the natural inclusion for* $d(\tilde{v}, v) < r$.

Proof. Let $v \in \mathcal{D}(Y)(0)$ be a vertex corresponding to an irreducible component Y_v of the special fibre Y and let $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ be a one-cycle on $\tilde{Y}$ which we can assume to be of the form (4.3). Note first that $d(\tilde{v}, v) < r$ if and only if v is a vertex of $\psi_*\tilde{v}$ by the construction in (5.4). Hence the right hand side of (5.11) is well-defined in $\mathrm{CH}_0(Y_v, \Lambda)$. We claim that the equality in (5.11) is an equality of cycles. If $r = 1$, then there is nothing to prove, as the base-change and the resolution are trivial, i.e. we can assume without loss of generality $r > 1$. We note that by the projection formula [Ful98, Proposition 2.3 (c)] for every $w \in A_{\mathcal{D}(Y)}(v)$

$$q_*\left(\tilde{\gamma}_v|_{(Y_v\cap Y_w)_L}\right) = (q_*\tilde{\gamma}_v)|_{Y_v\cap Y_w}\,. \tag{5.12}$$

Similarly, for every $\tilde{v} \in A_{\mathcal{D}(\tilde{Y})}(v)$ and $\sigma \in \mathcal{D}(Y)(1)$ such that $(\sigma, v) \in \mathcal{L}_{\psi}(\tilde{v})$ the projection formula yields

$$q_*\left(\tilde{\gamma}_v|_{Y_{\sigma,L}}\right) = (q_*\tilde{\gamma}_v)|_{Y_\sigma}\,. \tag{5.13}$$

Recall from (5.9) that $\Phi^{\Lambda}_{\tilde{\mathfrak{X}},\tilde{Y}_{\tilde{v}}}(\tilde{\gamma})$ consists of three sums. We consider the right hand side of (5.11) for each of the sums separately: The terms $\tilde{\iota}_{\tau,v\,*}q_*\tilde{\alpha}_\tau$ in (5.9) sum up on the right side of (5.11) to 0 by Lemma 5.3. The right hand side of (5.11) reads for the first sum of (5.9) as follows (using the notation for the inclusions from Remark 5.8(a))

$$\begin{aligned}
&\sum_{\tilde{v}\in\mathcal{D}(\tilde{Y})(0)} (r - d(\tilde{v}, v))\iota_{\psi_*\tilde{v},v}q_*\left(\sum_{w\in A_{\psi}(\tilde{v})} -\tilde{\iota}_{w\tilde{v},\tilde{v}\,*}\ \tilde{\gamma}_{\psi_*\tilde{v}}|_{(Y_w\cap Y_{\psi_*\tilde{v}})_L}\right)\\
&= \sum_{w\in A_{\mathcal{D}(Y)}(v)} -rq_*\tilde{\iota}_{w\psi^*v,\psi^*v\,*}\ \tilde{\gamma}_v|_{(Y_v\cap Y_w)_L}\\
&\overset{(5.12)}{=} \sum_{w\in A_{\mathcal{D}(Y)}(v)} -r\iota_{vw,v\,*}\ (q_*\tilde{\gamma}_v)|_{(Y_v\cap Y_w)}\,.
\end{aligned}$$

For second sum of (5.9) it reads as follows (using again the notation from Remark 5.8(a))

$$\begin{aligned}
&\sum_{\tilde{v}\in\mathcal{D}(\tilde{Y})(0)} (r - d(\tilde{v}, v))\iota_{\psi_*\tilde{v},v}q_*\left(\sum_{(\sigma,w)\in\mathcal{L}_{\psi}(\tilde{v})} \tilde{\iota}_{\sigma,\tilde{v}\,*}\ \tilde{\gamma}_w|_{Y_{\sigma,L}}\right)\\
&= \sum_{w\in\mathcal{D}(Y)(0)} \sum_{\substack{\sigma\in\mathcal{D}(Y)(1)\\ w\in\sigma}} (r - d(\tilde{v}(w, \sigma), v))\iota_{\psi_*\tilde{v}(w,\sigma),v\,*}q_*\tilde{\iota}_{\sigma,\tilde{v}(w,\sigma)\,*}\ \tilde{\gamma}_w|_{Y_{\sigma,L}}\\
&\overset{(5.13)}{=} \sum_{w\in\mathcal{D}(Y)(0)} \sum_{\substack{\sigma\in\mathcal{D}(Y)(1)\\ w\in\sigma}} (r - d(\tilde{v}(w, \sigma), v))\iota_{\sigma,v\,*}\ (q_*\tilde{\gamma}_w)|_{Y_\sigma}\\
&= \sum_{\substack{\sigma\in\mathcal{D}(Y)(1)\\ v\in\sigma}} (r - 1)\iota_{\sigma,v\,*}\ (q_*\tilde{\gamma}_v)|_{Y_\sigma} + \iota_{\sigma,v\,*}\ \left(q_*\tilde{\gamma}_{w(\sigma)}\right)\Big|_{Y_\sigma}\\
&= \sum_{w\in A_{\mathcal{D}(Y)}(v)} (r - 1)\iota_{vw,v\,*}\ (q_*\tilde{\gamma}_v)|_{Y_v\cap Y_w} + \iota_{vw,v\,*}\ (q_*\tilde{\gamma}_w)|_{Y_v\cap Y_w}\,,
\end{aligned}$$

where $\tilde{v}(w,\sigma) \in \mathcal{D}(\tilde{Y})(0)$ is the unique vertex of $\mathcal{D}(Y)$ which is contained in σ and adjacent to $\psi^* w$, $w(\sigma)$ is the unique different from v vertex of the 1-simplex σ. Note that we used in the second last equality the easy observation that $d(\tilde{v}(w,\sigma),v)$ is 0 if $v \notin \sigma$, $r-1$ if v is contained in σ and adjacent to $\tilde{v}(w,\sigma)$, and 1 otherwise, which follows directly from Construction 5.1.

Since $\Phi^{\Lambda}_{\mathfrak{X},Y_v}(q_*\tilde{\gamma})$ is given by

$$\Phi^{\Lambda}_{\mathfrak{X},Y_v}(q_*\tilde{\gamma}) = \sum_{w\in A_{\mathcal{D}(Y)}(v)} \iota_{vw,v\,*}\left((q_*\tilde{\gamma}_w)|_{Y_v\cap Y_w} - (q_*\tilde{\gamma}_v)|_{Y_v\cap Y_w} \right),$$

the above arguments shows the equality in (5.11). □

6. Proof of the main results

6.1. Exactness of the complex.

Theorem 6.1. *Let R be a dvr with algebraically closed residue field k. Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable R-scheme with special fibre Y, let $\tilde{R}/R$ be a finite extension of dvr's with ramification index r, and let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution of $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.1. Let Λ be a ring and let $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ be the subdivision from Lemma 4.9. Let A/R and $\tilde{A}/\tilde{R}$ be unramified extensions of dvr's with the same induced extension L'/k on residue fields. Then the following holds*

(a) If $\Phi^{\Lambda}_{\tilde{\mathfrak{X}}_{\tilde{A}}}$ is surjective, then so is $\Phi^{\Lambda}_{\mathfrak{X}_A}$.

(b) Assume additionally that $\mathfrak{X}$ is proper. If the complex

$$\mathrm{CH}_1(\tilde{Y}_{L'},\Lambda) \xrightarrow{\Phi^{\Lambda}_{\tilde{\mathfrak{X}}_{\tilde{A}}}} \bigoplus_{\tilde{v}\in\mathcal{D}(\tilde{Y})(0)} \mathrm{CH}_0(\tilde{Y}_{\tilde{v},L'},\Lambda) \xrightarrow{\sum_{\tilde{v}}\deg} \Lambda \tag{6.1}$$

is exact, then the complex

$$\mathrm{CH}_1(Y_{L'},\Lambda) \xrightarrow{\Phi^{\Lambda}_{\mathfrak{X}_A}} \bigoplus_{v\in\mathcal{D}(Y)(0)} \mathrm{CH}_0(Y_{v,L'},\Lambda) \xrightarrow{\sum_{v}\deg} \Lambda \tag{6.2}$$

is exact.

Remark 6.2. *Let R be a discrete valuation ring with residue field k and let L/k be a field extension. Then there exists an unramified extension of discrete valuation rings A/R such that the induced extension on residue field is L/k, see [Bou06, Théorème 1 and Corollaire on page AC IX.40-41].*

Proof of Theorem 6.1. Note that the families $\mathfrak{X}_A \to \operatorname{Spec} A$ and $\tilde{\mathfrak{X}}_{\tilde{A}} \to \operatorname{Spec} \tilde{A}$ are strictly semi-stable by [Har01, Proposition 1.3], as A/R and $\tilde{A}/\tilde{R}$ are unramified extension. Moreover, the natural morphism $q\colon \tilde{Y} \to Y$ induces a morphism of the base-change $q\colon \tilde{Y}_{L'} \to Y_{L'}$. We will prove the statement for ease of notation in the case $L' = k$. The general case follows verbatim by the same argument.

We show the following claim: If $\Phi^{\Lambda}_{\tilde{\mathfrak{X}}}$ is surjective or $\mathfrak{X}$ is proper and the complex (6.1) is exact, then

$$q_*\Phi^{\Lambda}_{\tilde{\mathcal{X}}}\left(\mathrm{CH}_1(\tilde{Y},\Lambda)\right) = \Phi^{\Lambda}_{\mathcal{X}}\left(\mathrm{CH}_1(Y,\Lambda)\right), \tag{6.3}$$

where $q\colon \tilde{Y} \to Y$ is the natural morphism induced by the morphism $\tilde{\mathfrak{X}} \to \mathfrak{X}$. The claim immediately implies items (a) + (b), as the push-forward map

$$q_*\colon \bigoplus_{\tilde{v}\in\mathcal{D}(\tilde{Y})(0)} \mathrm{CH}_0(\tilde{Y}_{\tilde{v}},\Lambda) \longrightarrow \bigoplus_{v\in\mathcal{D}(Y)(0)} \mathrm{CH}_0(Y_v,\Lambda) \tag{6.4}$$

is clearly surjective, see Proposition 4.12, and q_* commutes with the degree map. Hence it suffices to prove the claim.

We show first the inclusion "$\supseteq$" in (6.3). Let $\gamma \in \mathrm{CH}_1(Y, \Lambda)$ be a one-cycle. We aim to show that $\Phi^\Lambda_{\mathfrak{X}}(\gamma)$ lies in the image of $q_*\Phi^\Lambda_{\tilde{\mathfrak{X}}}$. It follows from Proposition 4.14 that there exists a $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ such that

$$q_*\tilde{\gamma} = \gamma \in \mathrm{CH}_1(Y).$$

(Note that we used here that k is the residue field of R and $\tilde{R}$, as k is algebraically closed.) The key formula (5.11) in Proposition 5.9 shows

$$\Phi^\Lambda_{\mathfrak{X}}(\gamma) = \Phi^\Lambda_{\mathfrak{X}}(q_*\tilde{\gamma}) \in q_*\Phi^\Lambda_{\tilde{\mathfrak{X}}}.$$

This concludes the proof of the inclusion "$\supseteq$".

We turn to the proof of the inclusion "$\subseteq$" in (6.3). Consider a collection of zero-cycles $\beta = (\beta_v)_v \in \bigoplus_{v\in\mathcal{D}(Y)(0)} \mathrm{CH}_0(Y_v, \Lambda)$ of the form

$$\beta_v = \begin{cases} \iota_{\sigma,v_1\,*}\alpha & \text{if } v = v_1, \\ -\iota_{\sigma,v_2\,*}\alpha & \text{if } v = v_2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.5}$$

for some $\sigma \in \mathcal{D}(Y)(1)$ with vertices v_1 and v_2, $\alpha \in \mathrm{CH}_0(Y_\sigma, \Lambda)$ and $\iota_{\sigma,v}\colon Y_\sigma \hookrightarrow Y_v$ the natural inclusion. We aim to show that $\beta \in \Phi^\Lambda_{\mathfrak{X}}(\mathrm{CH}_1(Y, \Lambda))$. To this end, consider $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ such that

$$\left(\Phi^\Lambda_{\tilde{\mathfrak{X}},\tilde{Y}_{\tilde{v}}}\right)(\tilde{\gamma}) = \begin{cases} \tilde{\iota}_{\sigma,\psi^*v_1\,*}\alpha & \text{if } \tilde{v} = \psi^* v_1, \\ -\alpha & \text{if } (\sigma, v_1) \in \mathcal{L}_\psi(\tilde{v}), \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\iota}_{\sigma,\psi^*v_1}$ is as in Remark 5.8(a) and $\mathcal{L}_\psi(\tilde{v})$ is defined in Definition 4.6(iii). Note that there exists a unique vertex $\tilde{v}$ such that $(\sigma, v_1) \in \mathcal{L}_\psi(\tilde{v})$. Hence such a $\tilde{\gamma}$ exists by our assumption that $\Phi^\Lambda_{\tilde{\mathfrak{X}}}$ is surjective or $\mathfrak{X}$ is proper and the complex (6.1) is exact. Then the formula (5.11) implies

$$\begin{aligned} \left(\Phi^\Lambda_{\mathfrak{X},Y_{v_1}}\right)(q_*\tilde{\gamma}) &= r\iota_{\sigma,v_1\,*}\alpha + \iota_{\sigma,v_1\,*}(r-1)(-\alpha) + 0 = \iota_{\sigma,v_1\,*}\alpha, \\ \left(\Phi^\Lambda_{\mathfrak{X},Y_{v_2}}\right)(q_*\tilde{\gamma}) &= 0 + 1 \cdot \iota_{\sigma,v_2\,*}(-\alpha) = -\iota_{\sigma,v_2\,*}\alpha, \\ \left(\Phi^\Lambda_{\mathfrak{X},Y_v}\right)(q_*\tilde{\gamma}) &= 0, \end{aligned}$$

for $v \in \mathcal{D}(Y) \setminus \{v_1, v_2\}$. Hence, $\beta = \Phi^\Lambda_{\tilde{\mathfrak{X}}}(q_*\tilde{\gamma}) \in \Phi^\Lambda_{\mathcal{X}}(\mathrm{CH}_1(Y, \Lambda))$.

Since $q_*\Phi^\Lambda_{\tilde{\mathfrak{X}}}(\tilde{\gamma}')$ for every $\tilde{\gamma}' \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ is a (finite) linear combination of zero-cycles of the form (6.5) by Remark 5.6, the above argument shows that

$$q_*\Phi^\Lambda_{\tilde{\mathcal{X}}}\left(\mathrm{CH}_1(\tilde{Y}, \Lambda)\right) \subseteq \Phi^\Lambda_{\mathcal{X}}(\mathrm{CH}_1(Y, \Lambda)).$$

Hence, we proved the equality (6.3), which finishes the proof of the theorem. □

Remark 6.3. *Note that the assumption on the residue field being algebraically closed is crucial for the argument. Because otherwise, the push-forward map (6.4) is in general not surjective. In fact, for a vertex $v \in \mathcal{D}(Y)(0)$ and $z \in \mathrm{CH}_0(Y_v, \Lambda)$, we have*

$$q_*z_L = [L : k] \cdot z \in \mathrm{CH}_0(Y_v, \Lambda),$$

where L is the residue field of $\tilde{R}$.

Using the same argument as in [PS23, Theorem 4.1], Theorem 6.1(b) implies the following result.

Corollary 6.4. *Let Λ be a ring. Let R be a dvr with algebraically closed residue field k, let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective R-scheme with special fibre $Y = \bigcup_{i\in I} Y_i$ such that the geometric generic fibre $\bar{X}$ of $\mathfrak{X}$ admits a Λ-decomposition of the diagonal. Then for any unramified extension A/R of dvr's with induced extension of residue fields L'/k, the complex*

$$\bigoplus_{i\in I} \mathrm{CH}_1\left(Y_{i,L'}, \Lambda\right) \xrightarrow{\Phi^{\Lambda}_{\mathfrak{X}_A}} \bigoplus_{i\in I} \mathrm{CH}_0\left(Y_{i,L'}, \Lambda\right) \xrightarrow{\deg} \Lambda \tag{6.6}$$

is exact, where is $\Phi^{\Lambda}_{\mathfrak{X}_A}$ as in (5.1), see also Definition 5.4 and Remark 5.6.

Proof. Since the homomorphism $\Phi^{\Lambda}_{\mathfrak{X}}$ depends only on the special fibre (see [PS23, Lemma 3.2]) and the special fibre does not change under the base-change to the completion of R, we can assume without loss of generality that R is complete. The Λ-decomposition of the diagonal for the geometric generic fibre holds after some finite field extension F/K of the fraction field of R.

Consider the integral closure $\tilde{R}$ of R in F. Since F is a finite field extension and R is complete, the ring $\tilde{R}$ is also a dvr and a finite extension over R. Note that the residue field of $\tilde{R}$ is again k because k is algebraically closed. Moreover, fix a resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ of the base-change $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.1.

Let A/R be a unramified extension of dvr's with induced extension L'/k of residue fields. By Remark 6.2 there exists an unramified extension $\tilde{A}/\tilde{R}$ whose extension of residue fields is L'/k. Since the generic fibre $\tilde{X} = X_F$ of $\tilde{\mathfrak{X}}$ admits a Λ-decomposition of the diagonal, the homomorphism $\Phi^{\Lambda}_{\tilde{\mathfrak{X}}_{\tilde{A}}}$ is surjective onto the kernel of the degree map by [PS23, Theorem 1.2 (1)]. (*Loc. cit.* shows it for $\Lambda = \mathbb{Z}$, but the same argument works for general Λ.) Hence, the complex (6.6) is exact by Theorem 6.1(b) and Remark 5.5. □

Proof of Theorem 1.1. Let R be a dvr with fraction field k and let L/k be a field extension. By [Bou06, Corollaire on page AC IX.41], there exists a unramified extension A/R whose induced extension of residue fields is L/k. Thus Theorem 1.1 is a reformulation of Corollary 6.4, because the degree map $\sum_i \deg$ is clearly surjective as $k = \bar{k}$. □

The assumption on the residue field in Theorem 1.1 and Corollary 6.4 is crucial.

Remark 6.5. *Theorem 1.1 and Corollary 6.4 show the exactness of the complex (1.1) for strictly semi-stable R-scheme if the residue field k of R is algebraically closed and the geometric generic fibre admits a decomposition of the diagonal.*

The exactness of the complex (1.1) at $\mathbb{Z}$ requires the existence of a zero-cycle of degree 1 on the special fibre. Such a zero-cycle does not exist in general over non-closed fields. The exactness of the complex (1.1) fails over non-closed fields in general also at $\bigoplus_i \mathrm{CH}_0(Y_i)$. Indeed, consider a smooth, projective geometrically rational variety Y over a field $k \neq \bar{k}$ such that $\mathrm{A}_0(Y) := \ker(\deg\colon \mathrm{CH}_0(Y) \to \mathbb{Z})$ is non-trivial. Then the smooth, projective family $\mathfrak{X} = Y \times_k k[[t]] \to \operatorname{Spec} k[[t]]$ is strictly semi-stable and the image of Φ is trivial. Hence (1.1) is not exact, but the geometric generic fibre is rational and thus admits a decomposition of diagonal. As an example for Y, we can consider the diagonal cubic surface $S_1 := \{x^3 + y^3 + z^3 + pw^3 = 0\} \subset \mathbb{P}^3_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ which satisfies $\mathrm{A}_0(S_1) \neq 0$ by

[CTS96, Example 2.8] and [SS14, Theorem 4.1.1]. By [CT18, Proposition 1.3] there is a cubic surface S_2 over $\mathbb{R}$ with $\mathrm{A}_0(S_2) \neq 0$, which yields another example for Y.

We also provide a counterexample over number fields. Consider a (diagonal) cubic surface S over a number field whose Brauer group $\mathrm{Br}(S)$ is non-trivial, see e.g. [CTKS87, Proposition 1] and [SD93]. This implies by [Mer08, Theorem 2.11] that there exists a field extension L/k such that $\mathrm{A}_0(S_L) \neq 0$. Hence, the complex (1.1) is not exact after base-change to the field extension L/k.

6.2. **Geometrically retract rational varieties over Laurent fields.** We prove Corollary 1.2 more generally over fraction fields of excellent, henselian dvr's with algebraically closed residue field and for Chow groups with Λ-coefficients. Recall that the exponential characteristic of a field k is p if the characteristic $\mathrm{char}\, k = p > 0$ and 1 if $\mathrm{char}\, k = 0$.

Corollary 6.6. *Let R be an excellent, henselian dvr with fraction field K and algebraically closed residue field k. Let X be a smooth, projective variety over K and assume that X admits a strictly semi-stable projective model over R. If X admits geometrically a Λ-decomposition of the diagonal, then the degree map $\deg\colon \mathrm{CH}_0(X, \Lambda) \to \Lambda$ is an isomorphism up to inverting the exponential characteristic of k.*

Proof. We first note that the degree map is surjective. Indeed, let $\mathfrak{X} \to \mathrm{Spec}\, R$ be a strictly semi-stable projective R-scheme with generic fibre X. Since k is algebraically closed, there exists a k-rational point in the smooth locus of the special fibre. By Hensel's lemma we can lift this k-point to a section of $\mathfrak{X} \to \mathrm{Spec}\, R$, see e.g. [EGAIV.4, Theorem 18.5.17]. The intersection of this section with X yields a zero-cycle of degree 1 on X, i.e. the degree map is surjective.

By assumption, the base-change $\bar{X}$ to the algebraic closure $\bar{K}$ admits an Λ-decomposition of the diagonal. This Λ-decomposition of the diagonal holds after some finite field extension F/K. By a pull/push argument, we find that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that

$$N[\Delta_X] = [X \times z] + [Z] \in \mathrm{CH}_{\dim X}(X \times_K X, \Lambda), \tag{6.7}$$

where $\Delta_X \subset X \times_K X$ is the diagonal, z is a zero-cycle in X of degree 1, and Z is a $\dim X$-cycle on $X \times_K X$ which does not dominate the first factor. Hence the kernel of the degree map

$$\mathrm{A}_0(X, \Lambda) := \ker\left(\deg\colon \mathrm{CH}_0(X, \Lambda) \longrightarrow \Lambda\right)$$

is N-torsion, in fact one can choose $N = [F : K]$, see e.g. [ACTP17, proof of Lemma 1.3].

Let Y denote the special fibre of the strictly semi-stable family $\mathfrak{X} \to \mathrm{Spec}\, R$ and let Y_i denote the irreducible components of Y with $i \in I$. Consider Fulton's localization exact sequence

$$\mathrm{CH}_1(Y, \Lambda) \longrightarrow \mathrm{CH}_1(\mathfrak{X}, \Lambda) \longrightarrow \mathrm{CH}_0(X, \Lambda) \longrightarrow 0,$$

see [Ful75, Section 4.4]. After tensoring this sequence with $\mathbb{Z}/l$ for some $l \in \mathbb{Z}_{\geq 0}$ coprime to the exponential characteristic of k, this sequence fits by [SS10, Corollary 0.9] into the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_1(Y,\Lambda)/l \arrow[r] \arrow[d, equal] & \mathrm{CH}_1(\mathfrak{X},\Lambda)/l \arrow[r] \arrow[d, "\cong"] & \mathrm{CH}_0(X,\Lambda)/l \arrow[r] \arrow[d, "\deg"] & 0 \\
\mathrm{CH}_1(Y,\Lambda)/l \arrow[r, "\deg_i \circ \Phi^\Lambda_{\mathfrak{X}}"] & \left(\bigoplus_{i \in I} \Lambda/l \cdot [Y_i]\right)^\vee \arrow[r, "\deg"] & \Lambda/l \arrow[r] & 0,
\end{tikzcd} \tag{6.8}$$

see also Bloch's argument in [EW16, Theorem A.1] or [EKW16]. The bottom row is exact by Corollary 6.4. Indeed, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_1(Y,\Lambda)/l \arrow[r, "\Phi_{\mathfrak{X}}"] \arrow[d, equal] & \bigoplus_{i\in I} \mathrm{CH}_0(Y_i,\Lambda)/l \arrow[r, "\deg"] \arrow[d, "\deg_i"] & \Lambda/l \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{CH}_1(Y,\Lambda)/l \arrow[r, "\deg_i \circ \Phi_{\mathfrak{X}}^{\Lambda}"] & \left(\bigoplus_{i\in I} \Lambda/l\cdot[Y_i]\right)^{\vee} \arrow[r, "\deg"] & \Lambda/l \arrow[r] & 0
\end{tikzcd}$$

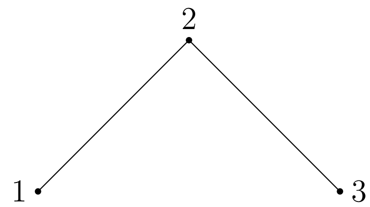
Since k is algebraically closed, the middle arrow is surjective. The top row is exact by Corollary 6.4. Hence the bottom row is exact by a simple diagram chase.

Thus, the right arrow in (6.8) is injective, in particular $\mathrm{A}_0(X,\Lambda)$ is divisible by l for each l coprime to the exponential characteristic of k. Since $\mathrm{A}_0(X,\Lambda)$ is torsion by (6.7), the corollary follows. $\square$

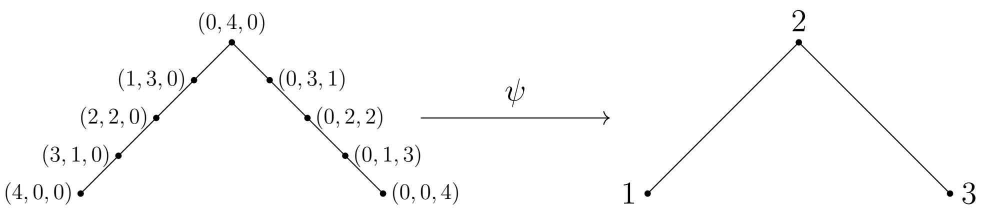
Appendix A. Two concrete resolutions

We illustrate the proof of the key proposition (Proposition 5.9) for $\Lambda = \mathbb{Z}$ in two concrete examples by spelling out the constructions and arguments in Sections 4 and 5. Recall the general setup from Setup 2.2: We consider a strictly semi-stable family $\mathfrak{X} \to \operatorname{Spec} R$ over a dvr with special fibre Y such that the irreducible components of Y and for simplicity all their intersections are geometrically integral. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution from Proposition 2.1 after a finite extension $\tilde{R}/R$ of dvr's of ramification index r with induced extension L/k of residue fields. We denote the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ by $\tilde{Y}$ and the natural morphism by $q\colon \tilde{Y} \to Y$.

Example A.1. *Assume that Y is a chain of three Cartier divisors and denote the three irreducible components by Y_1, Y_2, Y_3. The dual complex $\mathcal{D}(Y)$ of Y from Definition 4.7 is*



The subdivision from Lemma 4.9 is then given in the case $r = 4$ by



Recall that the subdivision $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ corresponds to a resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ of $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$. The irreducible components of the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ correspond to the vertices $(i_1, i_2, i_3) \in \mathbb{N}^3$ of $\mathcal{D}(\tilde{Y})$: The components $\tilde{Y}_{(4,0,0)}$, $\tilde{Y}_{(0,4,0)}$, and $\tilde{Y}_{(0,0,4)}$ are isomorphic to $Y_{1,L}$, $Y_{2,L}$ and $Y_{3,L}$, respectively. The component $\tilde{Y}_{(i_1,i_2,i_3)}$ corresponding to a vertex (i_1, i_2, i_3) with $i_2, i_1 + i_3 \neq 0$ in the above picture is a (Zariski) locally trivial $\mathbb{P}^1$-bundle over $Y_{j,L} \cap Y_{2,L}$ where $j \in \{1,3\}$ such that $i_j \neq 0$.

By Proposition 4.14, we have a surjection,

$$f\colon \bigoplus_{j=1}^{3} \mathrm{CH}_1(Y_{j,L}) \oplus \bigoplus_{\substack{i_1,i_2\geq 1\\ i_1+i_2=4}} \mathrm{CH}_0((Y_1\cap Y_2)_L) \oplus \bigoplus_{\substack{j_2,j_3\geq 1\\ j_2+j_3=4}} \mathrm{CH}_0((Y_2\cap Y_3)_L) \longrightarrow \mathrm{CH}_1(\tilde{Y}),$$

$$(\gamma_1,\gamma_2,\gamma_3,\alpha_{(i_1,i_2,0)},\alpha_{(0,j_2,j_3)}) \mapsto \gamma_1+\gamma_2+\gamma_3+\sum_{\substack{i_1,i_2\geq 1\\ i_1+i_2=4}} q^*_{(i_1,i_2,0)}\alpha_{(i_1,i_2,0)} + \sum_{\substack{j_2,j_3\geq 1\\ j_2+j_3=4}} q^*_{(0,j_2,j_3)}\alpha_{(0,j_2,j_3)}$$

where $q_{(i_1,i_2,0)}\colon \tilde{Y}_{(i_1,i_2,0)} \to (Y_1\cap Y_2)_L$ and $q_{(0,j_2,j_3)}\colon \tilde{Y}_{(0,j_2,j_3)} \to (Y_2\cap Y_3)_L$ are the natural (flat) projections of the projective bundles. For simplicity we left out the push-forwards along the natural inclusion of the irreducible components to $\tilde{Y}$. The distance function d from Construction 5.1 can be read of the indices of the components as follows:

$$d((i_1,i_2,i_3),(4,0,0)) = 4-i_1,\quad d((i_1,i_2,i_3),(0,4,0)) = 4-i_2,\quad d((i_1,i_2,i_3),(0,0,4)) = 4-i_3.$$

Note that the relative walls in this examples are the new vertices $(i_1,i_2,0)$ and $(0,j_2,j_3)$ with $i_1,\ i_2,\ j_2,\ j_3>0$; hence we denote the relative walls by the vertex. Then the function $I_\tau(\tilde{v})$ from Construction 5.2 is given for $\tau=(2,2,0)$ by

$$I_{(2,2,0)}((i_1,i_2,i_3)) = \begin{cases} 1 & \text{if } (i_1,i_2,i_3)=(1,3,0) \text{ or } (3,1,0),\\ -2 & \text{if } (i_1,i_2,i_3)=(2,2,0),\\ 0 & \text{otherwise}\end{cases}$$

and similarly for the other relative walls.

The explicit description of the morphism $\Phi_{\tilde{\mathfrak{X}}}$ from Lemma 5.7 is given as follows

$$\begin{aligned}
\Phi_{(4,0,0)}(\gamma) &= -\,\gamma_1|_{(Y_1\cap Y_2)_L} + \alpha_{(3,1,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(4,0,0)}),\\
\Phi_{(0,4,0)}(\gamma) &= -\,\gamma_2|_{(Y_1\cap Y_2)_L} - \gamma_2|_{(Y_2\cap Y_3)_L} + \alpha_{(0,3,1)} + \alpha_{(1,3,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,4,0)}),\\
\Phi_{(0,0,4)}(\gamma) &= -\,\gamma_3|_{(Y_2\cap Y_3)_L} + \alpha_{(0,1,3)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,0,4)}),\\
\Phi_{(3,1,0)}(\gamma) &= \gamma_1|_{(Y_1\cap Y_2)_L} + \alpha_{(2,2,0)} - 2\alpha_{(3,1,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(3,1,0)}),\\
\Phi_{(2,2,0)}(\gamma) &= \alpha_{(1,3,0)} + \alpha_{(3,1,0)} - 2\alpha_{(2,2,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(2,2,0)}),\\
\Phi_{(1,3,0)}(\gamma) &= \gamma_2|_{(Y_1\cap Y_2)_L} + \alpha_{(2,2,0)} - 2\alpha_{(1,3,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,3,0)}),\\
\Phi_{(0,3,1)}(\gamma) &= \gamma_2|_{(Y_2\cap Y_3)_L} + \alpha_{(0,2,2)} - 2\alpha_{(0,3,1)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,3,1)}),\\
\Phi_{(0,2,2)}(\gamma) &= \alpha_{(0,3,1)} + \alpha_{(0,1,3)} - 2\alpha_{(0,2,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,2,2)}),\\
\Phi_{(0,1,3)}(\gamma) &= \gamma_3|_{(Y_2\cap Y_3)_L} + \alpha_{(0,2,2)} - 2\alpha_{(0,1,3)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,1,3)}),
\end{aligned}$$

where $\Phi_{(i_1,i_2,i_3)} = \Phi_{\tilde{\mathfrak{X}},\tilde{Y}_{(i_1,i_2,i_3)}}$ and $\gamma = f\left(\gamma_1,\gamma_2,\gamma_3,\alpha_{(i_1,i_2,0)},\alpha_{(0,j_2,j_3)}\right) \in \mathrm{CH}_1(\tilde{Y})$.

The key formula (Proposition 5.9) reads in this example as follows:

$$\Phi_{\mathfrak{X},Y_1}(q_*\gamma) = \sum_{i_1=1}^{4} i_1\cdot q_*\Phi_{(i_1,4-i_1,0)}(\gamma),$$

$$\Phi_{\mathfrak{X},Y_2}(q_*\gamma) = 4q_*\Phi_{(0,4,0)}(\gamma) + \sum_{i_2=1}^{3} i_2\cdot q_*\Phi_{(4-i_2,i_2,0)}(\gamma) + i_2\cdot q_*\Phi_{(0,i_2,4-i_2)}(\gamma),$$

$$\Phi_{\mathfrak{X},Y_3}(q_*\gamma) = \sum_{j_3=1}^{4} j_3\cdot q_*\Phi_{(0,4-j_3,j_3)}(\gamma),$$

where we again leave out the push-forwards along the natural inclusions. This can be checked directly from the above description of $\Phi_{\tilde{\mathfrak{X}}}$ together with the observations

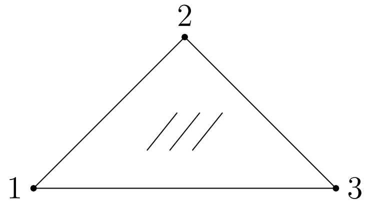
$$(q_*\gamma_i)|_{Y_i\cap Y_j} = q_*\left(\gamma_i|_{(Y_i\cap Y_j)_L}\right), \quad q_*q^*_{(i_1,i_2,i_3)}\alpha_{(i_1,i_2,i_3)} = 0,$$

see [Ful98, Proposition 2.3 (c)] for the first claim. Note that the explicit description of $\Phi_{\mathfrak{X}}$ in Lemma 5.7 reads in this example

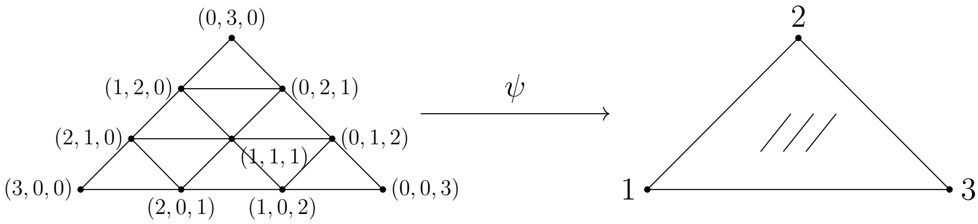
$$\begin{aligned}
\Phi_{\mathfrak{X},Y_1}(q_*\gamma) &= -\,(q_*\gamma_1)|_{Y_1\cap Y_2} + (q_*\gamma_2)|_{Y_1\cap Y_2}\,,\\
\Phi_{\mathfrak{X},Y_2}(q_*\gamma) &= -\,(q_*\gamma_2)|_{Y_1\cap Y_2} - (q_*\gamma_2)|_{Y_2\cap Y_3} + (q_*\gamma_1)|_{Y_1\cap Y_2} + (q_*\gamma_3)|_{Y_2\cap Y_3}\,,\\
\Phi_{\mathfrak{X},Y_3}(q_*\gamma) &= -\,(q_*\gamma_3)|_{Y_2\cap Y_3} + (q_*\gamma_2)|_{Y_2\cap Y_3}\,,
\end{aligned}$$

where we left out again the push forward along the natural inclusions.

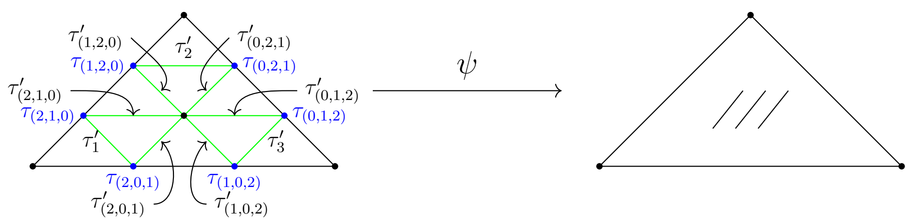
Example A.2. *Assume that Y is a complete intersection of three Cartier divisors Y_1, Y_2, and Y_3, i.e. the dual complex $\mathcal{D}(Y)$ of Y is the standard 2-simplex:*



For $r = 3$, a possible subdivision looks like:



This subdivision can be obtained by the blow-ups along the components corresponding to the vertices $(3,0,0),(3,0,0),(0,3,0),(1,2,0),(0,3,0),(1,2,0),(2,1,0)$ in that order. The component $\widetilde{Y}_{(1,1,1)}$ corresponding to the central vertex $(1,1,1)$ in the picture is a Zariski locally trivial $\mathbb{P}^1\times\mathbb{P}^1$-bundle blown up along two disjoint sections. The components indexed by $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$ are isomorphic to $Y_{1,L}$, $Y_{2,L}$, and $Y_{3,L}$, respectively. The other components are blow-ups of a Zariski locally trivial $\mathbb{P}^1$-bundle over the intersection $(Y_i\cap Y_j)_L$ of two components of Y along the smooth locus $(Y_1\cap Y_2\cap Y_3)_L$ viewed via a section as a subvariety in the projective bundle. There are two types of relative walls which are depicted below in two different colors:



To simplify notation, we consider the following action of the symmetric group S_3 on the vertices: $\sigma(i_1,i_2,i_3) = (i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)})$ for every $(i_1,i_2,i_3)\in\mathbb{N}_0^3$. By Proposition 4.14, we *can write any one-cycle $\gamma \in \mathrm{CH}_1(\tilde{Y})$ as*

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + (q_1')^*\alpha_1' + (q_2')^*\alpha_2' + (q_3')^*\alpha_3' + \sum_{\sigma \in S_3} \left(q_{\sigma(2,1,0)}\right)^* \alpha_{\sigma(2,1,0)} + \left(q'_{\sigma(2,1,0)}\right)^* \alpha'_{\sigma(2,1,0)}$$

in $\mathrm{CH}_1(\tilde{Y})$ for some $\gamma_i \in \mathrm{CH}_1(Y_{i,L})$, $\alpha_{(i_1,i_2,i_3)} \in \mathrm{CH}_0(\tilde{Y}_{(i_1,i_2,i_3)})$, and $\alpha_1', \alpha_2', \alpha_3', \alpha'_{(i_1,i_2,i_3)} \in \mathrm{CH}_0((Y_1 \cap Y_2 \cap Y_3)_L)$, where the zero-cycles $\alpha_{\dots}$ and $\alpha'_{\dots}$ correspond to the respective relative wall in the picture above and the morphisms $q_{\dots}$ and $q'_{\dots}$ are the natural projection of the associated $\mathbb{P}^1$-bundle. Note that we leave out the push-forwards along the respective natural inclusions.

As in Example A.1, the distance function d from Construction 5.1 can be read of the indices of the components as follows:

$$d((i_1,i_2,i_3),(4,0,0)) = 4 - i_1, \quad d((i_1,i_2,i_3),(0,4,0)) = 4 - i_2, \quad d((i_1,i_2,i_3),(0,0,4)) = 4 - i_3.$$

We write down the function I representing the intersection numbers by the matrix

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}$$

where the columns are labelled in the following order:

$\tau_{(2,1,0)}, \tau_{(1,2,0)}, \tau_{(2,0,1)}, \tau_{(1,0,2)}, \tau_{(0,2,1)}, \tau_{(0,1,2)}, \tau_1', \tau_2', \tau_3', \tau'_{(2,1,0)}, \tau'_{(1,2,0)}, \tau'_{(2,0,1)}, \tau'_{(1,0,2)}, \tau'_{(0,2,1)}, \tau'_{(0,1,2)}$,

and the rows represent the component $\tilde{Y}_{(i_1,i_2,i_3)}$ in the following order

$$(3,0,0), (0,3,0), (0,0,3), (2,1,0), (1,2,0), (2,0,1), (1,0,2), (0,2,1), (0,1,2), (1,1,1).$$

For simplicity, we denote the map $\Phi_{\tilde{\mathfrak{X}}, \tilde{Y}_{(i_1,i_2,i_3)}}$ by $\Phi_{(i_1,i_2,i_3)}$. Then, for $\gamma \in \mathrm{CH}_1(\tilde{Y})$ of the above form the map $\Phi_{\tilde{\mathfrak{X}}}$ is given

$$\begin{aligned}
\Phi_{(3,0,0)}(\gamma) &= -\gamma_1|_{(Y_1 \cap Y_2)_L} - \gamma_1|_{(Y_1 \cap Y_3)_L} + \alpha_{(2,1,0)} + \alpha_{(2,0,1)} + \alpha_1' && \in \mathrm{CH}_0(\tilde{Y}_{(3,0,0)}), \\
\Phi_{(0,3,0)}(\gamma) &= -\gamma_2|_{(Y_1 \cap Y_2)_L} - \gamma_2|_{(Y_2 \cap Y_3)_L} + \alpha_{(1,2,0)} + \alpha_{(0,2,1)} + \alpha_2' && \in \mathrm{CH}_0(\tilde{Y}_{(0,3,0)}), \\
\Phi_{(0,0,3)}(\gamma) &= -\gamma_3|_{(Y_1 \cap Y_3)_L} - \gamma_3|_{(Y_2 \cap Y_3)_L} + \alpha_{(1,0,2)} + \alpha_{(0,1,2)} + \alpha_3' && \in \mathrm{CH}_0(\tilde{Y}_{(0,0,3)}), \\
\Phi_{(2,1,0)}(\gamma) &= \gamma_1|_{(Y_1 \cap Y_2)_L} - 2\alpha_{(2,1,0)} + \alpha_{(1,2,0)} - \alpha_1' - \alpha'_{(2,1,0)} + \alpha'_{(1,2,0)} + \alpha'_{(2,0,1)} && \in \mathrm{CH}_0(\tilde{Y}_{(2,1,0)}), \\
\Phi_{(1,2,0)}(\gamma) &= \gamma_2|_{(Y_1 \cap Y_2)_L} + \alpha_{(2,1,0)} - 2\alpha_{(1,2,0)} - \alpha_2' + \alpha'_{(2,1,0)} - \alpha'_{(1,2,0)} + \alpha'_{(0,2,1)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,2,0)}), \\
\Phi_{(2,0,1)}(\gamma) &= \gamma_1|_{(Y_1 \cap Y_3)_L} - 2\alpha_{(2,0,1)} + \alpha_{(1,0,2)} - \alpha_1' + \alpha'_{(2,1,0)} - \alpha'_{(2,0,1)} + \alpha'_{(1,0,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(2,0,1)}), \\
\Phi_{(1,0,2)}(\gamma) &= \gamma_3|_{(Y_1 \cap Y_3)_L} + \alpha_{(2,0,1)} - 2\alpha_{(1,0,2)} - \alpha_3' + \alpha'_{(2,0,1)} - \alpha'_{(1,0,2)} + \alpha'_{(0,1,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,0,2)}), \\
\Phi_{(0,2,1)}(\gamma) &= \gamma_2|_{(Y_2 \cap Y_3)_L} - 2\alpha_{(0,2,1)} + \alpha_{(0,1,2)} - \alpha_2' + \alpha'_{(1,2,0)} - \alpha'_{(0,2,1)} + \alpha'_{(0,1,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,2,1)}), \\
\Phi_{(0,1,2)}(\gamma) &= \gamma_3|_{(Y_2 \cap Y_3)_L} + \alpha_{(0,2,1)} - 2\alpha_{(0,1,2)} - \alpha_3' + \alpha'_{(1,0,2)} + \alpha'_{(0,2,1)} - \alpha'_{(0,1,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,1,2)}), \\
\Phi_{(1,1,1)}(\gamma) &= \alpha_1' + \alpha_2' + \alpha_3' - \sum_{\sigma \in S_3} \alpha'_{\sigma(2,1,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,1,1)}),
\end{aligned}$$

where we left out the push forwards along the natural inclusions. Note that $\mathrm{CH}_0(\tilde{Y}_{(2,1,0)}) = \mathrm{CH}_0((Y_1 \cap Y_2)_L)$ *as* $\tilde{Y}_{(2,1,0)}$ *is a blow-up in smooth centers of a locally trivial* $\mathbb{P}^1$*-bundle over* $(Y_1 \cap Y_2)_L$ *(and similarly for the other components), i.e. we view the zero-cyles* $\alpha_{(1,2,0)}$ *as a zero-cycle on* $\tilde{Y}_{(2,1,0)}$*. The key lemma reads*

$$\sum_{\substack{i_1,i_2,i_3\\ i_1>0}} i_1 q_*\Phi_{(i_1,i_2,i_3)}(\gamma) = -\left.(q_*\gamma_1)\right|_{Y_1\cap Y_2} - \left.(q_*\gamma_1)\right|_{Y_1\cap Y_3} + \left.(q_*\gamma_2)\right|_{Y_1\cap Y_2} + \left.(q_*\gamma_3)\right|_{Y_1\cap Y_3} = \Phi_{\mathfrak{X},Y_1}(q_*\gamma),$$

$$\sum_{\substack{i_1,i_2,i_3\\ i_2>0}} i_2 q_*\Phi_{(i_1,i_2,i_3)}(\gamma) = -\left.(q_*\gamma_2)\right|_{Y_1\cap Y_2} - \left.(q_*\gamma_2)\right|_{Y_2\cap Y_3} + \left.(q_*\gamma_1)\right|_{Y_1\cap Y_2} + \left.(q_*\gamma_3)\right|_{Y_2\cap Y_3} = \Phi_{\mathfrak{X},Y_2}(q_*\gamma),$$

$$\sum_{\substack{i_1,i_2,i_3\\ i_3>0}} i_3 q_*\Phi_{(i_1,i_2,i_3)}(\gamma) = -\left.(q_*\gamma_3)\right|_{Y_1\cap Y_3} - \left.(q_*\gamma_3)\right|_{Y_2\cap Y_3} + \left.(q_*\gamma_1)\right|_{Y_1\cap Y_3} + \left.(q_*\gamma_2)\right|_{Y_2\cap Y_3} = \Phi_{\mathfrak{X},Y_3}(q_*\gamma),$$

which can be checked immediately by the above description together with the observations:

$$q_*\gamma = q_*\gamma_1 + q_*\gamma_2 + q_*\gamma_3, \text{ and } \left.(q_*\gamma_i)\right|_{Y_i\cap Y_j} = q_*\left(\left.\gamma_i\right|_{(Y_i\cap Y_j)_L}\right),$$

where the latter is again [Ful98, Proposition 2.3 (c)].

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