European Journal of Operational Research 39 (1989) 54–60
North-Holland

Theory and Methodology

Fuzzy input–output analysis

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Abstract: This paper generalizes Leontief's celebrated input–output analysis to incorporate fuzzy numbers.

Keywords: Economics, input–output analysis, fuzzy numbers

1. Introduction

We first, in this section, briefly review Leontief's open input–output model for an economy [10, 11, 12]. In the next section we generalize the model to allow for fuzzy numbers. We also present in the second section a very general sufficient condition (Theorem 1) for the existence of fuzzy input–output models for an economy. Three simple two industry input–output models are discussed in the third section. Two of these examples show that if the hypothesis of Theorem 1 is violated, then the fuzzy input–output model may fail to exist. The last section contains a brief symmary and our conclusions.

The open input–output model for a two industry economy is shown in Table 1. The constraints on the numbers in this table are that the a_{ij} and the o_j are in [0, 1], the F_i and T_i are non-negative numbers, and

$$a_{11}T_1 + a_{12}T_2 + F_1 = T_1, \tag{1}$$

$$a_{21}T_1 + a_{22}T_2 + F_2 = T_2, \tag{2}$$

$$a_{11} + a_{21} + o_1 = 1.00, \tag{3}$$

$$a_{12} + a_{22} + o_2 = 1.00. \tag{4}$$

We now explain the meaning of all the numbers in Table 1. T_1 (T_2) is the total annual output, measured possibly in dollars, of industry I (II). F_1 (F_2) is the amount of T_1 (T_2) consumed within the economy (government, individuals, etc.), excluding industries I and II, or is available for export. The a_{11} and a_{21} represent the percent [1] of the total input to industry I which comes from industry I and II, respectively. Industry I may have other inputs, besides from industries I and II, and this percentage is given by o_1. Usually labor comes under the o_j. Similarly, a_{12} (a_{22}) is the percent of

Table 1
Input–output table

	Industries		Final demands	Gross (total) output
	I	II		
Industry I	a_{11}	a_{12}	F_1 (New value $= B_1$)	T_1 (New value $= X_1$)
Industry II	a_{21}	a_{22}	F_2 (New value $= B_2$)	T_2 (New value $= X_2$)
Outside inputs (labor)	o_1	o_2		
Total	1.00	1.00		

[1] We will write a_{ij} both for a percent and its equivalent decimal. That is, $a_{ij} = 30\%$ and $a_{ij} = 0.30$. We do the same for the o_j.

Received November 1987; Revised March 1988

the total input of II arising from I (II) and o_2 is the percent originating outside industries I and II.

We therefore see the first two columns in Table 1 must sum to one (equations (3) and (4)). In equation (1) $a_{11}T_1$ ($a_{12}T_2$) is the total dollar input to I (II) coming from industry I. The left hand side of equation (1) shows how the total output of I is distributed between I, II and other consumption and export. Equation (2) has a similar interpretation for the distribution of the total output of industry II.

If the model does not allow for outside inputs, it is called a closed input–output model. We will consider only the open input–output model in this paper.

Let us write equations (1) and (2) in matrix form. Set $A=[a_{ij}]$, $F=[F_i]$, and $T=[T_i]$. Then we have

$$AT+F=T. \tag{5}$$

The matrix A is called the input–output, or technological, matrix for the economy. One use of the model is to predict total outputs for given different final demands. We assume the technological matrix A is known, or has been estimated, and does not change over the planning horizon. Let B be a vector of new final demands and let X be the new vector of total outputs. Then

$$AX+B=X, \tag{6}$$

must be solved for X. Solving for X we obtain

$$X=(I-A)^{-1}B, \tag{7}$$

assuming $I-A$ is nonsingular and the X in equation (7) turns out to be non-negative.

We have been considering a two industry economy so let us now extend these ideas to an m industry economy. Then $A=[a_{ij}]$ is a $m\times m$ matrix of percentages so that $0\leqslant a_{ij}\leqslant 1$, $O=[o_j]$ is a $1\times m$ vector of outside inputs so that $0\leqslant o_j\leqslant 1$, and F, B, T and X are all $m\times 1$ vectors of non-negative numbers. Equations (1) to (4) still hold for the m industry economy.

The purpose of this paper is to now allow all the numbers in the input–output table to be fuzzy numbers. In practice all the a_{ij} and B must be known exactly. We may now assume that some of these numbers are not precisely known and may then be modeled by an appropriate fuzzy number. For example, the a_{11} in Table 1 can now be around 30%, or between 20% and 25%. A fuzzy number can capture the vagueness in these statements. Therefore, A will be a $m\times m$ matrix of fuzzy percentages and B will be a $m\times 1$ vector of non-negative fuzzy final demands. The main problem is to solve equation (6) for the unknown $m\times 1$ vector X of non-negative fuzzy total outputs. A further restriction is that the equality (6) must be exact (crisp) and not some fuzzy equality. When equation (6) has a solution for fuzzy X we will say the fuzzy input–output model exists for this economy.

2. Fuzzy input–output analysis

Any fuzzy number [2] $\bar{N}$, with membership function $\mu(x\mid\bar{N})$, will be described by $(n_1\mid n_2,\ n_3\mid n_4)$ where: (1) $n_1\leqslant n_2\leqslant n_3\leqslant n_4$; (2) $\mu(x\mid\bar{N})$ is zero outside $(n_1,\ n_4)$ and equals one on $[n_2,\ n_3]$; (3) the graph of $y=\mu(x\mid\bar{N})$ is continuous and monotonically increasing, from zero to one, on $[n_1,\ n_2]$; and (4) the graph of $y=\mu(x\mid\bar{N})$ is continuous monotonically decreasing, from one to zero, on $[n_3,\ n_4]$. The graph of $y=\mu(x\mid\bar{N})$ has a trapezoidal shape when $n_2<n_3$ and it has a triangular shape if $n_2=n_3$. When $n_1=n_2=n_3=n_4=n$, then $\bar{N}=n$ is a real number. We will say $\bar{N}$ is non-negative if $n_1\geqslant 0$ and $\bar{N}$ is positive when $n_1>0$.

Let us now introduce the notation needed to describe the fuzzy input–output model. $\bar{A}=[\bar{a}_{ij}]$ is an $m\times m$ matrix of fuzzy numbers $\bar{a}_{ij}=(a_{ij1}\mid a_{ij2},\ a_{ij3}\mid a_{ij4})$ where $0\leqslant a_{ij1}\leqslant a_{ij2}\leqslant a_{ij3}\leqslant a_{ij4}\leqslant 1$ represent the fuzzy percentages. For example, approximately 70% could be (0.60 | 0.70 | 0.80) and between 30 and 40% might be (0.20 | 0.30, 0.40 | 0.50). When $\bar{N}=(n_1\mid n_2,\ n_3\mid n_4)$ and $n_2=n_3=n$ we simply write $\bar{N}=(n_1\mid n\mid n_4)$. Next $\bar{B}=[\bar{B}_i]$, where $\bar{B}_i=(b_{i1}\mid b_{i2},\ b_{i3}\mid b_{i4})$ with $\bar{B}_i$ non-negative. Also, $\bar{X}=[\bar{X}_i]$ and $\bar{X}_i=(x_{i1}\mid x_{i2},\ x_{i3}\mid x_{i4})$ and $\bar{X}_i$ must be non-negative. $\bar{A}$ is the fuzzy technological matrix for the economy, $\bar{B}$ is the $m\times 1$ fuzzy vector of new final demands, and $\bar{X}$ is a $m\times 1$ fuzzy vector of total

[2] Any symbol representing a fuzzy number, a fuzzy vector, or a fuzzy matrix will have a bar on it. $\bar{A}$, $\bar{B}$, ... are fuzzy while A, B, ... are not fuzzy.

outputs for the industries in this economy. We therefore must have

$$\overline{A}\overline{X}+\overline{B}=\overline{X}, \tag{8}$$

where we employ standard fuzzy multiplication and addition in equation (8). Let $\overline{O}=[\overline{O}_j]$, where $\overline{O}_j=(o_{j1}\,|\,o_{j2},\,o_{j3}\,|\,o_{j4})$ and $0\leqslant o_{j1}\leqslant o_{j2}\leqslant o_{j3}\leqslant o_{j4}\leqslant 1$, be the $1\times m$ fuzzy vector of outside inputs. If

$$\overline{C}_j=\sum_{i=1}^{m}\bar{a}_{ij}+\overline{O}_j, \tag{9}$$

then we require that $c_{j2}\leqslant 1\leqslant c_{j3}$ where $\overline{C}_j=(c_{j1}\,|\,c_{j2},\,c_{j3}\,|\,c_{j4})$. This means that the column sums in the fuzzy input–output table are approximately equal to 1.00 (100%).

Fuzzy arithmetic is more easily performed in terms of α-cuts. An α-cut of $\overline{N}$, $0<\alpha\leqslant 1$, is

$$\overline{N}^{\alpha}=\left[N_{\ell}^{\alpha},\,N_{\mathrm{u}}^{\alpha}\right], \tag{10}$$

where

$$\left[N_{\ell}^{\alpha},\,N_{\mathrm{u}}^{\alpha}\right]=\left\{x\,|\,\mu(x\,|\,\overline{N})\geqslant\alpha\right\}. \tag{11}$$

We separately define

$$\overline{N}^{0}=\left[n_1,\,n_4\right]. \tag{12}$$

All the α-cuts of fuzzy numbers are bounded, closed, intervals. We now define

$$\bar{a}_{ij}^{\alpha}=\left[a_{ij\ell}^{\alpha},\,a_{iju}^{\alpha}\right], \tag{13}$$

$$\overline{B}_i^{\alpha}=\left[b_{i\ell}^{\alpha},\,b_{iu}^{\alpha}\right], \tag{14}$$

$$\overline{X}_i^{\alpha}=\left[x_{i\ell}^{\alpha},\,x_{iu}^{\alpha}\right], \tag{15}$$

for $0\leqslant\alpha\leqslant 1$. Of course, $\overline{X}_i^0=[x_{i1},\,x_{i4}]$, $\overline{B}_i^0=[b_{i1},\,b_{i4}]$, etc.

We next set $A_{\ell}^{\alpha}=[a_{ij\ell}^{\alpha}]$, $A_{\mathrm{u}}^{\alpha}=[a_{iju}^{\alpha}]$, $B_{\ell}^{\alpha}=[b_{i\ell}^{\alpha}]$, $B_{\mathrm{u}}^{\alpha}=[b_{iu}^{\alpha}]$, $X_{\ell}^{\alpha}=[x_{i\ell}^{\alpha}]$ and $X_{\mathrm{u}}^{\alpha}=[x_{iu}^{\alpha}]$. Fuzzy arithmetic based on α-cuts becomes an interval arithmetic [1, 6, 8] and equation (8) is equal to

$$A_{\ell}^{\alpha}X_{\ell}^{\alpha}+B_{\ell}^{\alpha}=X_{\ell}^{\alpha}, \tag{16}$$

$$A_{\mathrm{u}}^{\alpha}X_{\mathrm{u}}^{\alpha}+B_{\mathrm{u}}^{\alpha}=X_{\mathrm{u}}^{\alpha}, \tag{17}$$

for $0\leqslant\alpha\leqslant 1$. Solving for the X's we obtain

$$X_{\ell}^{\alpha}=\left(I-A_{\ell}^{\alpha}\right)^{-1}B_{\ell}^{\alpha}, \tag{18}$$

$$X_{\mathrm{u}}^{\alpha}=\left(I-A_{\mathrm{u}}^{\alpha}\right)^{-1}B_{\mathrm{u}}^{\alpha}, \tag{19}$$

provided the inverses exist.

Even if the inverses exist and we may obtain X_{ℓ}^{α} and X_{u}^{α} from equations (18) and (19) much more is required to obtain bonafide fuzzy numbers for $\overline{X}_i$. We will say a solution exists for $\overline{X}_i$ whenever:

(1) $(I-A_{\ell}^{\alpha})^{-1}$ and $(I-A_{\mathrm{u}}^{\alpha})^{-1}$ exist;
(2) $0\leqslant x_{i\ell}^{\alpha}$ is an increasing function of α, all i;
(3) x_{iu}^{α} is a decreasing function of α, all i; and
(4) $x_{i\ell}^{\alpha}\leqslant x_{iu}^{\alpha}$ for $\alpha=1$, all i.

A_{ℓ}^{α} (A_{u}^{α}) is semi-positive if all the elements are non-negative and each row (column) has at least one positive element. We will asume [3] that both A_{ℓ}^{α} and A_{u}^{α} are semi-positive for all α in [0, 1].

Theorem 1 [4]. *If $\sum_{i=1}^{m} a_{ij4}<1$ all j, then the fuzzy input–output model exists for this economy.*

The proof is outlined in Appendix A. The theorem says that if each column of fuzzy percentages in the fuzzy technological matrix add up to be less than one, then we can find the fuzzy total output $\overline{X}$ for any (non-negative) fuzzy final demands $\overline{B}$. In other words, if the condition in the theorem is true, a fuzzy input–output model exists for the economy.

3. Examples

Example 1. The data for the fuzzy input–output model is given in Table 2. We notice that $a_{114}+a_{214}<1$ and $a_{124}+a_{224}<1$ so Theorem 1 guarantees a solution for $\overline{X}_1$ and $\overline{X}_2$. To simplify the computations we assume that the graph of the membership function for $\bar{a}_{ij}$ is a straight line on $[a_{ij1},\,a_{ij2}]$ and on $[a_{ij3},\,a_{ij4}]$ for all i and j. Similarly, we assume that the graphs of the membership functions for the $\overline{B}_i$ and the $\overline{O}_j$ consist of straight line segments. Therefore, all our given fuzzy numbers are triangular or trapezoidal in shape.

[3] If a whole row in $\overline{A}$ is zero, then we could delete this industry from consideration. We would then take the inputs to this industry (if any) and add them to the final demand column. If a whole column in $\overline{A}$ is zero we may delete this industry and place its inputs (if any) into the outside input row.

[4] We also need to assume the increasing and decreasing parts of the membership functions are differentiable. This is explained in the Appendix.

Table 2

	Industries		Final demands	Gross (total) output
	I	II		
Industry I	(0.25/0.3/0.35)	(0.3/0.4/0.5)	(60/65, 75/80)	$\bar{X}_1$
Industry II	(0.4/0.45, 0.55/0.6)	(0.2/0.25, 0.35/0.4)	(50/55, 65/70)	$\bar{X}_2$
Outside inputs	(0.1/0.2/0.3)	(0.2/0.3/0.4)		
Total	(0.75/0.95, 1.05/1.25)	(0.7/0.95, 1.05/1.3)		

We now solve equations (18) and (19) for $\alpha = 0.0, 0.1, \ldots, 0.9, 1.0$ producing the fuzzy numbers $\bar{X}_1$ and $\bar{X}_2$ for the needed gross output. The graphs of $\bar{X}_1$ and $\bar{X}_2$ are shown in Figure 1. To obtain $\bar{X}_1$ and $\bar{X}_2$, in general, for selected values of α, we simply solve a system of linear equations for their unique solution.

Example 2. We change the data in Table 2 to $\bar{a}_{11} = (0.2 \mid 0.3 \mid 0.4)$, $\bar{a}_{12} = (0.3 \mid 0.4 \mid 0.5)$, $\bar{a}_{21} =$

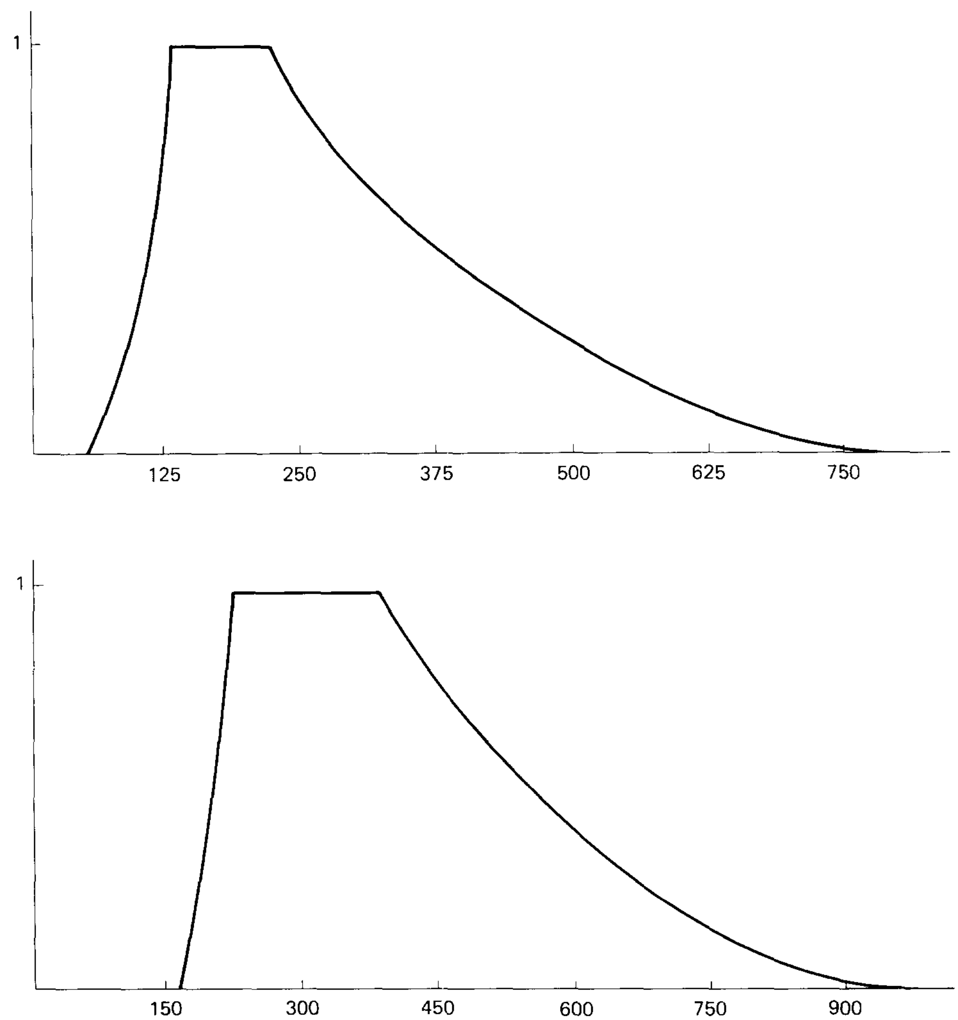

Fig. 1. Fuzzy total output for Industry I ($\bar{X}_1$, above) and for Industry II ($\bar{X}_2$, below)

$(0.4\,|\,0.5\,|\,0.6)$, and $\bar{a}_{22} = (0.3\,|\,0.4\,|\,0.5)$. We would also change $\bar{o}_1 = (0.1\,|\,0.2\,|\,0.3)$ and $\bar{o}_2 = (0.1\,|\,0.2\,|\,0.3)$ but do not change the $\bar{B}_i$. Now we have $a_{114} + a_{214} = 1$ and $a_{124} + a_{224} = 1$ so Theorem 1 will not guarantee a solution for $\bar{X}_1$ and $\bar{X}_2$.

What happens is that $I - A_u^\alpha$ is singular at $\alpha = 0$ and the system of equations in equation (17) does not have a solution for X_u^α at $\alpha = 0$. If we solve for X_u^α for $0 < \alpha \leqslant 1$ we obtain

$$\lim_{\alpha \to 0+} x_{iu}^\alpha = +\infty, \tag{20}$$

for $i = 1, 2$.

Therefore, $\bar{X}_1$ and $\bar{X}_2$ are not regular fuzzy numbers because they have infinite support. If we allow 'unbounded' fuzzy numbers, then we can solve for $\bar{X}_1$ and $\bar{X}_2$ in this example.

Example 3. In Table 2 let $\bar{a}_{11} = (0.3\,|\,0.4\,|\,0.5)$, $\bar{a}_{12} = (0.5\,|\,0.6\,|\,0.7)$, $\bar{a}_{21} = (0.4\,|\,0.5\,|\,0.6)$, and $\bar{a}_{22} = (0.2\,|\,0.3\,|\,0.4)$. Also let $\bar{o}_1 = (0\,|\,0.1\,|\,0.2) = \bar{o}_2$ but we keep the same $\bar{B}_i$. We see $a_{114} + a_{214} = 1.1$ and $a_{124} + a_{224} = 1.1$ so Theorem 1 does not apply.

We find that: (1) $I - A_u^\alpha$ is singular $\alpha = 0.5$ and equations (17) has no solution for X_u^α at $\alpha = 0.5$; (2) if $0 \leqslant \alpha < 0.5$, then $x_{iu}^\alpha < 0$ for $i = 1, 2$; (3) if $0.5 < \alpha \leqslant 1$, then $x_{iu}^\alpha > 0$ for $i = 1, 2$; (4) x_{iu}^α has an infinite limit as α approaches 0.5, $i = 1, 2$; and (5) x_{iu}^α cannot be a decreasing function of α for $i = 1, 2$. Obviously, the $\bar{X}_i$ cannot be fuzzy numbers in this example.

Examples 2 and 3 show that if the conditions of Theorem 1 are violated, then the economy may not have a fuzzy input–output model. The conditions of Theorem 1 are only sufficient and not necessary for a fuzzy input–output model to exist. For example, the sum $a_{1j4} + \cdots + a_{mj4}$ may equal one for some j and be less than one otherwise and a fuzzy input–output model can exist. However, in this initial paper on fuzzy input–output analysis, we will not search for the most general conditions guaranteeing a fuzzy input–output model, but instead be content with the general conditions stated in Theorem 1.

4. Summary and conclusions

This paper extends Leontief's well-known (open) input–output model for an economy to incorporate fuzzy numbers. Certain values in the model, for example, the percent of total input to industry I coming from industry J and the final demands for industry K, do not need to be known exactly but now may be approximated by an appropriate fuzzy number. We present a general sufficient condition for the fuzzy input–output model to exist for an economy. The sufficient condition says that if the column sums of the fuzzy percentages in the fuzzy technological matrix are less than one, then we can find fuzzy total outputs for all fuzzy final demands. Three examples of two industry economies are presented. The first example illustrates computing the fuzzy total outputs while the other two examples show that if the conditions of the theorem are not met, then the fuzzy input–output model may not exist.

We may now conclude that fuzzy input–output analysis is generally now worthwhile for Leontief's closed model. If the model is closed, there are not outside inputs (see Table 1) and then $a_{1j} + \cdots + a_{mj}$ must equal one for each column in the input–output matrix A. When we fuzzify A to $\bar{A}$ we would require that $\bar{a}_{1j} + \cdots + \bar{a}_{mj}$ to be approximately one for all j. This means that we would expect $a_{1j4} + \cdots + a_{mj4}$ to exceed one for all j. Example 3 shows what to expect in this case is that the $\bar{X}_i$ will not be fuzzy numbers.

Further research is required to obtain a more general sufficient condition for the existence of (open) fuzzy input–output models.

Appendix A. Proof of Theorem 1

Let $W = A_\ell^\alpha$, or A_u^α for any α in $[0, 1]$. We know that the column sums of W are less than one. Let $v = (1, \ldots, 1)$ be a $1 \times m$ vector whose elements are equal to one. Then

$$vW = s = (s_1, \ldots, s_m), \tag{1}$$

where $0 < s_i < 1$ for all i.

We now quote some results in economics for semipositive square matrices (see [2, 3, 4, 5, 7, 9, 13]. W has a particular characteristic root (eigenvalue) λ^*, with associated characteristic vector (eigenvector) x^* such that:

(1) λ^* is real and non-negative;

(2) no other characteristic root has modulus exceeding λ^*;

(3) x^* is non-negative ($x_i^* \geqslant 0$ all i, $x_j^* > 0$ for some j); and

(4) For any $\mu > \lambda^*$, $\mu I - W$ is nonsingular and $(\mu I - W)^{-1}$ is semipositive.

We now argue that $\lambda^* < 1$ implying that $I - A_\ell^\alpha$ and $I - A_\mathrm{u}^\alpha$ are nonsingular and their inverses are semipositive. We know

$$Wx^* = \lambda^* x^*, \tag{2}$$

so that

$$vWx^* = \lambda^*(vx^*). \tag{3}$$

Therefore

$$sx^* = \lambda^* \sum_{i=1}^{m} x_i^*. \tag{4}$$

But

$$sx^* < \sum_{i=1}^{m} x_i^*, \tag{5}$$

since $0 < s_i < 1$ for all i. Hence

$$\lambda^* \sum_{i=1}^{m} x_i^* < \sum_{i=1}^{m} x_i^* \tag{6}$$

and $\lambda^* < 1$ because the sum in equation (6) is positive.

We also have $x_{i\ell}^\alpha$ and $x_{i\mathrm{u}}^\alpha$ are non-negative since $\bar{B}_i$ is non-negative and the inverses are semipositive.

We next show that $x_{i\ell}^\alpha$ is an increasing function of α for all i. In order to do this we now need to introduce more notation. For any fuzzy number $\bar{N} = (n_1 \mid n_2, n_3 \mid n_4)$ we let $x = f(\alpha)$, $0 \leqslant \alpha \leqslant 1$, be the inverse of $\alpha = \mu(x \mid \bar{N})$ on $[n_i, n_2]$ and $x = g(\alpha)$, $0 \leqslant \alpha \leqslant 1$, be the inverse of $\alpha = \mu(x \mid \bar{N})$ on $[n_3, n_4]$. We therefore have: (1) $a_{ij\ell}^\alpha = f_{ij}(\alpha)$ the inverse of $\alpha = \mu(x \mid \bar{a}_{ij})$ on $[a_{ij1}, a_{ij2}]$ and $a_{ij\mathrm{u}}^\alpha = g_{ij}(\alpha)$ the inverse of $\alpha = \mu(x \mid \bar{a}_{ij})$ on $[a_{ij3}, a_{ij4}]$; (2) $b_{i\ell}^\alpha = f_i(\alpha)$ the inverse of $\alpha = \mu(x \mid \bar{B}_i)$ on $[b_{i1}, b_{i2}]$ and $b_{i\mathrm{u}}^\alpha = g_i(\alpha)$ the inverse of $\alpha = \mu(x \mid \bar{B}_i)$ on $[b_{i3}, b_{i4}]$. We now set $x_{i\ell}^\alpha = F_i(\alpha)$ to be the inverse of $\mu(x \mid \bar{X}_i)$ on $[x_{i1}, x_{i2}]$ and $x_{i\mathrm{u}}^\alpha = G_i(\alpha)$ to be the inverse of $\mu(x \mid \bar{X}_i)$ on $[x_{i3}, x_{i4}]$. We now assume that all of these inverse functions are differentiable functions of α. For $\bar{N}$ it is sufficient to assume that $\mu(x \mid \bar{N})$ is differentiable on (n_1, n_2) and (n_3, n_4) to guarantee that $f(\alpha)$ is differentiable on (n_1, n_2) and $g(\alpha)$ is differentiable on (n_3, n_4).

Equation (16) in the text may be written

$$\left(I - [f_{ij}(\alpha)]\right)[F_i(\alpha)] = [f_i(\alpha)], \tag{7}$$

where $[f_{ij}(\alpha)]$ is an $m \times m$ matrix and $[F_i(\alpha)]$, $[f_i(\alpha)]$ are $m \times 1$ vectors. Equation (7) implicitly defines $F_i(\alpha)$ so we differentiate both sides with respect to α giving

$$\begin{aligned}&[-f_{ij}'(\alpha)][F_i(\alpha)] + \left(I - [f_{ij}(\alpha)]\right)[F_i'(\alpha)]\\ &\quad = [f_i'(\alpha)],\end{aligned} \tag{8}$$

where the primes denote the derivative with respect to α. Solving equation (8) for $[F_i'(\alpha)]$ we obtain

$$[F_i'(\alpha)] = \left(I - [f_{ij}(\alpha)]\right)^{-1} E(\alpha), \tag{9}$$

where

$$E(\alpha) = [f_i'(\alpha)] + [f_{ij}'(\alpha)][F_i(\alpha)]. \tag{10}$$

We see that $F_i'(\alpha) > 0$ all i because $E(\alpha)$ is a positive $m \times 1$ vector ($f_i'(\alpha) > 0$ and $f_{ij}'(\alpha) > 0$) and the inverse is a semipositive matrix. Hence $x_{i\ell}^\alpha$ is an increasing function of α for all i.

Similarly we may show $x_{i\mathrm{u}}^\alpha$ is a decreasing function of α for all i since $g_i'(\alpha) < 0$ and $g_{ij}'(\alpha) < 0$.

The last thing to show is that $x_{i\ell}^\alpha \leqslant x_{i\mathrm{u}}^\alpha$ when $\alpha = 1$. If $a_{ij2} = a_{ij3}$ for all i and j and $b_{i2} = b_{i3}$ for all i, then $x_{i\ell}^\alpha = x_{i\mathrm{u}}^\alpha$ when $\alpha = 1$ for all i. Having some $a_{ij2} < a_{ij3}$ and/or some $b_{i2} < b_{i3}$ will produce some $x_{i\ell}^1 < x_{i\mathrm{u}}^1$ but we will always have $x_{il}^\alpha \leqslant x_{i\mathrm{u}}^\alpha$ when $\alpha = 1$ for all i.

References

[1] Buckley, J.J., and Chanas, S., "A fast method of ranking alternatives using fuzzy numbers", *Fuzzy Sets and Systems*, to appear.

[2] Debreu, G., and Herstein, N., "Nonnegative square matrices", *Econometrica* 21 (1953) 597–607.

[3] Dorfman, R., Samuelson, P.A., and Solow, R., *Linear Programming and Economic Analysis*, McGraw-Hill, New York, 1958.

[4] Gantmacher, F.R., *Applications of the Theory of Matrices*, Interscience, 1959.

[5] Gantmacher, F.R., *The Theory of Matrices*, Vol. I and II, Chelsea, 1960.

[6] Goetschel, R., and Voxman, W.,"Elementary fuzzy calculus", *Fuzzy Sets and Systems* 18 (1986) 31–43.

[7] Karlin, S., *Mathematical Methods and Theory in Games, Programming and Economics*, Vol. I, Addison-Wesley, New York, 1959.

[8] Kaufmann, A., and Gupta, M.M., *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold, New York, 1985.

[9] Lancaster, K., *Mathematical Economics*, Macmillan, New York, 1968.

[10] Leontief, W., "Quantitative input–output relations in the economic system of the United States", *The Review of Economics and Statistics* 18 (1936) 105–125.

[11] Leontief, W., *The Structure of the American Economy 1919–1929*, Harvard University Press, 1941.

[12] Leontief, W., *Input–Output Economics*, Oxford University Press, 1966.

[13] Yamane, T., *Mathematics for Economists*, Prentice-Hall, (Englewood Cliffs, NJ.) 1962.