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MU4MA006

Project 1

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Exercise 1

We show that the exact solution $\bar{u}$ remains 1-periodic if the initial condition u_{init} is 1-periodic.

$$\begin{aligned} \bar{u}(x+1,t) &= u_{\text{init}}(x-at+1) & (1)\\ &= u_{\text{init}}(x-at) & (2)\\ &= \bar{u}(x,t). & (3) \end{aligned}$$

Transition from (1) to (2) follows from u_{init} being 1-periodic. Thus, $\bar{u}$ is also 1-periodic.

Exercise 2

We verify that the initial condition $u_{\text{init}}(x) = \sin(2\pi x)$ is 1-periodic.

$$\begin{aligned} u_{\text{init}}(x+1) &= \sin(2\pi(x+1)) & (4)\\ &= \sin(2\pi x + 2\pi) & (5)\\ &= \sin(2\pi x) & (6)\\ &= u_{\text{init}}(x). & (7) \end{aligned}$$

The transition from (5) to (6) is due to the 2π-periodicity of the sine function, hence u_{init} is 1-periodic.

Exercise 3

Exercise 3.1

The spatial domain is discretized over one 1-period, specifically $[0,1)$, with J points, $(x_j)_{1\leq j<J}$. The time discretization depends on the ratio $\lambda = \frac{\Delta t}{\Delta x}$. The discretization does not include $x = 1$, so the last point is $1 - dx$.

```python
def discretize_space(J, lmbda, T):
    dx = 1/J
    dt = lmbda*dx
    M = int(T/dt)
    x = np.linspace(0, 1-dx, J)
    return dx, dt, x, M
```

Exercise 3.2

Next, we define a function `scheme` which iteratively multiplies $u^{(n)}$ by a matrix Q to compute the next time step's solution $u^{(n+1)}$, starting with $u^{(0)} = (u_{\text{init}}(x_j))_{1\leq j<J}$. The matrix Q depends on the chosen finite difference scheme (Exercise 3.3). The function returns the final approximation $u^{(M)}$ after M time steps, as well as the analytical solution $\bar{u}^{(M)}$.

```python
def scheme(dt, dx, x, M, cm1, c0, c1):
    Q = generate_Q(cm1, c0, c1, len(x))
    u_n = uinit(x)
    for n in range(0, M):
        u_n = Q.dot(u_n)
    ubar_n = ubar(x, M*dt)
    return u_n, ubar_n
```

Exercise 3.3

Each finite-difference scheme can be written in the form:

$$\begin{aligned} u_j^{n+1} &= c_{-1}u_{j-1}^n + c_0 u_j^n + c_1 u_{j+1}^n & (8)\\ &= H(u_{j-1}^n, u_j^n, u_{j+1}^n), & (9) \end{aligned}$$

where the coefficients c_{-1}, c_0, c_1 depend on the scheme.

The `compute_scheme_weights()` function calculates these coefficients and stores them in a dictionary, where each scheme corresponds to a set of weights $[c_{-1}, c_0, c_1]$. Using these weights, we construct the Q matrix, which represents the linear transformation H.

Considering the periodic boundary conditions, since $u_{j+J+1}^n = u_j^n$, we have:

$$\begin{aligned} u_0^{n+1} &= c_{-1}u_J^n + c_0 u_0^n + c_1 u_1^n, & (10)\\ u_J^{n+1} &= c_{-1}u_{J-1}^n + c_0 u_J^n + c_1 u_0^n. & (11) \end{aligned}$$

Thus, Q is a circulant matrix with the following form:

$$Q = \begin{pmatrix} c_0 & c_1 & 0 & \dots & 0 & c_{-1} \\ c_{-1} & c_0 & c_1 & 0 & \dots & 0 \\ 0 & c_{-1} & c_0 & c_1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & c_{-1} & c_0 & c_1 \\ c_1 & 0 & \dots & 0 & c_{-1} & c_0 \end{pmatrix}$$

A figure depicting the solutions for every method with $T = 0.75, a = 1, J = 20, \lambda = 0.8$, can be found in the appendix (A) and reproduced using the attached notebook file. These plots will be further commented in the Discussion section.

```python
def compute_scheme_weights(lmbda, a):
    schemes = {
        "centered": np.array([
            lmbda * a / 2,
            1,
```

```
            -lmbda * a / 2
        ]),
        "left": np.array([
            lmbda * a,
            1 - lmbda * a,
            0
        ]),
        "right": np.array([
            0,
            1 + lmbda * a,
            -lmbda * a
        ]),
        "Lax-Friedrichs": np.array([
            0.5 + 0.5 * lmbda * a,
            0,
            0.5 - 0.5 * lmbda * a
        ]),
        "Lax-Wendroff": np.array([
            0.5 * lmbda * a + 0.5 * lmbda**2 * a**2,
            1 - lmbda**2 * a**2,
            - 0.5 * lmbda * a + 0.5 * lmbda**2 * a**2
        ])
    }
    return schemes

def generate_Q(cm1, c0, c1, J):
    Q = np.zeros((J, J))
    # Fill in the diagonals with weights
    for j in range(J):
        Q[j, j] = c0   # Main diagonal
        Q[j, (j - 1) % J] = cm1   # Sub-diagonal (wrap around with % operator)
        Q[j, (j + 1) % J] = c1    # Super-diagonal (wrap around with % operator)
    return Q
```

Exercise 4

To analyze the stability of each numerical method, we compute the L_2 and L_∞ norms of the transformation matrix Q with $T = 0.75, a = 1, J = 20, \lambda = 0.8$ using the `compute_scheme_norms` function, defined in the code below:

```
def compute_scheme_norms(lmbda, a, J):
    weights = compute_scheme_weights(lmbda, a)
    norms_df = pd.DataFrame(columns=["infinity_norm", "2_norm"], index=weights.keys())
    # Loop over each scheme to generate Q and compute norms
    for scheme, coeffs in weights.items():
        # Generate Q using the scheme's weights
        cm1, c0, c1 = coeffs
        Q = generate_Q(cm1, c0, c1, J)
        # Compute norms and store in dataframe
        infinity_norm = norm(Q, ord=np.inf)   # Infinity norm
        two_norm = norm(Q, ord=2)             # 2-norm
        norms_df.loc[scheme] = [infinity_norm, two_norm]
    return norms_df
```

The resulting norms for each scheme are presented in Table 1:

Table 1: L_∞ and L_2 Norms for Different Schemes

Scheme	L_∞ Norm	L_2 Norm
Centered	1.8	1.2806
Left	1.0	1.0
Right	2.6	2.6
Lax-Friedrichs	1.0	1.0
Lax-Wendroff	1.16	1.0

From Table 1, we observe the following stability characteristics for each scheme:

- **Centered Scheme**: This method is unstable in both L_2 and L_∞ norms, exceeding the threshold of 1 in both cases.
- **Left Scheme**: Stable in both norms.
- **Right Scheme**: Unstable in both norms.
- **Lax-Friedrichs Method**: Stable in both L_2 and L_∞ norms.
- **Lax-Wendroff Method**: Stable in the L_2 norm, but marginally unstable in the L_∞ norm.

To further explore stability and explain these conclusions, we can derive the analytical stability criteria based on the L_∞ and the L_2 (von Neumann stability analysis) norms. These criteria provide insight into when each scheme is expected to be stable, given a, $\Delta x, \Delta t$.

Left-Sided Method

The left-sided method has coefficients:

$$c_0 = 1 - \lambda a, \quad c_{-1} = \lambda a, \quad c_1 = 0$$

Substitute these in to the Q matrix. The infinity norm $\|Q\|_\infty$ is given by:

$$\|Q\|_\infty = \max\left(|c_0| + |c_1| + |c_{-1}|\right)$$

Since $c_1 = 0$, the row sum simplifies to:

$$\|Q\|_\infty = |1 - \lambda a| + |\lambda a|$$

For stability, we require:

$$|1 - \lambda a| + |\lambda a| \leq 1$$

Case 1: $a \geq 0$

$$1 - \lambda a + \lambda a \leq 1 \implies 1 \leq 1$$

This condition is always satisfied.

Case 2: $a < 0$

$$1 - \lambda a - \lambda a \leq 1 \implies 1 - 2\lambda a \leq 1 \implies a \geq 0$$

This condition is not satisfied for negative a.

Thus, stability requires:

$$a \geq 0 \quad \text{and} \quad 1 - \lambda a \geq 0 \implies \lambda a \leq 1$$

If $a > 0$, this condition simplifies to:

$$a \cdot \frac{\Delta t}{\Delta x} \leq 1$$

for stability.

It can be shown through von Neumann stability analysis that this method is also stable in l_2 under the same CFL condition.

Right-Sided Method

The right-sided method has coefficients:

$$c_0 = 1 + \lambda a, \quad c_{-1} = 0, \quad c_1 = -\lambda a$$

Substitute these in to the Q matrix. The infinity norm $\|Q\|_\infty$ is given by:

$$\|Q\|_\infty = \max\left(|c_0| + |c_1| + |c_{-1}|\right)$$

Since $c_{-1} = 0$, the row sum simplifies to:

$$\|Q\|_\infty = |1 + \lambda a| + |-\lambda a|$$

For stability, we require:

$$|1 + \lambda a| + |-\lambda a| \leq 1$$

Case 1: $\lambda a \geq 0$

$$1 + \lambda a + \lambda a \leq 1 \implies 1 + 2\lambda a \leq 1 \implies \lambda a \leq 0$$

This condition is not satisfied for positive λa.

Case 2: $\lambda a < 0$

$$1 + \lambda a - \lambda a \leq 1 \implies 1 \leq 1$$

This condition is always satisfied.

Thus, stability requires:

$$\lambda a \leq 0 \quad \text{and} \quad 1 + \lambda a \geq 0 \implies -1 \leq \lambda a$$

If $a < 0$, this condition simplifies to:

$$|a| \cdot \frac{\Delta t}{\Delta x} \leq 1$$

for stability. It can be shown through von Neumann stability analysis that this method is also stable in l_2 under the same CFL condition.

Centered Method

The centered method has coefficients:

$$c_0 = 1, \quad c_1 = -\frac{\lambda a}{2}, \quad c_{-1} = \frac{\lambda a}{2}$$

Substitute these in to the Q matrix. The infinity norm $\|Q\|_\infty$ is given by:

$$\|Q\|_\infty = \max\left(|c_0| + |c_1| + |c_{-1}|\right)$$

Substituting the coefficients, we have:

$$\|Q\|_\infty = |1| + \left|-\frac{\lambda a}{2}\right| + \left|\frac{\lambda a}{2}\right|$$

Simplifying:

$$\|Q\|_\infty = 1 + \frac{\lambda |a|}{2} + \frac{\lambda |a|}{2} = 1 + \lambda |a|$$

For stability, we require:

$$\|Q\|_\infty \leq 1$$

Substituting:

$$1 + \lambda |a| \leq 1$$

Simplifying:

$$\lambda |a| \leq 0$$

Since $\lambda = \frac{\Delta t}{\Delta x} > 0$, this implies:

$$|a| = 0$$

So this method is unstable. It can be shown through von Neumann stability analysis that this method is also unstable in l_2 under the same CFL condition.

Lax-Friedrichs Method

The Lax-Friedrichs method has coefficients:

$$c_{-1} = \frac{1 + \lambda a}{2}, \quad c_0 = 0, \quad c_1 = \frac{1 - \lambda a}{2}.$$

Substitute these into the Q matrix.

The infinity norm $\|Q\|_\infty$ is given by:

$$\|Q\|_\infty = \max\left(|c_0| + |c_1| + |c_{-1}|\right).$$

Since $c_0 = 0$, the row sum simplifies to:

$$\|Q\|_\infty = \left|\frac{1 + \lambda a}{2}\right| + \left|\frac{1 - \lambda a}{2}\right|.$$

If all coefficients are non-negative (which they are if the CFL condition is satisfied $a\lambda \leq 1$) we have stability if:

$$\|Q\|_\infty \leq 1.$$

Substitute the values of c_{-1} and c_1:

$$\left|\frac{1 + \lambda a}{2}\right| + \left|\frac{1 - \lambda a}{2}\right| \leq 1.$$

Simplify to:

$$|1 + \lambda a| + |1 - \lambda a| \leq 2.$$

Case 1: $\lambda a \geq 0$ If $\lambda a \geq 0$, then:

$$|1+\lambda a| = 1+\lambda a, \quad |1-\lambda a| = 1-\lambda a.$$

Substitute into the inequality:

$$(1+\lambda a)+(1-\lambda a) \leq 2.$$

Simplify:

$$2 \leq 2.$$

This condition is satisfied.

Case 2: $\lambda a < 0$ If $\lambda a < 0$, then:

$$|1+\lambda a| = 1-\lambda a, \quad |1-\lambda a| = 1+\lambda a.$$

Substitute into the inequality:

$$(1-\lambda a)+(1+\lambda a) \leq 2.$$

Simplify:

$$2 \leq 2.$$

This condition is also satisfied.

Thus, the Lax-Friedrichs method is stable in l_∞ if the CFL condition is respected. It can be shown through von Neumann stability analysis that this method is also stable in l_2 under the same CFL condition.

The Lax-Wendroff Method

The Lax-Wendroff method has coefficients:

$$c_{-1} = \frac{\lambda a}{2}+\frac{\lambda^2 a^2}{2}, \quad c_0 = 1-\lambda^2 a^2, \quad c_1 = -\frac{\lambda a}{2}+\frac{\lambda^2 a^2}{2}$$

The update formula for this method in space and time is:

$$u_j^{n+1} = u_j^n - \frac{\lambda a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\lambda^2 a^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

where $\lambda = \frac{\Delta t}{\Delta x}$.

In this particular case we see that $\|Q\|_\infty > 1$ for any chosen λ and a. Since this method is unstable in l_∞, we analyze the stability in l_2. To analyze stability, we use von Neumann Stability Analysis, first writing the solution in a Fourier basis: $\hat{u}^{n+1} = \sum_j u_j^{n+1} e^{-ij\Delta x\xi}$.

Substitute that into the scheme noticing that

$$\sum_j e^{-ij\Delta x\xi} u_j^n = \hat{u}^n$$

$$\sum_j e^{-ij\Delta x\xi} u_{j+1}^n = \hat{u}^n e^{i\Delta x\xi},$$

$$\sum_j e^{-ij\Delta x\xi} u_{j-1}^n = \hat{u}^n e^{-i\Delta x\xi}.$$

we get:

$$\begin{aligned}\hat{u}^{n+1} = \hat{u}^n & \\ & - \frac{\lambda a}{2}\left(\hat{u}^n e^{i\Delta x\xi} - \hat{u}^n e^{-i\Delta x\xi}\right) \\ & + \frac{\lambda^2 a^2}{2}\left(\hat{u}^n e^{i\Delta x\xi} - 2\hat{u}^n \right. \\ & \left. + \hat{u}^n e^{-i\Delta x\xi}\right).\end{aligned}$$

Factor out $\hat{u}^n$:

$$\begin{aligned}\hat{u}^{n+1} = \hat{u}^n \Big[& 1 - \frac{\lambda a}{2}\left(e^{i\Delta x\xi} - e^{-i\Delta x\xi}\right) \\ & + \frac{\lambda^2 a^2}{2}\left(e^{i\Delta x\xi} - 2 + e^{-i\Delta x\xi}\right)\Big].\end{aligned}$$

Simplify using Euler's formulas:

$$e^{i\Delta x\xi} - e^{-i\Delta x\xi} = 2i\sin(\Delta x\xi),$$
$$e^{i\Delta x\xi} + e^{-i\Delta x\xi} = 2\cos(\Delta x\xi).$$

Substitute into the equation:

$$\begin{aligned}\hat{u}^{n+1} = \hat{u}^n \Big[& 1 - i\lambda a\sin(\Delta x\xi) \\ & + \lambda^2 a^2\big(\cos(\Delta x\xi) - 1\big)\Big].\end{aligned}$$

The amplification factor $A(\xi)$ is:

$$A(\xi) = 1 - i\lambda a\sin(\Delta x\xi) - \lambda^2 a^2\left(1 - \cos(\Delta x\xi)\right).$$

The squared magnitude is:

$$|A(\xi)|^2 = \left(1 - \lambda^2 a^2\left(1 - \cos(\Delta x\xi)\right)\right)^2 + \left(\lambda a\sin(\Delta x\xi)\right)^2.$$

To analyze stability, we consider the extreme (worst-case) values of the trigonometric functions:

- When $\cos(\Delta x\xi) = -1$, we have $1 - \cos(\Delta x\xi) = 2$.
- When $\cos(\Delta x\xi) = -1$, $\sin(\Delta x\xi) = 0$

Substitute these values into $|A(\xi)|^2$:

$$|A(\xi)|^2 = \left(1 - \lambda^2 a^2(2)\right)^2 + \left(\lambda a(0)\right)^2.$$

Simplify:

$$|A(\xi)|^2 = \left(1 - 2\lambda^2 a^2\right)^2.$$

For stability, we require:

$$|A(\xi)|^2 \leq 1.$$

Substitute the expression for $|A(\xi)|^2$:

$$\left(1 - 2\lambda^2 a^2\right)^2 \leq 1.$$

Take the square root of both sides (since both sides are positive):

$$\left|1 - 2\lambda^2 a^2\right| \leq 1.$$

This absolute value inequality can be split into two conditions:

$$-1 \leq 1 - 2\lambda^2 a^2 \leq 1.$$

The second inequality is always satisfied.

From the first inequality, we obtain:

$$\lambda^2 a^2 \leq 1.$$

or

$$|a| \frac{\Delta t}{\Delta x} \leq 1.$$

for stability in the l_2-norm.

The analytical stability results reveals the following conclusions for each method:

- The **centered method** is **unconditionally unstable** in both the L_∞ and L_2 norms.
- The **left and right methods** are **conditionally stable** depending on the sign of a and adherence to the CFL condition, $|a|\lambda \leq 1$. Specifically:
  - If $a > 0$, stability is guaranteed for the left method when $|a|\lambda \leq 1$ (as in this case where $|a|\lambda = 0.8 \leq 1$). The right method is unstable because it requires $a < 0$ for stability.
  - If $a < 0$, the right method would be stable under the CFL condition, and the left method would be unstable.
- The **Lax-Friedrichs method** is **conditionally stable** under the CFL condition $|a|\lambda \leq 1$. Since $|a|\lambda = 0.8 \leq 1$ here, the method is stable in both L_∞ and L_2 norms.
- The **Lax-Wendroff method** has a mixed stability profile:
  - It is stable in the L_2 norm when the CFL condition $|a|\lambda \leq 1$ is satisfied.
  - However, it is **unstable in the L_∞ norm** even when the CFL condition is met. In this case, although $|a|\lambda = 0.8 \leq 1$, the Lax-Wendroff method remains unstable in the L_∞ norm.

If we had $\lambda > 1$, all methods (Left, Lax-Friedrichs, and Lax-Wendroff) would be unstable in both L_∞ and L_2 norms, so we need to ensure that $\lambda = \frac{\Delta t}{\Delta x} \leq 1$.

Exercise 5

To study the convergence of the given methods, the function `scheme` has been modified to return the maximum values of the discrete L_∞ and L_2 norms of the errors over all timesteps up to M:

$$\epsilon^{(\infty)}_{\Delta t, \Delta x} = \max_{n \geq 0} \left(\|\overline{U}_{\Delta x} - U_{\Delta x}\|_{\infty,\Delta} \right) \tag{12}$$

$$\epsilon^{(2)}_{\Delta t, \Delta x} = \max_{n \geq 0} \left(\|\overline{U}_{\Delta x} - U_{\Delta x}\|_{2,\Delta} \right) \tag{13}$$

The modified function is then called with various values of Δx (and the corresponding grid parameters) to obtain error estimates as the grid is refined. This allows us to analyze the convergence rates of each method.

Given that the centered and right schemes are known to be unstable in both norms with given parameters, the convergence analysis will focus solely on the left, Lax-Friedrich and Lax-Wendroff methods. Specifically, the following observations can be made:

- The log-log plot (Figure 6) of the error norms for the **left method**, as a function of Δx, reveals a slope of approximately 0.98. This suggests that the method has a spatial convergence order of 1. Given that the time discretization is also first-order consistent with the time derivative, we conclude that the left method converges with an overall order of 1 in both time and space.

- Similarly, the log-log plot (Figure 7) for the **Lax-Friedrichs method** shows a slope of approximately 0.97, also indicating a spatial convergence order of 1. Since this method's time discretization is also first-order consistent, we deduce that the Lax-Friedrichs method has a convergence order of 1 in both spatial and temporal dimensions.

- For the **Lax-Wendroff method**, the log-log plot (Figure 8) reveals a slope of nearly 2, indicating a second-order convergence rate in space. This result aligns with expectations, as the Lax-Wendroff scheme uses a second-order centered finite difference approximation for the spatial derivative. Moreover, the method incorporates a correction term that compensates for first-order error terms in time. This adjustment makes the method second-order consistent in both time and space, yielding a higher convergence rate than the left and Lax-Friedrichs methods (given that stability conditions are fulfilled).

Also, all methods' log-log plots show that the errors are higher in the L_∞ norm, because the intercept of their lines are higher than in L_2.

Discussion

Based on the solution plots obtained in Exercise 3.3, along with the stability and convergence analysis, we can draw the following conclusions:

- The **right method** fails to approximate the solution. This is primarily because the direction of transport and flow of information is from left to right (since $a > 0$), while the difference scheme relies on information from the right, specifically u_{j+1}^n. Consequently, the analytical solution exhibits a "lag" behind the numerical solution.
- The **centered scheme** behaves similarly to the right method but to a lesser extent. This is because it utilizes information from both sides: u_{j-1}^n and u_{j+1}^n. However, this additional information does not fully compensate for the inherent inaccuracies.
- The **left method** effectively captures the behavior of the exact solution, as it predominantly relies on information from u_{j-1}^n for its approximation. This method aligns well with the direction of transport.
- Although the **Lax-Friedrichs method** also incorporates u_{j+1}^n for the approximation, it still manages to represent the solution well. This is due to its greater reliance on u_{j-1}^n compared to u_j^n, since u_j^n is replaced by the average of u_{j-1}^n and u_{j+1}^n. As a result, its approximation is superior to that of the centered and right methods, but still not as effective as the left method, which puts more weight on u_{j-1}^n.
- The **Lax-Wendroff method** demonstrates the best performance, largely due to its higher order of convergence. Even with a limited number of grid points J, it can closely approximate the exact solution.

In conclusion, given that we have selected λ such that the CFL condition is satisfied, we find that among the stable methods, the Lax-Wendroff method, with its superior convergence rates in both space and time, is the most favorable choice. The plots clearly illustrate that it achieves the highest accuracy in this context.

A Solution Plots

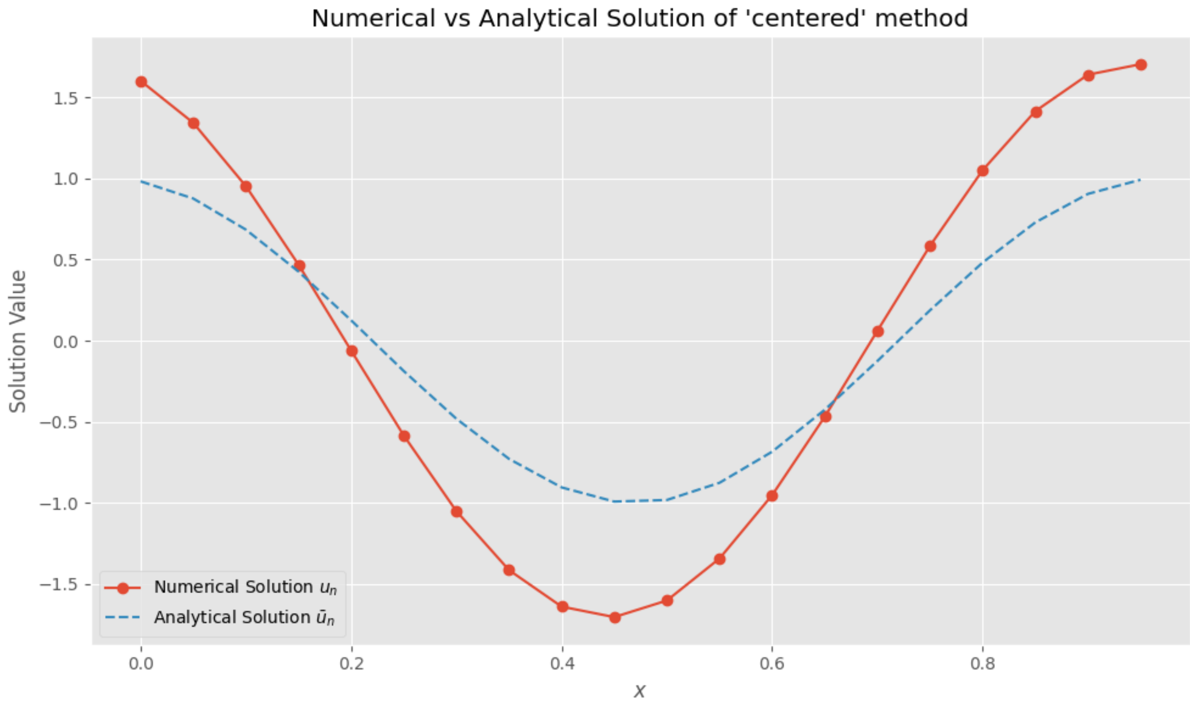

Figure 1: Comparison of Numerical and Analytical Solutions of ”centered” method

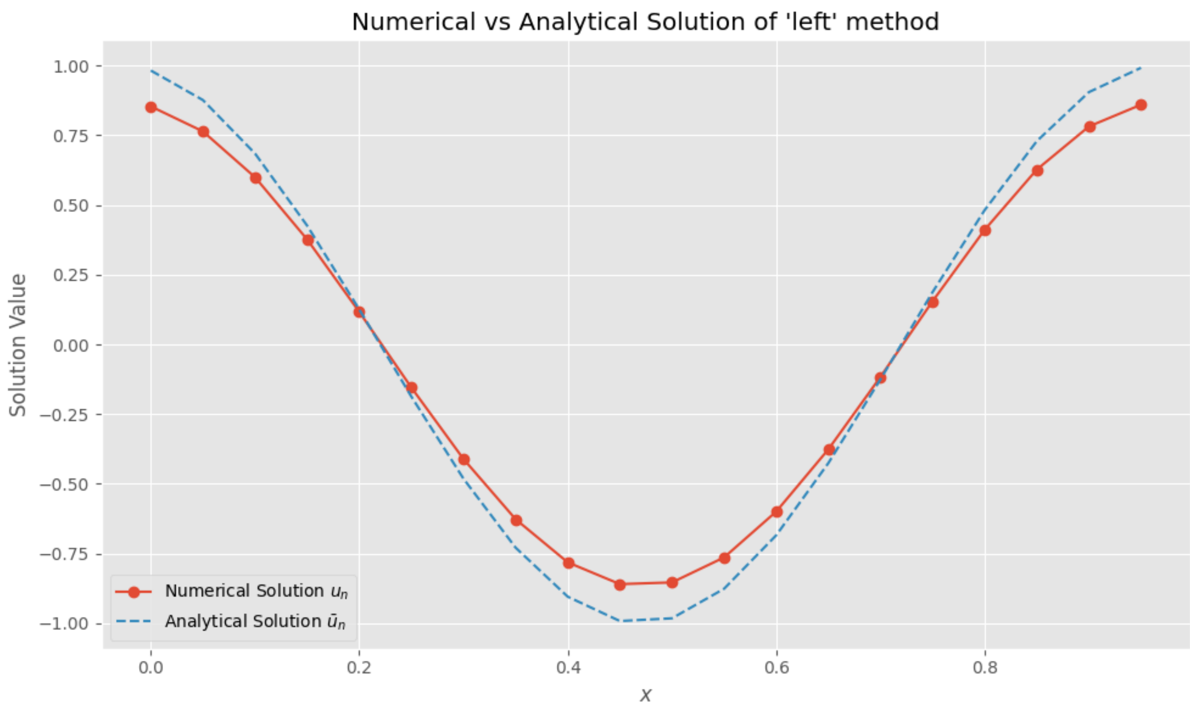

Figure 2: Comparison of Numerical and Analytical Solutions of ”left” method

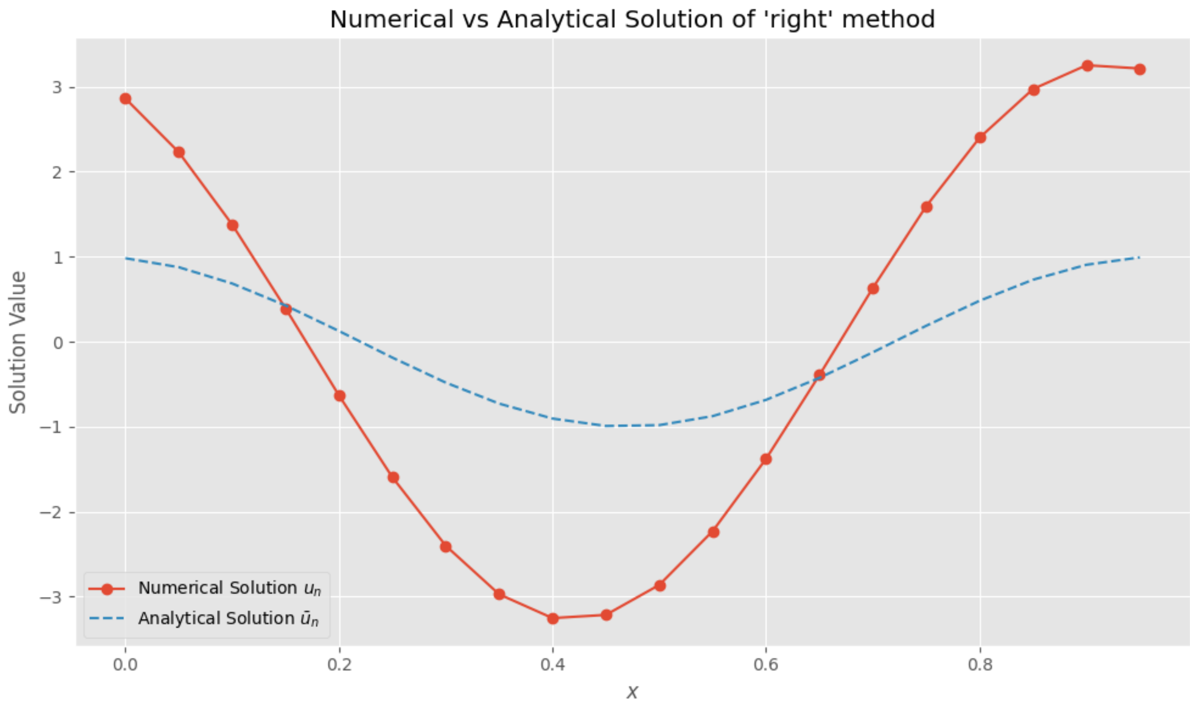

Figure 3: Comparison of Numerical and Analytical Solutions of ”right”

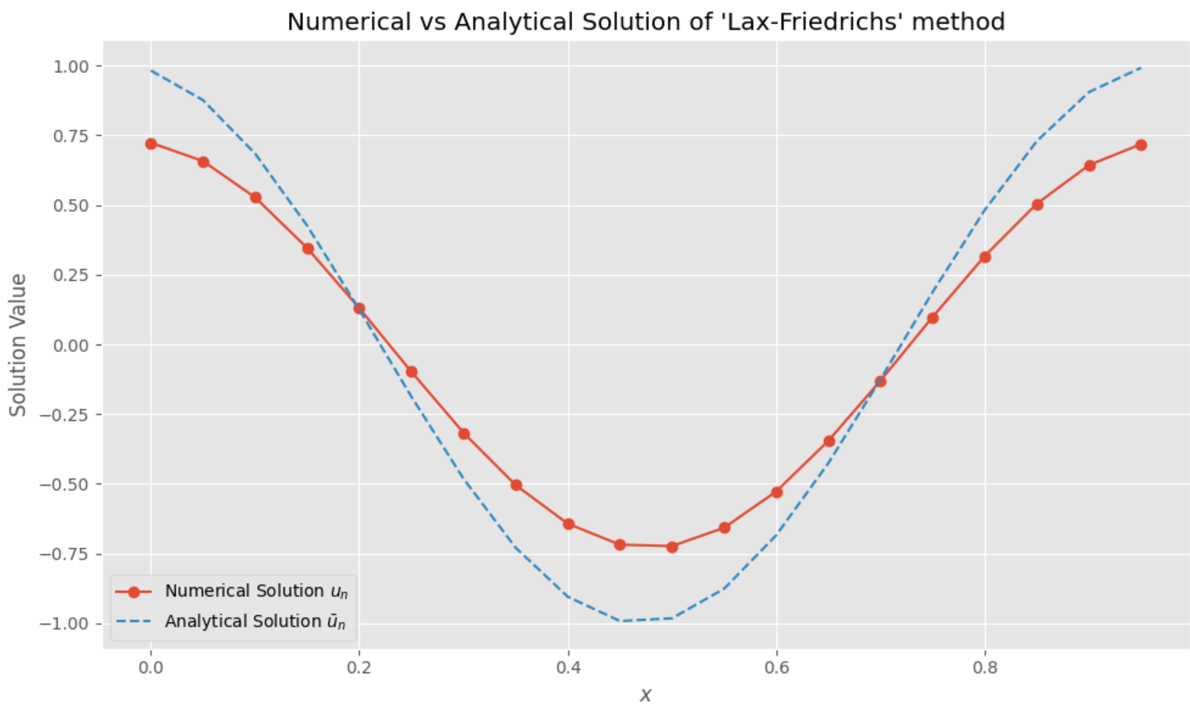

Figure 4: Comparison of Numerical and Analytical Solutions of ”Lax-Friedrichs” method

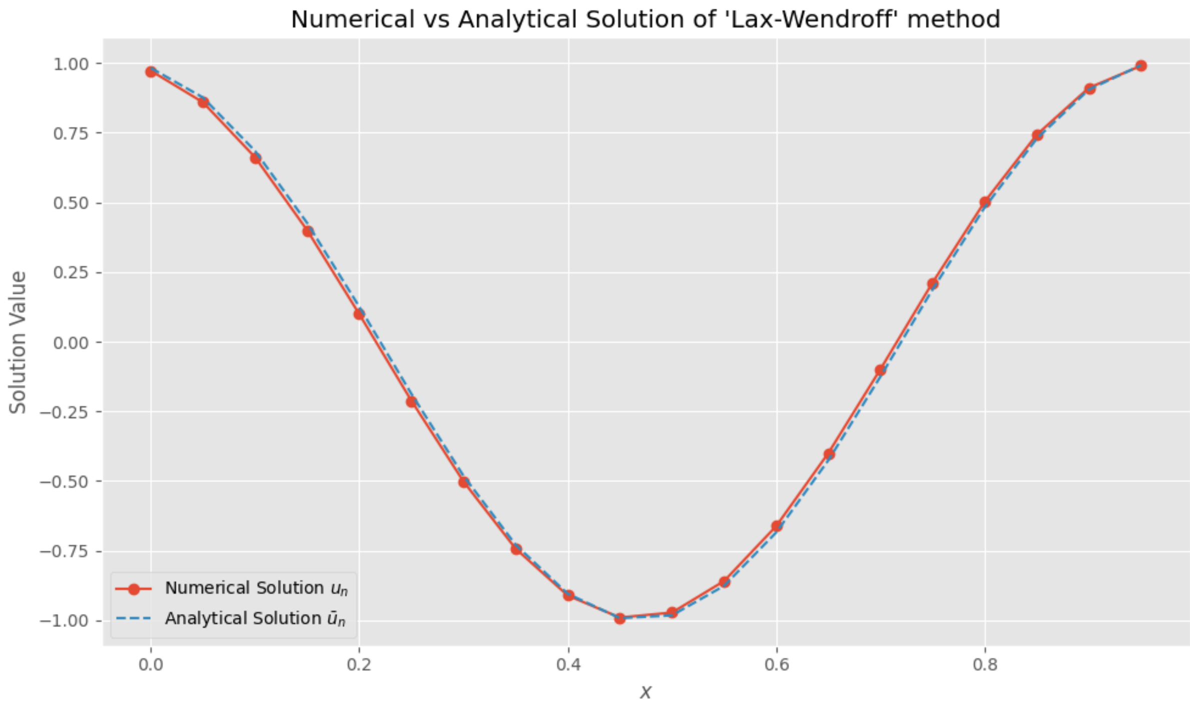

Figure 5: Comparison of Numerical and Analytical Solutions of "Lax-Wendroff" method

B Convergence Plots

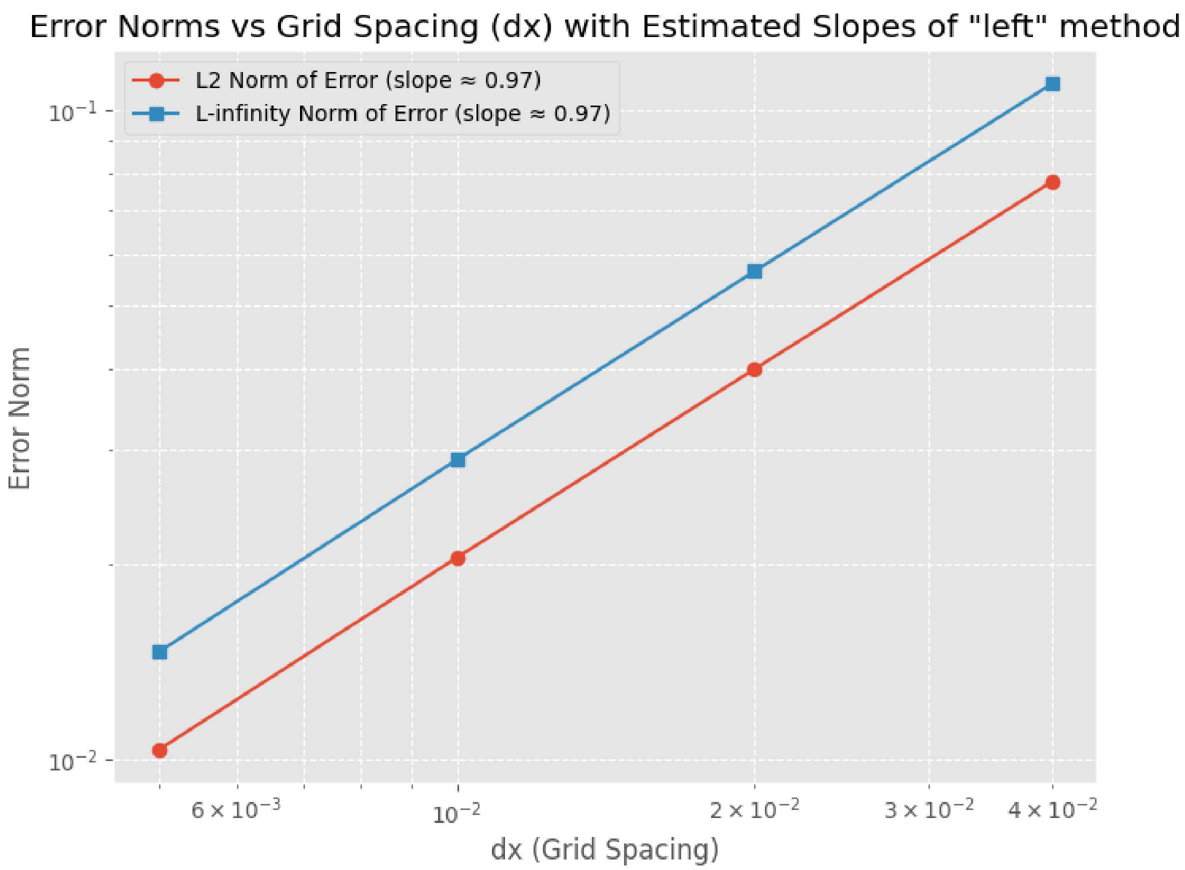

Figure 6: Convergence of "left" method

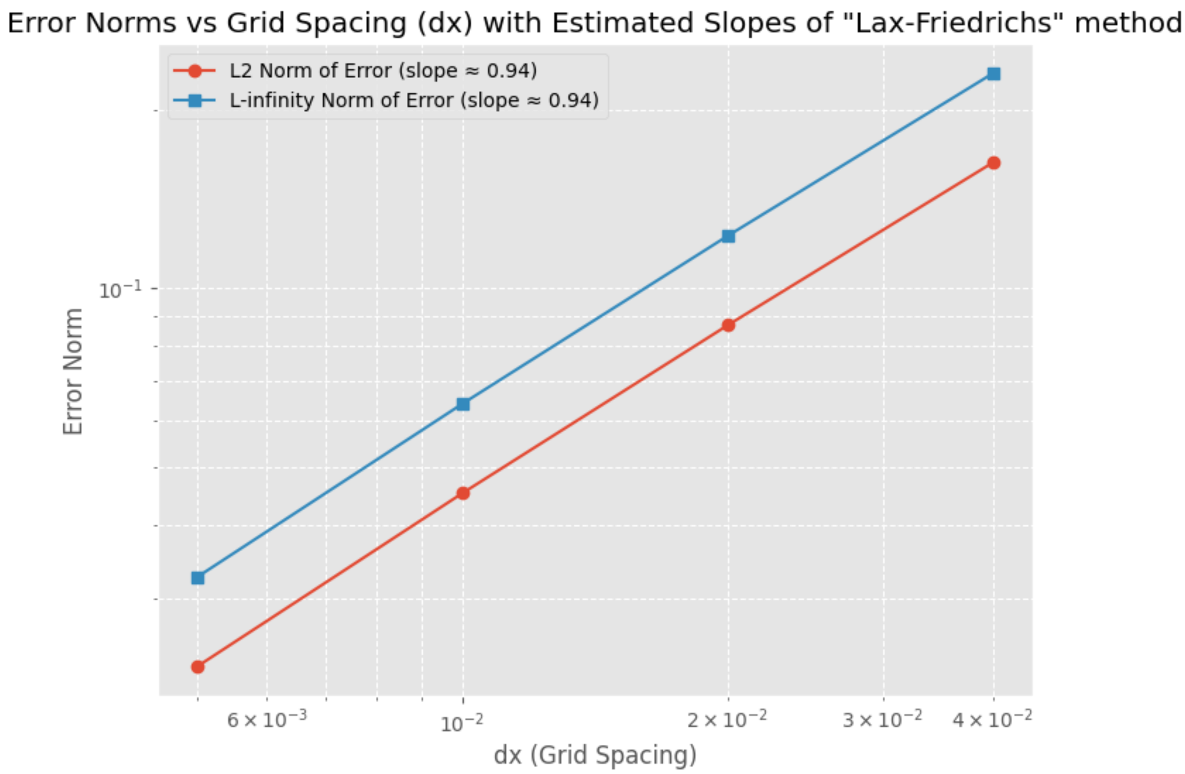

Figure 7: Convergence of "Lax-Friedrichs" method

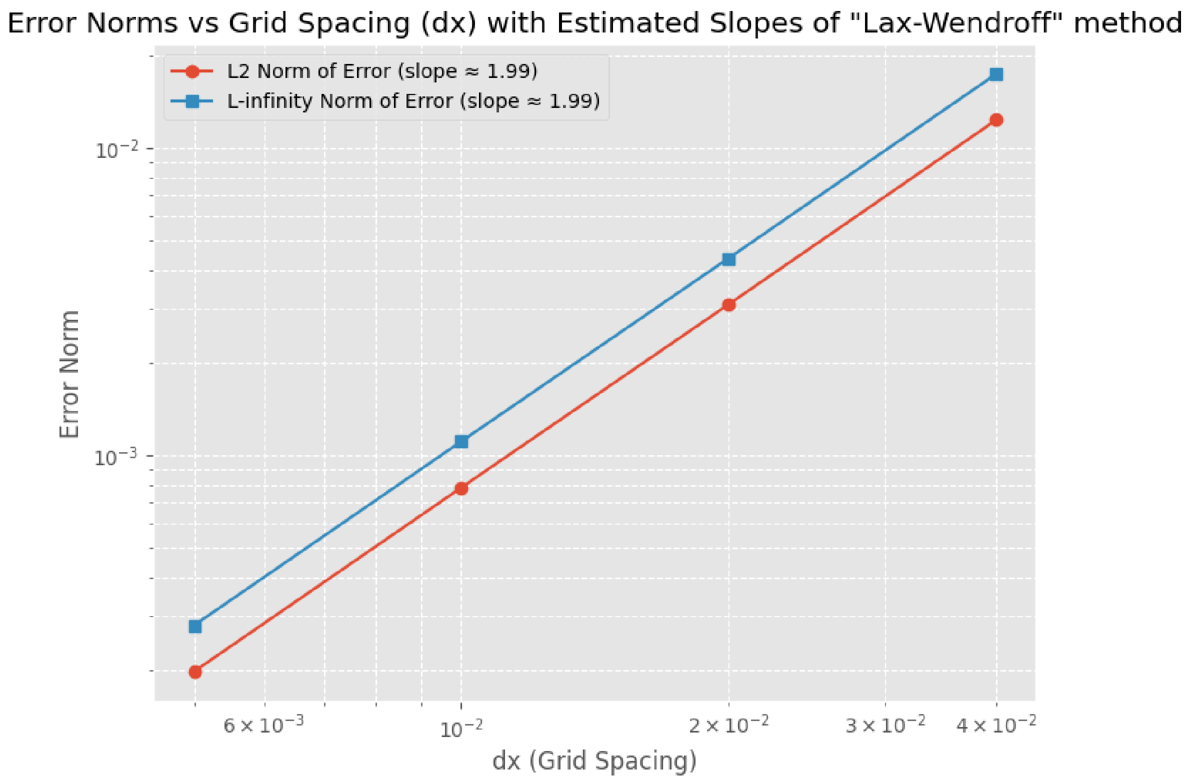

Figure 8: Convergence of "Lax-Wendroff" method