Rineke Verbrugge, manuscript, 1992



## Verzamelingen-Veltman frames en modellen

**Definitie 1.**

Een $IL_{Set}$*frame* is een L-frame $\langle W,R\rangle$ met een extra relatie $S_w$ voor iedere $w\in W$, met de volgende eigenschappen:

(I) $S_w \subseteq W[w]\times P(W[w])\setminus\emptyset$

(II) $S_w$ is "reflexief": Als $wRx$, dan $xS_w\{x\}$

$S_w$ is "transitief": Als $xS_wY$ dan geldt voor alle $y\in Y$ en alle $V\in P(W[w])$ : $yS_wV \Rightarrow xS_wV$.

(III)Als $wRw'Rw''$, dan $w'S_w\{w''\}$

English translation on page 1a

Een $IL_{Set}$ *model* bestaat uit een $IL_{Set}$ frame $\langle W,R,S\rangle$ met een forcing relatie $\Vdash$ die voldoet aan:

$u\Vdash\Box\varphi \iff \forall v(uRv\Rightarrow v\Vdash\varphi)$

$u\Vdash\varphi\rhd\psi \iff \forall v(uRv$ en $v\Vdash\varphi \Rightarrow \exists V(vS_uV$ en $\forall x\in V\ x\Vdash\psi))$

Het is niet moeilijk in te zien dat voor ieder $IL_{Set}$ model K, $K\models IL$.

## From $IL_{Set}$ models to IL models

**Theorem 2.**

Let $\langle W,R,S,\Vdash\rangle$ be an $IL_{Set}$ model. Then there is an IL model $\langle W',R',S',\Vdash'\rangle$ and a bijection $f:W\to P(W')$ such that for all $w\in W$ and all $w'\in f(w)$: $w\Vdash\varphi \iff w'\Vdash'\varphi$.

**Proof.**

Let W' consist of all points $\langle x,A\rangle$, where $x\in W$ and A is a set of ordered pairs such that:

For all w, V with $xS_wV$, there is a $v\in V$ with $\langle w,v\rangle\in A$;

and, conversely, if $\langle w,v\rangle\in A$, then there is a V such that $v\in V$ and $xS_wV$.

Define R', S' as follows:

$\langle x,A\rangle R'\langle y,B\rangle \iff xRy$ and for all w: such that $wRx$: and all z: if $\langle w,z\rangle\in B$, then $\langle w,z\rangle\in A$.

$\langle x,A\rangle S'_{\langle w,C\rangle}\langle y,B\rangle \iff \langle w,C\rangle R'\langle x,A\rangle$, $\langle w,C\rangle R'\langle y,B\rangle$ and for all v:

if $\langle w,v\rangle \in B$, then $\langle w,v\rangle \in A$
(thus in particular, because $S_w$'s being "reflexive" implies $\langle w,y\rangle \in B$, we have $\langle w,y\rangle \in A$).

Finally, define $\Vdash'$ as follows:
$\langle x,A\rangle \Vdash' p \iff x \Vdash p$.

We will prove that

(a) $\langle W',R',S'\rangle$ is an IL frame

(b) For all $\varphi$, $\langle x,A\rangle \Vdash' \varphi \iff x \Vdash \varphi$

Proof of (a):
First of all, it is not difficult to see that $\langle W',R'\rangle$ is an L-frame. Checking the clauses for $S'_{\langle w,C\rangle}$ requires a bit more work.

(I') If $\langle w,C\rangle \in W'$, then $S'_{\langle w,C\rangle}$ is a relation on $W'[\langle w,C\rangle]$; this follows immediately from the definition of $S'_{\langle w,C\rangle}$.

(II') $S'_{\langle w,C\rangle}$ is reflexive; for suppose $\langle w,C\rangle R' \langle x,A\rangle$, then by the definition of $W'$: $\langle x,A\rangle S'_{\langle w,C\rangle} \langle x,A\rangle$.
$S'_{\langle w,C\rangle}$ is transitive; for suppose $\langle x,A\rangle S'_{\langle w,C\rangle} \langle y,B\rangle S'_{\langle w,C\rangle} \langle z,D\rangle$, then $\langle w,C\rangle R' \langle x,A\rangle$, $\langle w,C\rangle R' \langle z,D\rangle$ and for all $v$ : if $\langle w,v\rangle \in D$, then $\langle w,v\rangle \in B$, and thus $\langle w,v\rangle \in A$; therefore $\langle x,A\rangle S'_{\langle w,C\rangle} \langle z,D\rangle$.

(III') If $\langle w,C\rangle R' \langle x,A\rangle R' \langle y,B\rangle$, then $wRx$ and by definition of $R'$: for all $z$, if $\langle w,z\rangle \in B$, then $\langle w,z\rangle \in A$; therefore $\langle x,A\rangle S'_{\langle w,C\rangle} \langle y,B\rangle$.

Proof of (b):
As usual, the only interesting case is the induction step for $\rhd$.

Suppose $w \Vdash \varphi \rhd \psi$ and $\langle w,C\rangle \in W'$. We want to prove $\langle w,C\rangle \Vdash' \varphi \rhd \psi$.
So suppose $\langle w,C\rangle R' \langle x,A\rangle$ and $\langle x,A\rangle \Vdash' \varphi$.
Then $wRx$ and, by the induction hypothesis, $x \Vdash \varphi$. Therefore, there is a $V$ with $xS_wV$ and $\forall y \in V\ y \Vdash \psi$. We want to find a $B$ such that $\langle x,A\rangle S'_{\langle w,C\rangle} \langle y,B\rangle$. This is possible because of the following two facts:

(1) By "transitivity" of $S_w$, we have $\forall V\ (yS_wV \Rightarrow xS_wV)$

(2) For any $b$ with $bRwRy$ we have, by (III), $wS_b\{y\}$, and thus by transitivity: if $yS_bV$, then $wS_bV$.

Therefore, we can take a $B$ such that $\langle y,B\rangle \in W'$ and, by (1), for all $v$, if $\langle w,v\rangle \in B$, then $\langle w,v\rangle \in A$; moreover $wRy$ and, by (2), if $bRw$, then for all $z$: if $\langle b,z\rangle \in B$, then $\langle b,z\rangle \in C$, so $\langle w,C\rangle R'\langle y,B\rangle$. In conclusion, $\langle x,A\rangle S'_{\langle w,C\rangle}\langle y,B\rangle$ and by induction hypothesis, $\langle y,B\rangle \Vdash' \psi$.

Suppose on the other hand that $w \Vdash \neg(\varphi \rhd \psi)$. We will prove $\langle w,C\rangle \Vdash' \neg(\varphi \rhd \psi)$.

First, $w \Vdash \neg(\varphi \rhd \psi)$ implies that there is an $x$ with $wRx$ and for all $V$ with $xS_wV$ there is a $y \in V$ such that $y \Vdash \neg\psi$.

Therefore, it is possible to take an $A$ such that $\langle x,A\rangle \in W'$ and for all $y$: if $\langle w,y\rangle \in A$, then $y \Vdash \neg\psi$, and moreover $\langle w,C\rangle R'\langle x,A\rangle$ (the extra clause for $R'$ doesn't interfere with our desiderations for $A$).

For this $\langle x,A\rangle$, we have by induction hypothesis $\langle x,A\rangle \Vdash \varphi$.

Moreover, if $\langle x,A\rangle S'_{\langle w,C\rangle}\langle y,B\rangle$, then $\langle w,y\rangle \in B$, and thus $\langle w,y\rangle \in A$ [see earlier remark], so by induction hypothesis $\langle y,B\rangle \Vdash \neg\psi$

**Def** $W[w] := \{w' \in W \mid wRw'\}$

**Def**

An $IL_{Set}$ frame is an $L$-frame $\langle W, R\rangle$ + for each $w \in W$, a relation $S_w$ with the following properties:

(I) $S_w \subseteq W[w] \times \mathcal{P}(W[w]) \setminus \{\emptyset\}$

(II) $S_w$ is "quasi-reflexive":

for all $w, x$, $wRx \longrightarrow x S_w \{x\}$

$S_w$ is "quasi-transitive":

$x S_w Y \longrightarrow \forall y \in Y, \forall V \in \mathcal{P}(W[w]) (y S_w V \Rightarrow x S_w V)$

(III) $wRw'Rw'' \longrightarrow w' S_w \{w''\}$.

An $IL_{Set}$ model consists of an $IL_{Set}$-frame $\langle W, R, \{S_w : w \in W\}\rangle$ + a forcing relation $\Vdash$ satisfying the following:

$u \Vdash \Box\varphi \Leftrightarrow \forall v (uRv \Rightarrow v \Vdash \varphi)$

$u \Vdash \varphi \rhd \psi \Leftrightarrow \forall v (uRv \wedge v \Vdash \varphi \Rightarrow \exists V (v S_u V \wedge \forall x \in V \; x \Vdash \psi$

It is not difficult to check that for any $IL_{Set}$-model $K$, $K \models IL$.

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It seems that for some applications, transitivity is more nicely defined as:

$x S_w Y \wedge \forall y \in Y (y S_w Z_y) \rightarrow x S_w \bigcup_{y \in Y} Z_y$

+ extra condition:

$x S_w Y \wedge Y \subseteq Z \longrightarrow x S_w Z$.

We didn't check what happens in this case, and in the rest of these pages we always use the first definition

$M: \underline{A \rhd B \longrightarrow A \wedge \Box C \rhd B \wedge \Box C}$

What property does $M$ characterize on set models?

$M*)$ $u S_w V \longrightarrow \exists V' \subseteq V, (u S_w V' \wedge \forall v \in V'(vRz \to uRz))$

Proof

1) Suppose our frame satisfies $M*$ and $w \Vdash A \rhd B$.

Suppose $wRu$, $u \Vdash A \wedge \Box C$. Because $w \Vdash A \rhd B$,

$\exists V \; u S_w V \;\&\; \forall v \in V \; v \Vdash B$.

Now by $*$,

$\exists V' \subseteq V \; (u S_w V' \wedge \forall v \in V'(vRz \to uRz))$

so $\forall v \in V'$, $v \Vdash B \wedge \Box C$.

Therefore $w \Vdash A \wedge \Box C \rhd B \wedge \Box C$.

2) Suppose our frame does not satisfy $M*$, i.e.

Suppose $u S_w V$ but $\forall V' \subseteq V (u S_w V' \to \exists v \in V'(vRz \wedge \neg uRz$

now take a valuation which:

- forces $p$ only in $u$
- forces $r$ in $V$ but nowhere else
- Does not force $q$ in those $v \in V$ with $vRz \wedge \neg uRz$, but does force $q$ everywhere else.

Then $w \nVdash p \wedge \Box q \rhd r \wedge \Box q$,

because $wRu$, $u \Vdash p \wedge \Box q$.

But for any $V'$ with $u S_w V'$ and $V' \Vdash r$, we have $V' \subseteq V$, so $V' \nVdash \Box q$.

$= KM1$

$\Pi$: $A \rhd \Diamond B \to \Box(A \to \Diamond B)$

What property is characterized by $\Pi$ on set frames (On normal frames it is the same as the $M$-property)

$\Pi *$ : If $u S_w V$, then $\exists v \in V \, \forall z (vRz \to uRz)$

Proof 1) Suppose oure frame satisfies $\Pi *$, and suppose $w \Vdash A \rhd \Diamond B$. Suppose $wRu$ and $u \Vdash A$
Then there is a $V$ such that $u S_w V$, and for all $v \in V \, \exists z_v \, vRz_v$ & $z_v \Vdash B$.
By $*$), $\exists v \in V$ such that $\forall z (vRz \to uRz)$,
So, especially $uRz_v$, thus $u \Vdash \Diamond B$.
Therefore $w \Vdash \Box(A \to \Diamond B)$.

2) Suppose our frame does not satisfy $\Pi *$, i.e. we suppose
$u S_w V$ & $\forall v \in V \, \exists z_v \, (vRz_v \wedge \neg uRz_v)$.
We take a valuation such that
- $p$ is forced only in $u$
- $q$ is forced only in the "$z_v$'s".

Then $w \Vdash p \rhd \Diamond q$, because $u S_w V$ and $\forall v \in V$, $v \Vdash \Diamond q$ (because $z_v \Vdash q$).
But not $w \Vdash \Box(p \to \Diamond q)$, because $wRu$ and $u \nVdash \Diamond q$ ($u$ has no $R$-arrow to any "$z_v$").

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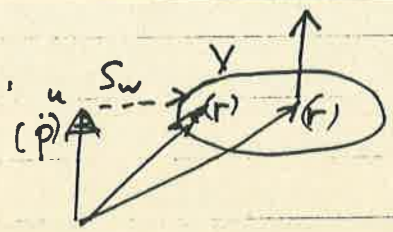
$IL\Pi \nvDash M$ :




Countermodel

This frame satisfies property $\Pi *$, but
$w \nVdash p \rhd r \to p \wedge \Box\bot \rhd r \wedge \Box\bot$

$P: \quad A \rhd B \to \Box(A \rhd B)$

What property does $P$ characterize on set frames?

$*P: \quad u S_w V \wedge w R w' R u \to \exists V' \subseteq V \;\; u S_{w'} V'$.

Proof

1) Suppose our frame satisfies $*P$,
and suppose $w \Vdash A \rhd B$.
Moreover suppose $wRw'$, $w'Ru$ and $u \Vdash A$.
We have by transitivity $wRu$, so there is a $V$
with $u S_w V$ and $\forall v \in V \; v \Vdash B$.
By $*P$, $\exists V' \subseteq V \;\; u S_{w'} V'$, and $\forall v \in V' \; v \Vdash B$.
So $w' \Vdash A \rhd B$, thus $w \Vdash \Box(A \rhd B)$.

2) Suppose our frame does not satisfy $*P$, e.g.
suppose
$u S_w V$, $wRw'Ru$, but $\forall V' \subseteq V$ not $u S_{w'} V'$.
Take a valuation such that
- $p$ is forced only in $u$
- $q$ is forced everywhere in $V$, but nowhere else.

Then $w \Vdash p \rhd q$, but
$w' \nVdash p \rhd q$, so $w \nVdash (p \rhd q) \to \Box(p \rhd q)$.

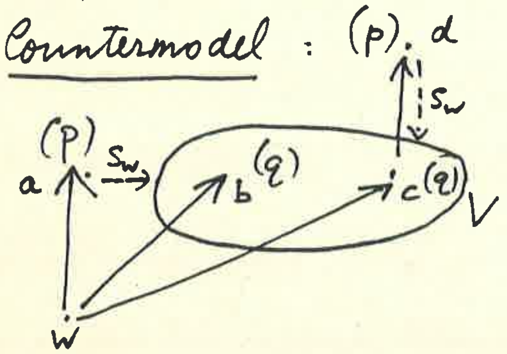
$F: \varphi \rhd \Diamond\varphi \longrightarrow \Box\neg\varphi$

$W: \varphi \rhd \psi \longrightarrow \varphi \rhd \psi \wedge \Box\neg\varphi$

$KW1: \varphi \rhd \Diamond\top \longrightarrow \top \rhd \neg\varphi$

<u>Th</u> $ILF \nvdash W$

<u>Countermodel</u> :




1) $w \nVdash p \rhd q \longrightarrow p \rhd q \wedge \Box\neg p$.

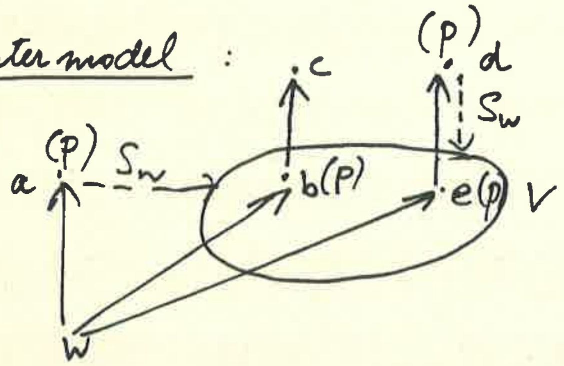
<u>Pf</u>. For, $w \Vdash p \rhd q$ : from both points in which $p$ holds (a and d), one can reach $V$ by $S_w$, and $V \Vdash q$.
Also $w \nVdash p \rhd q \wedge \Box\neg p$: from $a$, we cannot reach any $V'$ with $V' \Vdash q \wedge \Box\neg p$; because $q$ holds only in $V$, and $V \nVdash \Box\neg p$ because $cRd$, $d \Vdash p$.

2) for all $\varphi$, $w \Vdash \varphi \rhd \Diamond\varphi \longrightarrow \Box\neg\varphi$. So, $w \Vdash ILF$

<u>Pf</u>. Notice first that $a$ and $d$ are indistinguishable: They force exactly the same sentences (proof by induction).
Now suppose that $w \Vdash \varphi \rhd \Diamond\varphi$.
Suppose $a \Vdash \varphi$, then we must have $a \Vdash \Diamond\varphi$ or $V \Vdash \Diamond\varphi$ $\Sigma$
So $a \nVdash \varphi$ and $d \nVdash \varphi$.
Suppose $c \Vdash \varphi$, then $c \Vdash \Diamond\varphi$ and $d \Vdash \varphi$ $\Sigma$, so $c \nVdash \varphi$.
Suppose $b \Vdash \varphi$, then $b \Vdash \Diamond\varphi$ $\Sigma$, so $b \nVdash \varphi$.
Thus, $w \Vdash \Box\neg\varphi$.

<u>Th</u> $ILF \nvdash KW1$

<u>Counter model</u> :




1) $w \nVdash p \rhd \Diamond\top \longrightarrow \top \rhd \neg p$

<u>Pf</u> $w \Vdash p \rhd \Diamond\top$ : from a, we can reach by $S_w$ $V$, and $V \Vdash \Diamond\top$
from b, we reach $\{b\}$, and $b \Vdash \Diamond\top$
from e, we reach $\{e\}$ and $e \Vdash \Diamond\top$
from d, we reach $V$, and $V \Vdash \Diamond\top$.
$w \nVdash \top \rhd \neg p$ : from e, we can not reach by $S_w$ any set $V'$ with $V' \Vdash \neg p$.

2) For all $\varphi$, $w \Vdash \varphi \rhd \Diamond\varphi \to \Box\neg\varphi$.

<u>Pf</u>. Again, a and d are indistinguishable.
Suppose that $w \Vdash \varphi \rhd \Diamond\varphi$. Suppose $c \Vdash \varphi$. Then $c \Vdash \Diamond\varphi$ $\lightning$ : $c \nVdash \varphi$
Suppose $a \Vdash \varphi$, then either $a \Vdash \Diamond\varphi$ or $V \Vdash \Diamond\varphi$.
so especially $c \Vdash \varphi$ $\lightning$. So $a \nVdash \varphi$ and $d \nVdash \varphi$.
Suppose $b \Vdash \varphi$, then either $b \Vdash \Diamond\varphi$ or $c \Vdash \varphi$, in both cases $c \Vdash \varphi$ $\lightning$, so $b \nVdash \varphi$
Suppose $e \Vdash \varphi$, then $e \Vdash \Diamond\varphi$ or $d \Vdash \varphi$, in either case $d \Vdash \varphi$ $\lightning$, so $e \nVdash \varphi$.
We conclude $w \nVdash \Box\varphi$