Lab4 - Mnożenie macierzy hierarchicznych przez wektor i przez siebie (Jan Masternak, Jakub Mróz):

Importy potrzebnych paczek:

Poniższe wystarczy jednorazowo odkomentować i uruchomić w celu zainstalowania paczek. Egzekucja poniższego kodu może chwilę zająć, nawet do kilkunastu minut.

In [21]:
```
using Pkg
Pkg.add("FileIO")
# Pkg.add("CSV")
# Pkg.add("DataFrames")
# Pkg.add("Plots")
# Pkg.add("LinearAlgebra")
# Pkg.add("LaTeXStrings")
# Pkg.add("Images")
# Pkg.add("ImageIO")
# Pkg.add("ColorTypes")
# Pkg.add("ImageView
# Pkg.add("Printf")
# Pkg.add("WebP")
# Pkg.add("Statistics");
```

```
    Updating registry at `C:\Users\Jan\.julia\registries\General.toml`
   Resolving package versions...
    Updating `C:\Users\Jan\.julia\environments\v1.10\Project.toml`
  [5789e2e9] + FileIO v1.16.6
  No Changes to `C:\Users\Jan\.julia\environments\v1.10\Manifest.toml`
```

Poniższe uruchamiamy każdorazowo.

In [1]:
```
using FileIO
using CSV
using DataFrames
using Plots
using LinearAlgebra
# using LaTeXStrings
# using Images
# using ImageIO
# using ColorTypes
# using ImageView
```

```julia
using Printf
using Statistics;
```

Definicja zmiennych globalnych

In [2]:
```julia
error_eps = 1e-2;
```

W Julii aby przeciążać operatory i funkcje wbudowane należy zaimportować je z Base.

In [3]:
```julia
import Base: +, *, -, size, iterate
```

Macierze hierarchiczne z zadania 3:

W ramach tego ćwiczenia wprowadzono pewne drobne zmiany względem reprezentacji macierzy skompresowanych:

- Macierz skompresowana jest reprezentowana jako obiekt typu CompressedMatrix (a nie CompressedMatrixNode) który to jest tak na dobrą sprawę wrapper dla korzenia drzewa macierzy skompresowanej (typ CompressedMatrixNode), oraz dodatkowo zawiera jeszcze informacje o rozmiarach macierzy;
- Przeładowano funkcje $size(matrix)$ oraz $size(matrix, dimension)$ dla CompressedMatrix oraz CompressedMatrixNode - dla CompressedMatrix działanie jest oczywiste, natomiast dla CompressedMatrixNode zwraca rozmiary obszaru kompresowanej macierzy, której instancja CompressedMatrixNode odpowiada;
- Dodano funckję $break\_up\_compressed$, która dzieli skompresowany (a co za tym idzie bezdzietny) CompressedMatrixNode na czwórkę dzieci w następujący sposób:

$$A \text{ - CompressedMatrixNode do podzielenia}$$

$$U = A.U\ (U = U_{m,\gamma}),\ V = A.V\ (V = V_{\gamma,n})$$

$$b = \lfloor\frac{m+1}{2}\rfloor, c = \lfloor\frac{n+1}{2}\rfloor$$

$$leftUpperChild.U = U[1:b,:],\ leftUpperChild.V = V[:,1:c]$$

$$leftLowerChild.U = U[b+1:m,:],\ leftLowerChild.V = V[:,1:c]$$

$$rightUpperChild.U = U[1:b,:],\ rightUpperChild.V = V[:,c+1:n]$$

$$rightLowerChild.U = U[b+1:m,:],\ rightLowerChild.V = V[:,c+1:n]$$

gdzie zapis U[a:b, c:d] oznacza wycinek macierzy U obejmujący od a-wiersza do b-wiersza (obustronnie włącznie) oraz od c-kolumny do d-kolumny (obustronnie włącznie), natomiast ":" w U[:, c:d] oznacza, że bierzemy wszystkie wiersze (analogicznie dla kolumn). Warto zauważyć, że np. w przypadku gdy V składa się z jednej kolumny to w prawych dzieciach wyprodukujemy puste macierze, ten przypadek skrajny jest oczywiście wzięty pod uwagę.

- Dodano funkcję fix_addrs, która poprawnie ustawia jakim wycinkom z macierzy kompresowanej odpowiadają węzły CompressedMatrixNode.

In [4]:
```
mutable struct CompressedMatrixNode
    rank::Union{Int64, Nothing}
    addr::Union{Tuple{Int64, Int64, Int64, Int64}, Nothing}

    left_upper_child::Union{CompressedMatrixNode, Nothing}
    left_lower_child::Union{CompressedMatrixNode, Nothing}
    right_upper_child::Union{CompressedMatrixNode, Nothing}
    right_lower_child::Union{CompressedMatrixNode, Nothing}

    singular_values
    U_matrix::Union{Matrix{<:Number}, Nothing}
    V_tr_matrix::Union{Matrix{<:Number}, Nothing}

end


function size(cmn::CompressedMatrixNode)
    return (cmn.addr[2]-cmn.addr[1]+1, cmn.addr[4]-cmn.addr[3]+1)
end


function size(cmn::CompressedMatrixNode, dim::Int64)
    if dim == 1
        return cmn.addr[2]-cmn.addr[1]+1
    elseif dim == 2
        return cmn.addr[4]-cmn.addr[3]+1
    else
        throw(ArgumentError("Compressed matrix only has two dimansions"))
    end
end


function is_compressed(cmn::CompressedMatrixNode)
    return isnothing(cmn.left_upper_child) && isnothing(cmn.left_lower_child) &&
           isnothing(cmn.right_upper_child) && isnothing(cmn.right_lower_child) &&
           !isnothing(cmn.U_matrix)
end
```

```julia
function create_new_compressed_matrix_node()::CompressedMatrixNode
    return CompressedMatrixNode(
        nothing, nothing,
        nothing, nothing, nothing, nothing,
        nothing, nothing, nothing
    )
end


function break_up_compressed_cmn(cmn::CompressedMatrixNode)
    r_min, r_max, c_min, c_max = cmn.addr
    r_mid = div(r_min+r_max,2)
    c_mid = div(c_min+c_max,2)
    right_upper_child, left_lower_child, right_lower_child = nothing, nothing, nothing

    gamma = cmn.rank
    U_border = r_mid - r_min + 1
    V_border = c_mid - c_min + 1
    U_end = r_max - r_min + 1
    V_end = c_max - c_min + 1

    left_upper_child = create_new_compressed_matrix_node()
    left_upper_child.addr = (r_min, r_mid, c_min, c_mid)
    left_upper_child.rank = gamma
    left_upper_child.U_matrix = cmn.U_matrix[1:U_border , 1:end]
    if !isnothing(cmn.V_tr_matrix)
        if r_min == r_mid && c_min == c_mid
            left_upper_child.U_matrix *= cmn.V_tr_matrix[1:end , 1:V_border]
        else
            left_upper_child.V_tr_matrix = cmn.V_tr_matrix[1:end , 1:V_border]
        end
    end

    if U_border < U_end
        left_lower_child = create_new_compressed_matrix_node()
        left_lower_child.addr = (r_mid+1, r_max, c_min, c_mid)
        left_lower_child.rank = gamma
        left_lower_child.U_matrix = cmn.U_matrix[U_border+1:end , 1:end]
        if !isnothing(cmn.V_tr_matrix)
            if r_mid+1 == r_max && c_min == c_mid
                left_lower_child.U_matrix *= cmn.V_tr_matrix[1:end , 1:V_border]
            else
```

```
                left_lower_child.V_tr_matrix = cmn.V_tr_matrix[1:end , 1:V_border]
            end
        end

        if V_border < V_end
            right_lower_child = create_new_compressed_matrix_node()
            right_lower_child.addr = (r_mid+1, r_max, c_mid+1, c_max)
            right_lower_child.rank = gamma
            right_lower_child.U_matrix = cmn.U_matrix[U_border+1:end , 1:end]
            if !isnothing(cmn.V_tr_matrix)
                if r_mid+1 == r_max && c_mid+1 == c_max
                    right_lower_child.U_matrix *= cmn.V_tr_matrix[1:end , V_border+1:end]
                else
                    right_lower_child.V_tr_matrix = cmn.V_tr_matrix[1:end , V_border+1:end]
                end
            end

        end
    end

    if V_border < V_end
        right_upper_child = create_new_compressed_matrix_node()
        right_upper_child.addr = (r_min, r_mid, c_mid+1, c_max)
        right_upper_child.rank = gamma
        right_upper_child.U_matrix = cmn.U_matrix[1:U_border , 1:end]
        if !isnothing(cmn.V_tr_matrix)
            if r_min == r_mid && c_mid+1 == c_max
                right_upper_child.U_matrix *= cmn.V_tr_matrix[1:end , V_border+1:end]
            else
                right_upper_child.V_tr_matrix = cmn.V_tr_matrix[1:end , V_border+1:end]
            end
        end

    end

    return left_upper_child, left_lower_child, right_upper_child, right_lower_child
end


function fix_addrs(cmn::Union{CompressedMatrixNode, Nothing}, r_min::Int64, r_max::Int64, c_min::Int64, c_max::Int64)
    if isnothing(cmn)
        return
    end
```

```julia
    cmn.addr = (r_min, r_max, c_min, c_max)

    r_mid = div(r_min+r_max, 2)
    c_mid = div(c_min+c_max, 2)
    fix_addrs(cmn.left_upper_child, r_min, r_mid, c_min, c_mid)
    fix_addrs(cmn.left_lower_child, r_mid+1, r_max, c_min, c_mid)
    fix_addrs(cmn.right_upper_child, r_min, r_mid, c_mid+1, c_max)
    fix_addrs(cmn.right_lower_child, r_mid+1, r_max, c_mid+1, c_max)

end


mutable struct CompressedMatrix
    head::CompressedMatrixNode
    size::Tuple{Int64, Int64}

end


function size(cmn::CompressedMatrix)
    return cmn.size
end


function size(cmn::CompressedMatrix, dim::Int64)
    return cmn.size[dim]
end


function fix_addrs(cm::CompressedMatrix)
    fix_addrs(cm.head, 1, size(cm, 1), 1, size(cm, 2))
end


function compressed_matrix_drawer(target_matrix, node)
    r_min, r_max, c_min, c_max = node.addr

    target_matrix[r_min:r_max, c_min:c_max] .= 1
    if !isnothing(node.rank) && node.rank > 0
        for i in r_min:r_max
            target_matrix[i:i, c_min:c_min+node.rank - 1] .= 0
        end

        for j in c_min:c_max
```

```
            target_matrix[r_min:r_min+node.rank - 1, j:j] .= 0
        end
    end

    if !isnothing(node.left_upper_child)
        compressed_matrix_drawer(target_matrix, node.left_upper_child)
    end

    if !isnothing(node.right_upper_child)
        compressed_matrix_drawer(target_matrix, node.right_upper_child)
    end

    if !isnothing(node.left_lower_child)
        compressed_matrix_drawer(target_matrix, node.left_lower_child)
    end

    if !isnothing(node.right_lower_child)
        compressed_matrix_drawer(target_matrix, node.right_lower_child)
    end

end;
```

Truncated SVD:

Funkcja TSVD nie różni się niczym od wersji z poprzedniego laboratorium.

In [5]:

```
function trunc_svd(A::Matrix{Float64}; gamma = min(size(A,1), size(A,2)))
    gamma = min(gamma, size(A,1), size(A,2))
    U, D, V = svd(A)
    D = diagm(D)
    U = U[: , 1:gamma]
    D = D[1:gamma , 1:gamma]
    V = V[: , 1:gamma]
    return U, D, V'
end;
```

Operacje na macierzach hierarchicznych:

Kompresja macierzy (brak zmian względem poprzedniego zadania):

In [6]:
```
function compress_submatrix(A, r_min, r_max, c_min, c_max, U, D, V_tr, gamma)
    if all(abs.(A[r_min:r_max , c_min:c_max]) .< error_eps)
        v = create_new_compressed_matrix_node()
        v.rank = 0
        v.addr = (r_min, r_max, c_min, c_max)
        return v
    else
        sigmas = diag(D)
        v = create_new_compressed_matrix_node()
        v.rank = gamma
        v.addr = (r_min, r_max, c_min, c_max)
        v.singular_values = sigmas[1:gamma]
        v.U_matrix = U[: , 1:gamma]
        v.V_tr_matrix = D[1:gamma , 1:gamma] * V_tr[1:gamma , :]
        return v
    end
end


function compress_matrix(A, r_min=1, r_max=size(A,1), c_min=1, c_max=size(A,2); gamma=1, eps=1)
    if r_min == r_max && c_min == c_max
        v = create_new_compressed_matrix_node()
        v.rank = 1
        v.addr = (r_min, r_max, c_min, c_max)
        v.U_matrix = reshape([A[r_min, c_min]], 1, 1)
        return v
    end

    U, D, V_tr = trunc_svd(A[r_min:r_max , c_min:c_max], gamma=gamma+1)
    if gamma+1 > size(D,1)
        gamma = size(D,1)
        if D[gamma, gamma] - eps < error_eps
            v = compress_submatrix(A, r_min, r_max, c_min, c_max, U, D, V_tr, gamma)
            return v
        end
    elseif D[gamma+1, gamma+1] - eps < error_eps
        v = compress_submatrix(A, r_min, r_max, c_min, c_max, U, D, V_tr, gamma)
        return v
    end

    r_mid = div(r_min+r_max,2)
    c_mid = div(c_min+c_max,2)
    right_upper_child, left_lower_child, right_lower_child = nothing, nothing, nothing
```

```
    left_upper_child = compress_matrix(A, r_min, r_mid, c_min, c_mid, gamma=gamma, eps=eps)
    if c_mid < c_max
        right_upper_child = compress_matrix(A, r_min, r_mid, c_mid+1, c_max, gamma=gamma, eps=eps)
    end
    if r_mid < r_max
        left_lower_child = compress_matrix(A, r_mid+1, r_max, c_min, c_mid, gamma=gamma, eps=eps)
        if c_mid < c_max
            right_lower_child = compress_matrix(A, r_mid+1, r_max, c_mid+1, c_max, gamma=gamma, eps=eps)
        end
    end

    v = create_new_compressed_matrix_node()
    v.addr = (r_min, r_max, c_min, c_max)
    v.left_upper_child = left_upper_child
    v.right_upper_child = right_upper_child
    v.left_lower_child = left_lower_child
    v.right_lower_child = right_lower_child
    return v

end


function perform_matrix_compression(A, r_min=1, r_max=size(A,1), c_min=1, c_max=size(A,2); gamma=1, eps=1)
    return CompressedMatrix(compress_matrix(A, r_min, r_max, c_min, c_max; gamma=gamma, eps=eps), size(A))
end;
```

Budowanie macierzy postaci jawnej z macierzy skompresowanej:

In [7]:

```
function build_matrix_based_on_tree(target_matrix, node)
    r_min, r_max, c_min, c_max = node.addr

    if !isnothing(node.U_matrix)
        if !isnothing(node.V_tr_matrix)
            target_matrix[r_min:r_max , c_min:c_max] += node.U_matrix*node.V_tr_matrix
        else
            target_matrix[r_min:r_max , c_min:c_max] += node.U_matrix;
        end
    end

    if !isnothing(node.left_upper_child)
        build_matrix_based_on_tree(target_matrix, node.left_upper_child)
    end
```

```
    if !isnothing(node.right_upper_child)
        build_matrix_based_on_tree(target_matrix, node.right_upper_child)
    end

    if !isnothing(node.left_lower_child)
        build_matrix_based_on_tree(target_matrix, node.left_lower_child)
    end

    if !isnothing(node.right_lower_child)
        build_matrix_based_on_tree(target_matrix, node.right_lower_child)
    end

end


function build_matrix_from_compressed(matrix_size, compressed_matrix)
    built_matrix = zeros(matrix_size[1], matrix_size[2])
    build_matrix_based_on_tree(built_matrix, compressed_matrix.head)
    return built_matrix
end;
```

Dodawanie macierzy skompresowanych:

In [8]:
```
function +(cmn1::Union{CompressedMatrixNode, Nothing}, cmn2::Union{CompressedMatrixNode, Nothing})::Union{CompressedMatrixNode, Nothing}
    if isnothing(cmn1) && isnothing(cmn2)
        return nothing
    elseif isnothing(cmn1)
        return deepcopy(cmn2)
    elseif isnothing(cmn2)
        return deepcopy(cmn1)
    elseif !all(size(cmn1) .== size(cmn2))
        throw(ArgumentError("Cannot add matrix nodes of different sizes"))
    end

    add_cmn::Union{CompressedMatrixNode, Nothing} = nothing

    if cmn1.addr[1] == cmn1.addr[2] && cmn1.addr[3] == cmn1.addr[4]
        add_cmn = create_new_compressed_matrix_node()
        add_cmn.U_matrix = cmn1.U_matrix + cmn2.U_matrix
        add_cmn.rank = cmn1.rank
        add_cmn.addr = cmn1.addr

    elseif cmn1.rank == 0 && cmn2.rank == 0
```

```julia
        add_cmn = create_new_compressed_matrix_node()
        add_cmn.rank = 0
        add_cmn.addr = cmn1.addr

    elseif cmn1.rank == 0
        add_cmn = deepcopy(cmn2)

    elseif cmn2.rank == 0
        add_cmn = deepcopy(cmn1)

    elseif is_compressed(cmn1) && is_compressed(cmn2)
        u_wave = [cmn1.U_matrix cmn2.U_matrix]
        v_wave = [cmn1.V_tr_matrix ; cmn2.V_tr_matrix]
        gamma = max(cmn1.rank, cmn2.rank)
        if isnothing(cmn1.U_matrix) || isnothing(cmn2.U_matrix) || isnothing(cmn1.V_tr_matrix) || isnothing(cmn2.V_tr_matrix)
            println(cmn1.U_matrix, "\n\n", cmn2.U_matrix, "\n\n", cmn1.V_tr_matrix, "\n\n", cmn2.V_tr_matrix, "\n\n\n")
        end
        u_dash, d_dash, v_tr_dash = trunc_svd(u_wave*v_wave, gamma=gamma)
        sigmas = diag(d_dash)
        add_cmn = create_new_compressed_matrix_node()
        add_cmn.rank = gamma
        add_cmn.addr = cmn1.addr
        add_cmn.singular_values = sigmas[1:gamma]
        add_cmn.U_matrix = u_dash
        add_cmn.V_tr_matrix = d_dash*v_tr_dash

    elseif !is_compressed(cmn1) && !is_compressed(cmn2)
        add_cmn = create_new_compressed_matrix_node()
        add_cmn.rank = cmn1.rank
        add_cmn.addr = cmn1.addr
        add_cmn.left_upper_child = cmn1.left_upper_child + cmn2.left_upper_child
        add_cmn.left_lower_child = cmn1.left_lower_child + cmn2.left_lower_child
        add_cmn.right_upper_child = cmn1.right_upper_child + cmn2.right_upper_child
        add_cmn.right_lower_child = cmn1.right_lower_child + cmn2.right_lower_child

    elseif is_compressed(cmn1) && !is_compressed(cmn2)
        cmn1_left_upper, cmn1_left_lower, cmn1_right_upper, cmn1_right_lower = break_up_compressed_cmn(cmn1)
        add_cmn = create_new_compressed_matrix_node()
        add_cmn.rank = cmn1.rank
        add_cmn.addr = cmn1.addr
        add_cmn.left_upper_child = cmn1_left_upper + cmn2.left_upper_child
        add_cmn.left_lower_child = cmn1_left_lower + cmn2.left_lower_child
        add_cmn.right_upper_child = cmn1_right_upper + cmn2.right_upper_child
        add_cmn.right_lower_child = cmn1_right_lower + cmn2.right_lower_child
```

```
    elseif !is_compressed(cmn1) && is_compressed(cmn2)
        cmn2_left_upper, cmn2_left_lower, cmn2_right_upper, cmn2_right_lower = break_up_compressed_cmn(cmn2)
        add_cmn = create_new_compressed_matrix_node()
        add_cmn.rank = cmn2.rank
        add_cmn.addr = cmn2.addr
        add_cmn.left_upper_child = cmn1.left_upper_child + cmn2_left_upper
        add_cmn.left_lower_child = cmn1.left_lower_child + cmn2_left_lower
        add_cmn.right_upper_child = cmn1.right_upper_child + cmn2_right_upper
        add_cmn.right_lower_child = cmn1.right_lower_child + cmn2_right_lower
    end

    return add_cmn

end


function +(cm1::CompressedMatrix, cm2::CompressedMatrix)
    if !all(cm1.size .== cm2.size)
        throw(ArgumentError("Cannot add matrixes of different sizes"))
    end
    return CompressedMatrix(cm1.head + cm2.head, cm1.size)
end;
```

Mnożenie macierzy skompresowanej przez macierz skompresowaną:

Funkcja $remove\_unneccessary\_children$ odpowiedzialna jest za usuwanie niepotrzbenych ogonów - w wyniku działania algorytmu może zostać wyprodukowany node, który odpowiada jednemu elementowi w macierzy, ale zamiast przetrzymania tej jednej wartości w polu macierzowym U, ma jedno (konkretnie lewe górne dziecko), które odpowiada temu samemu obszarowi (1x1) i to ono przetrzymuje dane macierzowe.

In [9]:
```
function remove_unneccessary_children(cmn::Union{CompressedMatrixNode, Nothing})::Union{CompressedMatrixNode, Nothing}
    if isnothing(cmn)
        return nothing
    end

    new_cmn = cmn
    while all(size(new_cmn) .== (1,1)) && isnothing(new_cmn.U_matrix) && !isnothing(new_cmn.left_upper_child)
        new_cmn = cmn.left_upper_child
    end

    return new_cmn
end
```

```
function *(cmn1::Union{CompressedMatrixNode, Nothing}, cmn2::Union{CompressedMatrixNode, Nothing})::Union{CompressedMatrixNode, Nothing}
    if isnothing(cmn1) || isnothing(cmn2)
        return nothing
    elseif size(cmn1, 2) != size(cmn2, 1)
        throw(ArgumentError("Cannot multiply compressed matrix nodes of incompatible shapes"))
    end

    mult_cmn::Union{CompressedMatrixNode, Nothing} = nothing

    if all(size(cmn1) .== (1,1)) && all(size(cmn2) .== (1,1))
        mult_cmn = create_new_compressed_matrix_node()
        mult_cmn.addr = (1, 1, 1, 1)
        mult_cmn.rank = 1
        mult_cmn.U_matrix = cmn1.U_matrix * cmn2.U_matrix

    elseif cmn1.rank == 0 || cmn2.rank == 0
        mult_cmn = create_new_compressed_matrix_node()
        mult_cmn.rank = 0
        mult_cmn.addr = (1, size(cmn1, 1), 1, size(cmn2, 2))

    elseif is_compressed(cmn1) && is_compressed(cmn2)
        mult_cmn = create_new_compressed_matrix_node()
        V1_tr_matrix = cmn1.V_tr_matrix
        U_dash = nothing
        if !isnothing(V1_tr_matrix)
            U_dash = cmn1.U_matrix * (V1_tr_matrix * cmn2.U_matrix)
        else
            U_dash = cmn1.U_matrix * cmn2.U_matrix
        end
        mult_cmn.addr = (1, size(cmn1, 1), 1, size(cmn2, 2))
        mult_cmn.U_matrix = U_dash
        mult_cmn.V_tr_matrix = deepcopy(cmn2.V_tr_matrix)
        if sum(Int64.(size(mult_cmn.U_matrix)) .== (1,1)) == 1 && isnothing(mult_cmn.V_tr_matrix)
            mult_cmn.V_tr_matrix = reshape([1], 1, 1)
        end
        mult_cmn.rank = size(mult_cmn.U_matrix, 2)

    elseif !is_compressed(cmn1) && !is_compressed(cmn2)
        mult_cmn = create_new_compressed_matrix_node()
        mult_cmn.left_upper_child = cmn1.left_upper_child * cmn2.left_upper_child + cmn1.right_upper_child * cmn2.left_lower_child
        mult_cmn.right_upper_child = cmn1.left_upper_child * cmn2.right_upper_child + cmn1.right_upper_child * cmn2.right_lower_child
        mult_cmn.left_lower_child = cmn1.left_lower_child * cmn2.left_upper_child + cmn1.right_lower_child * cmn2.left_lower_child
        mult_cmn.right_lower_child = cmn1.left_lower_child * cmn2.right_upper_child + cmn1.right_lower_child * cmn2.right_lower_child
```

```
        mult_cmn.addr = (1, size(cmn1, 1), 1, size(cmn2, 2))

    elseif is_compressed(cmn1) && !is_compressed(cmn2)
        mult_cmn = create_new_compressed_matrix_node()
        cmn1_left_upper, cmn1_left_lower, cmn1_right_upper, cmn1_right_lower = break_up_compressed_cmn(cmn1)
        mult_cmn.left_upper_child = cmn1_left_upper * cmn2.left_upper_child + cmn1_right_upper * cmn2.left_lower_child
        mult_cmn.right_upper_child = cmn1_left_upper * cmn2.right_upper_child + cmn1_right_upper * cmn2.right_lower_child
        mult_cmn.left_lower_child = cmn1_left_lower * cmn2.left_upper_child + cmn1_right_lower * cmn2.left_lower_child
        mult_cmn.right_lower_child = cmn1_left_lower * cmn2.right_upper_child + cmn1_right_lower * cmn2.right_lower_child
        mult_cmn.addr = (1, size(cmn1, 1), 1, size(cmn2, 2))

    elseif !is_compressed(cmn1) && is_compressed(cmn2)
        mult_cmn = create_new_compressed_matrix_node()
        cmn2_left_upper, cmn2_left_lower, cmn2_right_upper, cmn2_right_lower = break_up_compressed_cmn(cmn2)
        mult_cmn.left_upper_child = cmn1.left_upper_child * cmn2_left_upper + cmn1.right_upper_child * cmn2_left_lower
        mult_cmn.right_upper_child = cmn1.left_upper_child * cmn2_right_upper + cmn1.right_upper_child * cmn2_right_lower
        mult_cmn.left_lower_child = cmn1.left_lower_child * cmn2_left_upper + cmn1.right_lower_child * cmn2_left_lower
        mult_cmn.right_lower_child = cmn1.left_lower_child * cmn2_right_upper + cmn1.right_lower_child * cmn2_right_lower
        mult_cmn.addr = (1, size(cmn1, 1), 1, size(cmn2, 2))

    end

    mult_cmn = remove_unneccessary_children(mult_cmn)

    return mult_cmn

end


function *(cm1::CompressedMatrix, cm2::CompressedMatrix)
    if cm1.size[2] != cm2.size[1]
        throw(ArgumentError("Cannot multiply matrixes of incompatible shapes"))
    end
    mult_cm = CompressedMatrix(cm1.head * cm2.head,  (cm1.size[1], cm2.size[2]))
    fix_addrs(mult_cm)
    return mult_cm
end;
```

Mnożenie wektora przez skalar:

Można zaobserwować, że przy mnożeniu macierzy U o rozmiarze $U_{m,\gamma}$ przez macierz V o rozmiarze $V_{\gamma,n}$ element w i-tym wierszu i j-tej kolumnie macierzy wynikowej W będzie równy:

$$W_{i,j} = \sum_{k=1}^{\gamma} U[i,k] * V[k,j]$$

a co za tym idzie wystarczy przemnożyć macierz U przez skalar α, aby osiągnąć macierz pomnożoną przez ten skalar:

$$W_{i,j} = \alpha * U * V = \sum_{k=1}^{\gamma} \alpha * U[i,k] * V[k,j] = \alpha * \sum_{k=1}^{\gamma} U[i,k] * V[k,j]$$

więc jak widać, aby otrzymać macierz skompresowną pomnożoną przez skalar wystarczy przez skalar mnożyć macierze U przetrzymywane w nodach.

In [13]:

```
function *(cmn::Union{CompressedMatrixNode, Nothing}, alpha::Number)::Union{CompressedMatrixNode, Nothing}
    if isnothing(cmn)
        return nothing
    end

    if cmn.rank == 0
        return cmn
    elseif is_compressed(cmn)
        cmn.U_matrix .*= alpha
    else
        cmn.left_upper_child = cmn.left_upper_child*alpha
        cmn.left_lower_child = cmn.left_lower_child*alpha
        cmn.right_upper_child = cmn.right_upper_child*alpha
        cmn.right_lower_child = cmn.right_lower_child*alpha
    end

    return cmn
end


function *(alpha::Number, cmn::Union{CompressedMatrixNode, Nothing})::Union{CompressedMatrixNode, Nothing}
    return cmn * alpha
end


function *(cm::CompressedMatrix, alpha::Number)
    return CompressedMatrix(alpha * deepcopy(cm.head), size(cm.head))
end


function *(alpha::Number, cm::CompressedMatrix)
```

```
    return cm * alpha
end;
```

Odejmowanie macierzy skompresowanych:

Odjęcie macierzy skompresowanych jest jednoznaczne z:

$$W = A - B = A + (-B)$$

więc jak widać wystarczą operacje przeciążone wyżej.

In [14]:
```
function -(cm::CompressedMatrix)
    return cm * -1
end


function -(cm1::CompressedMatrix, cm2::CompressedMatrix)
    return cm1 + (-cm2)
end;
```

Operacje na macierzach hierarchicznych i wektorach:

In [15]:
```
function *(v::CompressedMatrixNode, X::Vector{<:Number})::Vector{<:Number}
    if isnothing(v.left_upper_child) && isnothing(v.right_upper_child) && isnothing(v.left_lower_child) && isnothing(v.right_lower_child)
        if v.rank == 0
            return zeros(size(X, 1))
        end

        if !isnothing(v.V_tr_matrix)
            return v.U_matrix * (v.V_tr_matrix * X)
        end

        return v.U_matrix * X
    end

    rows = size(X, 1)
    X1 = X[1:div(rows + 1, 2)]
    X2 = X[div(rows + 1, 2)+1:rows]


    if !isnothing(v.left_upper_child) && size(X1, 1) != 0
        Y11 = v.left_upper_child * X1
```

```
    else
        Y11 = zeros(1)
    end

    if !isnothing(v.right_upper_child) && size(X2, 1) != 0
        Y12 = v.right_upper_child * X2
    else
        Y12 = zeros(1)
    end

    if !isnothing(v.left_lower_child) && size(X1, 1) != 0
        Y21 = v.left_lower_child * X1
    else
        Y21 = zeros(1)
    end

    if !isnothing(v.right_lower_child) && size(X2, 1) != 0
        Y22 = v.right_lower_child * X2
    else
        Y22 = zeros(1)
    end

    if size(v, 1) == 1
        return Y11 + Y12
    end

    if size(v, 2) == 1
        return vcat(Y11, Y21)
    end

    return vcat(Y11 + Y12, Y21 + Y22)
end


function *(v::CompressedMatrix, X::Vector{<:Number})
    if v.size[2] != size(X, 1)
        throw(ArgumentError("Incompatible shapes, inner dimensions must agree"))
    end
    return v.head * X
end;
```

Norma Frobeniusa

In [16]:
```julia
function forbinus_norm(m, n)
    return norm(m-n)^2
end;
```

Funkcja generująca macierze o strukturze opisującej topologię trójwymiarowej siatki zbudowanej z elementów sześciennych

In [17]:
```julia
using Random

function generate_3d_grid_matrix(k)
    size = 2^(3 * k)
    matrix = zeros(Float64, size, size)

    function index(x, y, z)
        return x + y * 2^k + z * 2^(2 * k) + 1
    end

    for x in 0:(2^k - 1)
        for y in 0:(2^k - 1)
            for z in 0:(2^k - 1)
                idx = index(x, y, z)
                if x > 0
                    matrix[idx, index(x - 1, y, z)] = rand()
                end
                if x < 2^k - 1
                    matrix[idx, index(x + 1, y, z)] = rand()
                end
                if y > 0
                    matrix[idx, index(x, y - 1, z)] = rand()
                end
                if y < 2^k - 1
                    matrix[idx, index(x, y + 1, z)] = rand()
                end
                if z > 0
                    matrix[idx, index(x, y, z - 1)] = rand()
                end
                if z < 2^k - 1
                    matrix[idx, index(x, y, z + 1)] = rand()
                end
            end
        end
    end
```

```
    return matrix
end;
```

Funkcja genreująca kolumnowy wektor o długości m o losowaych wartościach z przedziału $[a, b]$:

In [18]:
```
function generate_random_vector(m, a, b)
    A = zeros(m)
    for i in 1:1:m
        A[i] = a + (b-a)*rand()
    end
    return A
end;
```

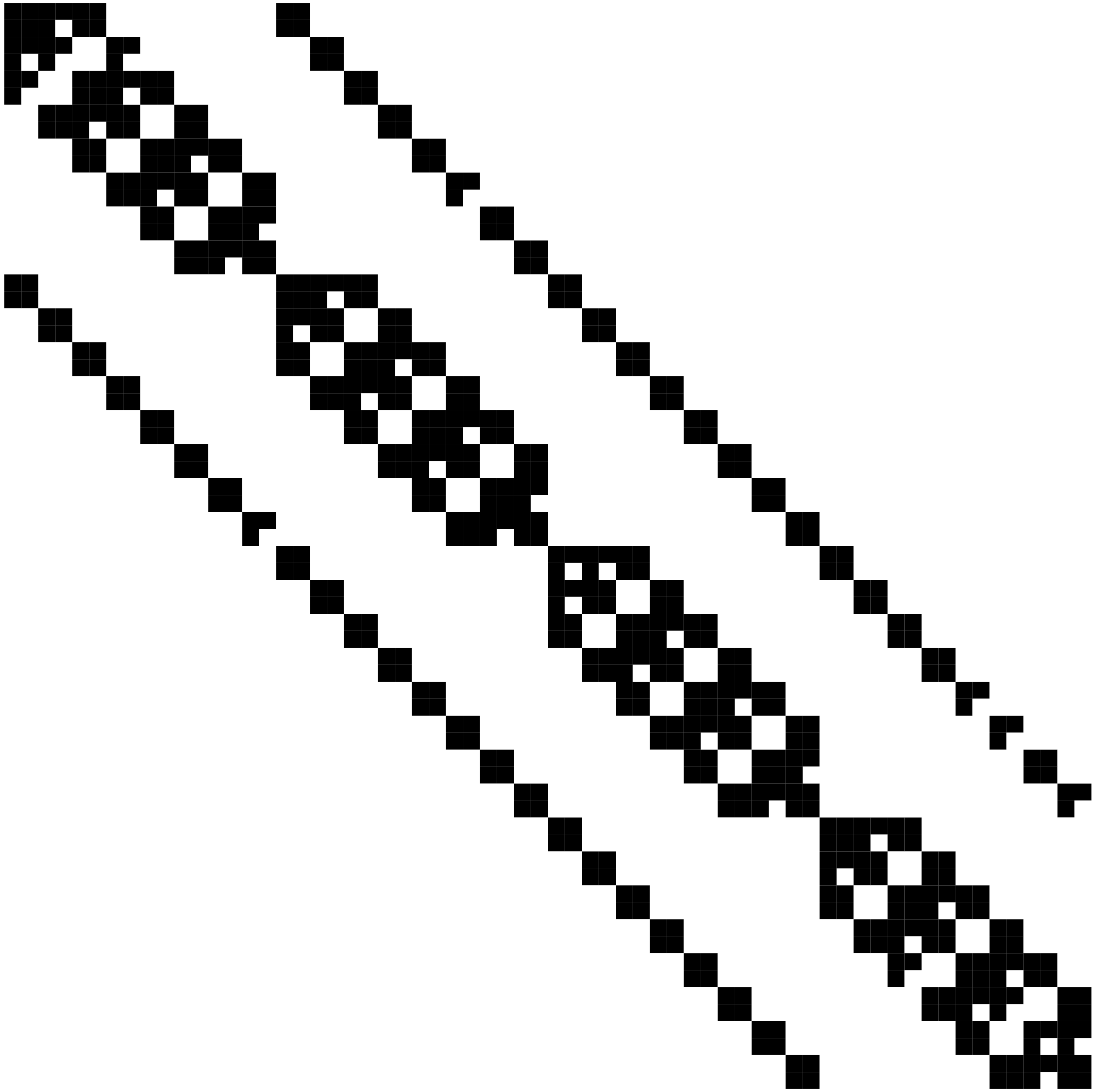
Funkcja wykonująca testy dla zadanego gamma i eps

In [19]:
```
function test(gamma, eps)
    filenames = ["cm_2^6_$gamma-$eps.png", "cm_2^9_$gamma-$eps.png", "cm_2^12_$gamma-$eps.png"]
    ks = [2, 3, 4]
    times_MM = []
    times_MV = []
    forbinus_norms_mv = []
    forbinus_norms_mm = []

    for (k, filename) in zip(ks, filenames)
        # generate random matrix and vector
        M = generate_3d_grid_matrix(k)
        V = generate_random_vector(2^(3*k), 0, 1)

        # perform matrix compression
        CM = perform_matrix_compression(M, gamma=gamma, eps=eps)

        # save compressed matrix drawing
        drawn_matrix = zeros(size(CM))
        compressed_matrix_drawer(drawn_matrix, CM.head)
        save(filename, drawn_matrix)

        # perform matrix vector multiplication and calculate time
        tik = time()
        CMV = CM * V
        tok = time()
```

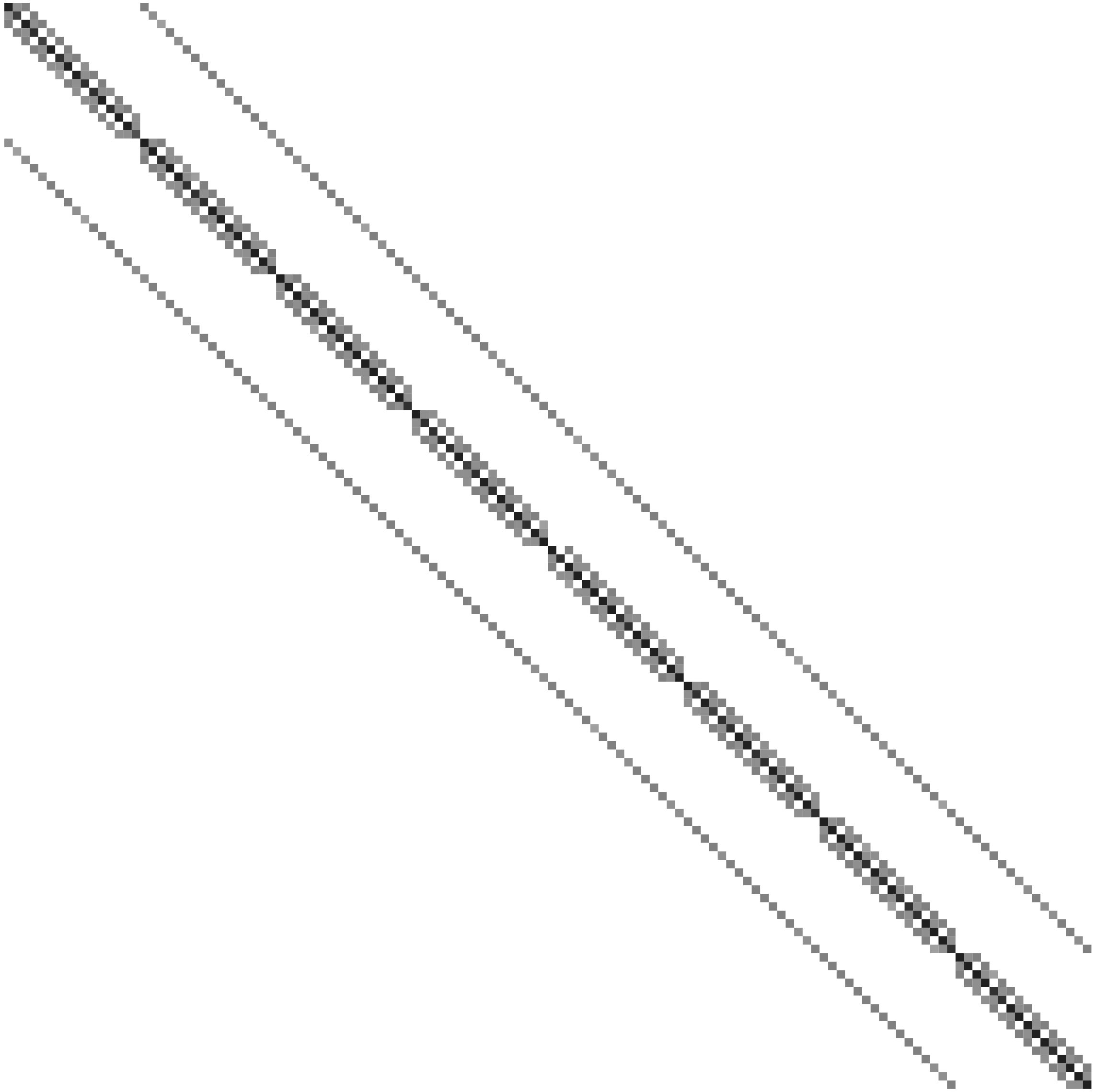
```
        push!(times_MV, tok-tik)

        # calculate forbinus norm
        push!(forbinus_norms_mv, forbinus_norm(M*V, CMV))

        # perform matrix multiplication and calculate time
        tik2 = time()
        CM2 = CM * CM
        tok2 = time()
        push!(times_MM, tok2-tik2)


        # calculate forbinus norm
        push!(forbinus_norms_mm, forbinus_norm(M*M, build_matrix_from_compressed(size(CM2), CM2)))
    end

    for filename in filenames
        println(filename)
        display(load(filename))
    end

    # plot results
    display(plot(2 .^ (3 .* ks), times_MV, label="Matrix vector multiplication", xlabel="N", ylabel="Time (s)", title="Time vs N (Matrix ve
    display(plot(2 .^ (3 .* ks), times_MM, label="Matrix multiplication", xlabel="N", ylabel="Time (s)", title="Time vs N (Matrix multiplica
    display(plot(2 .^ (3 .* ks), forbinus_norms_mv, label="Matrix vector multiplication", xlabel="N", ylabel="Forbinus norm", title="Forbin
    display(plot(2 .^ (3 .* ks), forbinus_norms_mm, label="Matrix multiplication", xlabel="N", ylabel="Forbinus norm", title="Forbinus norm

    # calculate α and β
    Ns = [2^(3 * n) for n in ks]
    B = log(Ns[3]/Ns[1], times_MV[3]/times_MV[1])
    a = times_MV[1] ./ (Ns[1] .^ B)

    # plot results with α and β for matrix vector multiplication
    plot(2 .^ (3 .* ks), times_MV, label="Matrix vector multiplication", xlabel="N", ylabel="Time (s)", title="Time vs N (Matrix vector mul
    display(plot!(2 .^ (3 .* ks), a .* Ns .^ B, label="αNβ α=$a, β=$B"))

    Ns = [2^(3*n) for n in ks]
    B = log(Ns[3]/Ns[1], times_MM[3]/times_MM[1])
    a = times_MM[1] ./ (Ns[1] .^ B)

    # plot results with α and β for matrix multiplication
    plot(2 .^ (3 .* ks), times_MM, label="Matrix multiplication", xlabel="N", ylabel="Time (s)", title="Time vs N (Matrix multiplication)")
    display(plot!(2 .^ (3 .* ks), a .* Ns .^ B, label="αNβ α=$a, β=$B"))
end;
```

Test dla gamma=1 i eps=0.1

In [20]:
```
test(1, 0.1)
```

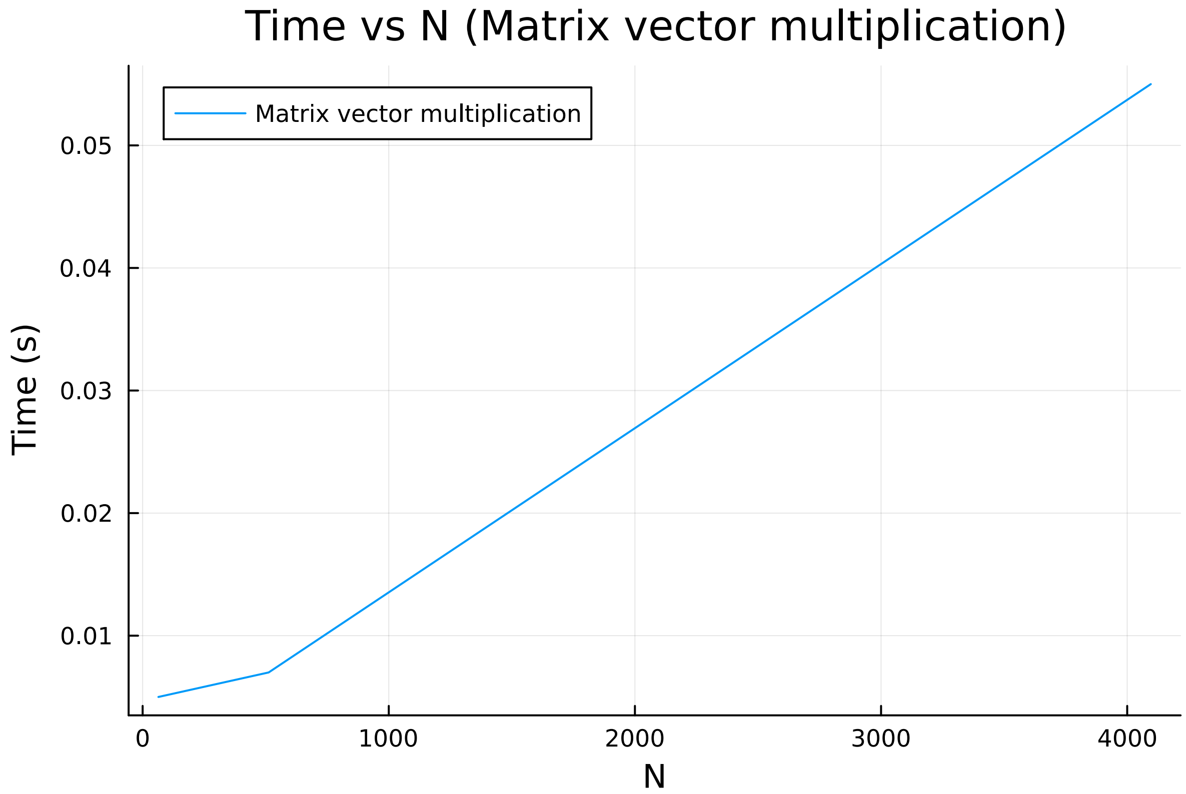


```
cm_2^6_1-0.1.png
cm_2^9_1-0.1.png
```

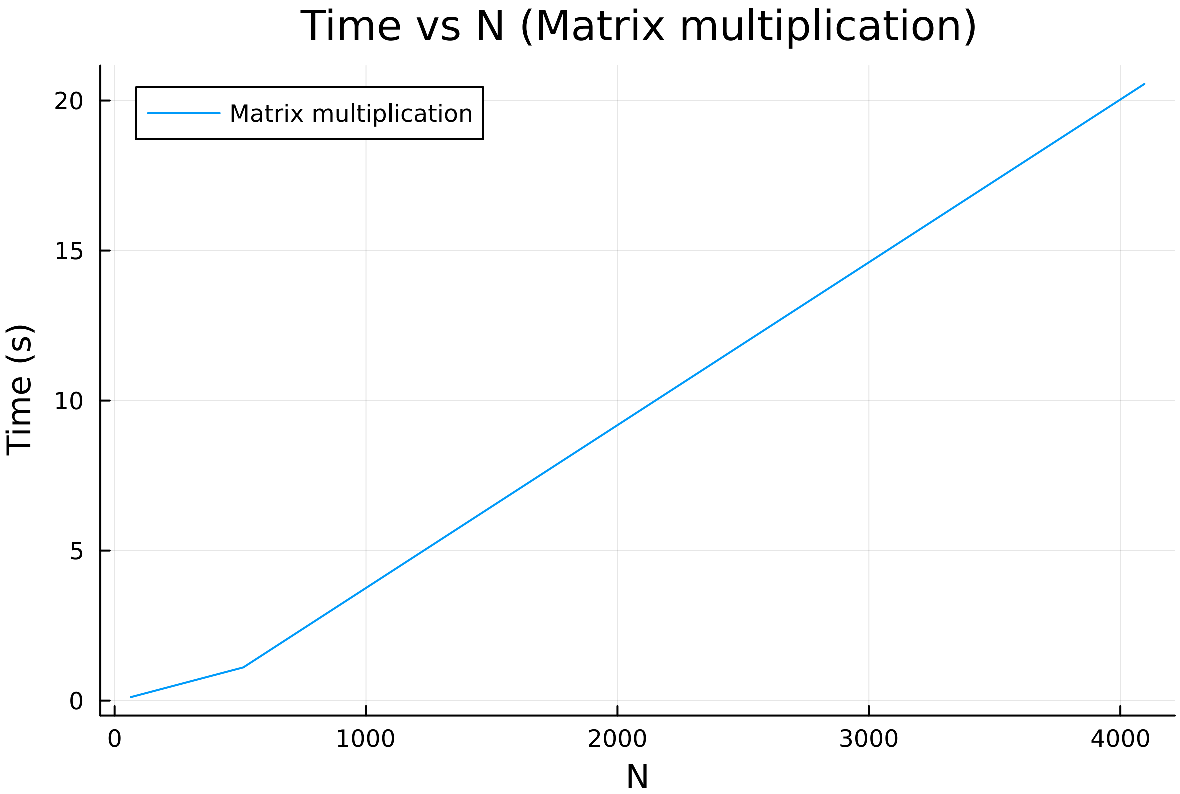


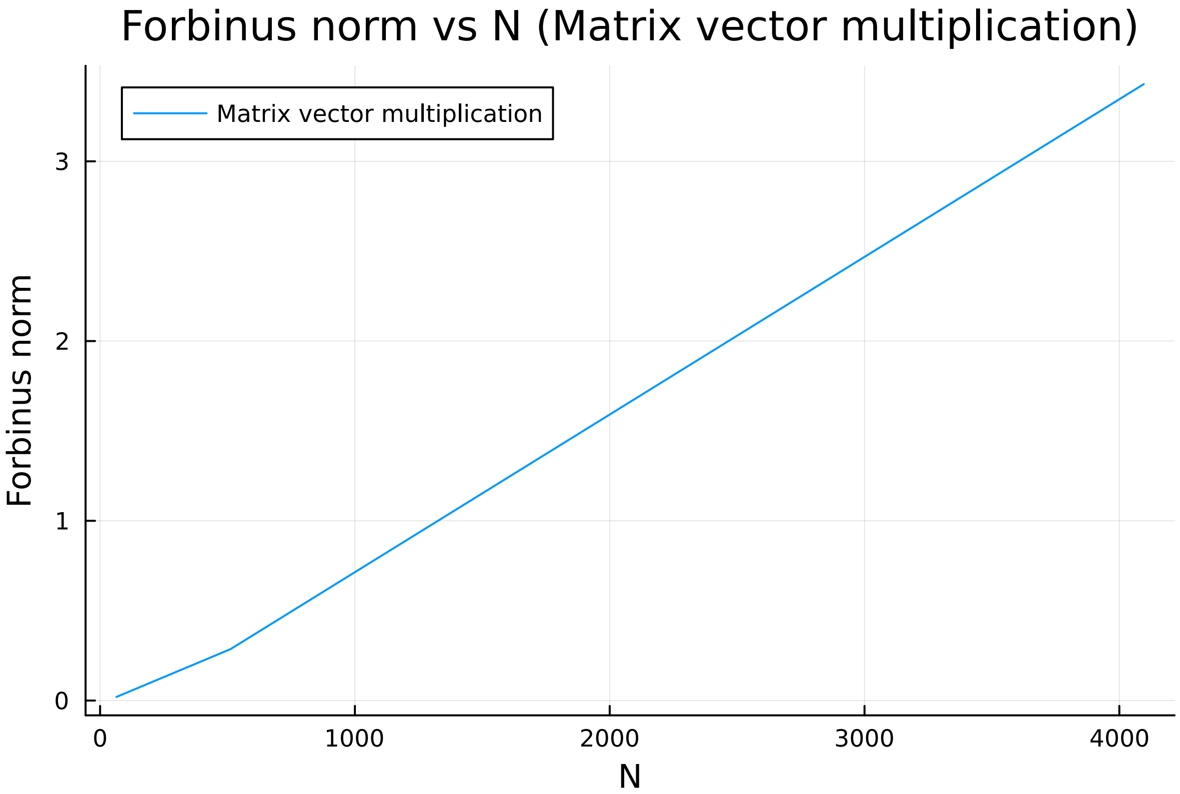
```
┌ Warning: Output swatches are reduced due to the large size (512×512).
│ Load the ImageShow package for large images.
└ @ Colors C:\Users\Jan\.julia\packages\Colors\itIUE\src\display.jl:159
```

cm_2^12_1-0.1.png

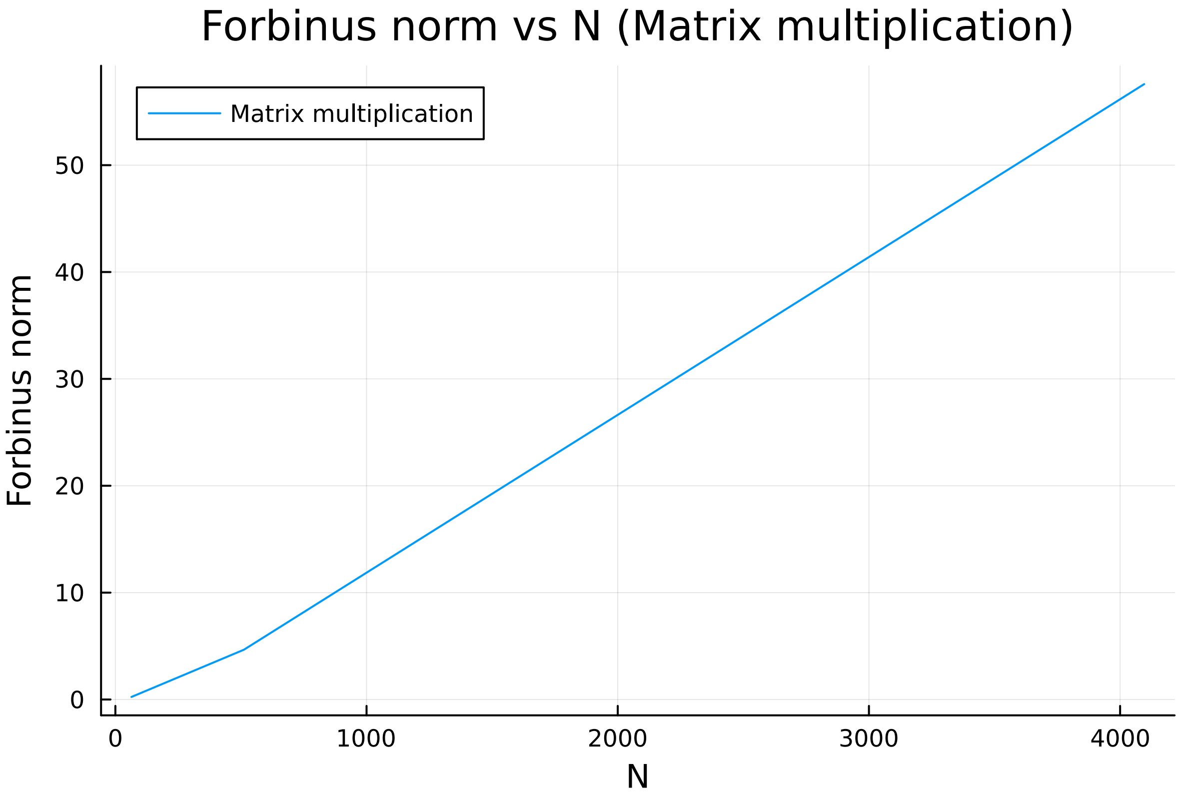


Time vs N (Matrix vector multiplication)
Matrix vector multiplication
Time (s)
0.05
0.04
0.03
0.02
0.01
0
1000
2000
3000
4000
N

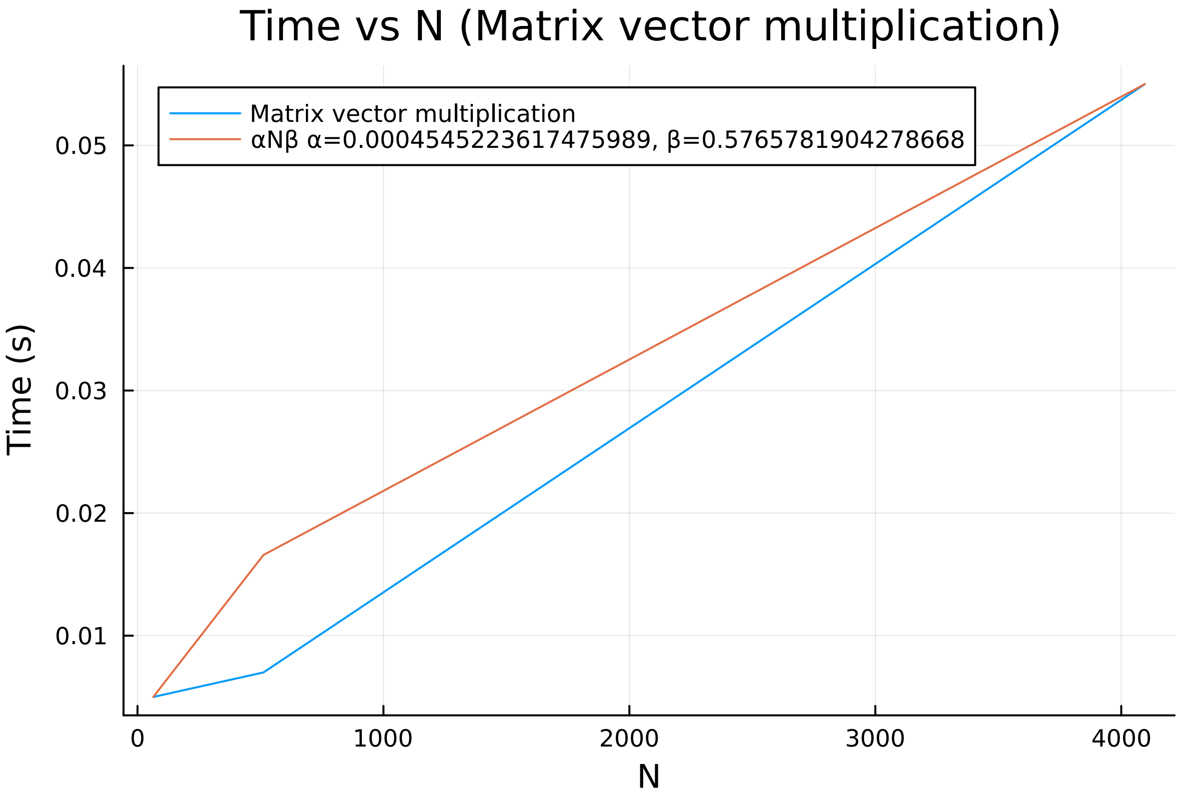

Time vs N (Matrix multiplication)
Matrix multiplication
Time (s)
0
5
10
15
20
0
1000
2000
3000
4000
N



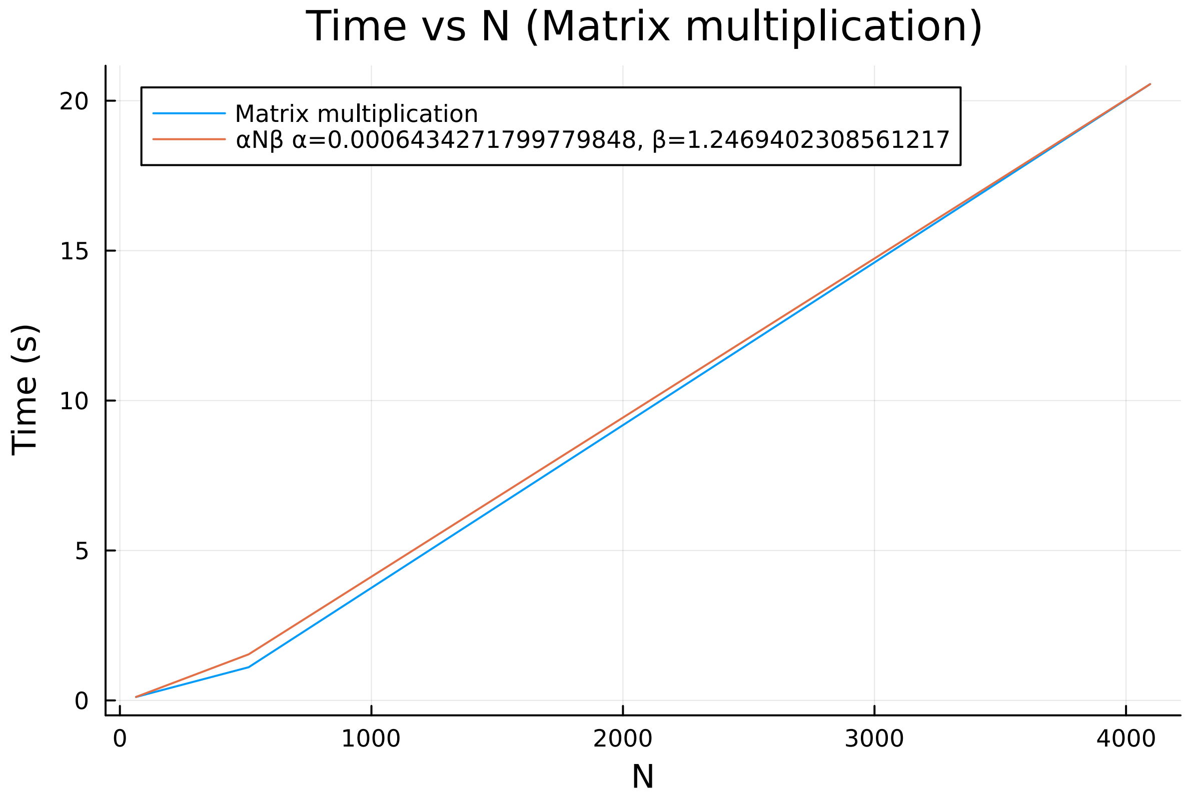
Forbinus norm vs N (Matrix vector multiplication)
Matrix vector multiplication
Forbinus norm
0
1
2
3
0
1000
2000
3000
4000
N



Forbinus norm vs N (Matrix multiplication)
Matrix multiplication
Forbinus norm
N
0
10
20
30
40
50
0
1000
2000
3000
4000



Time vs N (Matrix vector multiplication)
Matrix vector multiplication
αNβ α=0.0004545223617475989, β=0.5765781904278668
Time (s)
0.05
0.04
0.03
0.02
0.01
0
1000
2000
3000
4000
N

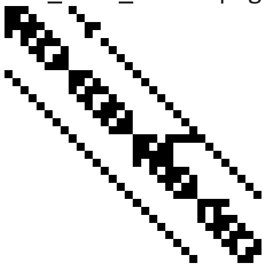


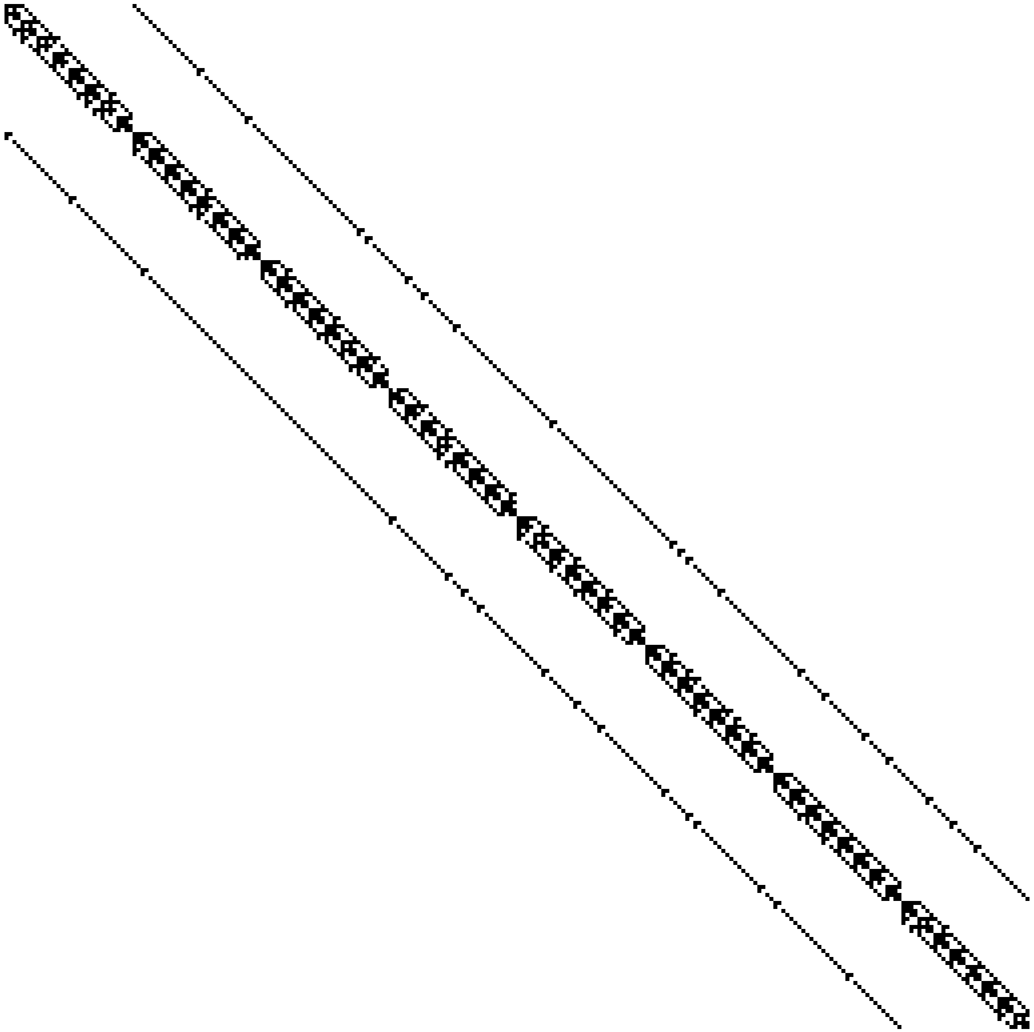

```
┌ Warning: Output swatches are reduced due to the large size (4096×4096).
│ Load the ImageShow package for large images.
└ @ Colors C:\Users\Jan\.julia\packages\Colors\itIUE\src\display.jl:159
```

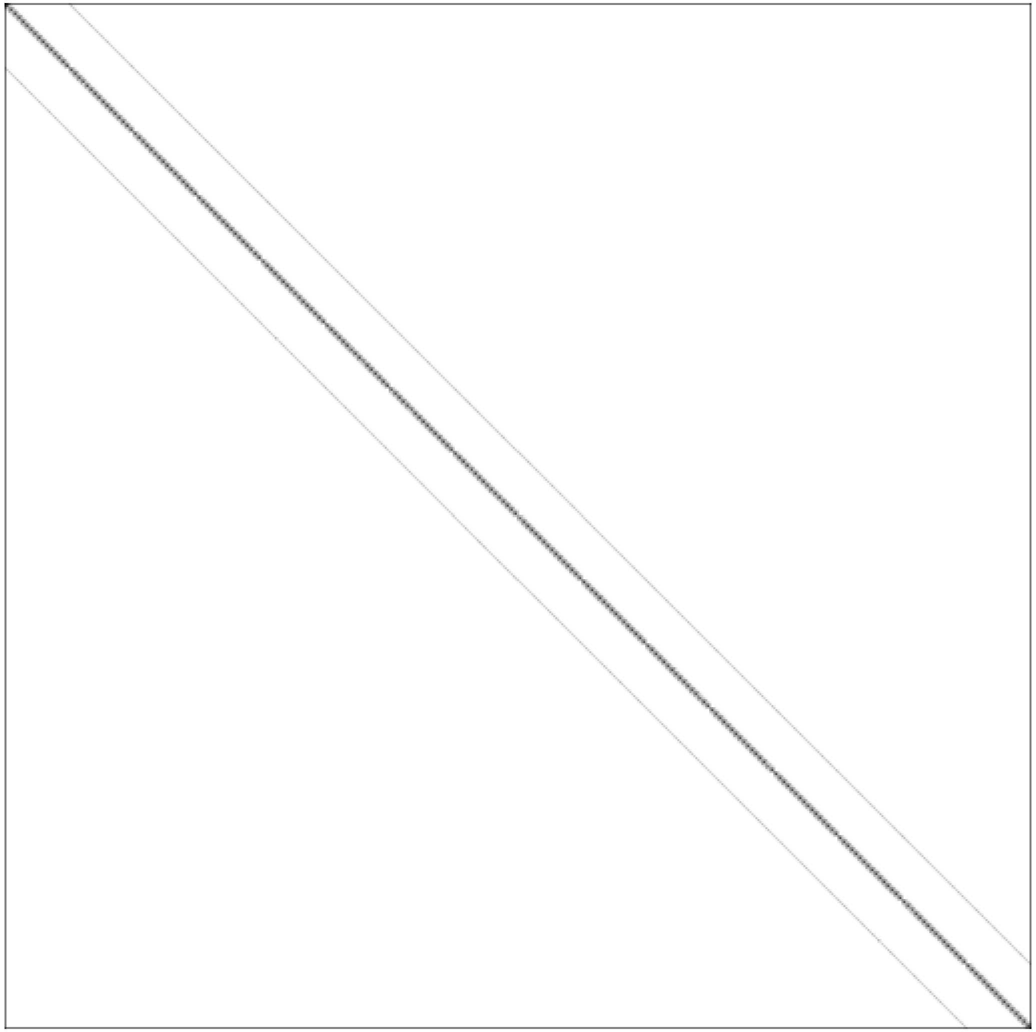
Test dla gamma=2 i eps=0.2

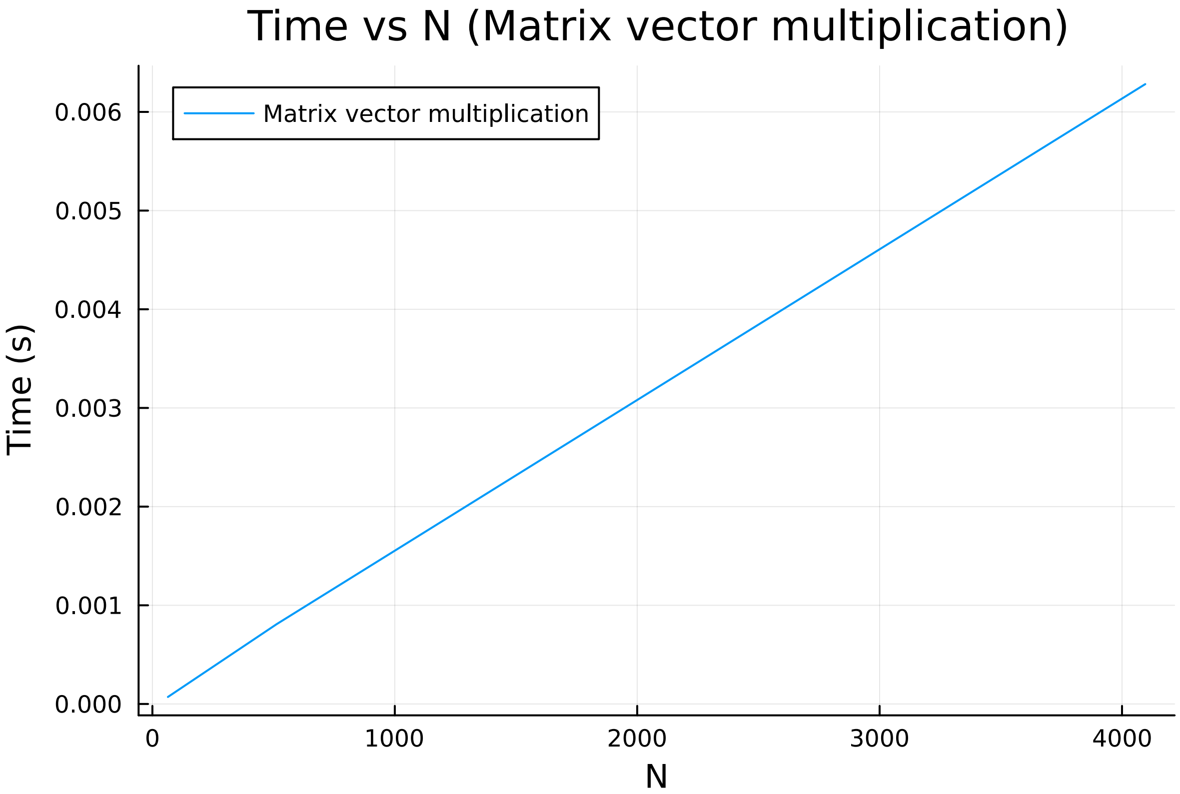
In [14]:
```
test(2, 0.2)
```

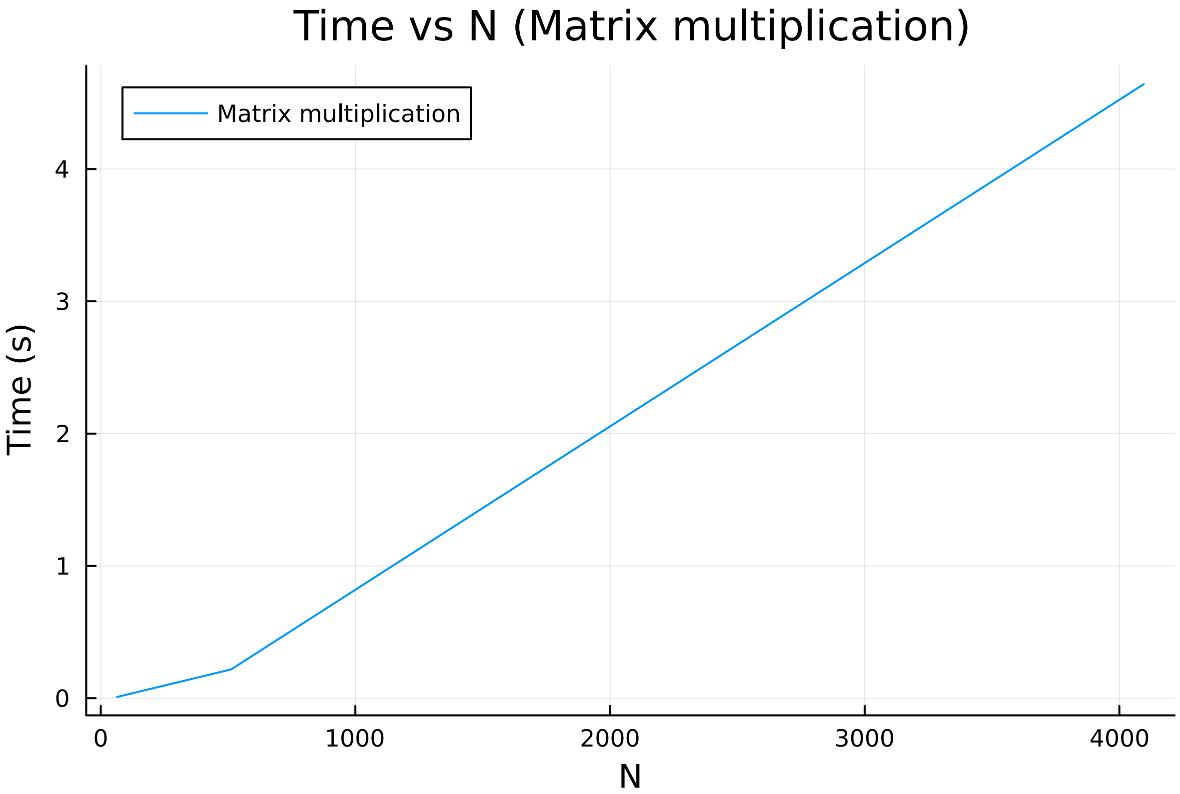
```
cm_2^6_2-0.2.png
cm_2^9_2-0.2.png
cm_2^12_2-0.2.png
```



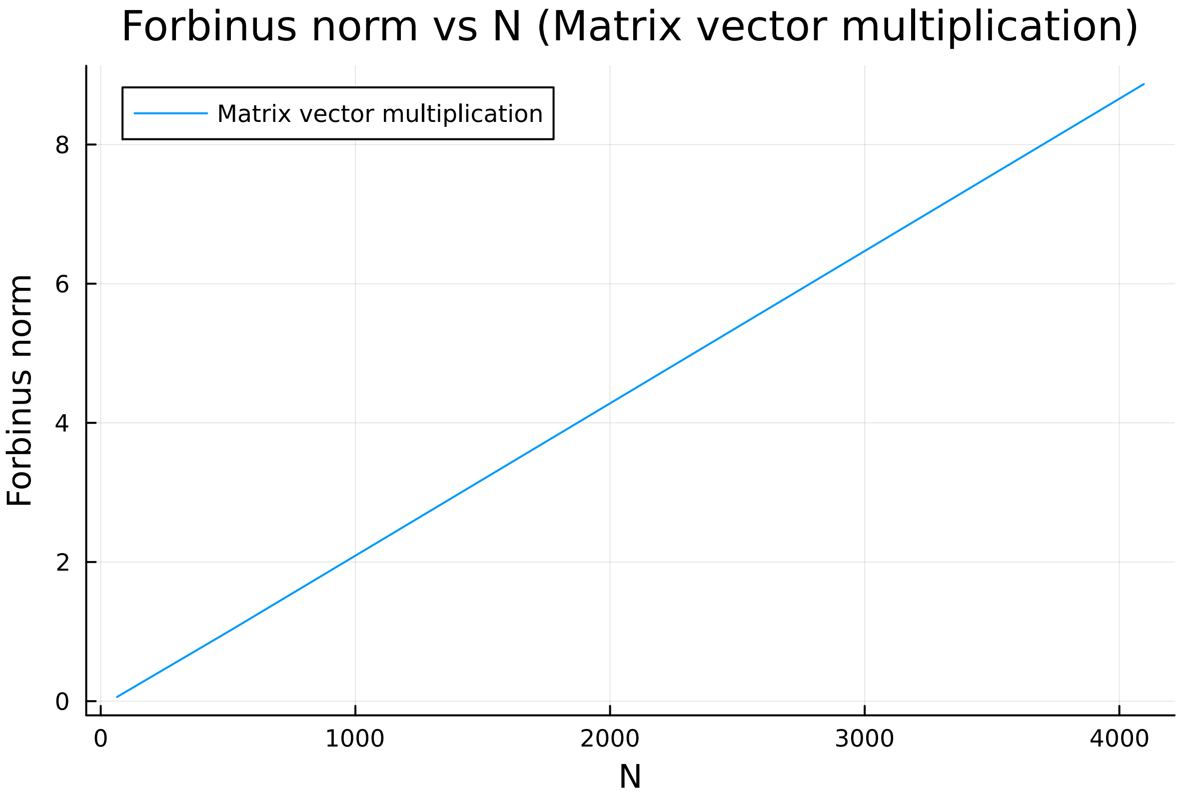


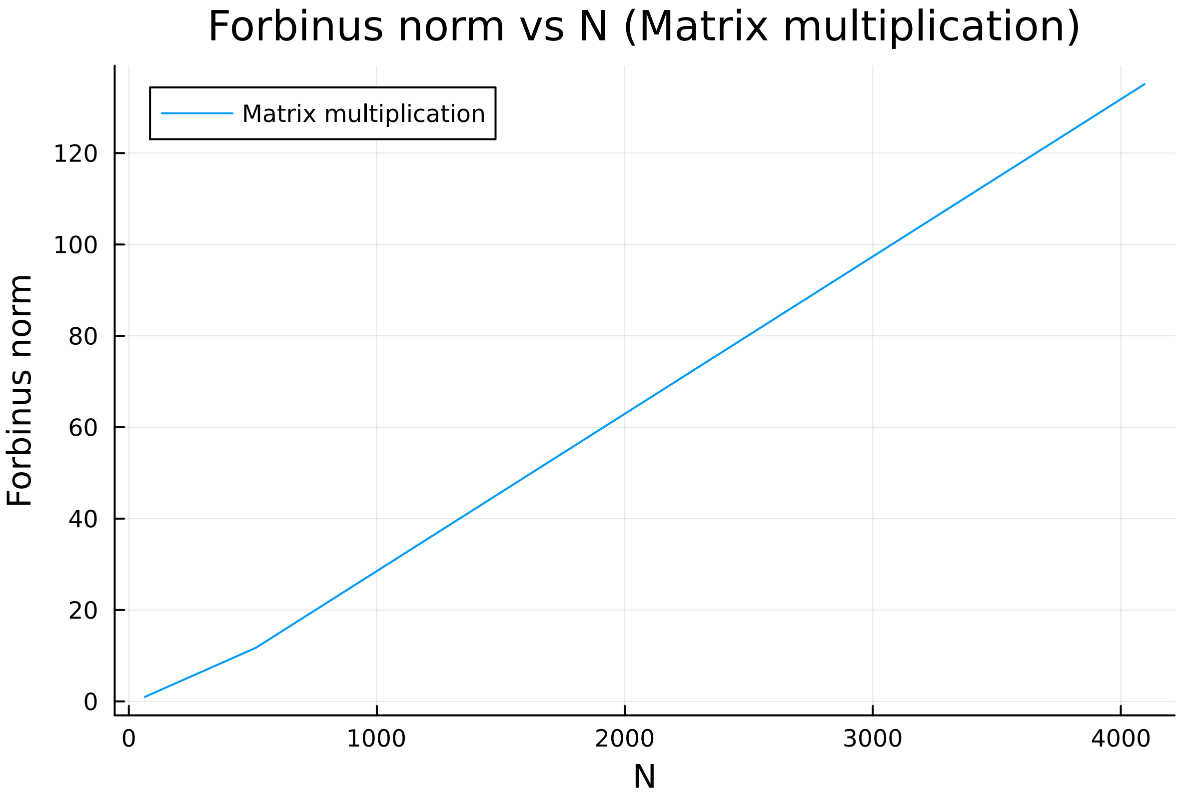



Time vs N (Matrix vector multiplication)
Matrix vector multiplication
Time (s)
0.000
0.001
0.002
0.003
0.004
0.005
0.006
0
1000
2000
3000
4000
N

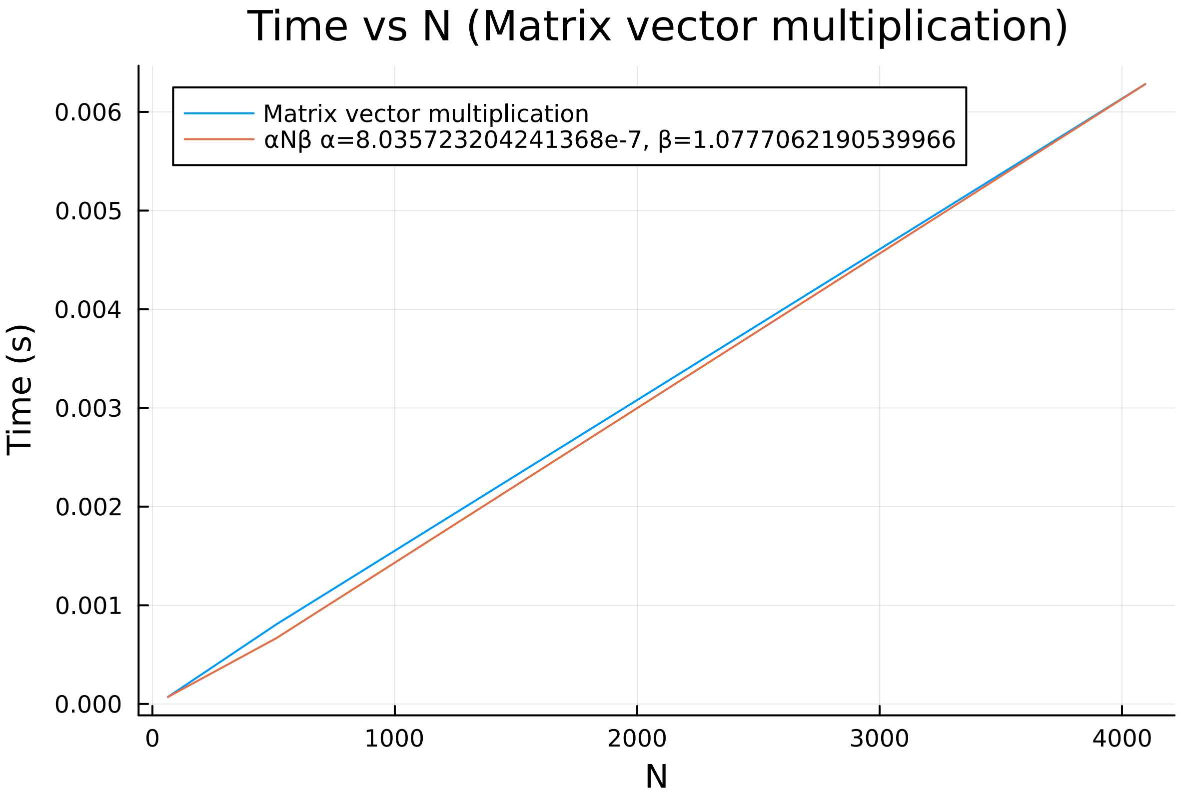


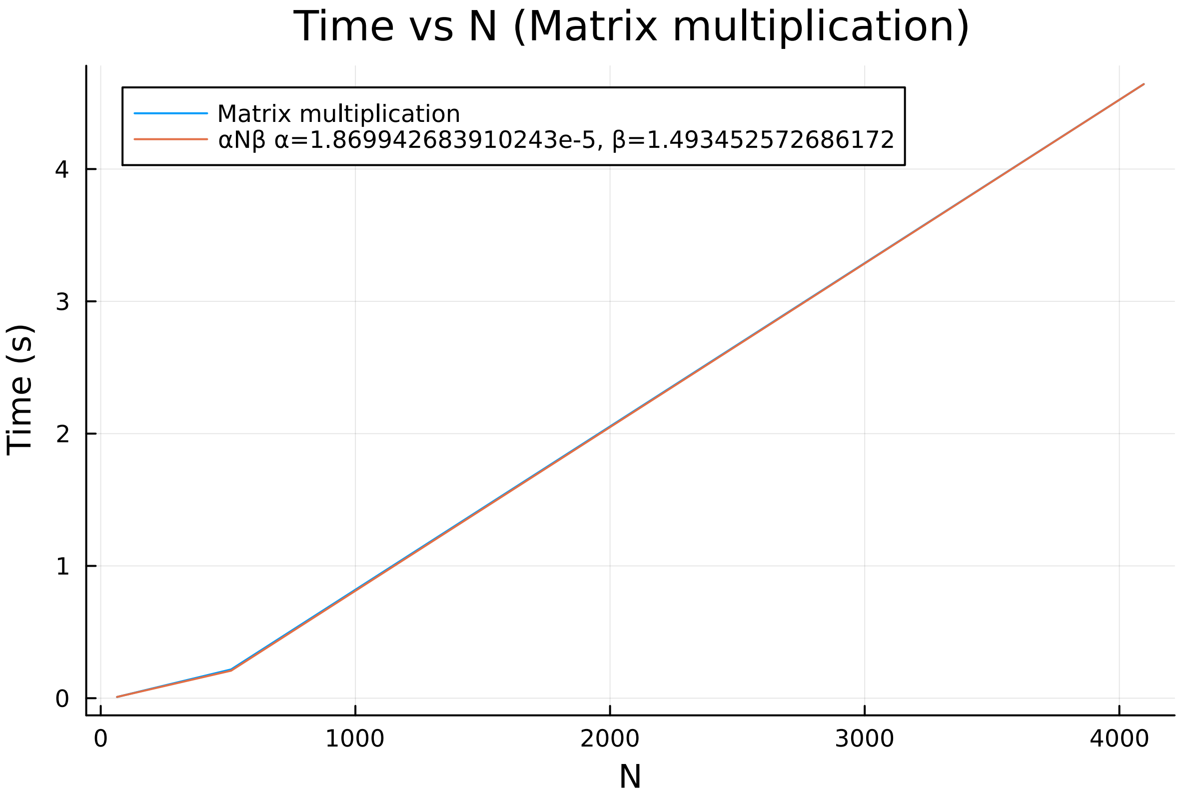
Time vs N (Matrix multiplication)
Matrix multiplication
Time (s)
0
1
2
3
4
0
1000
2000
3000
4000
N


Forbinus norm vs N (Matrix vector multiplication)
Matrix vector multiplication
Forbinus norm
N
0
2
4
6
8
0
1000
2000
3000
4000



Forbinus norm vs N (Matrix multiplication)
Matrix multiplication
Forbinus norm
N
0
20
40
60
80
100
120
0
1000
2000
3000
4000

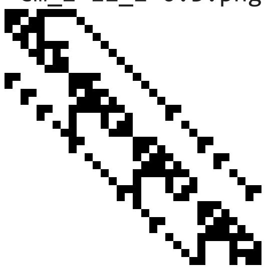

Time vs N (Matrix vector multiplication)
Matrix vector multiplication
αNβ α=8.035723204241368e-7, β=1.0777062190539966
Time (s)
0.006
0.005
0.004
0.003
0.002
0.001
0.000
0
1000
2000
3000
4000
N

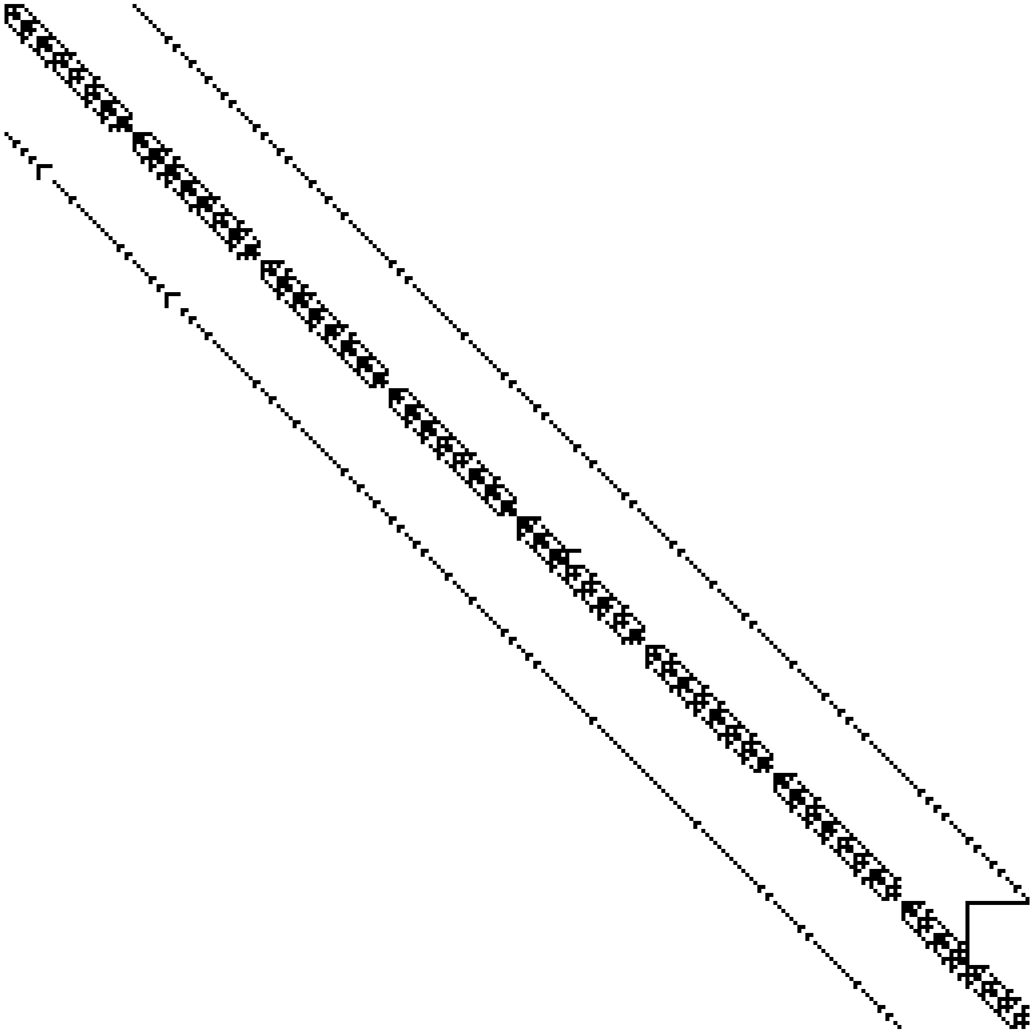


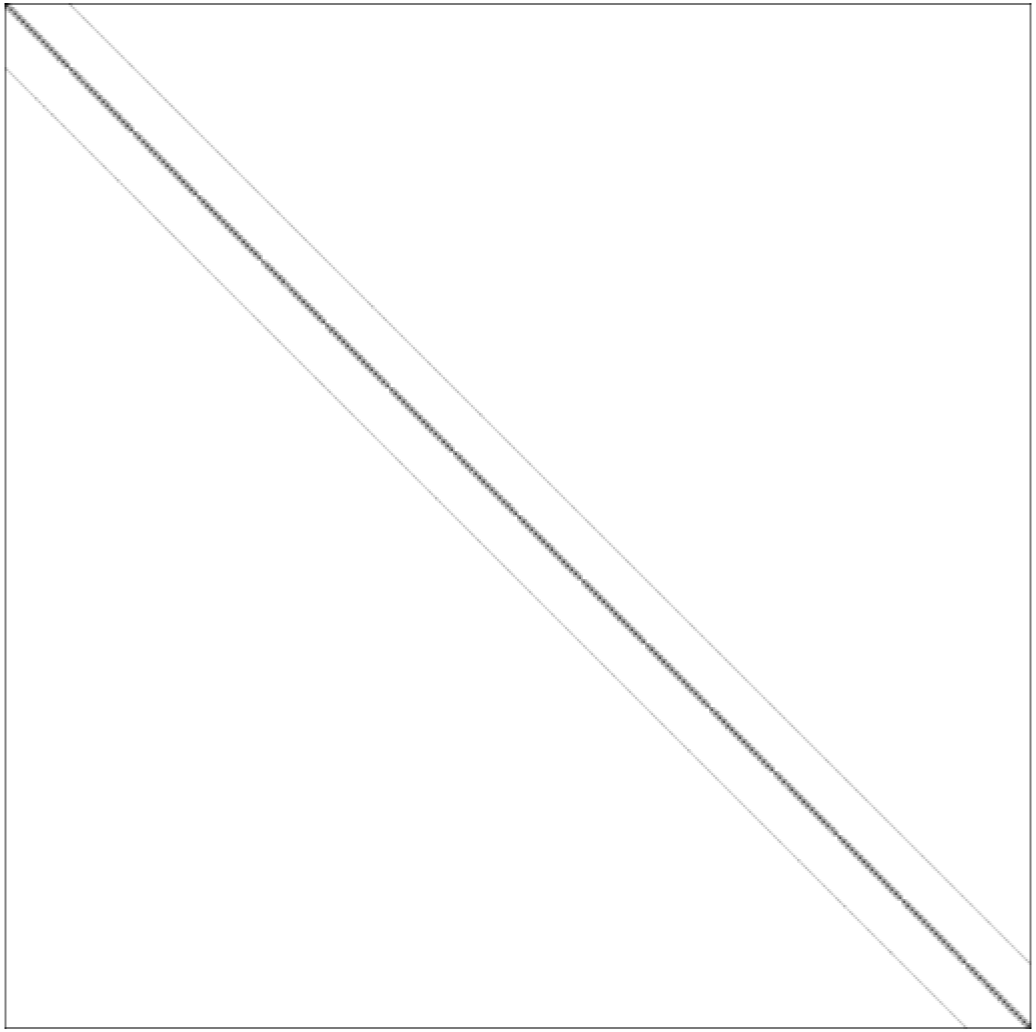
Test dla gamma=3 i eps=0.3

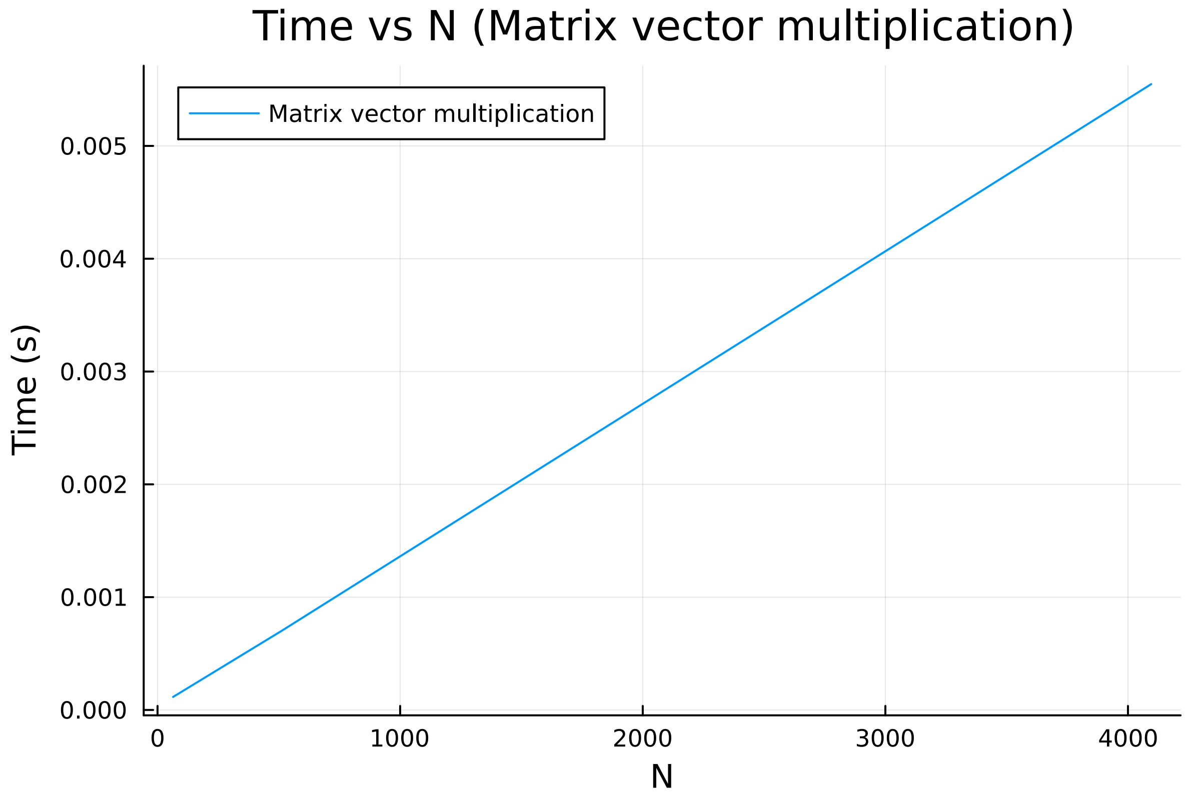
In [15]:
```
test(2, 0.3)
```

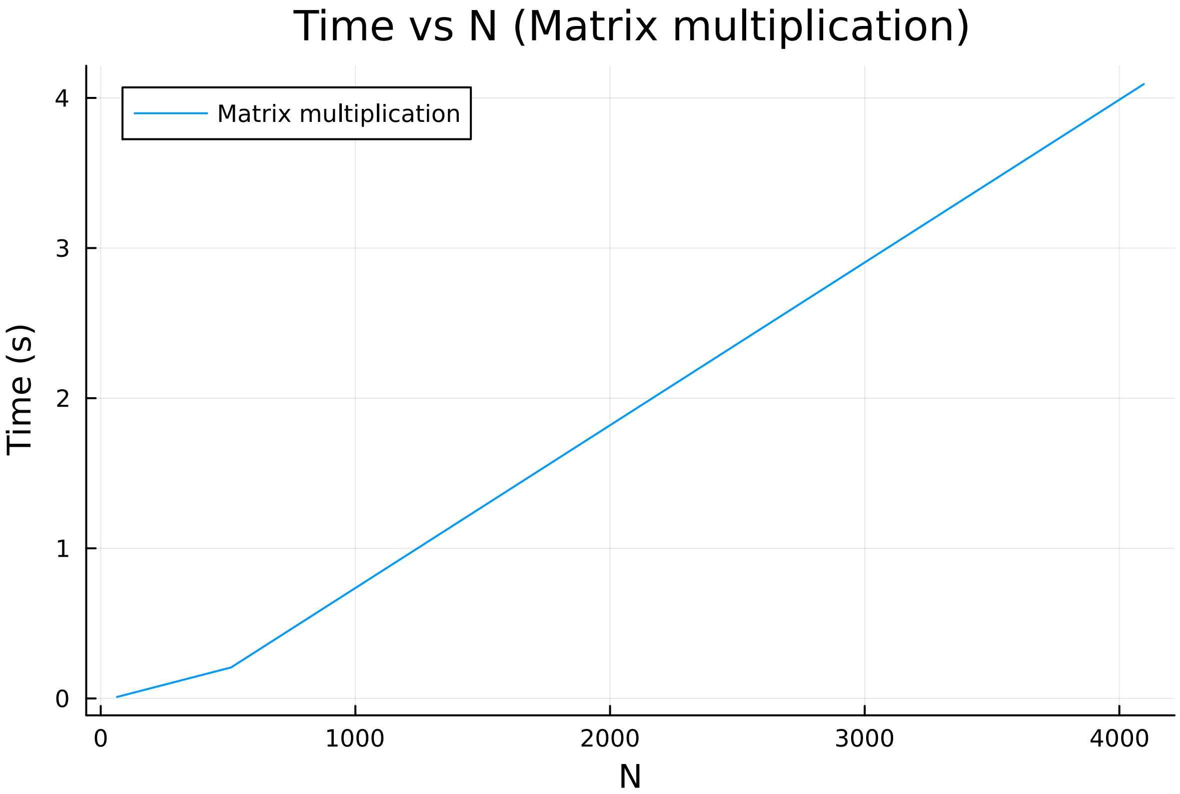
```
cm_2^6_2-0.3.png
cm_2^9_2-0.3.png
cm_2^12_2-0.3.png
```

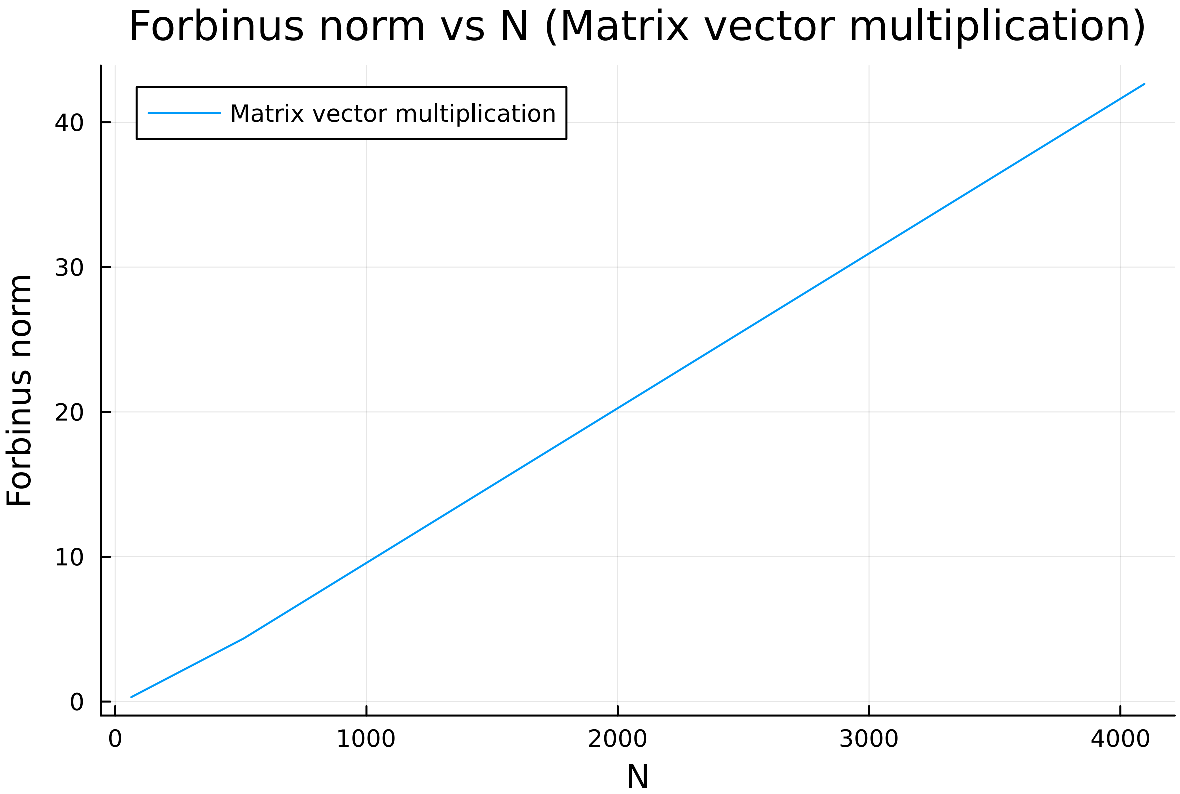




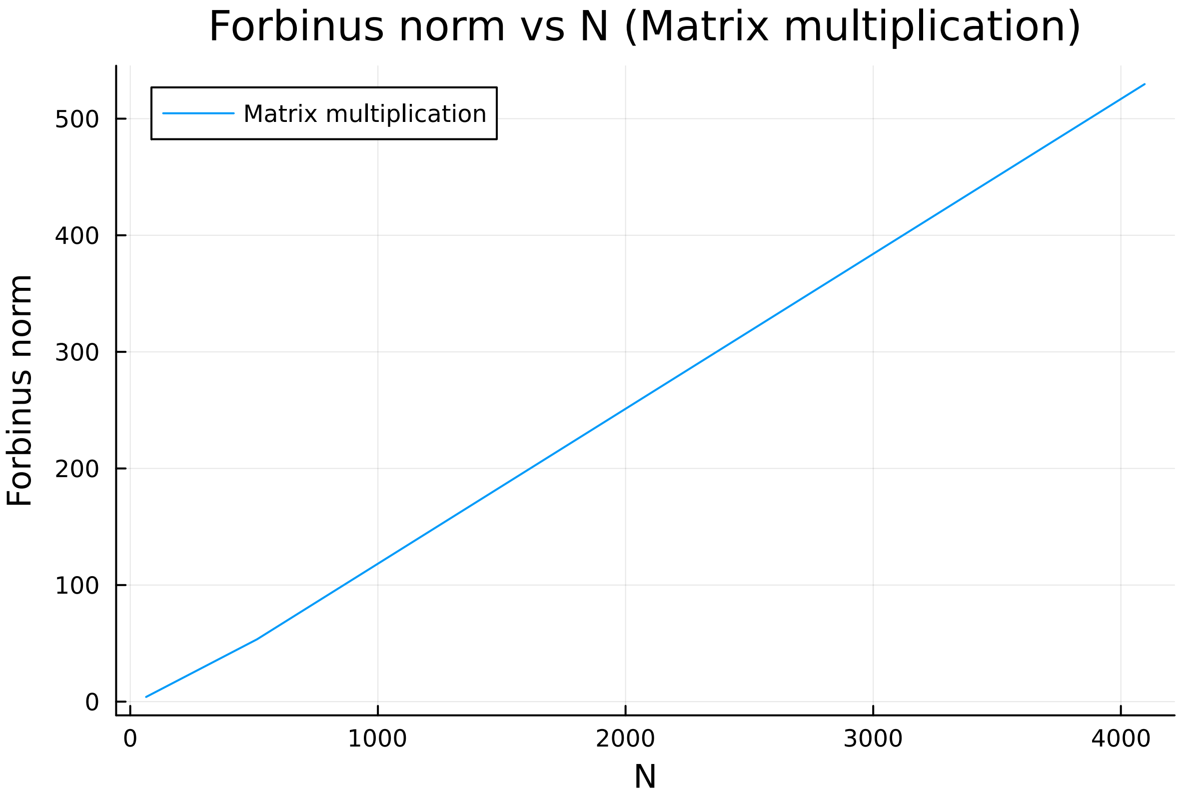


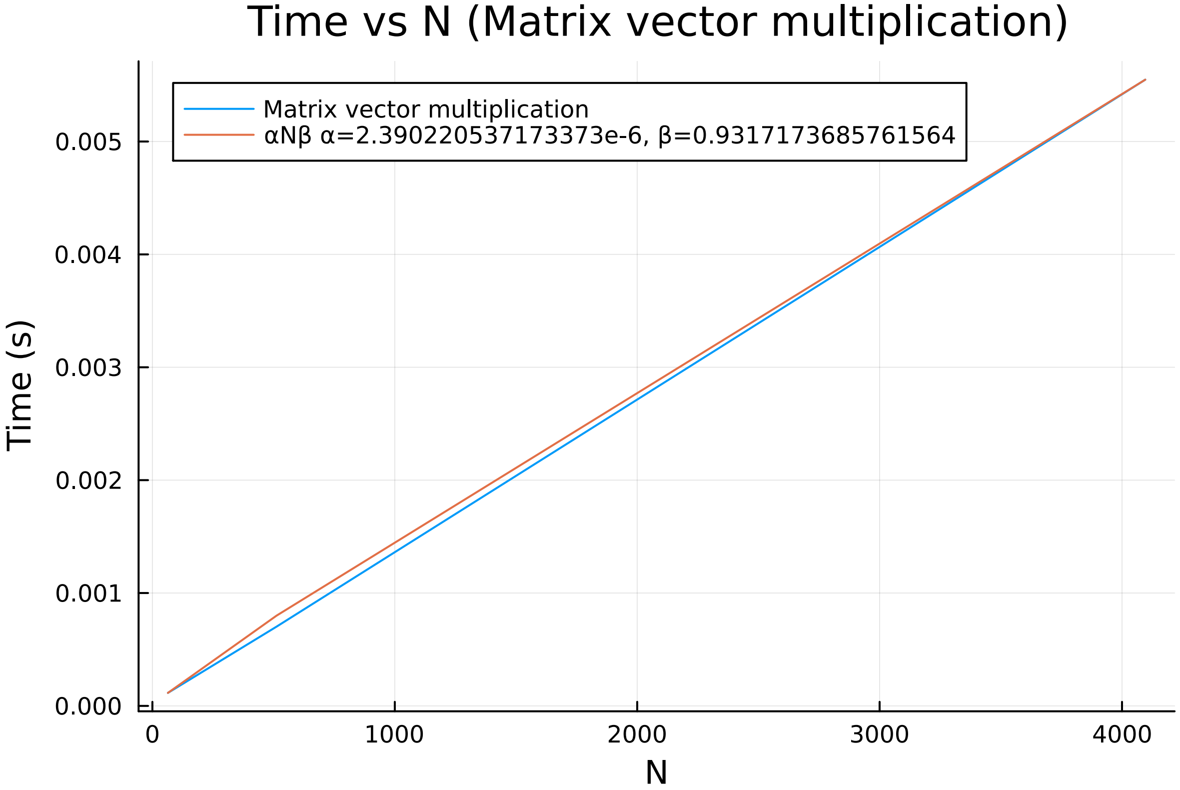

Time vs N (Matrix vector multiplication)
Matrix vector multiplication
Time (s)
0.000
0.001
0.002
0.003
0.004
0.005
0
1000
2000
3000
4000
N


Time vs N (Matrix multiplication)
Matrix multiplication
Time (s)
N
0
1
2
3
4
0
1000
2000
3000
4000

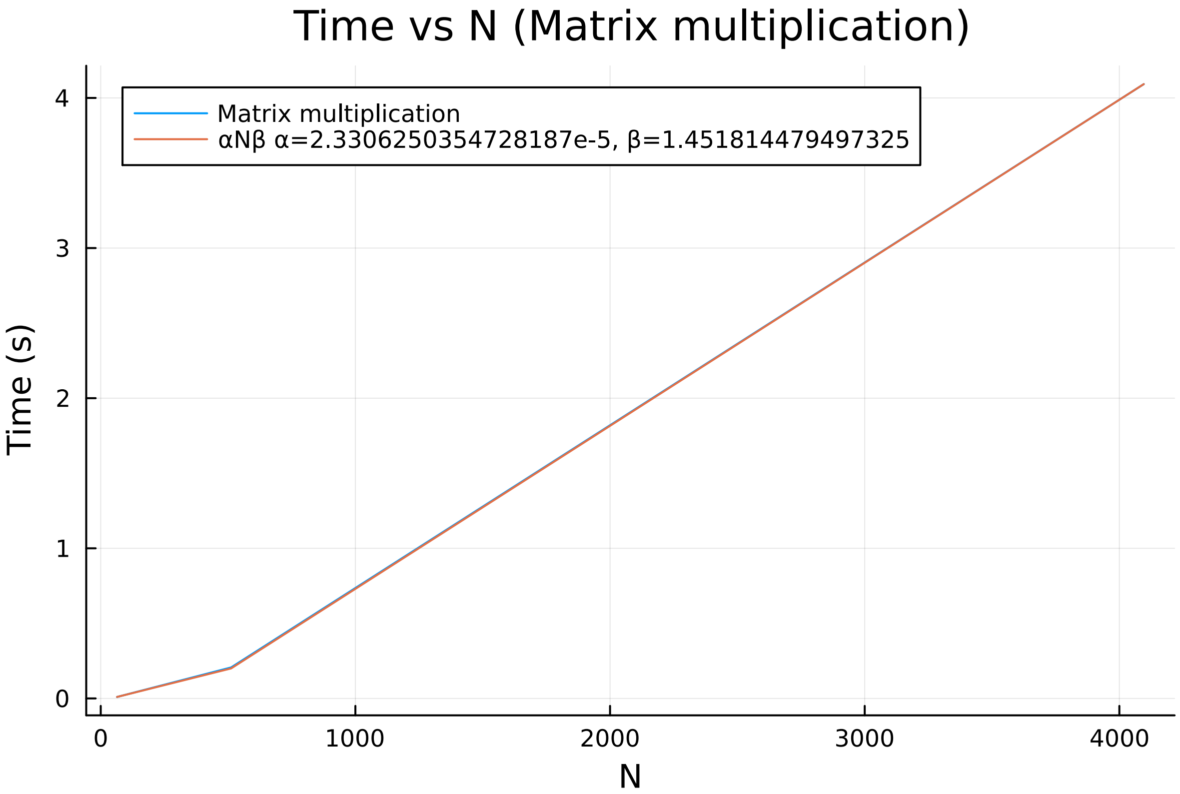


Forbinus norm vs N (Matrix vector multiplication)
Matrix vector multiplication
Forbinus norm
0
10
20
30
40
0
1000
2000
3000
4000
N


Forbinus norm vs N (Matrix multiplication)
Matrix multiplication
Forbinus norm
N
0
100
200
300
400
500
0
1000
2000
3000
4000



Time vs N (Matrix vector multiplication)
Matrix vector multiplication
αNβ α=2.390220537173373e-6, β=0.9317173685761564
0.005
0.004
0.003
0.002
0.001
0.000
Time (s)
0
1000
2000
3000
4000
N


Time vs N (Matrix multiplication)
Matrix multiplication
αNβ α=2.3306250354728187e-5, β=1.451814479497325
Time (s)
N
0
1
2
3
4
0
1000
2000
3000
4000