1 Basics of generating functions

- Introduction [Wilf 1–3]:
  - how to define a sequence: exact formula, recurrent relation (Fibonacci), algorithm (the sequence of primes); there are uncomputable sequences (programs that do not stop)
  - a new way: power series (members of the sequence as coefficients in the series)
  - advantages: many advanced tools from analytical theory of functions
  - very powerful: works on many sequences where nothing else is known to work
  - allows to get asymptotic formulas and statistical properties
  - powerful way to prove combinatorial identities
  - "Konečne vidím, že je tá matalýza aj na niečo dobrá. Keby mi to bol niekto predtým povedal..."
- Two examples [Wilf 3–7]:
  - $a_{n+1} = 2a_n + 1$ for $n \geq 0$, $a_0 = 0$
  - write few members, guess $a_n = 2^n - 1$, provable by induction
  - multiply by x^n, sum over all n, assign gf: $\dfrac{A(x)}{x} = 2A(x) + \dfrac{1}{1-x}$
  - partial fraction expansion: $A(x) = \dfrac{x}{(1-x)(1-2x)} = \dfrac{1}{1-2x} - \dfrac{1}{1-x}$
  - the method stays basically the same for harder problems
  - $a_{n+1} = 2a_n + n$ for $n \geq 0$, $a_0 = 1$
  - exact formula not obvious; no unqualified variables in the recurrence
  - obstacle: $\sum_{n\geq 0} nx^n = x/(1-x)^2$; solution: differentiation
  - concern: is differentiation allowed? discussed later, but in principle yes: in formal power series (as an algebraic ring) or via convergence (if we care about analytical properties)
  - $A(x) = \dfrac{1-2x+2x^2}{(1-x)^2(1-2x)} = \dfrac{A}{(1-x)^2} + \dfrac{B}{1-x} + \dfrac{C}{1-2x} = \dfrac{-1}{(1-x)^2} + \dfrac{2}{1-2x}$
  - $1/(1-x)^2$ is just $x/(1-x)^2$ (see above) shifted by 1
  - $a_n = 2^{n+1} - n - 1$
- The method [Wilf 8]:
  - 1. make sure variables in the recurrence are qualified (e.g. range for n)
  - 2. name and define the gf
  - 3. multiply by x^n, sum over all n in the range
  - 4. express both sides in terms of the gf
  - 5. solve the equation for gf
  - 6. calculate coefficients of gf power series expansion
  - useful notation: $[x^n]f(x)$; e.g.

$$[x^n]e^x = 1/n! \qquad [t^r]\frac{1}{1-3t} = 3^r \qquad [v^m](1+v)^s = \binom{s}{m}$$

- Solve $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$. [$G(x) = \frac{x}{(1-2x)(1-3x)}$; $a_n = 3^n - 2^n$]
- Fibonacci [Wilf 8–10]:
  - three-term recurrence: $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, $F_0 = 0$, $F_1 = 1$.

- apply the method ($r_\pm = (1 \pm \sqrt{5})/2$):

$$F(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-xr_+)(1-xr_-)} = \frac{1}{r_+ - r_-}\left(\frac{1}{1-xr_+} - \frac{1}{1-xr_-}\right)$$

- $F_n = \frac{1}{\sqrt{5}}(r_+^n - r_-^n)$
- the second term is <1 and goes to zero, so the first term $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$ gives a good approximation

- Find ogf for the following sequences (always $n \geq 0$) [W1.1]:

(a)	$a_n = n$	[introduce xD; $(x\mathrm{D})\frac{1}{1-x} = \frac{x}{(1-x)^2}$]
(b)	$a_n = \alpha n + \beta$	[$\alpha x/(1-x)^2 + \beta/(1-x)$]
(c)	$a_n = n^2$	[$(x\mathrm{D})^2 1/(1-x) = \frac{1+x}{(1-x)^3}$]
(d)	$a_n = n^3$	[$(x\mathrm{D})^3 1/(1-x)$]
(e)	$a_n = P(n)$; P is a polynomial of degree m	[$P(x\mathrm{D})\frac{1}{1-x}$]
(f)	$a_n = 3^n$	[$1/(1-3x)$]
(g)	$a_n = 5 \cdot 7^n - 3 \cdot 4^n$	[$\frac{5}{(1-7x)} - \frac{3}{1-4x}$]
(h)	$a_n = (-1)^n$	[$1/(1+x)$]

- Find the following coefficients [W1.5]:

(a)	$[x^n]\, e^{2x}$	[$2^n/n!$]
(b)	$[x^n/n!]\, e^{\alpha x}$	[α^n]
(c)	$[x^n/n!]\, \sin x$	[$(-1)^m$ if $n = 2m+1$ is odd, 0 otherwise]
(d)	$[x^n]\, 1/(1-ax)(1-bx)$ $(a \neq b)$	[$(a^{n+1} - b^{n+1})/(a-b)$]
(e)	$[x^n]\, (1+x^2)^m$	[$[2 \mid n]\binom{m}{n/2}$]

- Compute $\square_n = \sum_{k=1}^n k^2$.
  - assign ogf to the sequence $1^2, 2^2, \ldots, n^2$: $f(x) = \sum_{k=1}^n k^2 x^k$
  - $(x\mathrm{D})^2[(x^{n+1}-1)/(x-1)] = x\frac{-2n^2x^{n+1}+n^2x^{n+2}+n^2x^n-2nx^{n+1}+x^{n+1}+2nx^n+x^n-x-1)}{(x-1)^3}$
  - note that $\square_n = f(1) = \lim_{x\to 1}(xD)^2[(x^{n+1}-1)/(x-1)] = n(n+1)(2n+1)/6$

- Homework (use computer for Taylor series expansion and prove it by hand):
  Find the sequence with gf $1/(1-x)^3$.

- Find explicit formulas for the following sequences [W1.6, R2, R3, R7]:

(a)	$a_{n+1} = 3a_n + 2$ for $n \geq 0$; $a_0 = 0$	[$3x/(1-x)(1-3x)$; $\quad 3^n - 1$]
(b)	$a_{n+1} = \alpha a_n + \beta$ for $n \geq 0$; $a_0 = 0$	[$\beta x/(1-x)(1-\alpha x)$; $\quad \frac{\alpha^n - 1}{\alpha - 1}\beta$]
(c)	$a_{n+1} = a_n/3 + 1$ for $n \geq 0$; $a_0 = 1$	[$\frac{3/2}{1-x} - \frac{1/2}{1-x/3}$; $\quad \frac{3^{n+1}-1}{2\cdot 3^n}$]
(d)	$a_{n+2} = 2a_{n+1} - a_n$ for $n \geq 0$, $a_0 = 0$, $a_1 = 1$	[$x/(1-x)^2$; $\quad n$]
(e)	$a_{n+2} = 3a_{n+1} - 2a_n + 3$ for $n > 0$; $a_0 = 1$, $a_1 = 2$	[$\frac{4}{1-2x} - \frac{3}{(1-x)^2}$; $\quad 2^{n+2} - 3n - 3$]
(f)	$a_n = 2a_{n-1} - a_{n-2} + (-1)^n$ for $n > 1$; $a_0 = a_1 = 1$	[$\frac{1/2}{(1-x)^2} - \frac{1/4}{1-x} + \frac{1/4}{1+x}$; $\frac{2n+3+(-1)^n}{4}$]
(g)	$a_n = 2a_{n-1} - n \cdot (-1)^n$ for $n \geq 1$; $a_0 = 0$	[$\frac{x/9-2/9}{(1+x)^2} + \frac{2/9}{1-2x}$; $\frac{2^{n+1}-(3n+2)(-1)^n}{9}$]
(h)	$a_n = 3a_{n-1} + \binom{n}{2}$ for $n \geq 1$; $a_0 = 2$	[$\frac{1}{8}(19 \cdot 3^n - 2n(n+2) - 3)$]
(i)	$a_n = 2a_{n-1} - a_{n-2} - 2$ for $n \geq 1$; $a_0 = a_{10} = 0$	[$n(a_1 + 1 - n)$, so with a_{10}, $a_n = n(10-n)$]

2 Ordinary generating functions

- From the homework: solve $a_n = 2a_{n-1} - a_{n-2} - 2$ for $n \geq 1$; $a_0 = a_{10} = 0$.
  Applying the standard method, while keeping a_1 as a parameter, we get

$$A(x) = \frac{a_1 x - a_1 x^2 - 2x^2}{(1-x)^3} = \frac{a_1 x}{(1-x)^2} + \frac{x(1-x)}{(1-x)^3} - \frac{x^2+x}{(1-x)^3},$$

  so $a_n = (a_1+1)n - n^2$. From $a_{10} = 0$ we get $a_1 = 9$, thus $a_n = n(10-n)$.

- Another way for boundary problems (this particular example is motivated by splines, Wilf 10–11):
  - consider $au_{n+1} + bu_n + cu_{n-1} = d_n$ for $1 \leq n \leq N-1$; $u_0 = u_N = 0$.
  - similar to Fibonacci with two given non-consecutive terms (but more general)
  - define $U(x) = \sum_{j=0}^{N} u_j x^j$ (unknown); $D(x) = \sum_{j=1}^{N-1} d_j x^j$ (known)
  - derive $a \cdot \dfrac{U(x) - u_1 x}{x} + bU(x) + cx(U(x) - u_{N-1}x^{N-1}) = D(x)$
  - $(a + bx + cx^2)U(x) = xD(x) + au_1 x + cu_{N-1}x^N$ (*)
  - plug in suitable values of x (roots r_+ and r_- of the quadratic polynomial on the LHS)
  - solve the system of two linear equations and two uknowns u_1, u_{N-1}
  - if the roots are equal, differentiate (*) to obtain the second equation

- Mutually recursive sequences [Knuth 343, Example 3]
  - consider the number u_n of tilings of $3 \times n$ board with 2×1 dominoes
  - define v_n as the number of tilings of $3 \times n$ board without a corner
  - $u_n = 2v_{n-1} + u_{n-2}$; $u_0 = 1$; $u_1 = 0$
  - $v_n = v_{n-2} + u_{n-1}$; $v_0 = 0$; $v_1 = 1$
  - derive
$$U(x) = \frac{1-x^2}{1-4x^2+x^4}, \qquad V(x) = \frac{x}{1-4x^2+x^4}$$
  - consider $W(z) = 1/(1-4z+z^2)$; $U(x) = (1-x^2)W(x^2)$, so $u_{2n} = w_n - w_{n-1}$
  - hence $u_{2n} = \frac{(2+\sqrt{3})^n}{3-\sqrt{3}} + \frac{(2-\sqrt{3})^n}{3+\sqrt{3}} = \left\lceil \frac{(2+\sqrt{3})^n}{3-\sqrt{3}} \right\rceil$ (derivation as a homework)

- Given $f(x) \stackrel{\text{ogf}}{\longleftrightarrow} (a_n)_{n\geq 0}$, express ogf for the following sequences in terms of f [W1.3]:

(a) $(a_n + c)_{n\geq 0}$ — $[\ f(x) + c/(1-x)\ ]$

(b) $(na_n)_{n\geq 0}$ — $[\ x\mathrm{D}f(x)\ ]$; napísať im $(P(n)a_n)_{n\geq 0} \longleftrightarrow P(x\mathrm{D})f(x)$

(c) $0, a_1, a_2, a_3, \ldots$ — $[\ f(x) - a_0\ ]$

(d) $0, 0, 1, a_3, a_4, a_5, \ldots$ — $[\ f(x) - a_0 - a_1 x + (1-a_2)x^2\ ]$

(e) $(a_{n+2} + 3a_{n+1} + a_n)_{n\geq 0}$ — $[\ (f - a_0 - a_1 x)/x^2 + 3(f - a_0)/x + f\ ]$

(f) $a_0, 0, a_2, 0, a_4, 0, a_6, 0 \ldots$ — $[\ (f(x) + f(-x))/2\ ]$

(g) $a_0, 0, a_1, 0, a_2, 0, a_3, 0 \ldots$ — $[\ f(x^2)\ ]$

(h) $a_1, a_2, a_3, a_4, \ldots$ — $[\ (f(x) - a_0)/x\ ]$

(i) $a_0, a_2, a_4, \ldots$ — $[\ (f(\sqrt{x}) + f(-\sqrt{x}))/2\ ]$

GFs of two variables

- introducing a new variable and changing the order of summation can help

$$\begin{aligned}\sum_{n\geq 0}\binom{n}{k}x^n &= [y^k]\sum_{m\geq 0}\left(\sum_{n\geq 0}\binom{n}{m}x^n\right)y^m = [y^k]\sum_{n\geq 0}(1+y)^n x^n \\ &= [y^k]\frac{1}{1-x(1+y)} = \frac{1}{1-x}[y^k]\frac{1}{1-\frac{x}{1-x}y} = \frac{x^k}{(1-x)^{k+1}} \qquad (1)\end{aligned}$$

- alternatively, one can use binomial theorem (Knuth 199/5.56 and 5.57):

$$\begin{aligned}\frac{1}{(1-z)^{n+1}} &= (1-z)^{-n-1} = \sum_{k\geq 0}\binom{-n-1}{k}(-z)^k \\ &= \sum_{k\geq 0}\frac{(-n-1)(-n-2)\dots(-n-k)}{k!}(-z)^k = \sum_{k\geq 0}\binom{n+k}{n}z^k\end{aligned}$$

Formal power series [Wilf chapter 2]

- a ring with addition and multiplication $\sum_n a_n x^n \sum_n b_n x^n = \sum_n \sum_k (a_k b_{n-k})x^n$
- if $f(0)\neq 0$, then f has a unique reciprocal $1/f$ such that $f\cdot 1/f = 1$
- composition $f(g(x))$ defined iff $g(0)=0$ or f is a polynomial (cf. e^{e^x-1} vs. e^{e^x})
- formal derivative D: $\mathrm{D}\sum_n a_n x^n = \sum n a_n x^{n-1}$; usual rules for sum, product etc.
- exercise: find all f such that $\mathrm{D}f = f$

Rules for manipulation [Wilf 2.1, Knuth 334]. Assume that $f \overset{\text{ogf}}{\longleftrightarrow} (a_n)_{n=0}^{\infty}$.

- **Rule 1**: for a positive integer h, $(a_{n+h}) \overset{\text{ogf}}{\longleftrightarrow} (f - a_0 - \ldots - a_{h-1}x^{h-1})/x^h$
- **Rule 2**: if P is a polynomial, then $P(x\mathrm{D})f \overset{\text{ogf}}{\longleftrightarrow} (P(n)a_n)_{n\geq 0}$
  - example: $(n+1)a_{n+1} = 3a_n + 1$ for $n\geq 0$, $a_0 = 1$; thus $f' = 3f + 1/(1-x)$
  - example: $\sum_{n\geq 0}\frac{n^2+4n+5}{n!}$; thus $f = \sum_{n\geq 0}(n^2+4n+5)\frac{x^n}{n!} = ((x\mathrm{D})^2+4x\mathrm{D}+5)e^x = (x^2+5x+5)e^x$
    we need $f(1) = 11e$; works because the resulting f is analytic in a disk
    containing 1 in the complex plane (that is, it converges to its Taylor series)
- **Rule 3**: if $g \overset{\text{ogf}}{\longleftrightarrow} (b_n)$, then $fg \overset{\text{ogf}}{\longleftrightarrow} (\sum_{k=0}^n a_k b_{n-k})_{n\geq 0}$

$$\sum_{k=0}^{n}(-1)^k k = (-1)^n\sum_{k=0}^{n}k\cdot(-1)^{n-k} = (-1)^n[x^n]\frac{x}{(1-x)^2}\cdot\frac{1}{1+x} = \frac{(-1)^n}{4}(2n+1-(-1)^n)$$

- **Rule 4**: for a positive integer k, we have $f^k \overset{\text{ogf}}{\longleftrightarrow} \left(\sum_{n_1+n_2+\cdots+n_k=n} a_{n_1}a_{n_2}\dots a_{n_k}\right)_{n\geq 0}$
  - example: let $p(n,k)$ be the number of ways n can be written as an ordered sum of k nonnegative integers
  - according to R4, $(p(n,k))_{n\geq 0} \overset{\text{ogf}}{\longleftrightarrow} 1/(1-x)^k$, so $p(n,k) = \binom{n+k-1}{n}$ thanks to (1)
- **Rule 5**: $\dfrac{f}{(1-x)} \overset{\text{ogf}}{\longleftrightarrow} \left(\sum_{k=0}^{n} a_k\right)_{n\geq 0}$
  - example: $(\square_n)_{n\geq 0} \overset{\text{ogf}}{\longleftrightarrow} \frac{1}{1-x}\cdot(x\mathrm{D})^2\frac{1}{1-x} = \frac{x(1+x)}{(1-x)^4}$, so by (1), $\square_n = \binom{n+2}{3} + \binom{n+1}{3}$

1. Using Rule 5, prove that $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$ for $n \geq 0$ [Wilf 38, example 6].
   [Compare gfs of both sides, left is $f/(1-x)$, where $f = x/(1-x-x^2)$, i.e. Fibonacci.]

2. Solve $g_n = g_{n-1} + g_{n-2}$ for $n \geq 2$, $g_0 = 0$, $g_{10} = 10$.
   [$g_n = \frac{G_{10}}{F_{10}} F_n$, try the "boundary method" described above, computer necessary]

3. Solve $a_n = \sum_{k=0}^{n-1} a_k$ for $n > 0$; $a_0 = 1$. [R16]
   [$a_n = 2^{n-1}$ for $n \geq 1$]

4. Solve $f_n = 2f_{n-1} + f_{n-2} + f_{n-3} + \cdots + f_1 + 1$ for $n \geq 1$, $f_0 = 0$ [Knuth 349/(7.41)]
   [$F(x) = x/(1-3x+x^2)$; $f_n = F_{2n}$]

5. Solve $g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0$ for $n > 0$, $g_0 = 1$. [K7.7]
   [$G(x) = 1 + x/(1-3x+x^2)$; $g_n = F_{2n} + [n=0]$]

6. Solve $g_n = \sum_{k=1}^{n-1} \frac{g_k + g_{n-k} + k}{2}$ for $n \geq 2$, $g_1 = 1$.

7. Solve $g_n = g_{n-1} + 2g_{n-2} + (-1)^n$ for $n \geq 2$, $g_0 = g_1 = 1$. [Knuth 341, example 2]
   [$G(x) = \frac{1+x+x^2}{(1-2x)(1+x)^2}$; $g_n = \frac{7}{9}2^n + \frac{1}{9}(3n+2)(-1)^n$]

8. Solve $a_{n+2} = 3a_{n+1} - 2a_n + n + 1$ for $n \geq 0$; $a_0 = a_1 = 1$. [R24]
   [$A(z) = \frac{2}{1-2z} - \frac{1}{(1-z)^3}$; $a_n = 2^{n+1} - \binom{n+2}{2}$]

9. Prove that $\ln \dfrac{1}{1-x} = \displaystyle\sum_{n \geq 1} \frac{1}{n} x^n$. [consider $\int \frac{1}{1-x}$]

3 prednaska

- Discovering combinatorial identities via gfs [Knuth 198, Vandermonde and 5.55]
  - $(1+x)^r = \sum_{k \geq 0} \binom{r}{k} x^k$; consider $(1+x)^r(1+x)^s = (1+x)^{r+s}$
  - comparison of coefficients yields $\sum_{k \geq 0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$ — Vandermonde
  - by considering $(1-x)^r(1+x)^r = (1-x^2)^r$, we obtain

  $$\sum_{k=0}^{n} \binom{r}{k}\binom{r}{n-k}(-1)^k = (-1)^{n/2}\binom{r}{n/2}[2 \mid n]$$

  TODO tahak oficialny, pridat nan tabulky z knutha

- Catalan numbers [Knuth 357, example 4]
  - consider the number of possibilities c_n of how to specify the multiplication order of $A_0 A_1 \ldots A_n$ by parentheses
  - split it by the place of last multiplication; $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n > 0$; $c_0 = 1$
  - three ways for the right-hand side: shift before convolution (thus we have $C(x) \cdot xC(x)$), shift after convolution (thus $x \cdot C(x)^2$), or rewrite through sums and change the order of summation:

  $$\sum_{n \geq 1} x^n \sum_{k=0}^{n-1} c_k c_{n-1-k} = \sum_{k=0}^{\infty} x^k c_k \sum_{n \geq k+1} c_{n-1-k} x^{n-k} = \sum_{k=0}^{\infty} x^k c_k x C(x) = xC(x) \cdot C(x)$$

  - consequently, $C(x) - 1 = xC(x)^2$ and thus $C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$
  - we want C continuous and $C(0) = 1$, so we choose the minus sign (note that the resulting function below is analytical since $\binom{2n}{n}/(n+1) < 2^{2n}$; it would be analytical also if we choose the plus sign)

– binomial theorem yields

$$\sqrt{1-4x} = \sum_{k\ge 0}\binom{1/2}{k}(-4x)^k \quad = \quad 1+\sum_{k\ge 1}\frac{1}{2k\cdot(-4)^{k-1}}\binom{2k-2}{k-1}(-4)^k x^k$$
$$= \quad 1-\sum_{k\ge 1}\frac{2}{k}\binom{2k-2}{k-1}x^k$$

– we used $\binom{1/2}{k} = \frac{1/2}{k}\binom{-1/2}{k-1} = \frac{1}{2k(-4)^{k-1}}\binom{2k-2}{k-1}$ because $\binom{-1/2}{m} = \frac{1}{(-4)^m}\binom{2m}{m}$

– therefore,

$$C(x) = \frac{1}{2x}\sum_{k\ge 1}\frac{2}{k}\binom{2k-2}{k-1}x^k = \sum_{n\ge 0}\frac{1}{n+1}\binom{2n}{n}x^n$$

- Every third binomial coefficient [Wilf 51, example 4]
  - why $(f(x)+f(-x))/2 \overset{\text{ogf}}{\longleftrightarrow} a_0, 0, a_2, 0, a_4, \ldots$ works: $\frac{1}{2}(1^n+(-1)^n) = [2 \mid n]$
  - in general, $\sum_{j=0}^{r-1}\omega_r^n = \sum_{j=0}^{r-1}e^{2\pi ijn/r} = r\cdot[r \mid n]$ (just a geometric progression)
  - problem: find $\lambda_n = \sum_k (-1)^k\binom{n}{3k}$
  - if we knew $f(x) = \sum_k \binom{n}{3k}x^{3k}$, we would have $\lambda_n = f(-1)$
  - for $F(x) = (1+x)^n$, we have $f(x) = (F(x)+F(x\omega^1)+F(x\omega^2))/3$
  - and so $\lambda_n = f(-1) = \frac{1}{3}[(1-\omega)^n + (1-\omega^2)^n)] =$

$$= \frac{1}{3}\left[\left(\frac{3-\sqrt{3}i}{2}\right)^n + \left(\frac{3+\sqrt{3}i}{2}\right)^n\right] = 2\cdot 3^{n/2-1}\cos(\pi n/6)$$

- Exercises
  1. Find ogf for $H_n = 1+1/2+1/3+\ldots$. $[\ -\ln(1-x)/(1-x)\ ]$
  2. Evaluate $S_n = \sum_{k=0}^{n}(-1)^k k$. [GF11] $[\ f(x) = \frac{-x}{(1+x)^2(1-x)} = \frac{1/2}{(1+x)^2} - \frac{1/4}{1+x} - \frac{1/4}{1-x}$; $S_n = \frac{1}{4}(2(n+1)(-1)^n - (-1)^n - 1)\ ]$
  3. samplesort Knuth 354 ex. 2: convolution...
  4. nejake ulohy zo zbierky: R23, R21?, R19?

- Snake oil method [Wilf 118, chapter 4.3] – external method vs. internal manipulations
  1. identify the free variable and give the name to the sum, e.g. $f(n)$
  2. let $F(x) = \sum f(n)x^n$
  3. interchange the order of summation; solve the inner sum in closed form
  4. find coefficients of $F(x)$

- Example 0
  - let's evaluate $f(n) = \sum_k \binom{n}{k}$; after Step 2, $F(x) = \sum_{n\ge 0}x^n\sum_k\binom{n}{k}$
  - $F(x) = \sum_k\sum_n\binom{n}{k}x^n = \sum_k\frac{x^k}{(1-x)^{k+1}} = \frac{1}{1-x}\cdot\frac{1}{1-\frac{x}{1-x}} = \frac{1}{1-2x}$

- Example 1 [Wilf 121]
  - let's evaluate $f(n) = \sum_{k\ge 0}\binom{k}{n-k}$
  - after Step 2, $F(x) = \sum_n x^n\sum_{k\ge 0}\binom{k}{n-k}$
  - $F(x) = \sum_{k\ge 0}\sum_n\binom{k}{n-k}x^n = \sum_{k\ge 0}x^k\sum_n\binom{k}{n-k}x^{n-k} = \sum_{k\ge 0}x^k(1+x)^k = \frac{1}{1-x-x^2}$
  - so $f(n) = F_{n+1}$

- Example 2 [Wilf 122]
  - let's evaluate $f(n) = \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$, where m, n are nonnegative integers

$$\begin{aligned} F(x) &= \sum_{n\ge 0} x^n \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \\ &= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \sum_{n\ge 0}\binom{n+k}{m+2k} x^{n+k} \\ &= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+2k}}{(1-x)^{m+2k+1}} \\ &= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k}\frac{1}{k+1}\left(\frac{-x}{(1-x)^2}\right)^k \\ &= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left(1-\sqrt{1+\frac{4x}{(1-x)^2}}\right) = \frac{x^m}{(1-x)^m} \end{aligned}$$

  - so $f(n) = \binom{n-1}{m-1}$
- Example 6 [Wilf 127]
  - prove that $\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k$, where m, n are nonnegative integers
  - the ogf of the left-hand side is

$$L(x) = \sum_k \binom{m}{k} x^{-k} \sum_{n\ge 0}\binom{n+k}{m} x^{n+k} = \frac{(1+x)^m}{(1-x)^{m+1}}$$

  - we get the same for the right-hand side
- Introducing additional free variable [W4.13]
  - Let's prove that $\sum_k (-1)^{n-k}\binom{2n}{k}^2 = \binom{2n}{n}$.
  - We evaluate $\sum_k (-1)^k \binom{n}{k}\binom{n}{n-m+k}$ by multiplying by x^m etc.
- Exercises
  1. Prove that $\sum_k k\binom{n}{k} = n2^{n-1}$ via the snake oil method. $\left[\ L(x) = P(x) = \frac{x}{(1-2x)^2}\ \right]$
  2. Evaluate $f(n) = \sum_{k\ge 0}\binom{k}{n-k}t^k$. [W4.11(a)]

     $\left[\ F(x) = 1/(1-tx+x^2)\ \right]$
  3. Evaluate $f(n) = \sum_k \binom{n+k}{2k} 2^{n-k}$, $n \ge 0$. [Wilf 125, Example 4]

     $\left[\ F(x) = \frac{1-2x}{(1-x)(1-4x)} = \frac{2}{3(1-4x)} + \frac{1}{3(1-x)};\ f(n) = (2^{2n+1}+1)/3\ \right]$
  4. Evaluate $f(n) = \sum_k k^2\binom{n}{k}3^k$.

     $\left[\ F(x) = \frac{3x(1+2x)}{(1-4x)^3} = \frac{3/8}{1-4x} - \frac{3/2}{(1-4x)^2} + \frac{9/8}{(1-4x)^3};\ f(n) = 3\cdot 4^{n-2}\cdot n(1+3n)\ \right]$
  5. Evaluate $f(n) = \sum_{k\le n/2}(-1)^k\binom{n-k}{k}y^{n-2k}$. [Wilf 122, Example 3]

     $\left[\ F(x) = 1/(1-xy+x^2)\ \right]$
  6. Evaluate $f(n) = \sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k}$. [W4.11(c)]

     [replace $2n+1$ by m and solve for $f(m) = \binom{m-p-1}{p}2^{m-2p-1}$; $f(2n+1) = \binom{2n-p}{p}4^{n-p}$;

     $$F(x) = \frac{x}{(1-x)^2}\sum_{k\ge 0}\binom{p+k}{p}\left(\frac{x}{1-x}\right)^{2(p+k)} = \frac{x^{p+1}}{2^p}\cdot\frac{(2x)^p}{(1-2x)^{p+1}}$$]

4 prednaska

- Purpose of asymptotics [Knuth 439]
  - sometimes we do not have a closed form or it is hard to compare it to other quantities
  - $S_n = \sum_{k=0}^{n} \binom{3n}{k} \sim 2\binom{3n}{n}$; $S_n = \binom{3n}{n}\left(2 - \frac{4}{n} + O\left(\frac{1}{n^2}\right)\right)$
  - how to compare it with F_{4n}? we need to approximate the binomial coefficient
  - purpose is to find *accurate* and *concise* estimates:
    H_n is $\sum_{k\geq 1}^{n} 1/k$ vs. $O(\log n)$ vs. $\ln n + \gamma + O(n^{-1})$
- Hierarchy of log-exp functions [Hardy, see Knuth 442]
  - the class $\mathcal{L}$ of logarithmico-exponential functions: the smallest class that contains constants, identity function $f(n) = n$, difference of any two functions from $\mathcal{L}$, e^f for every $f \in \mathcal{L}$, $\ln f$ for every $f \in \mathcal{L}$ that is "eventually positive"
  - every such function is identically zero, eventually positive or eventually negative
  - functions in $\mathcal{L}$ form a hierarchy (every two of them are comparable by $\prec$ or $\asymp$)
- Notations
  - $f(n) = O(g(n))$ iff $\exists c : |f(n)| \leq c|g(n)|$ (alternatively, for $n \geq n_0$ for some n_0)
  - $f(n) = o(g(n))$ iff $\lim_{n\to\infty} f(n)/g(n) = 0$
  - $f(n) = \Omega(g(n))$ iff $\exists c : |f(n)| \geq c|g(n)|$ (alternatively, $\ldots$)
  - $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
  - basic manipulation: $O(f) + O(g) = O(|f| + |g|)$, $O(f)O(g) = O(fg) = fO(g)$ etc.
  - meaning of O in sums
  - *relative* vs. *absolute* error
- Warm-ups
  1. Prove or disprove: $O(f + g) = f + O(g)$ if f and g are positive. [K9.5] $[$ false $]$
  2. Multiply $\ln n + \gamma + O(1/n)$ by $n + O(\sqrt{n})$. [K9.6] $[\ n \ln n + \gamma n + O(\sqrt{n} \ln n)\ ]$
  3. Compare $n^{\ln n}$ with $(\ln n)^n$. $[\ \prec\ ]$
  4. Compare $n^{\ln \ln \ln n}$ with $(\ln n)!$. $[\ \prec\ ]$
  5. Prove or disprove: $O(x + y)^2 = O(x^2) + O(y^2)$. [K9.11] $[$ true $]$
- Common tricks
  - cut off series expansion (works for convergent series, Knuth 451)
  - substitution, e.g. $\ln(1 + 2/n^2)$ with precision of $O(n^{-5})$ $[\ \frac{2}{n^2} - \frac{4}{n^4} + O(n^{-6})\ ]$
  - factoring (pulling the large part out), e.g. $\frac{1}{n^2+n} = \frac{1}{n^2}\frac{1}{1+\frac{1}{n}} = \frac{1}{n^2} - \frac{1}{n^3} + O(n^{-4})$
  - division, e.g. $\dfrac{H_n}{\ln(n+1)} = \dfrac{\ln n + \gamma + O(n^{-1})}{(\ln n)(1 + O(n^{-1}))} = 1 + \dfrac{\gamma}{\ln n} + O(n^{-1})$
  - exp-log, i.e. $f(x) = e^{\ln f(x)}$
- Typical situations for approximation
  - Stirling formula: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \dfrac{1}{12n} + \dfrac{1}{288n^2} + O(n^{-3})\right)$
  - harmonic numbers: $H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(n^{-4})$
  - rational functions, e.g. $\frac{n}{n+2} = \frac{1}{1+\frac{2}{n}} = 1 - \frac{2}{n} + \frac{4}{n^2} + O(n^{-3})$

- exponentials: $e^{H_n} = ne^\gamma e^{O(1/n)} = ne^\gamma(1+O(1/n)) = ne^\gamma + O(1)$
- rational function powered to n, e.g.

$$\left(1-\frac{1}{n}\right)^n = e^{n\ln\left(1-\frac{1}{n}\right)} = \exp\left(n\left(\frac{-1}{n}+O\left(n^{-2}\right)\right)\right) = e^{-1+O(n^{-1})} = \frac{1}{e}+O(n^{-1})$$

- binomial coefficient, e.g. $\binom{2n}{n}$: factorials and Stirling formula

$$\binom{2n}{n} = \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}(1+O(n^{-1}))}{2\pi n\left(\frac{n}{e}\right)^{2n}(1+O(n^{-1}))^2} = \frac{2^{2n}}{\sqrt{\pi n}}(1+O(n^{-1}))$$

- Exercises

1. Estimate $\ln(1+1/n)+\ln(1-1/n)$ with abs. error $O(n^{-3})$ [$-1/n^2+O(n^{-4})$]
2. Estimate $\ln(2+1/n)-\ln(3-1/n)$ with abs. error $O(n^{-2})$ [$\ln\frac{2}{3}+\frac{5}{6n}+O(n^{-2})$]
3. Estimate $\lg(n-2)$, abs. error $O(n^{-2})$ [$\frac{\ln n}{\ln 2}-\frac{2}{n\ln 2}+O(n^{-2})$]
4. Evaluate H_n^2 with abs. error $O(n^{-1})$. [$(\ln n)^2+2\gamma\ln n+\gamma^2+(\ln n)/n+O(1/n)$]
5. Estimate $n^3/(2+n+n^2)$ with abs. error $O(n^{-3})$ [$n-1-\frac{1}{n}+\frac{3}{n^2}+O(n^{-3})$]
6. Prove or disprove: [K9.20] (b) $e^{(1+O(1/n))^2}=e+O(1/n)$ (c) $n! = O\left(((1-1/n)^n n)^n\right)$ [yes, no]
7. Evaluate $(n+2+O(n^{-1}))^n$ with rel. error $O(n^{-1})$. [K9.13] [$n^n\cdot e^2(1+O(n^{-1}))$]
8. Compare H_{F_n} with $F^2_{\lceil H_n\rceil}$ [K9.2] [$H_{F_n}\sim n\ln\varphi$, $F^2_{\lceil H_n\rceil}=O(n^{\ln\varphi^2})=o(n)$]
9. Estimate $\sum_{k\ge 0} e^{-k/n}$ with abs. error $O(n^{-1})$. [K9.7] [$n+1/2+O(n^{-1})$]
10. Estimate $H_n^5/\ln(n+5)$ with abs. error $O(n^{-2})$. [$2+\frac{\gamma}{\ln n}-\frac{6}{n\ln n}-\frac{3\gamma}{n\ln^2 n}+O(n^{-2})$]
11. Estimate $\binom{2n}{n}$ with relative error $O(n^{-2})$. [A1] [$\frac{2^{2n}}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+O(n^{-2})\right)$]
12. Estimate $\binom{2n+1}{n}$ with relative error $O(n^{-2})$. [A2] [$\frac{2^{2n+1}}{\sqrt{\pi n}}\left(1-\frac{1}{5n}+O(n^{-2})\right)$]
13. Compare $(n!)!$ with $((n-1)!)!\cdot(n-1)!^{n!}$. [K9.2c] (Homework if not enough time is left.)

5 prednaska

- Warm-ups

1. Let $f(n)=\sum_{k=1}^n\sqrt{k}$. Show that $f(n)=\Theta(n^{3/2})$. Find $g(n)$ such that $f(n)=g(n)+O(\sqrt{n})$. [$\int_0^n\sqrt{x}\,\mathrm{d}x\le S_n\le\int_1^{n+1}\sqrt{x}\,\mathrm{d}x$; $g(n)=\frac{2}{3}n\sqrt{n}$]
2. Estimate $(n-2)!/(n-1)$ with abs. error $O(n^{-2})$. [TODO consider $\frac{n!}{n(n-1)^2}$]
3. For a constant integer k, estimate $n^{\underline{k}}/n^k$ with abs. error $O(n^{-3})$. [A5]

$$\left[\ 1-\binom{k}{2}\frac{1}{n}+\frac{3k^4-10k^3+9k^2-2k}{24}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\ \right]$$

- Find a good estimate of $P_n=\frac{(2n-1)!!}{n!}$.
  - obviously $1.5^{n-1}\le\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{5}{3}\cdot\dots\cdot\frac{(2n-1)}{n}\le 2^{n-1}$
  - we split the product into a "small" part (first k terms, each at least $3/2$ except the first one) and a "large" part (remaining $n-k$ terms); then $P_n\ge\left(\frac{2k+1}{k+1}\right)^{n-k}\cdot 1.5^{k-1}=Q_n\cdot 1.5^{k-1}$; we estimate Q_n
  - if we try $k=\alpha n$, then

$$Q_n=2^{n-\alpha n}\exp\left((n-\alpha n)\ln\left(1-\frac{1}{2(\alpha n+1)}\right)\right)=2^{n(1-\alpha)}e^{\frac{\alpha-1}{2\alpha}}(1+O(n^{-1})),$$

so $P_n\ge(2^{1-\alpha}\cdot 1.5^\alpha)^n\Theta(1)$

– if we try $k = \ln n$, then

$$Q_n = \exp\left((n - \ln n)\left[\ln 2 + \ln\left(1 - \frac{1}{2(1+\ln n)}\right)\right]\right);$$

if we expand ln into Taylor series, the error will be $1/\ln^k n = \omega(n^{-1})$, so we can get relative error $O(1)$ at best;
anyway, if we carry it through, we get $P_n = \Omega(2^n n^{-c} e^{-0.5n/\ln n})$

– if we try $k = \sqrt{n}$, then

$$\begin{aligned} Q_n &= \exp\left((n - \sqrt{n})\left[\ln 2 + \ln\left(1 - \frac{1}{2(1+\sqrt{n})}\right)\right]\right) \\ &= 2^{n-\sqrt{n}} \exp\left((n - \sqrt{n})\left[\frac{-1}{2\sqrt{n}} + \frac{3}{8n} - \frac{7}{24n^{3/2}} + O(n^{-2})\right]\right) \\ &= 2^{n-\sqrt{n}} \exp\left(-\frac{\sqrt{n}}{2} + \frac{7}{8} - \frac{2}{3\sqrt{n}} + O(n^{-1})\right), \end{aligned}$$

thus $P_n \geq 2^n \cdot 0.75^{\sqrt{n}} \cdot e^{\frac{-\sqrt{n}}{2} + \frac{7}{8} - \frac{2}{3\sqrt{n}}}(1 + O(n^{-1})) = \Omega\left(2^n c^{\sqrt{n}}\right)$ for $c \in (0, 1)$.

– TODO compare with previous estimate from $k = \ln n$; which is better?

– another approach: $P_n = \frac{(2n)!}{n! 2^n n!} = \binom{2n}{n}/2^n = \frac{2^n}{\sqrt{\pi n}}(1 + O(n^{-1}))$

6 prednaska

- Estimate $S_n = \sum_{k=1}^{n} \frac{1}{n^2+k}$ with absolute error (a) $O(n^{-3})$, (b) $O(n^{-7})$. [Knuth 458/Problem 4] First approach: $\frac{1}{n^2+k} = \frac{1}{n^2(1+k/n^2)}$ etc.; second approach: $S_n = H_{n^2+n} - H_n$. (DU)

- Sums — gross bound on the tail: $S_n = \sum_{0\le k\le n} k! = n!\left(1 + \frac{1}{n} + \frac{1}{n(n-1)} + \dots\right)$, all the terms except the first two are at most $1/n(n+1)$, so $S_n = n!(1 + \frac{1}{n} + n\frac{1}{n(n-1)}) = n!(1+O(n^{-1}))$

- Sums — make the tail infinite:

$$\begin{aligned} n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} &= n!\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} - \sum_{k\ge n+1}\frac{(-1)^k}{k!}\right) \\ &= n!\left(e^{-1} - O\left(\frac{1}{(n+1)!}\right)\right) = \frac{n!}{e} + O(n^{-1}) \end{aligned}$$

- Estimate $S_n = \sum_{k=0}^{n}\binom{3n}{k}$ with relative error $O(n^{-2})$. We split the sum into a "small" and a "large" part at b (which is yet to be determined).

$$\begin{aligned} \sum_{k=0}^{n}\binom{3n}{k} &= \sum_{k=0}^{n}\binom{3n}{n-k} = \sum_{0\le k<b}\binom{3n}{n-k} + \sum_{b\le k\le n}\binom{3n}{n-k}. \\ \binom{3n}{n-k} &= \binom{3n}{n}\frac{n(n-1)\cdot\ldots\cdot 1}{(2n+1)(2n+2)\ldots(2n+k)} = \\ &= \binom{3n}{n}\cdot\frac{n^k}{(2n)^k}\frac{\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)}{\prod_{j=1}^{k}\left(1+\frac{j}{2n}\right)} = \binom{3n}{n}\cdot\frac{1}{2^k}\cdot\left[1 - \frac{3k^2-k}{4n} + O\left(\frac{k^4}{n^2}\right)\right]. \\ \sum_{b\le k\le n}\binom{3n}{n-k} &\le n\cdot\binom{3n}{n-b} = \binom{3n}{n}\cdot\frac{1}{2^b}O(n) = \binom{3n}{n}\cdot O\left(n^{-2}\right) \text{ if } \sqrt{n}\succ b\ge 3\lg n. \\ \sum_{0\le k<3\lg n}\frac{1}{2^k} &= 2 - \frac{1}{2^{3\lg n}} = 2 + O(n^{-3}). \\ -\frac{3}{4n}\sum_{0\le k<3\lg n}\frac{k^2}{2^k} &= \frac{-9}{2n} + O(n^{-3}). \\ +\frac{1}{4n}\sum_{0\le k<3\lg n}\frac{k}{2^k} &= \frac{1}{2n} + O(n^{-3}). \\ O(n^{-2})\cdot\sum_{0\le k<3\lg n}\frac{k^4}{2^k} &= O(n^{-2}) \end{aligned}$$

$$\sum_{k=0}^{n}\binom{3n}{k} = \binom{3n}{n}\cdot\left[2 - \frac{4}{n} + O(n^{-2})\right]$$

- Estimate $S_n = \sum_{k=0}^{n}\binom{4n+1}{k+1}$ with relative error $O(n^{-2})$.

$$\binom{4n+1}{k+1} = \binom{4n}{k+1} + \binom{4n}{k};$$

$$S_n = \sum_{k=0}^{n}\binom{4n+1}{k+1} = \sum_{k=0}^{n}\binom{4n}{k} + \sum_{k=0}^{n}\binom{4n}{k+1} = \sum_{k=0}^{n}\binom{4n}{k} + \sum_{k=1}^{n+1}\binom{4n}{k};$$

$$S_n = 2\sum_{k=0}^{n}\binom{4n}{k} + \binom{4n}{n+1} - \binom{4n}{0}.$$

$$Q_n = \sum_{k=0}^{n}\binom{4n}{k} = \sum_{k=0}^{n}\binom{4n}{n-k};$$

$$\binom{4n}{n-k} = \binom{4n}{n}\cdot\frac{\prod_{j=0}^{k-1}(n-j)}{\prod_{j=1}^{k}(3n+j)} = \binom{4n}{n}\cdot\left(\frac{1}{3}\right)^3\cdot\frac{\prod_{j=0}^{k-1}(1-j/n)}{\prod_{j=1}^{k}(1+j/3n)}$$

$$Q_n = \sum_{0\le k\le 2\log_3 n} \binom{4n}{n-k} + \sum_{2\log_3 n\le k<n} \binom{4n}{n-k}$$

$$\sum_{2\log_3 n\le k<n} \binom{4n}{n-k} = O\left(n\cdot\binom{4n}{n-\lceil 2\log_3 n\rceil}\right) = O\left(\binom{4n}{n}\cdot\frac{1}{n}\right).$$

$$\frac{\prod_{j=0}^{k-1}(1-j/n)}{\prod_{j=1}^{k}(1+j/3n)} = \frac{1-\frac{1}{n}\cdot\sum_{0\le j<k} j + O\left(\frac{k^4}{n^2}\right)}{1+\frac{1}{3n}\cdot\sum_{1<j\le k} j + O\left(\frac{k^4}{n^2}\right)} = 1 + \frac{2k^2+k}{3n} + O\left(\frac{\log^n}{n^2}\right),$$

$$\sum_{0\le k\le 2\log_3 n} \binom{4n}{n-k} = \binom{4n}{n}\cdot\sum_{0\le k\le 2\log_3 n}\left(\frac{1}{3}\right)^k\cdot[1+\frac{2k^2+k}{3n}+O\left(\frac{\log^n}{n^2}\right)] =$$

$$= \frac{3}{2}\cdot\binom{4n}{n}(1+O(n^{-1})).$$

$$\binom{4n}{n+1} = \binom{4n}{n}\cdot\frac{3n}{n+1} = 3\cdot\binom{4n}{n}(1+O(n^{-1}));$$

$$S_n = 6\cdot\binom{4n}{n}(1+O(n^{-1})).$$

- How many bits are needed to represent a binary tree with n internal nodes?
  - we need just the internal vertices to capture the structure; what is the relation between the number of internal vertices and total number of vertices?
  - imagine labeling the vertices by $1, 2, \ldots, n$ in such a way that we get a binary search tree (descendants in the left subtree are smaller, in the right subtree are larger); by summing over possible roots of the tree we get $t_n = \sum_{i=1}^{n} t_{i-1}t_{n-i}$; $t_0 = 1$
  - this is the same as for Catalan numbers, so $t_n = \binom{2n}{n}\frac{1}{n+1}$
  - and so we need $\log_2 t_n \sim 2n - 1.5\lg n - 0.5\lg\pi + O(n^{-1})$ bits