1 Basics of generating functions

Introduction [Wilf 1–3]

- how to define a sequence: exact formula, recurrent relation (Fibonacci), algorithm (the sequence of primes); there are uncomputable sequences (e.g. a list of programs that stop)
- a new way: power series (members of the sequence as coefficients in the series)
- advantages: many advanced tools from analytical theory of functions
- very powerful: works on many sequences where nothing else is known to work
- allows to get asymptotic formulas and statistical properties
- powerful way to prove combinatorial identities
- "Konečne vidím, že je tá matalýza aj na niečo dobrá. Keby mi to bol niekto predtým povedal…"

First-order linear recurrences [Wilf 3–7]

U: $a_{n+1} = 2a_n + 1$ for $n \geq 0$, $a_0 = 0$

- write few members, guess $a_n = 2^n - 1$, provable by induction
- multiply by x^n, sum over all n, assign gf: $\dfrac{A(x)}{x} = 2A(x) + \dfrac{1}{1-x}$
- partial fraction expansion: $A(x) = \dfrac{x}{(1-x)(1-2x)} = \dfrac{1}{1-2x} - \dfrac{1}{1-x}$
- the method stays basically the same for harder problems

U: $a_{n+1} = 2a_n + n$ for $n \geq 0$, $a_0 = 1$

- exact formula not obvious; no unqualified variables in the recurrence
- obstacle: $\sum_{n\geq 0} nx^n = x/(1-x)^2$; solution: differentiation
- concern: is differentiation allowed? discussed later, but in principle yes: in formal power series (as an algebraic ring) or via convergence (if we care about analytical properties)
- $A(x) = \dfrac{1-2x+2x^2}{(1-x)^2(1-2x)} = \dfrac{A}{(1-x)^2} + \dfrac{B}{1-x} + \dfrac{C}{1-2x} = \dfrac{-1}{(1-x)^2} + \dfrac{2}{1-2x}$
- $1/(1-x)^2$ is just $x/(1-x)^2$ (see above) shifted by 1
- $a_n = 2^{n+1} - n - 1$

The method [Wilf 8]:

1. Make sure variables in the recurrence are qualified (e.g. range for n).
2. Name and define the gf.
3. Multiply by x^n, sum over all n in the range.
4. Express both sides in terms of the gf.
5. Solve the equation for the gf.
6. Calculate coefficients in the gf power series expansion.

Useful notation: $[x^n]f(x)$; e.g.

$$[x^n]e^x = 1/n! \qquad [t^r]\frac{1}{1-3t} = 3^r \qquad [v^m](1+v)^s = \binom{s}{m}$$

U: Solve $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$. $\Big[$ $G(x) = \frac{x}{(1-2x)(1-3x)}$; $a_n = 3^n - 2^n$ $\Big]$

The Fibonacci sequence [Wilf 8–10]

- three-term recurrence: $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, $F_0 = 0$, $F_1 = 1$.
- apply the method ($r_\pm = (1 \pm \sqrt{5})/2$):

$$F(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-xr_+)(1-xr_-)} = \frac{1}{r_+ - r_-}\left(\frac{1}{1-xr_+} - \frac{1}{1-xr_-}\right)$$

- $F_n = \frac{1}{\sqrt{5}}(r_+^n - r_-^n)$
- the second term is <1 and goes to zero, so the first term $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$ gives a good approximation

Exercises

U: Find ogf for the following sequences (always $n \geq 0$) [W1.1]:

(a)	$a_n = n$	$\Big[$ introduce xD; $(x\mathrm{D})\frac{1}{1-x} = \frac{x}{(1-x)^2}$ $\Big]$
(b)	$a_n = \alpha n + \beta$	$\Big[$ $\alpha x/(1-x)^2 + \beta/(1-x)$ $\Big]$
(c)	$a_n = n^2$	$\Big[$ $(x\mathrm{D})^2 1/(1-x) = \frac{1+x}{(1-x)^3}$ $\Big]$
(d)	$a_n = n^3$	$\Big[$ $(x\mathrm{D})^3 1/(1-x)$ $\Big]$
(e)	$a_n = P(n)$; P is a polynomial of degree m	$\Big[$ $P(x\mathrm{D})\frac{1}{1-x}$ $\Big]$
(f)	$a_n = 3^n$	$\Big[$ $1/(1-3x)$ $\Big]$
(g)	$a_n = 5 \cdot 7^n - 3 \cdot 4^n$	$\Big[$ $\frac{5}{(1-7x)} - \frac{3}{1-4x}$ $\Big]$
(h)	$a_n = (-1)^n$	$\Big[$ $1/(1+x)$ $\Big]$

U: Find the following coefficients [W1.5]:

(a)	$[x^n]\, e^{2x}$	$\Big[$ $2^n/n!$ $\Big]$
(b)	$[x^n/n!]\, e^{\alpha x}$	$\Big[$ α^n $\Big]$
(c)	$[x^n/n!]\, \sin x$	$\Big[$ $(-1)^m$ if $n = 2m+1$ is odd, 0 otherwise $\Big]$
(d)	$[x^n]\, 1/(1-ax)(1-bx)$ $(a \neq b)$	$\Big[$ $(a^{n+1} - b^{n+1})/(a-b)$ $\Big]$
(e)	$[x^n]\, (1+x^2)^m$	$\Big[$ $[2 \mid n]\binom{m}{n/2}$ $\Big]$

U: Find the sequence with gf $1/(1-x)^2$. $\Big[$ Differentiate $1/(1-x)$ and divide by x, which corresponds to an index shift by 1. $\Big]$

U: Compute $\square_n = \sum_{k=1}^n k^2$.

- assign ogf to the sequence $1^2, 2^2, \ldots, n^2$: $f(x) = \sum_{k=1}^n k^2 x^k$
- $(x\mathrm{D})^2[(x^{n+1}-1)/(x-1)] = x\frac{-2n^2x^{n+1}+n^2x^{n+2}+n^2x^n-2nx^{n+1}+x^{n+1}+2nx^n+x^n-x-1)}{(x-1)^3}$
- note that $\square_n = f(1) = \lim_{x\to 1}(xD)^2[(x^{n+1}-1)/(x-1)] = n(n+1)(2n+1)/6$

DU: Find a linear recurrence of second order (going back two sequence members) that has a solution that contains $n \cdot 3^n$ (possibly plus some linear combination of other exponential or polynomial factors).

DU: Find explicit formulas for the following sequences [W1.6, R2, R3, R7]:

(a) $a_{n+1} = 3a_n + 2$ for $n \geq 0$; $a_0 = 0$ $\quad [\ 3x/(1-x)(1-3x);\quad 3^n - 1\ ]$

(b) $a_{n+1} = \alpha a_n + \beta$ for $n \geq 0$; $a_0 = 0$ $\quad [\ \beta x/(1-x)(1-\alpha x);\quad \frac{\alpha^n-1}{\alpha-1}\beta\ ]$

(c) $a_{n+1} = a_n/3 + 1$ for $n \geq 0$; $a_0 = 1$ $\quad [\ \frac{3/2}{1-x} - \frac{1/2}{1-x/3};\quad \frac{3^{n+1}-1}{2\cdot 3^n}\ ]$

(d) $a_{n+2} = 2a_{n+1} - a_n$ for $n \geq 0$, $a_0 = 0$, $a_1 = 1$ $\quad [\ x/(1-x)^2;\quad n\ ]$

(e) $a_{n+2} = 3a_{n+1} - 2a_n + 3$ for $n \geq 0$; $a_0 = 1$, $a_1 = 2$ $\quad [\ \frac{4}{1-2x} - \frac{3}{(1-x)^2};\quad 2^{n+2} - 3n - 3\ ]$

(f) $a_n = 2a_{n-1} - a_{n-2} + (-1)^n$ for $n > 1$; $a_0 = a_1 = 1$ $\quad [\ \frac{1/2}{(1-x)^2} - \frac{1/4}{1-x} + \frac{1/4}{1+x};\ \frac{2n+3+(-1)^n}{4}\ ]$

(g) $a_n = 2a_{n-1} - n\cdot(-1)^n$ for $n \geq 1$; $a_0 = 0$ $\quad [\ \frac{x/9-2/9}{(1+x)^2} + \frac{2/9}{1-2x};\ \frac{2^{n+1}-(3n+2)(-1)^n}{9}\ ]$

(h) $a_n = 3a_{n-1} + \binom{n}{2}$ for $n \geq 1$; $a_0 = 2$ $\quad [\ \frac{1}{8}(19\cdot 3^n - 2n(n+2) - 3)\ ]$

(i) $a_n = 2a_{n-1} - a_{n-2} - 2$ for $n \geq 2$; $a_0 = a_{10} = 0$ $\quad [\ n(a_1 + 1 - n)$, so with a_{10}, $a_n = n(10-n)\ ]$

(j) $a_n = 4(a_{n-1} - a_{n-2}) + (-1)^n$ for $n \geq 2$; $a_0 = 1$, $a_1 = 4$ $\quad [\ \frac{1+x+x^2}{(1+x)(1-2x)^2};\ \frac{(-1)^n}{9} - \frac{5}{18}\cdot 2^n + \frac{7}{6}(n+1)2^n\ ]$

(k) $a_n = -3a_{n-1} + a_{n-2} + 3a_{n-3}$ for $n \geq 3$; $a_0 = 20$, $a_1 = -36$, $a_2 = 60$ $\quad [\ 5(-3)^n + 18(-1)^n - 3\ ]$

2 Ordinary generating functions

U: From the homework: solve $a_n = 2a_{n-1} - a_{n-2} - 2$ for $n \geq 1$; $a_0 = a_{10} = 0$.
Applying the standard method, while keeping a_1 as a parameter, we get

$$A(x) = \frac{a_1 x - a_1 x^2 - 2x^2}{(1-x)^3} = \frac{a_1 x}{(1-x)^2} + \frac{x(1-x)}{(1-x)^3} - \frac{x^2 + x}{(1-x)^3},$$

so $a_n = (a_1+1)n - n^2$. From $a_{10} = 0$ we get $a_1 = 9$, thus $a_n = n(10-n)$.

Another way for boundary problems (Wilf 10–11)

U: Find (u_n) if $au_{n+1} + bu_n + cu_{n-1} = d_n$ for $1 \leq n \leq N-1$; $u_0 = u_N = 0$.

- Motivated by splines (cubic curves used e.g. to model font shapes).
- similar to Fibonacci with two given non-consecutive terms (but more general)
- define $U(x) = \sum_{j=0}^{N} u_j x^j$ (unknown); $D(x) = \sum_{j=1}^{N-1} d_j x^j$ (known)
- derive $a \cdot \dfrac{U(x) - u_1 x}{x} + bU(x) + cx(U(x) - u_{N-1}x^{N-1}) = D(x)$
- $(a + bx + cx^2)U(x) = xD(x) + au_1 x + cu_{N-1}x^N$ (*)
- plug in suitable values of x (roots r_+ and r_- of the quadratic polynomial on the LHS)
- solve the system of two linear equations and two uknowns u_1, u_{N-1}
- if the roots are equal, differentiate (*) to obtain the second equation

GFs of two variables

U: Find a formula for $\displaystyle\sum_{n\geq 0}\binom{n}{k}x^n$.

Introducing a new variable and changing the order of summation can help

$$\begin{aligned}\sum_{n\geq 0}\binom{n}{k}x^n &= [y^k]\sum_{m\geq 0}\left(\sum_{n\geq 0}\binom{n}{m}x^n\right)y^m = [y^k]\sum_{n\geq 0}(1+y)^n x^n \\ &= [y^k]\frac{1}{1-x(1+y)} = \frac{1}{1-x}[y^k]\frac{1}{1-\frac{x}{1-x}y} = \frac{x^k}{(1-x)^{k+1}}. \qquad (1)\end{aligned}$$

Alternatively, one can use binomial theorem (Knuth 199/5.56 and 5.57):

$$\begin{aligned}\frac{1}{(1-x)^{n+1}} &= (1-x)^{-n-1} = \sum_{k\geq 0}\binom{-n-1}{k}(-x)^k \\ &= \sum_{k\geq 0}\frac{(-n-1)(-n-2)\dots(-n-k)}{k!}(-x)^k = \sum_{k\geq 0}\binom{n+k}{n}x^k.\end{aligned}$$

U: Given $f(x) \overset{\text{ogf}}{\longleftrightarrow} (a_n)_{n\geq 0}$, express ogf for the following sequences in terms of f [W1.3]:

(a) $(a_n + c)_{n\geq 0}$ $\quad$ [$f(x) + c/(1-x)$]

(b) $(na_n)_{n\geq 0}$ $\quad$ [$x\mathrm{D}f(x)$]; $\quad$ napísať im $(P(n)a_n)_{n\geq 0} \longleftrightarrow P(x\mathrm{D})f(x)$

(c) $0, a_1, a_2, a_3, \dots$ $\quad$ [$f(x) - a_0$]

(d) $0, 0, 1, a_3, a_4, a_5, \dots$ $\quad$ [$f(x) - a_0 - a_1x + (1-a_2)x^2$]

(e) $(a_{n+2} + 3a_{n+1} + a_n)_{n\geq 0}$ $\quad$ [$(f - a_0 - a_1x)/x^2 + 3(f - a_0)/x + f$]

(f) $a_0, 0, a_2, 0, a_4, 0, a_6, 0\dots$ $\quad$ [$(f(x) + f(-x))/2$]

(g) $a_0, 0, a_1, 0, a_2, 0, a_3, 0\dots$ $\quad$ [$f(x^2)$]

(h) $a_1, a_2, a_3, a_4, \dots$ $\quad$ [$(f(x) - a_0)/x$]

(i) $a_0, a_2, a_4, \dots$ $\quad$ [$(f(\sqrt{x}) + f(-\sqrt{x}))/2$]

Formal power series [Wilf chapter 2]

- a ring with addition and multiplication $\sum_n a_nx^n \sum_n b_nx^n = \sum_n \sum_k (a_kb_{n-k})x^n$
- if $f(0) \neq 0$, then f has a unique reciprocal $1/f$ such that $f \cdot 1/f = 1$
- composition $f(g(x))$ defined iff $g(0) = 0$ or f is a polynomial (cf. e^{e^x-1} vs. e^{e^x})
- formal derivative D: $\mathrm{D}\sum_n a_nx^n = \sum na_nx^{n-1}$; usual rules for sum, product etc.
- **U:** Find all f such that $\mathrm{D}f = f$.

Rules for manipulation [Wilf 2.1, Knuth 334]. Assume that $f \overset{\text{ogf}}{\longleftrightarrow} (a_n)_{n=0}^{\infty}$.

- **Rule 1**: for a positive integer h, $(a_{n+h}) \overset{\text{ogf}}{\longleftrightarrow} (f - a_0 - \dots - a_{h-1}x^{h-1})/x^h$
- **Rule 2**: if P is a polynomial, then $P(x\mathrm{D})f \overset{\text{ogf}}{\longleftrightarrow} (P(n)a_n)_{n\geq 0}$
  - example: $(n+1)a_{n+1} = 3a_n + 1$ for $n \geq 0$, $a_0 = 1$; thus $f' = 3f + 1/(1-x)$
  - example: $\sum_{n\geq 0}\frac{n^2+4n+5}{n!}$; thus $f = \sum_{n\geq 0}(n^2+4n+5)\frac{x^n}{n!} = ((x\mathrm{D})^2+4x\mathrm{D}+5)e^x = (x^2+5x+5)e^x$
    we need $f(1) = 11e$; works because the resulting f is analytic in a disk
    containing 1 in the complex plane (that is, it converges to its Taylor series)
- **Rule 3**: if $g \overset{\text{ogf}}{\longleftrightarrow} (b_n)$, then $fg \overset{\text{ogf}}{\longleftrightarrow} (\sum_{k=0}^{n} a_kb_{n-k})_{n\geq 0}$

$$\sum_{k=0}^{n}(-1)^k k = (-1)^n\sum_{k=0}^{n} k\cdot(-1)^{n-k} = (-1)^n[x^n]\frac{x}{(1-x)^2}\cdot\frac{1}{1+x} = \frac{(-1)^n}{4}(2n+1-(-1)^n)$$

- **Rule 4**: for a positive integer k, we have $f^k \overset{\text{ogf}}{\longleftrightarrow} \left(\sum_{n_1+n_2+\dots+n_k=n} a_{n_1}a_{n_2}\dots a_{n_k}\right)_{n\geq 0}$
  - example: let $p(n,k)$ be the number of ways n can be written as an ordered sum of k nonnegative integers
  - according to R4, $(p(n,k))_{n\geq 0} \overset{\text{ogf}}{\longleftrightarrow} 1/(1-x)^k$, so $p(n,k) = \binom{n+k-1}{n}$ thanks to (1)

- **Rule 5**: $\dfrac{f}{(1-x)} \overset{\text{ogf}}{\longleftrightarrow} \left(\displaystyle\sum_{k=0}^{n} a_k\right)_{n\geq 0}$
  - example: $(\Box_n)_{n\geq 0} \overset{\text{ogf}}{\longleftrightarrow} \dfrac{1}{1-x}\cdot(x\mathrm{D})^2\dfrac{1}{1-x} = \dfrac{x(1+x)}{(1-x)^4}$, so by (1), $\Box_n = \binom{n+2}{3}+\binom{n+1}{3}$

Exercises

1. Using Rule 5, prove that $F_0+F_1+\cdots+F_n = F_{n+2}-1$ for $n\geq 0$ [Wilf 38, example 6].
   [Compare gfs of both sides, left is $f/(1-x)$, where $f = x/(1-x-x^2)$, i.e. Fibonacci.]
2. Solve $g_n = g_{n-1}+g_{n-2}$ for $n\geq 2$, $g_0 = 0$, $g_{10} = 10$.
   [$g_n = \frac{g_{10}}{F_{10}}F_n$, try the "boundary method" described above, computer necessary]
3. Solve $a_n = \sum_{k=0}^{n-1} a_k$ for $n > 0$; $a_0 = 1$. [R16]
   [$a_n = 2^{n-1}$ for $n\geq 1$]
4. Solve $f_n = 2f_{n-1}+f_{n-2}+f_{n-3}+\cdots+f_1+1$ for $n\geq 1$, $f_0 = 0$ [Knuth 349/(7.41)]
   [$F(x) = x/(1-3x+x^2)$; $f_n = F_{2n}$]
5. Solve $g_n = g_{n-1}+2g_{n-2}+\cdots+ng_0$ for $n > 0$, $g_0 = 1$. [K7.7]
   [$G(x) = 1+x/(1-3x+x^2)$; $g_n = F_{2n}+[n=0]$]
6. Solve $g_n = \sum_{k=1}^{n-1}\frac{g_k+g_{n-k}+k}{2}$ for $n\geq 2$, $g_1 = 1$.
7. Solve $g_n = g_{n-1}+2g_{n-2}+(-1)^n$ for $n\geq 2$, $g_0 = g_1 = 1$. [Knuth 341, example 2]
   [$G(x) = \frac{1+x+x^2}{(1-2x)(1+x)^2}$; $g_n = \frac{7}{9}2^n+\frac{1}{9}(3n+2)(-1)^n$]
8. Solve $a_{n+2} = 3a_{n+1}-2a_n+n+1$ for $n\geq 0$; $a_0 = a_1 = 1$. [R24]
   [$A(z) = \frac{2}{1-2z}-\frac{1}{(1-z)^3}$; $a_n = 2^{n+1}-\binom{n+2}{2}$]
9. Prove that $\ln\dfrac{1}{1-x} = \displaystyle\sum_{n\geq 1}\frac{1}{n}x^n$. [consider $\int\frac{1}{1-x}$]

3 Skipping sequence elements, Catalan numbers

Every third binomial coefficient [Wilf 51, example 4]

- Why $\frac{1}{2}(A(x)+A(-x)) \overset{\text{ogf}}{\longleftrightarrow} a_0, 0, a_2, 0, a_4, \ldots$ works: $\frac{1}{2}(1^n+(-1)^n) = [2\mid n]$.
- We know that if $(a_n) \overset{\text{ogf}}{\longleftrightarrow} A(x)$, then $(a_{2n})_{i=0}^{\infty} \overset{\text{ogf}}{\longleftrightarrow} (A(\sqrt{x})+A(-\sqrt{x}))/2$.
- Let's generalize: $(a_{rk})_{k=0}^{\infty} \overset{\text{ogf}}{\longleftrightarrow} A_r(x) = \frac{1}{r}\sum_{j=0}^{r-1} A\left(\omega^j\sqrt[r]{x}\right)$ where ω is the primitive r-th root of unity.
- Key step in the proof: $\frac{1}{r}\sum_{j=0}^{r-1}(\omega^j)^n = \frac{1}{r}\sum_{j=0}^{r-1} e^{2\pi ijn/r} = [r\mid n]$
  — just a geometric progression, or a consequence of $t^r-1 = (t-1)(t^{r-1}+\cdots+t+1)$;

$$\begin{aligned}
A_r(x) = \frac{1}{r}\sum_{j=0}^{r-1} A\left(\omega^j\sqrt[r]{x}\right) &= \frac{1}{r}\sum_{j=0}^{r-1}\sum_{n=0}^{\infty} a_n\left(\omega^j\sqrt[r]{x}\right)^n \\
&= \frac{1}{r}\sum_{n=0}^{\infty} a_n x^{n/r}\sum_{j=0}^{r-1}\omega^{jn} \\
&= \sum_{n=0}^{\infty} a_n x^{n/r}[r\mid n] = \sum_{k=0}^{\infty} a_{rk}x^k
\end{aligned}$$

- problem: find $S_n = \sum_k (-1)^k \binom{n}{3k}$
- if we knew $f(x) = \sum_k \binom{n}{3k} x^{3k}$, we would have $S_n = f(-1)$
- for $A(x) = (1+x)^n$, we have $f(x) = \frac{1}{3}\big(A(x) + A(x\omega^1) + A(x\omega^2)\big)$ for $\omega = e^{2\pi i/3}$
- and so $S_n = f(-1) = \frac{1}{3}[(1-\omega)^n + (1-\omega^2)^n)] =$

$$= \frac{1}{3}\left[\left(\frac{3-\sqrt{3}i}{2}\right)^n + \left(\frac{3+\sqrt{3}i}{2}\right)^n\right] = 2\cdot 3^{\frac{n}{2}-1}\cos\left(\frac{\pi n}{6}\right)$$

Mutually recursive sequences [Knuth 343, Example 3]

- consider the number u_n of tilings of $3\times n$ board with 2×1 dominoes
- define v_n as the number of tilings of $3\times n$ board without a corner
- $u_n = 2v_{n-1} + u_{n-2}$; $u_0 = 1$; $u_1 = 0$
- $v_n = v_{n-2} + u_{n-1}$; $v_0 = 0$; $v_1 = 1$
- derive

$$U(x) = \frac{1-x^2}{1-4x^2+x^4}, \qquad V(x) = \frac{x}{1-4x^2+x^4}$$

- consider $W(z) = 1/(1-4z+z^2)$; $U(x) = (1-x^2)W(x^2)$, so $u_{2n} = w_n - w_{n-1}$
- hence $u_{2n} = \frac{(2+\sqrt{3})^n}{3-\sqrt{3}} + \frac{(2-\sqrt{3})^n}{3+\sqrt{3}} = \left\lceil \frac{(2+\sqrt{3})^n}{3-\sqrt{3}} \right\rceil$ (derivation as a homework)

Discovering combinatorial identities via gfs [Knuth 198, Vandermonde and 5.55]

- $(1+x)^r = \sum_{k\ge 0}\binom{r}{k}x^k$; consider $(1+x)^r(1+x)^s = (1+x)^{r+s}$
- comparison of coefficients yields $\sum_{k\ge 0}^n \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$ — Vandermonde
- by considering $(1-x)^r(1+x)^r = (1-x^2)^r$, we obtain

$$\sum_{k=0}^n \binom{r}{k}\binom{r}{n-k}(-1)^k = (-1)^{n/2}\binom{r}{n/2}[2\mid n]$$

Catalan numbers [Knuth 357, example 4]

- consider the number of possibilities c_n of how to specify the multiplication order of $A_0A_1\ldots A_n$ by parentheses; let $C(x) = \sum_{n\ge 0} c_n x^n$
- divide possibilities by the place of last multiplication; $c_n = \sum\limits_{k=0}^{n-1} c_k c_{n-1-k}$ for $n > 0$; $c_0 = 1$
- many ways to deal with the recurrence:
  (1) shift the recurrence to $c_{n+1} = \sum_{k=0}^n c_k c_{n-k}$ and use Rules 1 and 3; $\frac{C(x)-1}{x} = C(x)^2$
  (2) RHS as a convolution of c_n with c_{n-1}, i.e. $C(x)\cdot xC(x)$
  (3) RHS as a convolution of c_n with c_n shifted by Rule 1, i.e. $x\cdot C(x)^2$
  (4) rewriting through sums and changing the order of summation:

$$\sum_{n\ge 1} x^n \sum_{k=0}^{n-1} c_k c_{n-1-k} = \sum_{k=0}^{\infty} x^k c_k \sum_{n\ge k+1} c_{n-1-k}x^{n-k} = \sum_{k=0}^{\infty} x^k c_k x C(x) = xC(x)\cdot C(x)$$

- consequently, $C(x) - 1 = xC(x)^2$ and thus $C(x) = \frac{1\pm\sqrt{1-4x}}{2x} = \frac{1}{2x}\left(1-\sqrt{1-4x}\right)$

- we want C continuous and $C(0)=1$, so we choose the minus sign (note that the resulting function below is analytical since $\binom{2n}{n}/(n+1) < 2^{2n}$; it would be analytical also if we chose the plus sign)
- binomial theorem yields

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{k\geq 0}\binom{1/2}{k}(-4x)^k \;=\; 1+\sum_{k\geq 1}\frac{1}{2k\cdot(-4)^{k-1}}\binom{2k-2}{k-1}(-4)^k x^k$$
$$= 1-\sum_{k\geq 1}\frac{2}{k}\binom{2k-2}{k-1}x^k$$

- we used $\binom{1/2}{k} = \frac{1/2}{k}\binom{-1/2}{k-1} = \frac{1}{2k(-4)^{k-1}}\binom{2k-2}{k-1}$ because $\binom{-1/2}{m} = \frac{1}{(-4)^m}\binom{2m}{m}$
- therefore,

$$C(x) = \frac{1}{2x}\sum_{k\geq 1}\frac{2}{k}\binom{2k-2}{k-1}x^k = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n$$

Exercises

1. Assume that $A(x) \stackrel{\text{ogf}}{\longleftrightarrow} (a_n)$. Express the generating function for $\sum_{n\geq 0} a_{3n}x^n$ in terms of $A(x)$.
   [$\frac{1}{3}(A(x^{1/3}) + A(\omega x^{1/3})) + A(\omega^2 x^{1/3})$, where $\omega = e^{2\pi i/3}$]
2. Compute $S_n = \sum_{n\geq 0} F_{3n}\cdot 10^{-n}$ (by plugging a suitable value into the generating function for F_{3n}).
   [The gf is $\frac{2x}{1-4x-x^2}$ and $S_n = 20/59$.]
3. Compute $\sum_k \binom{n}{4k}$. [$2^{\frac{n}{2}-2}\left(2^{\frac{n}{2}} + \cos\left(\frac{1}{4}n\pi\right) + (-1)^n\cos\left(\frac{3}{4}n\pi\right)\right)$]
4. Compute $\sum_k \binom{6m}{3k+1}$. [Compute it for general n and then plug in $n=6m$; $(2^{6m}-1)/3$]
5. Evaluate $S_n = \sum_{k=0}^{n}(-1)^k k^2$. [$f(x) = \frac{-x}{(1+x)^3}$; $S_n = \frac{1}{2}(-1)^n n(n+1)$]
6. Find ogf for $H_n = 1 + 1/2 + 1/3 + \dots$. [$-\ln(1-x)/(1-x)$]
7. Find the number of ways of cutting a convex n-gon with labelled vertices into triangles.
   [C_{n-2} (shifted Catalan numbers)]

4 Snake Oil

The Snake Oil method [Wilf 118, chapter 4.3] – external method vs. internal manipulations within a sum.

1. identify the free variable and give the name to the sum, e.g. s_n
2. let $S(x) = \sum s_n x^n$
3. interchange the order of summation; solve the inner sum in closed form
4. find coefficients of $S(x)$

U: Evaluate $s_n = \sum_k \binom{n}{k}$.

After Step 2, $S(x) = \sum_{n\geq 0} x^n \sum_k \binom{n}{k}$.

$$S(x) = \sum_k\sum_n \binom{n}{k}x^n = \sum_{k\geq 0}\frac{x^k}{(1-x)^{k+1}} = \frac{1}{1-x}\cdot\frac{1}{1-\frac{x}{1-x}} = \frac{1}{1-2x}$$

U: Evaluate $s_n = \sum_{k\geq 0}\binom{k}{n-k}$ [Wilf 121].

After Step 2, $S(x) = \sum_n x^n \sum_{k\geq 0} \binom{k}{n-k}$

$$S(x) = \sum_{k\geq 0}\sum_n \binom{k}{n-k} x^n = \sum_{k\geq 0} x^k \sum_n \binom{k}{n-k} x^{n-k} = \sum_{k\geq 0} x^k (1+x)^k = \frac{1}{1-x-x^2}$$

thus $s_n = F_{n+1}$.

U: Evaluate $s_n = \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$, where $m, n \in \mathbb{Z}_0^+$ [Wilf 122].

$$\begin{aligned}
S(x) &= \sum_{n\geq 0} x^n \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \sum_{n\geq 0}\binom{n+k}{m+2k} x^{n+k} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+2k}}{(1-x)^{m+2k+1}} \\
&= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k}\frac{1}{k+1}\left(\frac{-x}{(1-x)^2}\right)^k \\
&= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left(1-\sqrt{1+\frac{4x}{(1-x)^2}}\right) = \frac{x^m}{(1-x)^m},
\end{aligned}$$

thus $s_n = \binom{n-1}{m-1}$.

U: Prove that $\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k} 2^k$, where $m, n \in \mathbb{Z}_0^+$. [Wilf 127] The ogf of the left-hand side is

$$L(x) = \sum_k \binom{m}{k} x^{-k} \sum_{n\geq 0}\binom{n+k}{m} x^{n+k} = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

We get the same for the right-hand side

Exercises

1. Prove that $\sum_k k\binom{n}{k} = n2^{n-1}$ via the snake oil method. [$L(x) = P(x) = \frac{x}{(1-2x)^2}$]
2. Evaluate $f(n) = \sum_k k^2 \binom{n}{k} 3^k$.

   [$F(x) = \frac{3x(1+2x)}{(1-4x)^3} = \frac{3/8}{1-4x} - \frac{3/2}{(1-4x)^2} + \frac{9/8}{(1-4x)^3}$; $f(n) = 3\cdot 4^{n-2}\cdot n(1+3n)$]
3. Find a closed form for $\sum_{k\geq 0}\binom{k}{n-k} t^k$. [W4.11(a)]

   [$F(x) = 1/(1-tx-tx^2)$]
4. Evaluate $f(n) = \sum_k \binom{n+k}{2k} 2^{n-k}$, $n \geq 0$. [Wilf 125, Example 4]

   [$F(x) = \frac{1-2x}{(1-x)(1-4x)} = \frac{2}{3(1-4x)} + \frac{1}{3(1-x)}$; $f(n) = (2^{2n+1}+1)/3$]
5. Evaluate $f(n) = \sum_{k\leq n/2} (-1)^k \binom{n-k}{k} y^{n-2k}$. [Wilf 122, Example 3]

   [$F(x) = 1/(1-xy+x^2)$]
6. Evaluate $f(n) = \sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k}$. [W4.11(c)]

   [replace $2n+1$ by m and solve for $f(m) = \binom{m-p-1}{p} 2^{m-2p-1}$; $f(2n+1) = \binom{2n-p}{p} 4^{n-p}$;

   $$F(x) = \frac{x}{(1-x)^2}\sum_{k\geq 0}\binom{p+k}{p}\left(\frac{x}{1-x}\right)^{2(p+k)} = \frac{x^{p+1}}{2^p}\cdot\frac{(2x)^p}{(1-2x)^{p+1}}$$]

7. Try to prove that $\sum_k \binom{n}{k}\binom{2n}{n+k} = \binom{3n}{n}$ via the snake oil method in three different ways: consider the sum

$$\sum_k \binom{n}{k}\binom{m}{r-k}$$

and the free variable being one of n, m, r.