

★ GENERATING FUNCTIONS

♣ Definitions, notation

$$\{a_n\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} a_n x^n$$

$$D := \lambda f. f' \quad (D(f)(x) = f'(x))$$

$$xD := \lambda f x. xD(f)(x) \quad (xD(f)(x) = x f'(x))$$

$$[x^n]F(x) := \text{coef. of } x^n \text{ in T. expansion of } F(x)$$

$$\left[\frac{x^n}{\alpha} \right] F(x) := \alpha [x^n] F(x)$$

$$(P(n) \text{ is (finite) polynomial})$$

♣ Basic facts

$$\{1\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

$$\{r^n\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} r^n x^n = \frac{1}{1-rx}$$

$$\{P(n)\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} P(n)x^n = P(xD) \left(\frac{1}{1-x} \right)$$

$$\{1, 2, 3, 4, 5, \dots\} \xleftrightarrow{ogf} \sum_{n \geq 0} (n+1)x^n = \frac{1}{(1-x)^2}$$

$$\left\{ \binom{n}{q} \right\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} \binom{n}{q} x^n = \frac{x^q}{(1-x)^{q+1}}$$

$$\left\{ \binom{n+m-1}{n} \right\}_{n \geq 0} \xleftrightarrow{ogf} \frac{1}{(1-x)^m}$$

$$\left\{ \binom{c}{n} \right\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} \binom{c}{n} x^n = (1+x)^c$$

$$\left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \xleftrightarrow{ogf} \sum_{n \geq 0} \frac{1}{n} x^n = \ln \frac{1}{1-x}$$

$$\left\{ 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots \right\} \xleftrightarrow{ogf} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} x^n = \ln(1+x)$$

$$\left\{ \frac{1}{n+1} \binom{2n}{n} \right\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\{F_n\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{(x-\phi)(x-\psi)}$$

$$\sqrt[q]{1_k} = \sqrt[q]{\exp 2\pi i_k} = \exp \frac{2\pi k i}{q}$$

$$[q | n] = \frac{1}{q} \sum_{k=0}^{q-1} \left(\sqrt[q]{1_k} \right)^n = \frac{1}{q} \sum_{k=0}^{q-1} \left(\exp \frac{2\pi n i}{q} \right)^k$$

♣ GF/sequences transformations

$$\text{Let } \{a_n\}_{n \geq 0} \xleftrightarrow{ogf} f(x)$$

$$\{\alpha a_n + \beta\}_{n \geq 0} \xleftrightarrow{ogf} \alpha f(x) + \frac{\beta}{1-x}$$

$$\{P(n)a_n\}_{n \geq 0} \xleftrightarrow{ogf} P(xD)(f)$$

$$\{na_n\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} na_n x^n = x f'(x)$$

$$\{0, a_0, a_1, \dots\} \xleftrightarrow{ogf} x f(x)$$

$$\{a_{n+k}\}_{n \geq 0} \xleftrightarrow{ogf} \frac{f(x) - a_0 - a_1 x - \dots - a_{k-1} x^{k-1}}{x^k}$$

$$\{a_n [q | n]\}_{n \geq 0} \xleftrightarrow{ogf} \sum_{n \geq 0} a_{qn} x^{qn} = \frac{1}{q} \sum_{k=0}^{q-1} f(x \sqrt[q]{1_k})$$

$$\{a_0, 0, a_2, 0, a_4, \dots\} \xleftrightarrow{ogf} \sum_{n \geq 0} a_{2n} x^{2n} = \frac{f(x) + f(-x)}{2}$$

$$\{a_0, 0, a_1, 0, a_2, 0, \dots\} \xleftrightarrow{ogf} f(x^2)$$

$$\left\{ \sum_{k=0}^n a_k b_{n-k} \right\}_{n \geq 0} \xleftrightarrow{ogf} f g \quad (\text{if } \{b_n\}_{n \geq 0} \xleftrightarrow{ogf} g)$$

$$\left\{ \sum_{n_1+n_2+\dots+n_k=n} a_{n_1} a_{n_2} \dots a_{n_k} \right\}_{n \geq 0} \xleftrightarrow{ogf} f^k$$

$$\left\{ \sum_{k=0}^n a_k \right\}_{n \geq 0} \xleftrightarrow{ogf} \frac{f}{1-x}$$

$$D \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} n a_n x^{n-1}$$

$\frac{1}{f}$ exists and is unique, iff $f(0) \neq 0$ ($\Leftrightarrow [x^0]f \neq 0$)

$f(g(x))$ exists, if $g(0) = 0$ or $f(x)$ has finite Taylor expansion.

♣ Method for destroying recurrences

1. prerequisites: no free variables, known conditions for recurrent relations
2. define GF
3. multiply both sides of recurrent relation with x^n , sum for all possible n
4. rewrite both sides as functions of GF
5. solve equation for GF
6. find coefficient for x^n in Taylor expansion of GF

♣ SNAKE OIL method for destroying sums

1. identify free variable in given sum, define given sum as function $f(n)$
2. let $F(x) := \sum_{n \geq 0} f(n)x^n$
3. change order of summation, find closed form of inner sum
4. find $[x^n]F(x)$

★ ASYMPTOTICS

♣ Definitions

$$f(n) \in O(g(n)) \xleftrightarrow{\text{def}} \exists c \forall n \geq n_0 : |f(n)| \leq c|g(n)|$$

$$f(n) \in o(g(n)) \xleftrightarrow{\text{def}} \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) \in \Omega(g(n)) \stackrel{\text{def}}{\iff} g(n) \in O(f(n))$$

$$f(n) \in \omega(g(n)) \stackrel{\text{def}}{\iff} \lim_{n \rightarrow \infty} g(n) \in o(f(n))$$

$$f(n) \in \Theta(g(n)) \stackrel{\text{def}}{\iff} f(n) \in O(g(n)) \wedge g(n) \in O(f(n))$$

$$B(n) := \sum_{k \in T_n} b_k(n)$$

$$C(n) := \sum_{k \in D_n} |c_k(n)|$$

and prove all three are small.

4.

$$\begin{aligned} \sum_{k \in D_n \cup T_n} a_k(n) &= \\ &= \sum_{k \in D_n \cup T_n} b_k(n) + O(A(n)) + O(B(n)) + O(C(n)) \end{aligned}$$

source: https://github.com/japdlsd/kombat1_cheatsheet

♣ Basic approximations

- Taylor polynoms
- **Stirling** $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O(n^{-3})\right)$
- $H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(n^{-4})$
- $\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + O(n^{-2})\right)$

♣ Basic technics

- take away tail of Taylor expansion
- substitution
- if expression is too big to converge, take out bigger part and then apply Taylor expansion technics
- $\frac{1}{1-x} = 1 + O(x) \implies \frac{1}{1+O(n^{-1})} = 1 + O(n^{-1})$
- $f = e^{\ln f}$
- $[x] = x + O(1)$
- given precision limit, you can omit any part of expression with smaller magnitude (e.g. multiplication of two big sums)
- $\sum_{a \leq k < b} f(k) = \int_a^b f(x) dx + R$, where $R \leq \sum_{a \leq k < b} \max_{x \in [k, k+1)} |f(x) - f(k)|$. If f is monotonic, then $R \leq |f(b) - f(a)|$
- **[bootstrapping]** Find rough estimate for recurrence and plug it into recurrence to get better one
- **[dominant/tail]** separate sum into two parts and analyze them separately. Advantage is ability to approximate tail part very loosely.

♣ TAIL SWITCHING method for destroying sums

Given a sum $\sum_{k \in M} a_k(n)$

1. separate sum into two disjoint ranges, *dominant* D_n and *tail* T_n (i.e. $D_n \cup T_n = M$, $D_n \cap T_n = \emptyset$).
2. find asymptotic estimate $a_k(n) = b_k(n) + O(c_k(n))$ for $k \in D_n$
3. Let

$$A(n) := \sum_{k \in T_n} a_k(n)$$