★ GENERATING FUNCTIONS

♣ Definitions, notation

$$\{a_n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} a_n x^n$$
$$D := \lambda f.f' \quad (D(f)(x) = f'(x))$$
$$xD := \lambda fx.xD(f)(x) \quad (xD(f)(x) = xf'(x))$$
$$[x^n]F(x) := \text{coef. of } x^n \text{ in T. expansion of } F(x)$$
$$\left[\frac{x^n}{\alpha}\right] F(x) := \alpha\,[x^n]\,F(x)$$

($P(n)$ is (finite) polynom)

♣ Basic facts

$$\{1\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} x^n = \frac{1}{1-x}$$
$$\{r^n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} r^n x^n = \frac{1}{1-rx}$$
$$\{P(n)\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} P(n)x^n = P(xD)\left(\frac{1}{1-x}\right)$$
$$\{1,2,3,4,5,\ldots\} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0}(n+1)x^n = \frac{1}{(1-x)^2}$$
$$\left\{\binom{n}{q}\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0}\binom{n}{q}x^n = \frac{x^q}{(1-x)^{q+1}}$$
$$\left\{\binom{n-m+1}{n}\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \frac{1}{(1-x)^m}$$
$$\left\{\binom{c}{n}\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0}\binom{c}{n}c^n = (1+x)^c$$
$$\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0}\frac{1}{n}x^n = \ln\frac{1}{1-x}$$
$$\left\{0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots\right\} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}x^n = \ln(1+x)$$
$$\left\{\frac{1}{n+1}\binom{2n}{n}\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n = \frac{1-\sqrt{1-4x}}{2x}$$
$$\{F_n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0}F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{(x-\phi)(x-\psi)}$$
$$\sqrt[q]{1}_k = \sqrt[q]{\exp 2\pi i}_k = \exp\frac{2\pi k i}{q}$$
$$\Big[q \mid n\Big] = \frac{1}{q}\sum_{k=0}^{q-1}\left(\sqrt[q]{1}_k\right)^n = \frac{1}{q}\sum_{k=0}^{q-1}\left(\exp\frac{2\pi n i}{q}\right)^k$$

♣ GF/sequences transformations

Let $\{a_n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} f(x)$

$$\{\alpha a_n + \beta\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \alpha f(x) + \frac{\beta}{1-x}$$
$$\{P(n)a_n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} P(xD)(f)$$
$$\{na_n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} na_n x^n = xf'(x)$$
$$\{0,a_0,a_1,\ldots\} \overset{ogf}{\longleftrightarrow} xf(x)$$
$$\{a_{n+k}\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \frac{f(x)-a_0-a_1x-\ldots-a_{k-1}x^{k-1}}{x^k}$$
$$\left\{a_n\Big[q \mid n\Big]\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} a_{qn}x^{qn} = \frac{1}{q}\sum_{k=0}^{q-1} f\left(x\sqrt[q]{1}_k\right)$$
$$\{a_0,0,a_2,0,a_4,\ldots\} \overset{ogf}{\longleftrightarrow} \sum_{n\geq 0} a_{2n}x^{2n} = \frac{f(x)+f(-x)}{2}$$
$$\{a_0,0,a_1,0,a_2,0,\ldots\} \overset{ogf}{\longleftrightarrow} f(x^2)$$
$$\left\{\sum_{k=0}^{n} a_k b_{n-k}\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} fg \quad (\text{if } \{b_n\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} g)$$
$$\left\{\sum_{n_1+n_2+\ldots+n_k=n} a_{n_1}a_{n_2}\ldots a_{n_k}\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} f^k$$
$$\left\{\sum_{k=0}^{n} a_k\right\}_{n\geq 0} \overset{ogf}{\longleftrightarrow} \frac{f}{1-x}$$
$$D\sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} na_n x^{n-1}$$

$\frac{1}{f}$ exists and is unique, iff $f(0) \neq 0$ ($\Leftrightarrow [x^0]f \neq 0$)

$f(g(x))$ exists, if $g(0) = 0$ or $f(x)$ has finite Taylor expansion.

♣ Method for destroying recurrences

1. prerequisites: no free variables, known conditions for reccurent relations
2. define GF
3. multiply both sides of recurrent relation with x^n, sum for all possible n
4. rewrite both sides as functions of GF
5. solve equation for GF
6. find coefficient for x^n in Taylor expansion of GF

♣ SNAKE OIL method for destroying sums

1. identify free variable in given sum, define given sum as function $f(n)$
2. let $F(x) := \sum_{n\geq 0} f(n)x^n$
3. change order of summation, find closed form of inner sum

4. find $[x^n]F(x)$

★ ASYMPTOTICS

♣ Definitions

$$
\begin{aligned}
f(n) \in O(g(n)) &\overset{\text{def}}{\iff} \exists c\ \forall n \geq n_0 : |f(n)| \leq c|g(n)| \\
f(n) \in o(g(n)) &\overset{\text{def}}{\iff} \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \\
f(n) \in \Omega(g(n)) &\overset{\text{def}}{\iff} g(n) \in O(f(n)) \\
f(n) \in \omega(g(n)) &\overset{\text{def}}{\iff} \lim_{n\to\infty} g(n) \in o(f(n)) \\
f(n) \in \Theta(g(n)) &\overset{\text{def}}{\iff} f(n) \in O(g(n)) \wedge g(n) \in O(f(n))
\end{aligned}
$$

$$
\begin{aligned}
\text{absolute error: } & X + O(n^{-k}) \\
\text{relative error: } & X(1 + O(n^{-k}))
\end{aligned}
$$

♣ Basic approximations

- Taylor polynoms
- **Stirling** $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} - \frac{1}{288n^2} + O(n^{-3})\right)$
- $H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(n^{-4})$
- $\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + O\left(n^{-2}\right)\right)$

♣ Basic technics

- take away tail of Taylor expansion
- substitution
- if expression is too big to converge, take out bigger part and then apply Taylor expansion technics
- $\frac{1}{1-x} = 1 + O(x) \Longrightarrow \frac{1}{1+O(n^{-1})} = 1 + O(n^{-1})$
- $f = e^{\ln f}$
- $[x] = x + O(1)$
- given precision limit, you can omit any part of expression with smaller magnitude (e.g. multiplication of two big sums)
- $\sum_{a\leq k<b} f(k) = \int_a^b f(x)dx + R$, where $R \leq \sum_{a\leq k<b} \max_{x\in[k,k+1)} |f(x) - f(k)|$. If f is monotonic, then $R \leq |f(b) - f(a)|$
- **[bootstrapping]** Find rough estimate for recurrence and plug it into recurrence to get better one
- **[dominant/tail]** separate sum into two parts and analyze them separately. Advantage is ability to approximate tail part very loosely.

♣ TAIL SWITCHING method for destroying sums

Given a sum $\sum_{k\in M} a_k(n)$

1. separate sum into two disjoint ranges, *dominant* D_n and *tail* T_n (i.e. $D_n \cup T_n = M$, $D_n \cap T_n = \varnothing$).
2. find asymptotic estimate $a_k(n) = b_k(n) + O(c_k(n))$ for $k \in D_n$
3. Let
$$
\begin{aligned}
A(n) &:= \sum_{k\in T_n} a_k(n) \\
B(n) &:= \sum_{k\in T_n} b_k(n) \\
C(n) &:= \sum_{k\in D_n} |c_k(n)|
\end{aligned}
$$
and prove all three are small.
4.
$$
\begin{aligned}
&\sum_{k\in D_n\cup T_n} a_k(n) = \\
&= \sum_{k\in D_n\cup T_n} b_k(n) + O(A(n)) + O(B(n)) + O(C(n))
\end{aligned}
$$

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source: `https://github.com/japdlsd/kombat1_cheatsheet`