Section 1. Introduction, Definitions and Notations

1. A **graph** is a pair of sets $G = (V, E)$ where V is a set of vertices and E is a collection of edges whose endpoints are in V. It is possible that a graph can have infinitely many vertices and edges. Unless stated otherwise, we assume that all graphs are simple. [1]

2. Two vertices v, w are said to be **adjacent** if there is an edge joining v and w. An edge and a vertex are said to be **incident** if the vertex is an endpoint of the edge.

3. Given a vertex v, the **degree** of v is defined to be the number of edges containing v as an endpoint.

4. A **path** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}. (A graph itself can also be called a path.) The **length** of a path is defined to be the number of edges in the path.

5. A **cycle** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1} where the indices are taken modulo $t+1$. (A graph itself can also be called a cycle.) The **length** of a cycle is defined to be the number of vertices (or edges) in the path.

6. A graph is said to be **connected** if for any pair of vertices, there exists a path joining the two vertices. Otherwise, a graph is said to be **disconnected**.

7. The **distance** between two vertices u, v in a graph is defined to be the length of the shortest path joining u, v. (In the case the graph is disconnected, this may not be well-defined.)

8. Let $G = (V, E)$ be a graph. The **complement** $\overline{G}$ of G is a graph with the same vertex set as G and $E(\overline{G}) = \{e \notin E(G)\}$. i.e. $\overline{G}$ has edges exactly where there are no edges in G.

9. Let $G = (V, E)$ be a finite graph. A graph G is said to be **complete** if every pair of vertices in G is joined by an edge. A complete graph on n vertices is denoted by K_n.

10. A graph G is said to be **bipartite** if $V(G)$ can be partitioned into two non-empty disjoint sets A, B such that no edge has both endpoints in the same set. A graph is said to be **complete bipartite** if G is bipartite and all possible edges between the two sets A, B are drawn. In the case where $|A| = m, |B| = n$, such a graph is denoted by $K_{m,n}$.

11. Let $k \geq 2$. A graph G is said to be k-partite if $V(G)$ can be partitioned into k pairwise disjoint sets $A_1, \cdots, A_k$ such that no edge has both endpoints in the same set. A **complete** k**-partite** graph is defined similarly as a complete bipartite. In the case where $|A_i| = n_i$, such a graph is denoted by $K_{n_1, n_2, \cdots, n_k}$. (Note that a 2-partite graph is simply a bipartite graph.)

[1] An edge whose endpoints are the same is called a **loop**. A graph where there is more than one edge joining a pair of vertices is called a **multigraph**. A graph without loops and is not a multigraph is said to be **simple**.

Section 1 Exercises

The exercises in this section, while not of the olympiad nature, will familiarize you with the techniques that might be required to solve olympiad problems. It is important that you know how to solve all of these problems.

1. Let G be a graph with n vertices, m edges and the degrees of the n vertices are $d_1, d_2, \cdots, d_n$. Prove that
$$\sum_{i=1}^{n} d_i = 2m.$$
2. For any graph G, let $\Delta(G)$ be the maximum degree amongst the vertices in G. Characterize all graphs with $\Delta(G) \leq 2$. Characterize all graphs with $\Delta(G) = 2$.
3. Let G be a disconnected graph. Prove that its complement $\overline{G}$ is connected.
4. Let G be a connected graph. Prove that two paths which are both a longest path in the graph, contain at least one vertex in common.
5. Let G be a connected. An edge e is said to be a **cut-edge** if its removal disonnects the graph. Prove that e is not a cut-edge if and only if e is an edge of a cycle.
6. A graph is said to be **planar** if it can be drawn such that a pair of edges can only cross at a vertex.
   (a) Convince yourself that K_5 and $K_{3,3}$ are not planar. [2]
   (b) Suppose a (simple) planar graph G has $n \geq 3$ vertices. Prove that G has at most $3n-6$ edges. (Hint: Doesn't a planar graph look like a polyhedron to you?)
7. Prove that a graph is bipartite if and only if it does not contain an odd cycle.
8. Let G be a connected graph with an even number of vertices. Prove that you can select a subset of edges of G such that each vertex is incident to an odd number of selected edges.
9. (Italy 2007) Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to colour the computers and cables such that no two computers have the same colour, no two cables joined to a common computer have the same colour, and no computer is assigned the same colour as any cable joined to it. Prove that this can be done using n colours.
10. Given a graph G, let $\chi(G)$ be the minimum number of colours required to colour the vertices of G such that no two adjacent vertices are assigned the same colour. Let m be the number of edges in G. Prove that
$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

[2]Kuratowski's Theorem states that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor (a certain type of subgraph). The proof of this theorem is beyond the scope of this training.