Turan's Theorem and a Nice Application of it
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1 Introduction

(This is written to celebrate Paul Turan who, if he was alive, would be 100 years old on Aug 18, 2010.)

Paul Turan had many theorems in Analysis, Number Theory, and Analytic Number Theory. However, his name is attached to Turan's theorem (duh). Turan proved this in 1941 [Tur41]. There is a proof on Wikipedia which might be his.

We present Turan's theorem and a different proof of it from The Alon-Spencer book on the probabilistic method [AS92]. The proof is due to Ravi Boppana. We also present two application of Turan's theorem.

2 Turan's Theorem

Theorem 2.1 *If G is a graph with n vertices and e edges then there is an independent set of size at least*

$$\frac{n}{\frac{2e}{n}+1}.$$

Proof:

We use the Prob Method. Let G be a graph on n vertices. Randomly permute the vertices. Draw the graph with the vertices in that order. Let the vertices in order be $(v_1, \ldots, v_n)$.

We call a vertex v_i *cool* if there are no edges from v_i to any v_j with $i < j$. Note that the set of cool vertices forms an independent set. Let $COOL$ be that set.

We show that the expected size of $COOL$ is at least $\frac{n}{(2e/n)+1}$, hence there must be some ordering of the vertices where $COOL$ is at least this big. Let v be a vertex. What is the probability that $v \in COOL$? Let d_v be the degree of v. Look at v and its neighbors $\{v, u_1, \ldots, u_{d_v}\}$. There are $(d_v + 1)!$ permutations of this set. Vertex v is cool iff v is the rightmost vertex. There are $d_v!$ permutations where v is the rightmost vertex. Hence the probability that v is cool is $\frac{d_v!}{(d_v+1)!} = \frac{1}{d_v+1}$.

The expected size of $COOL$ is

$$\sum_{v\in V} \text{Prob}(v \in COOL) = \sum_{v\in V} \frac{1}{d_v + 1}.$$

We want to lower bound this. Note that $\sum_{v\in V} d_v = 2e$. So the problem is now to

Minimize (over the reals) $\sum_{v\in V} \frac{1}{d_v+1}$ relative to constraint $\sum_{v\in V} d_v = 2e$.

One can show that the sum is minimized when all of the d_v's are the same. So we set $d_v = \frac{2e}{n}$.

$$\sum_{v\in V} \text{Prob}(v \in COOL) = \sum_{v\in V} \frac{1}{d_v + 1} \geq \sum_{v\in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

∎

Note 2.2 The fact that any graph must have an independent set of size at least $\sum_{v \in V} \frac{1}{1+d_v}$ is due to Caro (1979) nad Wei (1981).

3 An Application of Turan's Theorem to Discrete Geometry

This is an application of Turan's theorem that is also due to Turan. Recall that the unit disc is the region enclosed by the circle of radius 1, and includes the boundary.

Theorem 3.1 *Let S be a set of n points in the unit disc. There will be at least $\frac{n^2}{6} - \frac{n}{2}$ pairs of points that are $\leq \sqrt{2}$ apart.*

Proof:

Let G be the graph formed by taking $V = S$ and E is the set of all pairs that are MORE THAN $\sqrt{2}$ apart.

Claim: G has no cliques of size 4.

Proof of Claim: Let w, x, y, z be four points of G.

If some $p \in \{w, x, y, z\}$ is the center of C then for all $q \in \{w, x, y, z\} - \{p\}$

$$d(p, q) \leq 1 < \sqrt{2}.$$

Hence there is no edge between p and any element of $\{w, x, y, z\} - \{p\}$.

We now assume that none of $\{w, x, y, z\}$ are at the center of C. Let c be the center of C. For $p \in \{w, x, y, z\}$ let L_p be the line segment from c to p. There exists $p, q \in \{w, x, y, z\}$ such that the angle between L_p and L_q is $\leq \frac{\pi}{2}$ (Figure 1). The max distance between p and q happens when

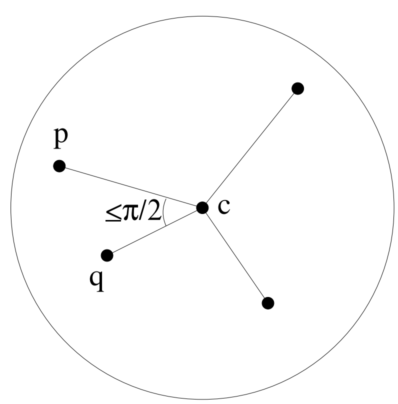

Figure 1: Two points must have angle with the center at most $\pi/2$.

the angle is $\frac{\pi}{2}$ and both points are on the circle. In this case, by the Pythagorean theorem, the distance is exactly $\sqrt{2}$ (Figure 2). Hence there is no edge between p and q.

End of Proof of Claim

G has no clique of size 4. Hence $\overline{G}$ has no independent set of size 4. Therefore the largest independent set of $\overline{G}$ is of size 3. Let e be the number of edges in $\overline{G}$. Note that (1) e is the number of pairs of points that are $\leq \sqrt{2}$ apart (AH- just the quantity we want), and (2) By Turan's theorem

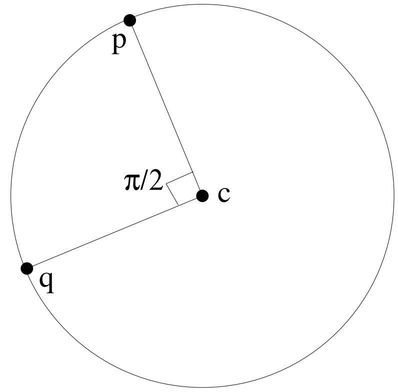

Figure 2: Two points must have distant apart at most $\sqrt{2}$.

$$\frac{n}{\frac{2e}{n}+1} \leq 3$$

$$n \leq \frac{6e}{n} + 3$$

$$n^2 \leq 6e + 3n$$

$$e \geq \frac{n^2 - 3n}{6} = \frac{n^2}{6} - \frac{n}{2}.$$

$\blacksquare$

We leave the following to the reader.

Theorem 3.2 *For all* n *there is a way to place* n *points on the unit disc such that there are* $\frac{n^2}{6} - \frac{n}{2}$ *pairs of points that are less than* $\sqrt{2}$ *apart.*

Theorem 3.1 has a generalization. We state it but leave it to the reader to find the proper constants and prove it.

Theorem 3.3 *Let* $d \in (0, \sqrt{2}]$. *There exists positive constants* A *and* B *depending on* d *such that the following is true:*

1. *Let* S *be a set of* n *points in the unit disc. There will be at least* $An^2 - Bn$ *pairs of points that are* $\leq d$ *apart.*

2. *For all* n *there is a way to place* n *points on the unit disc such that there are* $An^2 - Bn - O(1)$ *pairs of points that are less than* d *apart.*