K-theory seminar: Q-construction for exact categories

Jan McGarry Furriol

Abstract

This are notes for my expository talk following part of [Qui73]: we define the higher algebraic K-theory of exact categories—this includes the K-theory of rings and of schemes—; we show that it extends the definition of the Grothendieck group; and, finally, we prove dévissage and applications of it.

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1 Introduction

We want to define a K-theory space $K(\mathcal{E})$ for exact categories $\mathcal{E}$ which recovers the previously defined lower K-groups by taking homotopy groups $K_n(A) = \pi_n K(A)$. We would like this theory to satisfy *additivity*, *resolution*, *dévissage* and *localization*, which enable to do a lot of K-theory, such as for the Grothendieck group K_0. In this talk, we cover dévissage. The other results will be covered by the following talks.

Dévissage roughly says the following. Let $\mathcal{B} \subseteq \mathcal{A}$ be an abelian subcategory such that every object $A \in \mathcal{A}$ can be "broken up" into objects of $\mathcal{B}$ via short exact sequences

$$\begin{tikzcd}
0 = A_0 \arrow[r, hook] & A_1 \arrow[r, hook] \arrow[d, two heads] & A_2 \arrow[d, two heads] & \cdots & A_{n-1} \arrow[r, hook] \arrow[d, two heads] & A_n = A \arrow[d, two heads] \\
& A_1 & A_2/A_1 & \cdots & A_{n-1}/A_{n-2} & A/A_{n-1}.
\end{tikzcd}$$

with all the quotients in the second row being objects of $\mathcal{B}$—note that these objects are *subquotients* of A. Then we want an equivalence of K-theories $K(\mathcal{B}) \simeq K(\mathcal{A})$.

We saw such a result for K_0, and

$$[A] = [A_1] + [A_2/A_1] + \cdots + [A/A_{n-1}]$$

in $K_0(\mathcal{B}) \simeq K_0(\mathcal{A})$

To an exact category $\mathcal{E}$ we will associate an auxiliary category $Q\mathcal{A}$, functorialy, which packages the information of how objects of $\mathcal{E}$ can be "broken up". We will define its K-theory space by applying the classifying space functor B followed by taking loops Ω. In this step, results of §1 of [Qui73] such as theorems A and B, come into play.

2 Exact categories

Definition 2.1. An *exact category* is a pair $(\mathcal{E}, \mathcal{S})$ where $\mathcal{E}$ is an additive category, and $\mathcal{S}$ is a class of *(short) exact sequences*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

in $\mathcal{S}$, where such i and j will be called an *admissible monomorphism* and *admissible epimorphism* respectively, such that

- any sequence in $\mathcal{E}$ isomorphic to one in $\mathcal{S}$ is also in $\mathcal{S}$. All canonical sequences $0 \to A \to A \oplus B \to B \to 0$ are in $\mathcal{S}$. For any sequence in $\mathcal{S}$, i is the kernel of j, and j is the cokernel of i, in $\mathcal{E}$;l
- admissible epimorphisms (respectively, admissible monomorphisms) are closed under composition and pull-back (respectively push-out) by arbitrary maps in $\mathcal{E}$.

An *exact functor* between exact categories is an additive functor that preserves exact sequences.

Every exact category can be embedded as a full subcategory of an abelian category $\mathcal{A}$ such that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, and $\mathcal{S}$ is the class of sequences $0 \to A \to B \to C \to 0$ which are exact in $\mathcal{A}$. One can prove this following [Gab62]. Conversely, any such additive category is an exact category with exact sequences those that are exact in the ambient abelian category.

Example 2.1. Any abelian category $\mathcal{A}$ is an exact category with exact sequences the short exact sequences in $\mathcal{A}$; in this talk, we will consider abelian categories as exact categories in this way. Another class of sequences that can be picked in $\mathcal{A}$ to yield an exact category are split exact sequences.

Example 2.2. For a (commutative) ring R ring, the category $\mathbf{P}(R)$ of finitely generated projective R-modules is an exact category, viewed as full additive subcategory of the abelian category $\mathrm{Mod}(R)$ of R-modules.

Example 2.3. The category of vector bundles $\mathbf{VB}(X)$ over a ringed space X is an exact category, viewed as full additive subcategory of the abelian category of $\mathcal{O}_X$-modules.

We have two ways of expressing A as a "piece" of B in $\mathcal{E}$:

- $A \hookrightarrow B$ admissible monomorphism;
- $A \twoheadleftarrow B$ admissible epimorphism.

We want to construct a category with the objects of $\mathcal{E}$ and where morphisms express "piece" relations combining these two above.

3 The Q-construction

Let $\mathcal{E}$ be an exact category. We form a new category $Q\mathcal{E}$, known as the *Q-construction*, described as follows:

- Objects are the ones of $\mathcal{E}$.
- A morphism from A to B is an isomorphism class of diagrams

$$\begin{tikzcd} & M \arrow[dl, two heads] \arrow[dr, hook] & \\ A & & B \end{tikzcd}$$

  in $\mathcal{E}$, under isomorphisms which are the identity on A and B.
- Composition is given by taking pull-back:

$$\begin{tikzcd} & & M'' \arrow[dl, two heads] \arrow[dr, hook] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & M \arrow[dl, two heads] \arrow[dr, hook] & & M' \arrow[dl, two heads] \arrow[dr, hook] & \\ A & & B & & C. \end{tikzcd}$$

A square such as the middle one above is a push-out if and only if it is a pull-back in an ambient abelian category $\mathcal{A}$; thus the same is true in $\mathcal{E}$. We will call these *stable squares*, and denote them with a square $\square$ in the middle.

Notice that we took isomorphism classes of diagrams as morphisms for this to be a 1-category, because a pull-back is only uniquely determined up to isomorphism.

An admissible monomorphism $i\colon A \hookrightarrow B$ gives a morphism $i_! \colon A = A \hookrightarrow B$ in $Q\mathcal{E}$, witnessing A as a *subobject* of B in $\mathcal{E}$. An admissible epimorphism $j\colon B \twoheadrightarrow A$ gives a morphism $j^! \colon A \twoheadleftarrow B = B$ in $Q\mathcal{E}$, witnessing A as a *quotient* of B in $\mathcal{E}$. By construction, every morphism in $Q\mathcal{E}$ is factored as $i_! \circ j^!$, and a morphism from A to B in $Q\mathcal{E}$ precisely witnesses A as a *subquotient* of B in $\mathcal{E}$.

Universal property of $Q\mathcal{E}$. It is the initial category $\mathcal{C}$ with a covariant functorial assignment of morphisms $i_!$ in $\mathcal{C}$ to admissible monomorphisms i of $\mathcal{E}$, and a contravariant functorial assignment of morphisms $j^!$ in $\mathcal{C}$ to admissible epimorphisms j of $\mathcal{E}$, such that these are compatible in the sense that if

$$\begin{tikzcd} & M \arrow[dl, two heads, "j"'] \arrow[dr, hook, "i"] & \\ A \arrow[dr, hook, "\bar{i}"'] \arrow[rr, phantom, "\square"] & & B \arrow[dl, two heads, "\bar{j}"] \\ & N & \end{tikzcd}$$

is a stable square in $\mathcal{E}$, then $i_! \circ j^! = \bar{j}^! \circ \bar{i}_!$ in $\mathcal{C}$. In particular, an exact functor induces a functor between the Q-constructions.

4 The Grothendieck group

The purpose of this section is to retrieve the Grothendieck group K_0 from the Q-construction with the following result. We recall the definition of the Grothendieck group of an exact category from a previous talk.

Definition 4.1. Let $\mathcal{E}$ be a small exact category. The *Grothendieck group* K_0 of $\mathcal{E}$ is the free group on the objects $[A]$ of $\mathcal{E}$ with the following relations: if $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathcal{E}$ then $[B] = [A] \cdot [C]$.

Note that $K_0(\mathcal{E})$ is abelian because of the axiom for an exact category asking for all the canonical sequences $0 \to A \to A \oplus B \to B \to 0$ to be an exact sequence of $\mathcal{E}$.

Theorem 4.1. *Let $\mathcal{E}$ be a small exact category. The fundamental group $\pi_1(BQ\mathcal{E}, 0)$ is canonically isomorphic to the Grothendieck group of $\mathcal{E}$.*

Proof. The fundamental group of $BQ\mathcal{E}$ is the free group on finite sequences of not necessarily composable maps in $Q\mathcal{E}$, or *zig-zags*, starting and ending at 0, with relations given by composition in the category $Q\mathcal{E}$.

By construction, every map $A \to B$ in $Q\mathcal{E}$ can be factored as an admissible epimorphism composed with an admissible monomorphism, so a zig-zag is just a product of these, and composition in $Q\mathcal{E}$ works as follows. Informally, we distinguish two kinds of composition: (1) when we compose admissible epimorphisms with each other, or admissible monomorphisms with each other, the composition in $Q\mathcal{E}$ is just the composition in $\mathcal{E}$; (2) what tells us how to compose an admissible epimorphism with an admissible monomorphism are the stable squares—one can see this reflected in the universal property of the Q-construction above.

First we make a reduction using the relations from the first kind of composition. For every admissible monomorphism $A = A \hookrightarrow B$ and admissible epimorphism $A \twoheadleftarrow B = B$, we have factorizations (thus relations)

$$\begin{tikzcd} A \arrow[r, hook] & B \\ 0 \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[d, two heads] & B \arrow[l, two heads] \arrow[dl, two heads] \\ 0. & \end{tikzcd}$$

Consequently, the fundamental group of $BQ\mathcal{E}$ is the free group on the zig-zags $[A] := [0 \hookrightarrow A \twoheadrightarrow 0]$ in $Q\mathcal{E}$, for all $A \in \mathcal{E}$, with relations given by composition in $Q\mathcal{E}$ of the kind (2), i.e., stable squares. But if we look at what such a relation is, we find that it is just an exact sequence: for objects A, B and C of $\mathcal{E}$, the product $[A] \cdot [C]$ is identified with $[B]$ in $\pi_1(BQ\mathcal{E}, 0)$ precisely if there is a stable square such as the middle one below—the square gives a cell in $BQ\mathcal{E}$ and one can see it as witnessing a homotopy between the two mentioned elements of $\pi_1(BQ\mathcal{E}, 0)$.

$$\begin{tikzcd}
 & & 0 \arrow[dr, hook] & & \\
 & A \arrow[ur, two heads] \arrow[dr, hook] \arrow[rr, phantom, "\square"] & & C \arrow[dr, two heads] & \\
0 \arrow[ur, hook] \arrow[rr, hook] & & B \arrow[ur, two heads] \arrow[rr, two heads] & & 0.
\end{tikzcd}$$

Such a square is stable if and only if $0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$ is an exact sequence. $\square$

5 The K-theory space of an exact category

Theorem 4.1 makes sure that the upcoming definition extends that of the Grothendieck group of an exact category.

Definition 5.1. Let $\mathcal{E}$ be a small exact category. We define the *K-theory space* of $\mathcal{E}$ as

$$K(\mathcal{E}) := \Omega BQ\mathcal{E},$$

and its i-th *K-group*, for $i \geq 0$, as $K_i(\mathcal{E}) := \pi_i K(\mathcal{E})$.

More generally, if an exact category has a set of isomorphism classes of objects, we define its K-theory to be that of any equivalent small subcategory. When we apply results of §1 of [Qui73], e.g. Theorems A and B, it will be tacitly assumed that we have replaced any such category by an equivalent small one.

Properties of K($\mathcal{E}$).

— K is a functor from exact categories and exact functors to spaces.

— Isomorphic exact functors induce isomorphic functors under Q. Thus they induce homotopic maps on K-theories. In particular, equivalent exact categories (via exact functors) have equivalent K-theories.

— *K-groups commute with finite products*. Let $\mathcal{E}$ and $\mathcal{E}'$ be exact categories. Then $\mathcal{E} \oplus \mathcal{E}'$ is also an exact category and $Q(\mathcal{E} \oplus \mathcal{E}') \simeq Q\mathcal{E} \times Q\mathcal{E}'$. Thus

$$K(\mathcal{E} \oplus \mathcal{E}') \simeq K(\mathcal{E}) \times K(\mathcal{E}').$$

where we consider the right hand side with the compactly generated topology. The homotopy groups of this space are the same as for the product space because the compact sets are the same in both topologies. Consequently, for every $i \geq 0$,

$$K_i(\mathcal{E} \oplus \mathcal{E}') \simeq K_i(\mathcal{E}) \oplus K_i(\mathcal{E}').$$

— *K-groups commute with filtered colimits*. Consider a filtered diagram of exact categories and exact functors. The colimit is also an exact category. Then, the K-theory functor preserves its colimit, i.e.,

$$\mathrm{colim}_{j \in J} K(\mathcal{E})_j \xrightarrow{\simeq} K(\mathrm{colim}_{j \in J} \mathcal{E}_j),$$

because $Q\, \mathrm{colim}_{j \in J} \mathcal{E}_j \simeq \mathrm{colim}_{j \in J} Q\mathcal{E}_j$ and the classifying space functor commutes with filtered colimits. Since the i-th fundamental group functor π_i also commutes with filtered colimits, we have

$$\mathrm{colim}_{j \in J} K_i(\mathcal{E})_j \xrightarrow{\simeq} K_i(\mathrm{colim}_{j \in J} \mathcal{E}_j).$$

Definition 5.2. Let R be a ring with 1. We define the *K-theory space* of R as

$$K(R) := K(\mathbf{P}(R)),$$

and its i-th *K-group*, for $i \geq 0$, as $K_i(R) := K_i(\mathbf{P}(R))$.

The previous properties apply also coming from the category of rings, although one has to adjust filtered colimits (see [Wei13, pp. IV, 6.4]). In particular, *Morita equivalent* rings have equivalent K-theories.

Definition 5.3. Let X be a scheme. We define the *K-theory space* of X as

$$K(X) := K(\mathbf{VB}(X)),$$

and its i-th *K-group*, for $i \geq 0$, as $K_i(X) := K_i(\mathbf{VB}(R))$.

6 Dévissage

Let $\mathcal{A}$ be an abelian category with a set of isomorphism classes. Consider a non-empty full subcategory $\mathcal{B} \subseteq \mathcal{A}$ closed under subobjects, quotients and products. Then $\mathcal{B}$ is also an abelian subcategory of $\mathcal{A}$, i.e., it is abelian and the inclusion functor is exact.

Theorem 6.1 (Dévissage)**.** *Suppose that every object A of $\mathcal{A}$ admits a finite filtration $0 = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n = A$ such that $A_j/A_{j-1} \in \mathcal{B}$ for all $1 \leq j \leq n$. Then the map*

$$K(\mathcal{B}) \longrightarrow K(\mathcal{A})$$

induced by the inclusion functor is a homotopy equivalence.

Proof. Consider the inclusion functor $Q\iota : Q\mathcal{B} \longrightarrow Q\mathcal{A}$. We show that, for every $A \in \mathcal{A}$, the over category $Q\iota/A$ is contractible, and conclude with Theorem A.

Fix $A \in \mathcal{A}$ and $0 = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n = A$ as in the statement. We prove that this gives a sequence of homotopy equivalences

$$* \simeq BQ\iota/A_1 \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} BQ\iota/A_{n-1} \xrightarrow{\simeq} BQ\iota/A.$$

The first space is contractible because $A_1 \in \mathcal{B}$ by assumption, thus $\mathrm{id}_{A_1} \in Q\iota/A_1$ is terminal. For the others, given $A' \hookrightarrow A$, the property $A/A' \in \mathcal{B}$ and the assumptions on $\mathcal{B} \subseteq \mathcal{A}$ allow to formally construct (see reference) a retraction

$$Q\iota/A' \xrightarrow{\iota} Q\iota/A$$

and natural transformations $\iota r \Rightarrow \Leftarrow \mathrm{Id}_{Q\iota/A}$. This gives a deformation retraction $BQ\iota/A' \simeq BQ\iota/A$. □

Addendum. For the formal part at the end of the proof, it helps to view the over categories $Q\iota/A$ as categories of "layers" over A in $Q\mathcal{A}$, as Quillen did. We explain this in the appendix.

7 Applications of Dévissage

Corollary 7.1. *Let $\mathcal{A}$ be an abelian category such that every object A in $\mathcal{A}$ has finite length, i.e., a finite filtration $0 = A_0 \to A_1 \to \cdots \to A_n = A$ such that A_j/A_{j-1} is simple for all $1 \leq j \leq n$. Then*

$$K_i(\mathcal{A}) \simeq \bigoplus_{j \in \mathrm{Simple}} K_i(D_j),$$

where j indexes the set of isomorphism classes of simple objects, and D_j is the skew field $End(X_j)$ for a representative X_j in such set.

Proof. Let $\mathcal{A}_{\mathrm{ss}}$ be the abelian subcategory of $\mathcal{A}$ consisting of semi-simple objects. By Dévissage, the inclusion functor induces an equivalence on K-theories:

$$K(\mathcal{A}_{\mathrm{ss}}) \xrightarrow{\simeq} K(\mathcal{A}). \tag{1}$$

For each j let $\mathcal{A}_j \subseteq \mathcal{A}_{ss}$ be the full subcategory containing, as the unique simple object, a representative X_j of the corresponding isomorphism class. Then

$$\mathcal{A}_{ss} \simeq \bigoplus_{j \in \mathrm{Simple}} \mathcal{A}_j.$$

Fix a representative X_j. We have an equivalence of categories $\mathcal{A}_j \longrightarrow \mathrm{Mod}_{\mathrm{fg}}(\mathrm{End}(X_j))$, by sending $\bigoplus_1^r X_j$ to $\bigoplus_1^r \mathrm{End}(X_j)$. Since X_j is simple, $D_j := \mathrm{End}(X_j)$ is a skew field. Then the category of finitely generated modules over it is just $\mathbf{P}(D_j)$. All together,

$$\mathcal{A}_{\mathrm{ss}} \simeq \bigoplus_{j \in \mathrm{Simple}} \mathbf{P}(D_j) \tag{2}$$

Since we assumed that there is a set of isomorphism classes of simple objects, the direct sum over this indexing set can be written as filtered colimit of finite products. Then, using on (2) that the functors K_i commute with finite products and filtered colimits, and putting this together with (1), we obtain the result. □

Example 7.2. Let R be a Dedekind domain, and $\mathcal{A} = \mathrm{Mod}_{\mathrm{fg}}^{\mathrm{tor}}(R)$ the full subcategory of the abelian category of finitely generated R-modules $\mathrm{Mod}_{\mathrm{fg}}(R)$ consisting of torsion modules. Since torsion modules are closed under kernels, cokernels, products and coproducts, $\mathcal{A}$ is an abelian subcategory.

The hypothesis are satisfied as a consequence of the structure theorem for finitely generated torsion R-modules: every object A in $\mathcal{A}$ is of the form

$$R/p_1^r \oplus \cdots \oplus R/p_n^{r_n},$$

for prime ideals $p_1, \ldots, p_n$. Then, $R/p^r \supset pR/p^r \supset \cdots \supset p^{r-1}R/p^r \supset 0$, with simple quotients R/p. By the corollary,

$$K_i(\mathrm{Mod}_{\mathrm{fg}}^{\mathrm{tor}}(R)) \simeq \bigoplus_{p \text{ prime}} K_i(R/p).$$

Recall that for a Noetherian ring R, its G-theory was defined as $G(R) := K(\mathrm{Mod}_{\mathrm{fg}}(R))$. These agree for noetherian regular rings.

Corollary 7.3. *Let R be a Noetherian ring and I a nilpotent ideal. Consider* $\mathrm{Mod}_{\mathrm{fg}}(R/I) \subseteq \mathrm{Mod}_{\mathrm{fg}}(R)$. *Then we have an equivalence of G-theories*

$$G(R/I) \xrightarrow{\simeq} G(R).$$

Example 7.4. Consider $R = \mathbb{Z}/p^r$ and $I = (p)$. Then $R/I \simeq \mathbb{F}_p$ and $\mathrm{Mod}_{\mathrm{fg}}(\mathbb{F}_p) = \mathbf{P}(\mathbb{F}_p)$, so

$$K_i(\mathbb{F}_p) = G(\mathbb{F}_p) \xrightarrow{\simeq} G(\mathbb{Z}/p^r).$$

The K-theory of finite fields was computed by Quillen using the $+$-construction. The $+$-construction and the group completion K-theory for rings yield the same K-theory (see [Wei13, Corollary 4.11.1]). The later agrees with the Q-construction for split exact categories, by the comparison theorem in [Gra76]; see also [Wei13, IV, §7]. In particular, they all produce the same K-theory for rings and the calculation of Quillen transfers:

$$K_i(\mathbb{F}_p) \simeq \begin{cases} \mathbb{Z}/(p^k - 1) & i = 2k-1 \\ 0 & i = 2k. \end{cases}$$

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