Synthetic spectra II

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January 2022

Abstract

This is an expository note on synthetic spectra following [Pst18] and [PP21]. It has been written within the course Topics in Algebraic Topology 2021-2022, at the University of Copenhagen. It is the second part of a two-part exposition with João Fernandes. Comments and suggestions are very welcome.

Contents

Introduction

In this note, E will always be an Adams-type homotopy associative ring spectrum, the kind of spectra for which Adams wrote the Künneth, universal coefficients, and Adams spectral sequences. To define what this means, let us start with the following.

Definition 0.1. We say that a spectrum X is *E-projective* if E_*X is projective as a π_*E-module. We call X *finite E-projective* if it is finite and E_*X is finitely generated and projective as a π_*E-module. We denote by $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ the full subcategory of the ∞-category of spectra spanned by finite E-projective spectra.

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For example, the spheres are finite E-projective for any choice of ring spectrum E. In fact, as pointed out in the first part, given a map of spectra $E \to E'$ realizing to one of algebras in the homotopy category, every (finite) E-projective spectrum is also (finite) E'-projective.

Definition 0.2. We say that a homotopy ring spectrum E is *Adams-type* if it can be written as a filtered colimit $E \simeq \operatorname{colim} E_\alpha$ of finite S-projective spectra such that for each E_α the natural map

$$E^* E_\alpha \longrightarrow \operatorname{Hom}_{\pi_* E}(E_* E_\alpha, \pi_* E)$$

is an isomorphism.

Examples include the sphere spectrum S, $H\mathbb{F}_p$, MO, MU, MSp, KU, and KO – see [Ada74, 13.4]. Also any field, even ring spectrum, and any Landweber exact homology theory.

In the first part to this of this two-part exposition, the ∞-category of synthetic spectra

$$\mathcal{S}\mathrm{yn}_E := \mathrm{Sh}_\Sigma(\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}; \mathcal{S}\mathrm{p})$$

was presented, together with its most important properties and structures. The above denotes product-preserving sheaves of spectra on the excellent ∞-site $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$, where a map $Q \to P$ of finite E-projective spectra is a covering if the map induced on E-homology is an epimorphism [Pst18, 3.20].

The synthetic analogue

$$\nu \colon \mathcal{S}\mathrm{p} \longrightarrow \mathcal{S}\mathrm{yn}_E$$

was defined by $\nu X := \Sigma^\infty_+ y(X)$ for any spectrum X – we will see that it is fully faithful –, where $y(X)$ is the presheaf defined on a finite projective P by $y(X) = \mathrm{Map}_{\mathcal{S}\mathrm{p}}(P, X)$, and Σ^∞_+ is the left adjoint to the functor Ω^∞:

$$\mathcal{S}\mathrm{yn}_E \simeq \mathrm{Sp}(\mathrm{Sh}_\Sigma(\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}; \mathcal{S}\mathrm{paces})) \xrightarrow{\Omega^\infty} \mathrm{Sh}_\Sigma(\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}; \mathcal{S}\mathrm{paces});$$

the composition is given by taking infinite loop spaces pointwise.

It was observed that $y(X)$ is canonically a sheaf of infinite loop spaces, by the additivity of $\mathcal{S}\mathrm{yn}_E$, and then Σ^∞_+ is computed by delooping it to a sheaf of connective spectra, and then sheafifying. Then, one can see that it may also be described as the sheafification of the presheaf of connective spectra defined by

$$P \longmapsto \tau_{\geq 0} \operatorname{map}_{\mathcal{S}\mathrm{p}}(P, X).$$

Theorem 0.1 ([Pst18, 4.2, 4.5, 4.16])**.** *The ∞-category $\mathcal{S}\mathrm{yn}_E$ is presentable and stable. It can be promoted to the unique symmetric monoidal ∞-category that preserves colimits in each variable and promotes the restriction $\nu \colon \mathcal{S}\mathrm{p}_E^{\mathrm{fp}} \to \mathcal{S}\mathrm{yn}_E$ to a symmetric monoidal functor. The ∞-category of synthetic spectra admits a right complete t-structure in which the coconnective part is the ∞-category of product-preserving sheaves valued in coconnective spectra.*

Let us introduce the following important family of synthetic spectra.

Definition 0.3. Let t, w be integers. The *bigraded sphere* $\mathbb{S}^{t,w}$ is defined to be $\Sigma^{t-w} \nu \mathbb{S}^w$.

Now we can highlight another characteristic of the ∞-category of synthetic spectra, that is, tensoring with the bigraded spheres defines autoequivalences $\Sigma^{t,w} := -\otimes S^{t,w} \colon \mathcal{S}\mathrm{yn}_E \to \mathcal{S}\mathrm{yn}_E$. This is due to the fact that, on the one hand, the suspension functor commutes with the cocontinuous tensor product and is an autoquivalence of the stable ∞-category $\mathcal{S}\mathrm{yn}_E$; on the other hand, tensoring with spheres defines an autoequivalence of $\mathcal{S}\mathrm{p}$, and ν underlies a symmetric monoidal functor when restricted to finite E-projective spectra.

In this note, we start by introducing the *spectral Yoneda embedding*, defined for a spectrum X as the presheaf of spectra $Y(X)$ given by the formula

$$Y : P \longmapsto Y(X)(P) := \mathrm{map}_{\mathcal{S}\mathrm{p}}(P, X)$$

for P finite E-projective. It is clearly product-preserving, and a sheaf by the recognition principle 1.4, so that $Y(X)$ is an object of $\mathcal{S}\mathrm{yn}_E$.

In section 2 we show that the filtration of $Y(X)$ by connective covers in $\mathcal{S}\mathrm{yn}_E$ agrees with the tower over it by powers of a certain map τ. From this we deduce Proposition 3.3, which says that there is a map $\nu X \to Y(X)$ realizing both a connective cover with respect to the t-structure on $\mathcal{S}\mathrm{yn}_E$, and an inversion of the map τ. Furthermore, in section 3 we identify spectra as the synthetic spectra on which τ acts invertibly, and show how this fits in a stable recollement.

Finally, in section 5 we show that the spectral sequences induced by the mentioned filtrations of $Y(X)$ agree with the E-based Adams spectral sequence. For this, we will describe in section 4 the relation between synthetic spectra and the category $\mathrm{co}\mathcal{M}\mathrm{od}_{E_*E}$ of comodules over the graded Hopf algebroid given by $[n] \mapsto \pi_*(E^{\otimes[n]})$.

1. Sheaves of spectra

In this section, we provide some results that will be useful when working with the sheaves of spectra which we consider, including a couple of results playing the role of the Yoneda Lemma.

Lemma 1.1. *If P is a finite E-projective spectrum and X a synthetic spectrum, then there is a natural equivalence* $\mathrm{Map}(\nu P, X) \simeq \Omega^\infty X(P)$.

Proof. Recall that by definition $\nu P \simeq \Sigma^\infty_+ y(P)$. Then using first the adjunction $\Sigma^\infty_+ \dashv \Omega^\infty$, and then the Yoneda Lemma, we have

$$\mathrm{Map}(\nu P, X) \simeq \mathrm{Map}(y(P), \Omega^\infty X) \simeq (\Omega^\infty X)(P) \simeq \Omega^\infty X(P),$$

where the latter is the fact that Ω^∞ is computed pointwise. □

Lemma 1.2. *Synthetic spectra of the form $\Sigma^{k,0}\nu P$, where k varies in the integers and P varies in finite E-projective spectra, form a family of dualizable compact generators of $\mathcal{S}\mathrm{yn}_E$.*

Proof. It is immediate from the definition and Lemma 1.1 that the mentioned objects are generators. These are dualizable because they are as objects of $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ and the restriction of ν to this category underlies a symmetric monoidal functor.

For compactness, note that filtered colimits in spectra are computed pointwise, so that it is enough to show that objects of the form $y(P) \in \mathrm{Sh}_\Sigma(\mathcal{C})$ are compact, where $\mathcal{C} = \mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ or more generally an additive ∞-site, and y is the Yoneda embedding. This holds because the subcategory $\mathrm{Sh}_\Sigma(\mathcal{C}) \subset \mathrm{Ph}(\mathcal{C})$ is closed under filtered colimits, and in the latter, these are computed pointwise. Indeed, the spherical sheaf condition on an additive ∞-site $\mathcal{C}$ can be formulated in terms of finite limits by [Pst18, 2.8]. $\square$

Lemma 1.3. *If P is a finite E-projective spectrum and X a synthetic spectrum, then there is a natural equivalence* $\mathrm{map}(\nu P, X) \simeq X(P)$.

Proof. By Lemma 1.1 followed by Proposition 2.1, the spaces and maps forming each spectrum are identified:

$$\mathrm{Map}_{\mathcal{S}\mathrm{yn}_E}(\nu P, \Sigma^{n,0} X) \simeq \Omega^{\infty+n} X(P),$$

for every integer n. $\square$

Lemma 1.4 (Recognition principle for product-preserving sheaves of spectra)**.** *Consider the additive ∞-site $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$. A product-preserving presheaf of spectra X on $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ is a sheaf if and only if for every fiber sequence $P' \to P \to P''$ in $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ where the second map is a covering, i.e., induces an epimorphism on E_*-homology, the induced sequence*

$$X(P'') \longrightarrow X(P) \longrightarrow X(P')$$

of spectra is a fiber sequence.

Proof. By [Pst18, 2.8] the analogous claim is true for presheaves of spaces. Since

$$\mathcal{S}\mathrm{p} \simeq \lim(\cdots \xrightarrow{\Omega} \mathcal{S}\mathrm{paces}_* \xrightarrow{\Omega} \mathcal{S}\mathrm{paces}_* \xrightarrow{\Omega} \mathcal{S}\mathrm{paces}_*),$$

and sheaf conditions and fiber sequences are compatible with limits, these can be checked pointwise. Then, the cited result finishes the proof. $\square$

2. The map τ

A synthetic spectrum X does not in general take suspensions to loops. For each finite E-projective P we can consider the comparison map $X(\Sigma P) \to \Omega X(P)$, induced by the square

$$\begin{tikzcd} X(\Sigma P) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X(P). \end{tikzcd}$$

This square is obtained by evaluating the sheaf X at the square given by tensoring the colimit square of ΣS with P. We will show that this map is induced by a universal one τ, by tensoring in the category of synthetic spectra.

For this, we will need the following description of tensoring with bigraded spheres, which is essentially the fact that tensoring a sheaf of spaces with a representable on an excellent ∞-site, with respect to the Day convolution symmetric monoidal structure, is naturally equivalent to precomposition along tensoring with the monoidal dual of the representing object [Pst18, 2.26].

Proposition 2.1. *Let $\Sigma^{t,w}X$ be the bigraded suspension of a synthetic spectrum X. There is a canonical equivalence of sheaves $(\Sigma^{t,w}X)(P) \simeq \Sigma^{t-w}X(\Sigma^{-w}P)$ where P varies in $\mathcal{S}\mathrm{p}_E^{fp}$.*

As a consequence of the proposition, the square above can be identified with the one we obtain by evaluating the following at P:

$$\begin{tikzcd}
\nu(\Omega S)\otimes X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \nu(S)\otimes X,
\end{tikzcd}$$

where these synthetic analogues are the bigraded spheres $S^{-1,-1}$ and $S^{0,0}$ respectively. This shows that the comparison map $X(\Sigma P) \to \Omega X(P)$ can be identified with the map obtained by tensoring X with the limit comparison map $\nu(\Omega S) \to \Omega\nu(S)$, and evaluating at P.

Notation. We denote by $\tau\colon S^{-1,-1} \to S^{-1,0}$ the canonical limit comparison map. Next comes a result about the suspension of τ, for which we use the same notation.

Lemma 2.2. *If X is a spectrum, then the map $\tau\otimes X$ fits into a cofiber sequence*

$$\Sigma^{0,-1}\nu X \xrightarrow{\tau\otimes X} \nu X \longrightarrow \tau_{\leq 0}\nu X.$$

In other words, the map $\tau\otimes\nu X$ is a 1-connective cover.

Proof. Based on the description of 2.1 the discussion above concluded that $\Sigma^{0,-1}\nu X \to \nu X$ can be identified with the sheafification of the map

$$\Sigma\tau_{\geq 0}\operatorname{map}_{\mathcal{S}\mathrm{p}}(\Sigma P, X) \longrightarrow \tau_{\geq 0}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X);$$

now we are using that sheafification is exact, and therefore is preserves suspensions – it is a left exact functor between stable ∞-categories. Note that, for any natural number n, there is a natural equivalence of functors $\tau_{\geq n}\Omega \to \Omega\tau_{\geq n+1}$. Then, the above map identifies with

$$\tau_{\geq 1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X) \longrightarrow \tau_{\geq 0}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X).$$

The connectivity claim follows because it can be checked before sheafifying, since sheafification is t-exact [Lur18, 1.3.4.7]. $\square$

Remark 2.3. Observe that the proof shows that the cofiber $\nu X/\tau$ is in $\mathcal{S}\mathrm{yn}_E^{\heartsuit}$ and is equivalent to the sheafification of the presheaf whose value at a finite E-projective spectrum P is the Eilenberg-Mac Lane spectrum $H\pi_0\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X)$.

Proposition 2.4. *Let X be a spectrum. Consider the tower*

$$\cdots \longrightarrow \Sigma^{0,-1}\nu X \longrightarrow \nu X \longrightarrow \Sigma^{0,1}\nu X \longrightarrow \cdots$$

in which each map is given by multiplication by τ. Up to essentially unique equivalence, this tower is the Postnikov tower for the synthetic t-structure.

Proof. As in Lemma 2.2, this tower is identified with the sheafification of the tower

$$\cdots \longrightarrow \tau_{\geq 1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X) \longrightarrow \tau_{\geq 0}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X) \longrightarrow \tau_{\geq -1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,X) \longrightarrow \cdots. \qquad \square$$

3. τ-INVERTIBLE SYNTHETIC SPECTRA AND A STABLE RECOLLEMENT

In this section we identify spectra with the full subcategory of $\mathcal{S}\mathrm{yn}_E$ spanned by the synthetic spectra on which τ acts invertibly, and we show how this fits in a stable recollement.

Definition 3.1. We say that a synthetic spectrum X is *τ-invertible* if the multiplication by τ map $\tau\colon \Sigma^{0,-1}X \to X$ is an equivalence. We denote the full subcategory of $\mathcal{S}\mathrm{yn}_E$ spanned by the τ-invertible spectra by $\mathcal{S}\mathrm{yn}_E[\tau^{-1}]$.

Proposition 3.1. *The inclusion $i_*\colon \mathcal{S}\mathrm{yn}_E[\tau^{-1}] \hookrightarrow \mathcal{S}\mathrm{yn}_E$ admits a left adjoint $i^*\colon \mathcal{S}\mathrm{yn}_E \to \mathcal{S}\mathrm{yn}_E[\tau^{-1}]$.*

Proof. One checks that the formula

$$X[\tau^{-1}] := \operatorname{colim}(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \xrightarrow{\tau} \cdots),$$

where X is a synthetic spectrum, defines the unit $X \to i_*i^*X =: X[\tau^{-1}]$ of an adjunction. □

Corollary 3.2. *The canonical map $S^{0,0} \to S^{0,0}[\tau^{-1}]$ is an idempotent $\mathbb{E}_0$-algebra in $\mathcal{S}\mathrm{yn}_E^{\otimes}$. Therefore it can be uniquely promoted to a commutative algebra in $\mathcal{S}\mathrm{yn}_E^{\otimes}$, and the functor*

$$-\otimes S^{0,0}[\tau^{-1}]\colon \mathcal{S}\mathrm{yn}_E \longrightarrow \mathcal{S}\mathrm{yn}_E$$

is a localization which, together with its essential image, may be promoted to a symmetric monoidal functor. Moreover, the forgetful functor

$$\mathcal{M}\mathrm{od}_{S^{0,0}[\tau^{-1}]}(\mathcal{S}\mathrm{yn}_E^{\otimes}) \hookrightarrow \mathcal{S}\mathrm{yn}_E$$

is fully faithful and it restricts to an equivalence $\mathcal{M}\mathrm{od}_{S^{0,0}[\tau^{-1}]}(\mathcal{S}\mathrm{yn}_E^{\otimes}) \simeq \mathcal{S}\mathrm{yn}_E[\tau^{-1}]$.

Proof. It follows from the equivalence $X[\tau^{-1}] \simeq -\otimes S^{0,0}[\tau^{-1}]$, using the formula in the proof of Proposition 3.1, and the discussion on idempotent algebras on Marius Nielsen's notes for this course. □

Proposition 3.3. *If X is a spectrum, then the canonical map $\nu X \to Y(X)$ is both a τ-inversion and a connective cover.*

Proof. This is immediate after Proposition 2.4. □

In fact, either of the two properties in the proposition characterizes the map uniquely, up to contractible ambiguity. The following important theorem can be interpreted as saying that non-topological phenomena in synthetic spectra are necessarily "τ-torsion".

Theorem 3.4 ([Pst18, 4.36])**.** *The spectral Yoneda embedding $Y\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{yn}_E$ is fully faithful and its essential image is the full subcategory of τ-invertible spectra. The equivalence $\mathcal{S}\mathrm{p} \simeq \mathcal{S}\mathrm{yn}_E[\tau^{-1}]$ induced by Y can be canonically promoted to a symmetric monoidal equivalence.*

Corollary 3.5. *The synthetic analogue functor $\nu\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{yn}_E$ is fully faithful.*

Proof. Observe that, by Proposition 3.3, $Y(X) \simeq (\nu X)[\tau^{-1}]$ for any spectrum X. Now, by Theorem 3.4, it is enough to show that for any spectra A and B, the following composite induced by τ-inversion is an equivalence:

$$\mathrm{Map}(\nu A, \nu B) \longrightarrow \mathrm{Map}(\nu A, Y(B)) \longrightarrow \mathrm{Map}(Y(A), Y(B)).$$

The first map is an equivalence because νA is connective and $\nu B \to Y(B)$ is a connective cover, by Proposition 3.3; and the second map is an equivalence because $Y(B)$ is τ-invertible and $\nu A \to Y(B)$ is a τ-inversion, by Proposition 3.3. $\square$

Stable recollement. Recall that $\mathcal{S}\mathrm{yn}_E^{\otimes}$ is a presentably symmetric monoidal stable ∞-category and $\eta\colon S^{0,0} \to S^{0,0}[\tau^{-1}]$ is an idempotent $\mathbb{E}_0$-algebra. Therefore, we get the following triple of adjoint functors

$$\begin{tikzcd} \mathcal{S}\mathrm{yn}_E \arrow[r, bend left=60, "i^*"] \arrow[r, bend right=60, "i^!"'] & \mathcal{S}\mathrm{yn}_E[\tau^{-1}] \arrow[l, hook', "i_* \simeq i_!"'] \end{tikzcd}$$

where

$$i_* i^* \simeq - \otimes S^{0,0}[\tau^{-1}] \quad \text{and} \quad i_! i^! \simeq \underline{\mathrm{map}}(S^{0,0}[\tau^{-1}], -).$$

The right adjoint $i^!$ exists because i_* is a functor between presentable ∞-categories which preserves all colimits. Consequently, we formally get the following stable recollement:

$$\begin{tikzcd} \mathcal{S}\mathrm{yn}_E^{\tau-\mathrm{nil}} \arrow[dr, hook, "j_!"] \arrow[dd, shift left, "j^! j_*"] & & \\ & \mathcal{S}\mathrm{yn}_E \arrow[ul, "j^!"] \arrow[dl, "j^*"] \arrow[r, bend left=60, "i^*"] \arrow[r, bend right=60, "i^!"'] & \mathcal{S}\mathrm{yn}_E[\tau^{-1}] \arrow[l, hook', "i_* \simeq i_!"'] \\ \mathcal{S}\mathrm{yn}_E^{\tau-\mathrm{cpl}} \arrow[uu, shift left, "j^* j_!"] \arrow[uu, phantom, "\wr"] \arrow[ur, hook, "j_*"'] & & \end{tikzcd}$$

where we denote by $\mathcal{S}\mathrm{yn}_E^{\tau-\mathrm{nil}}$ and $\mathcal{S}\mathrm{yn}_E^{\tau-\mathrm{cpl}}$ the full subcategories of $\mathcal{S}\mathrm{yn}_E$ spanned by the synthetic spectra X with the property that

$$\mathrm{map}_{\mathcal{S}\mathrm{yn}_E}(X, i_* Y) \simeq 0 \quad \text{and} \quad \mathrm{map}_{\mathcal{S}\mathrm{yn}_E}(i_* Y, X) \simeq 0$$

respectively, for every τ-invertible synthetic spectrum Y. There are fiber sequences involving the units and counits:

$$j_! j^! \longrightarrow \mathrm{id} \longrightarrow i_* i^* \quad \text{and} \quad i_! i^! \longrightarrow \mathrm{id} \longrightarrow j_* j^*.$$

We now unravel what the above means in our case, which is completely analogous to the case of ordinary spectra and the endomorphism of the sphere spectrum given by multiplication by a prime. The τ-nilpotence condition of a synthetic spectrum X is equivalent to asking for

$$i_* i^* X \simeq X \otimes S^{0,0}[\tau^{-1}] \simeq \mathrm{colim}(X \xrightarrow{\tau} \Sigma^{0,1} X \xrightarrow{\tau} \Sigma^{0,2} X \xrightarrow{\tau} \cdots)$$

to be zero. For a synthetic spectrum X, the first cofiber sequence above has the form

$$X \otimes \Sigma^{-1,0} S^{0,0}[\tau^{-1}]/S^{0,0} \longrightarrow X \longrightarrow X \otimes S^{0,0}[\tau^{-1}].$$

The τ-completeness condition of a synthetic spectrum X is asking for

$$i_! i^! X \simeq \underline{\mathrm{map}}(S^{0,0}[\tau^{-1}], X) \simeq \lim(X \xleftarrow{\tau} \Sigma^{0,-1} X \xleftarrow{\tau} \Sigma^{0,-2} X \xleftarrow{\tau} \cdots)$$

to be zero. In fact, being τ-complete is equivalent to being $S^{0,0}/\tau$-local. The second cofiber sequence above takes the form

$$\underline{\mathrm{map}}(S^{0,0}[\tau^{-1}], X) \longrightarrow \underline{\mathrm{map}}(S^{0,0}, X) \longrightarrow \underline{\mathrm{map}}(\Sigma^{-1,0} S^{0,0}[\tau^{-1}]/S^{0,0}, X).$$

Observe that there is a canonical equivalence $S^{0,0}[\tau^{-1}]/S^{0,0} \simeq \operatorname{colim} S^{0,n}/\tau^n$, where the colimit is over the natural numbers and the maps are induced by multiplication by τ. Inputing this into the descriptions that we have given above for $j_! j^! X$ and $j_* j^* X$, that is, τ-nilpotent completion and τ-completion, respectively, we recover well-known formulas.

4. Modules over the cofiber of τ

We describe the relation between synthetic spectra and $\mathrm{co}\mathcal{M}\mathrm{od}_{E_*E}$, aspects of which we will use in the upcoming section. By [Pst18, 2.62, 3.7], we have an identification

$$\begin{tikzcd}
\mathcal{S}\mathrm{table}_{E_*E} \arrow[r, "\simeq"] & \mathrm{Sh}_\Sigma(\mathrm{co}\mathcal{M}\mathrm{od}^{\mathrm{fp}}_{E_*E}; \mathcal{S}\mathrm{p}) \\
\mathcal{D}(\mathrm{co}\mathcal{M}\mathrm{od}_{E_*E}) \arrow[u, hook] \arrow[r, "\simeq"] & \mathrm{Sh}^{\mathrm{hyp}}_\Sigma(\mathrm{co}\mathcal{M}\mathrm{od}^{\mathrm{fp}}_{E_*E}; \mathcal{S}\mathrm{p}). \arrow[u, hook]
\end{tikzcd}$$

Here, the superscript *hyp* denotes hypercomplete spectra.

Lemma 4.1 ([Pst18, 4.42])**.** *The morphism of ∞-sites $E_* \colon \mathcal{S}\mathrm{p}^{\mathrm{fp}}_E \to \mathrm{co}\mathcal{M}\mathrm{od}^{\mathrm{fp}}_{E_*E}$ given by taking E-homology induces an adjunction on categories of sheaves*

$$\epsilon_! : \mathcal{S}\mathrm{yn}_E \rightleftarrows \mathcal{S}\mathrm{table}_{E_*E} : \epsilon^*$$

where $\epsilon_!$ is the unique cocontinuous extension of E_* along ν and the inclusion $\mathrm{co}\mathcal{M}\mathrm{od}^{\mathrm{fp}}_{E_*E} \hookrightarrow \mathcal{S}\mathrm{table}_{E_*E}$ of dualizables into the heart of $\mathcal{S}\mathrm{table}_{E_*E}$.

Lemma 4.2. *The left adjoint $\epsilon_!$ can be canonically promoted to a symmetric monoidal functor. It follows that the right adjoint ϵ^* can be promoted to a lax symmetric monoidal functor.*

Proof. The functor E_* in the statement underlies a symmetric monoidal functor, and the left adjoint is the unique extension of it to the categories of sheaves, which we consider with the symmetric monoidal structure given by Day convolution. The statement about the right adjoint is [Lur17, 7.3.2.7]. □

Lemma 4.3. *If P is a finite E-projective spectrum, then there is a canonical equivalence $\nu P/\tau \simeq \epsilon^*\epsilon_!(P)$. Consequently, there is a canonical commutative algebra structure on $S^{0,0}/\tau$ and the right adjoint ϵ^* has a canonical lift χ^* to $S^{0,0}/\tau$-modules*

$$\epsilon^* \colon \mathcal{S}\mathrm{table}_{E_*E} \xrightarrow{\chi^*} \mathcal{M}\mathrm{od}_{S^{0,0}/\tau}(\mathcal{S}\mathrm{yn}_E) \longrightarrow \mathcal{S}\mathrm{yn}_E.$$

Proof. For the first part, it is enough to show that the unit map $\nu P \to \epsilon^*\epsilon_!(\nu P)$ of the adjunction identifies with the cofiber map $\nu P \to \nu P/\tau$. For this, [Pst18, 4.44] shows that both are a 0-truncation of νP.

Consider $P = S$ the sphere spectrum, and observe that $\nu S = S^{0,0}$ is the unit object of $\mathcal{S}\mathrm{yn}_E^{\otimes}$; and all objects of $\mathcal{S}\mathrm{yn}_E$ can be canonically promoted to modules over this commutative algebra. Since $\epsilon_!$ and ϵ^* underlie lax symmetric monoidal functors, they preserve commutative algebra structures and modules over these. Finally $\epsilon^*\epsilon_!(\nu P) \simeq S^{0,0}/\tau$; in fact, the unit for its algebra structure is the unit of the adjunction, which we identified with the cofiber map $S^{0,0} \to S^{0,0}/\tau$. □

We will need to know when does χ^* restrict to an equivalence. By [Pst18, 2.12], ϵ^* commutes with hypercompletion, so it restricts to a functor between hypercomplete sheaves.

Theorem 4.4 ([Pst18, 4.54])**.** *The functor* $\mathcal{S}\mathrm{table}_{E_*E} \xrightarrow{\chi^*} \mathcal{M}\mathrm{od}_{S^{0,0}/\tau}(\mathcal{S}\mathrm{yn}_E)$ *restricts to an equivalence*

$$\mathcal{D}(\mathrm{co}\mathcal{M}\mathrm{od}_{E_*E}) \xrightarrow{\simeq} \mathcal{M}\mathrm{od}_{S^{0,0}/\tau}(\mathcal{S}\mathrm{yn}_E^{\mathrm{hyp}}).$$

5. Postnikov towers and Adams filtrations

In this section we show that, for a spectrum X, the tower of 2.4

$$\cdots \to \tau_{\geq 1}Y(X) \to \tau_{\geq 0}Y(X) \to \tau_{\geq -1}Y(X) \to \cdots \simeq \cdots \to \Sigma^{0,-1}\nu X \to \nu X \to \Sigma^{0,1}\nu X \to \cdots$$

induces a spectral sequence isomorphic up to regrading to the E-based Adams spectral sequence. The main result is Theorem 5.3. We follow [PP21, §5.4]. Note that the above assembles into a functor $\mathbb{Z}^{\mathrm{op}} \to \mathcal{S}\mathrm{yn}_E$, a decreasingly filtered synthetic spectrum.

Construction 5.1. Let A, X be spectra. Consider the following filtered spectrum given by mapping into the above filtration:

$$F_E^\star(A,X) := \mathrm{map}_{\mathcal{S}\mathrm{yn}_E}(\nu A, \tau_{\geq \star}Y(X)).$$

Note that $\tau_{\geq n}Y(X)$ is the sheafification of $\tau_{\geq n}\mathrm{map}_{\mathcal{S}\mathrm{p}}(-,X)$, and that if $A = P$ is finite E-projective, then Lemma 1.3 gives

$$F_E^n(P,X) \simeq \tau_{\geq n}\mathrm{map}_{\mathcal{S}\mathrm{p}}(P,X)^{sh}.$$

Next, a hint that this filtration will recover the E-based Adams spectral sequence.

Lemma 5.2. *If X, Y a spectra, then there is an equivalence of mapping spectra*

$$\mathrm{gr}^t F_E^\star(Y,X) \simeq \mathrm{map}_{E_*E}(E_*Y, \Sigma^t E_*(X)[-t]).$$

In particular, the homotopy groups of the associated graded terms compute Ext *groups in* $\mathrm{co}\mathcal{M}\mathrm{od}_{E_*E}$.

Proof. Using the description of Proposition 2.1, we identify the degree t associated graded term as the mapping spectrum into $\Sigma^t \nu(\Omega^t X)/\tau$. In turn, we may rewrite this cofiber as tensoring with $S^{0,0}/\tau$, so that

$$\mathrm{gr}^t F_E^\star(Y,X) \simeq \mathrm{map}_{\mathrm{Syn}_E}(\nu Y, S^{0,0}/\tau \otimes \Sigma^t \nu(\Omega^t X)) \simeq \mathrm{map}_{S^{0,0}/\tau}(S^{0,0}/\tau \otimes \nu Y, S^{0,0}/\tau \otimes \Sigma^t \nu(\Omega^t X)).$$

In section 4 we have seen that $S^{0,0}/\tau \otimes \nu Y \simeq E_* Y$ for any spectrum Y. Also notice that $S^{0,0}/\tau \otimes \nu Y$ is hypercomplete, because it is in the heart of synthetic spectra – see Remark 2.3 –, and the latter is equivalent to that of hypercomplete spectra under the inclusion [Pst18, 2.17]. Consequently, the mapping spectrum above is equivalent to the one in the statement under the equivalence of Theorem 4.4.

The s-th homotopy group of the above is

$$\pi_0 \mathrm{Map}_{E_*E}(E_* Y, \Sigma^{t-s} E_*(X)[-t]) \simeq \mathrm{Ext}^{t-s,t}_{E_*E}(E_* Y, E_*(X)).$$

where the first superscript denotes the $(t-s)$-th derived functor, and the second denotes the t-graded part in Ext^{t-s}. □

Theorem 5.3. *Let X, P be spectra, with P finite E-projective. Then the filtration $F_E^\star(P,X)$ has the following properties:*

1. *there is a canonical equivalence* $\mathrm{colim}_{\mathbb{Z}^{op}} F_E^\star(P,X) \xrightarrow{\simeq} \mathrm{map}_{\mathrm{Sp}}(P,X)$*;*
2. *for every $s, t \in \mathbb{Z}$,*
$$\mathrm{im}(\pi_s F_E^t(P,X)) \subseteq \pi_s \mathrm{map}_{\mathrm{Sp}}(P,X)$$
coincides with the subgroup of elements of E-Adams filtration at least $t-s$;
3. *the spectral sequence*
$$\pi_{s+t}\, \mathrm{gr}^t(F_E^\star(P,X)) \Longrightarrow \pi_{s+t} \mathrm{map}_{\mathrm{Sp}}(P,X)$$
induced by the filtration $F_E^\star(P,X)$ coincides up to regrading with the E-based Adams spectral sequence.

Proof of (1). Since νP is compact by Lemma 1.2, the colimit of

$$\ldots \longrightarrow \mathrm{map}_{\mathrm{Syn}_E}(\nu P, \tau_{\geq n+1} Y(X)) \longrightarrow \mathrm{map}_{\mathrm{Syn}_E}(\nu P, \tau_{\geq n} Y(X)) \longrightarrow \mathrm{map}_{\mathrm{Syn}_E}(\nu P, \tau_{\geq n-1} Y(X)) \longrightarrow \ldots$$

can be taken in the target variable. Thus we compute the colimit to be $\mathrm{map}_{\mathrm{Sp}}(P,X)$, by the spectral Yoneda Lemma 1.3. □

Proof of (2). For $k > 0$ we look at

$$\ldots \longrightarrow \pi_s F_E^{s+k} \longrightarrow \ldots \longrightarrow \pi_s F_E^s \xrightarrow{\simeq} \ldots \xrightarrow{\simeq} \pi_s F_E^{s-k} \xrightarrow{\simeq} \ldots \xrightarrow{\simeq} \pi_s \mathrm{map}_{\mathrm{Sp}}(P,X).$$

An element in $\pi_s \mathrm{map}_{\mathrm{Sp}}(P,X)$ lifts to $\pi_s F_E^{s+k}$ if and only if it vanishes in $\pi_s \mathrm{gr}^{s+i}$ for each $i < k$; but by Lemma 5.2, $\pi_s \mathrm{gr}^{s+i} \simeq \mathrm{Ext}^i_{E_*E}$, so this is the same as being of E-Adams filtration at least k. □

The proof of part (3). We begin by presenting the needed results for the proof, which will be given at the end of this section.

Lemma 5.4. *The augmented cosimplicial diagram of filtered spectra*

$$F_E^\star(A,X) \longrightarrow F_E^\star(A,I^0) \rightrightarrows F_E^\star(A,I^1) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots, \tag{1}$$

where $X \to I^\bullet$ is the Adams cosimplicial resolution of Remark 5.8, is a limit diagram on the associated graded terms. In other words, the degree t graded term associated to $F_E^\star(A,X)$ is the totalization of the cosimplicial spectrum of degree t graded terms.

Proof. By Lemma 5.2, the diagram of filtered spectra in the statement looks as follows on the degree t associated graded term:

$$\mathrm{map}_{E_*E}(E_*Y, \Sigma^t E_*(X)[-t]) \longrightarrow \mathrm{map}_{E_*E}(E_*Y, \Sigma^t E_*(I^\bullet)[-t]).$$

This is a limit diagram because it is taking mapping spectra into the diagram

$$E_*X \longrightarrow E_*I^0 \rightrightarrows E_*I^1 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots,$$

which is a limit diagram in $\mathcal{D}(\mathrm{co}\mathcal{M}\mathrm{od}_{E_*E})$ by Remark 5.8. $\square$

Consequently, the map into the levelwise (at each $\star$) totalization

$$F_E^\star(A,X) \longrightarrow \lim_{\Delta} F_E^\star(A,I^\bullet)$$

is an equivalence on the associated graded terms. We conclude that these filtrations induce isomorphic spectral sequences.

Lemma 5.5. *If X is a spectrum which satisfies equation 2, then the canonical map to the sheafification $\tau_{\geq 0}\,\mathrm{map}_{\mathcal{S}\mathrm{p}}(-,X) \to \nu X$ is an equivalence.*

Proof. We use the recognition principle for spherical sheaves of spectra 1.4. Consider an arbitrary cofiber sequence $P'' \to P' \to P$ where the second map induces an epimorphism after applying E_*. We show that

$$\tau_{\geq 0}\,\mathrm{map}_{\mathcal{S}\mathrm{p}}(P'',X) \longrightarrow \tau_{\geq 0}\,\mathrm{map}_{\mathcal{S}\mathrm{p}}(P',X) \longrightarrow \tau_{\geq 0}\,\mathrm{map}_{\mathcal{S}\mathrm{p}}(P,X)$$

is a fiber sequence of spectra. It is in the full subcategory of connective spectra, because we can view it as a fiber sequence of spectra before applying the right adjoint $\mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_{\geq 0}$ to the inclusion $\mathcal{S}\mathrm{p}_{\geq 0} \subset \mathcal{S}\mathrm{p}$. Therefore, the sequence above is fiber if the fiber in $\mathcal{S}\mathrm{p}$ of the second map is connective. By the homotopy long exact sequence, this is equivalent to the map on π_0, $[P',X] \to [P,X]$, being an epimorphism. Since X satisfies equation 2, this map is

$$\mathrm{Hom}_{E_*E}(E_*P', E_*X) \longrightarrow \mathrm{Hom}_{E_*E}(E_*P, E_*X),$$

which is an epimorphism, because $E_*P \to E_*P'$ is a monomorphism by the E-homology long exact sequence. $\square$

We obtain the following consequence.

Lemma 5.6. *Let P, I be finite E-projective spectra. If I satisfies equation 2, then the filtration $F_E^\star(P,I)$ from Construction 5.1 is given by*

$$\cdots \longrightarrow \tau_{\geq n+1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I) \longrightarrow \tau_{\geq n}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I) \longrightarrow \tau_{\geq n-1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I) \longrightarrow \cdots$$

Proof. The presheaf of spectra $P \mapsto \tau_{\geq n}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I)$ is already a sheaf by Lemma 5.5. Recall from Construction 5.1 that the degree n term of the filtered spectrum $F_E^\star(P,I)$ is the sheafification of the mentioned presheaf. □

Proof of (3). It followed from Lemma 5.4 that the filtration $F_E^\star(P,X)$ in the statement of the theorem induces a spectral sequence isomorphic to that of $\lim_\Delta F_E^\star(P,I^\bullet)$. By Lemma 5.6, the latter unfolds to the filtered spectrum

$$\cdots \longrightarrow \lim_{\Delta}(\tau_{\geq n+1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I^\bullet)) \longrightarrow \lim_{\Delta}(\tau_{\geq n}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I^\bullet)) \longrightarrow \lim_{\Delta}(\tau_{\geq n-1}\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I^\bullet)) \longrightarrow \cdots,$$

where the connective covers are taken degree-wise in each cosimplicial spectrum. This is the *décalage* of the spectral sequence given by the cosimplicial spectrum

$$\operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I^0) \rightrightarrows \operatorname{map}_{\mathcal{S}\mathrm{p}}(P,I^1) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots.$$

The spectral sequences associated to the décalage and to this cosimplicial spectrum agree up to reindexing [Lev15, 6.3]. The latter is exactly what gives rise to the E-based Adams spectral sequence – see Remark 5.8. □

5.1. **Remarks on the Adams resolution**

Remark 5.7. Let $\mathcal{A}$ be an abelian category, in our case compactly generated Grothendieck abelian, so that $\mathcal{D}(\mathcal{A})$ exists. The dual Dold-Kan correspondence gives an equivalence

$$N\colon \operatorname{Fun}(\Delta,\mathcal{A}) \xrightarrow{\simeq} \operatorname{Ch}_{\leq 0}(\mathcal{A}).$$

Under the canonical map $\operatorname{Ch}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$, the image of a cosimplicial object M in $\mathcal{A}$ identifies with the totalization $\lim_\Delta M^\bullet$ in $\mathcal{D}(\mathcal{A})$.

To see this, consider the homotopy spectral sequence associated to the cosimplicial object:

$$E_1^{s,t} \simeq H_{t-s}(\Omega^s N^s M^\bullet) \simeq N^s H_t(M^\bullet) \Longrightarrow H_{t-s}(\lim_{\Delta} M^\bullet).$$

Since $M^\bullet$ takes values in $\mathcal{D}(\mathcal{A})^\heartsuit \simeq \mathcal{A}$, the E_1-page is concentrated on the 0-th row as the chain complex

$$N(M^\bullet) := \quad 0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots$$

where the differentials are the alternating sum of the face maps of M. This is the object of $\operatorname{Ch}(\mathcal{A})$ which corresponds to M under the Dold-Kan correspondence. We are done, because the convergence of the spectral sequence gives a quasi-isomorphism between the above complex and $\lim_\Delta M^\bullet \in \mathcal{D}(\mathcal{A})$.

Remark 5.8. Let X a spectrum. The filtration of X that Adams constructs in [Ada74, 15.1] – yielding the Adams spectral sequence by mapping into it – corresponds to a cosimplicial spectrum $X \otimes E^{\otimes[-]}\colon \Delta \to \mathcal{S}\mathrm{p}$ under the Dold-Kan correspondence – see [MNN17, §2.1].

After applying E_*-homology, the filtration of X gives an acyclic resolution of E_*X and yields a computation of Ext groups, as Adams showed – using Remark 5.9. Therefore, the limit of the cosimplicial diagram $E_*(X \otimes E^{\otimes[-]})$ in $\mathcal{D}(\mathcal{A})$ identifies with the mentioned acyclic resolution of E_*X, by Remark 5.7, and thus with E_*X itself. In other words, the augmented cosimplicial diagram $E_*(X \otimes E^{\otimes[-]})\colon \Delta_+ \to \mathcal{D}(\mathcal{A})$ is a limit diagram.

Remark 5.9. The spectra I^n are spectra I which have the property that for any E-projective spectrum P, the canonical morphism sending a map to the one it induces on E-homology,

$$I^\star(P) \xrightarrow{\simeq} \mathrm{Hom}^\star_{E_*E}(E_*P, E_*I), \tag{2}$$

is an isomorphism. This holds for any spectrum of the form $I = E \otimes Y$, because of the assumption that E is Adams-type: the map factors as the composite of

$$(E \otimes Y)^\star(P) \xrightarrow{\simeq} \mathrm{Hom}^\star_{\pi_*E}(E_*P, E_*Y),$$

which is an isomorphism because it is an edge-homomorphism in a spectral sequence

$$\mathrm{Ext}^{p,\star}_{\pi_*E}(E_*X, E_*Y) \Longrightarrow (E \otimes Y)^\star(P)$$

which converges by the assumption that E is Adams-type, and collapses because of the assumption that E_*X is projective over π_*X – see [Ada74, 13.6]; and the inverse of

$$\mathrm{Hom}^\star_{\pi_*E}(E_*P, E_*Y) \xleftarrow{\simeq} \mathrm{Hom}^\star_{E_*E}(E_*P, E_*(E \otimes Y)),$$

which is an isomorphism due to the fact that $E_*(E \otimes Y) \simeq E_*E \otimes_{\pi_*E} E_*Y$, using that E_*E is flat as a right module over π_*E, implied by the Adams-type assumption, and using the Künneth spectral sequence similarly.

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