SDS 383D, Exercises 2: Bayes and the Gaussian Linear Model

Jan-Michael Cabrera

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A simple Gaussian location model

(A) We have

$$p(\theta) \propto \left(1 + \frac{1}{\nu}\frac{(x-m)^2}{s^2}\right)^{-\frac{\nu+1}{2}}$$

$$p(\theta, \omega) \propto \omega^{(d+1)/2-1}\exp\left(-\omega\frac{k(\theta-\mu)^2}{2}\right)\exp\left(-\omega\frac{\eta}{2}\right)$$

$$p(\theta) \propto \int p(\theta, \omega)d\omega$$

$$\propto \int \omega^{a-1}\exp(-\omega b)d\omega$$

$a = \frac{d+1}{2}, b = \frac{k(\theta-\mu)^2}{2} + \frac{\eta}{2}$

$$p(\theta) \propto \frac{\Gamma(a)}{b^a}\int \frac{b^a}{\Gamma(a)}\omega^{a-1}\exp(-\omega b)d\omega$$

$$\propto \Gamma\left(\frac{d+1}{2}\right)\left[\frac{\eta}{2} + \frac{k(\theta-\mu)^2}{2}\right]^{-\frac{d+1}{2}}$$

$$\propto \left[\frac{\eta}{2}\left(1 + \frac{kd(\theta-\mu)^2}{2d\eta/2}\right)\right]^{-\frac{d+1}{2}}$$

$$\propto \left(1 + \frac{1}{d}\frac{(\theta-\mu)^2}{(d\eta/k)}\right)^{-\frac{d+1}{2}}$$

$$\nu = d$$
$$x = \theta$$
$$s^2 = \frac{d\eta}{k}$$
$$m = \mu$$

(B) Show that $p(\theta, \omega|\mathbf{y})$ has the form

$$p(\theta, \omega|\mathbf{y}) \propto \omega^{(d^*+1)/2-1}\exp\left[-\omega\frac{k^*(\theta-\mu^*)^2}{2}\right]\exp\left[-\omega\frac{\eta^*}{2}\right]$$

let

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$$p(\mathbf{y}|\theta, \omega) \propto \omega^{n/2}\exp\left[-\omega\left(\frac{S_y + n(\bar{y} - \theta)^2}{2}\right)\right]$$

$$p(\theta, \omega|\mathbf{y}) \propto \omega^{(d+1)/2-1}\exp\left[-\omega\frac{(\theta - \mu)^2}{2}\right]\exp\left[-\omega\frac{\eta}{2}\right]\omega^{n/2}\exp\left[-\omega\left(\frac{S_y + n(\bar{y} - \theta)^2}{2}\right)\right]$$

$$\propto \omega^{(d+n+1)/2-1}\exp\left[-\frac{\omega}{2}\left(k(\theta - \mu)^2 + \eta + S_y + n(\bar{y} - \theta)^2\right)\right]$$

$$k(\theta - \mu)^2 + \eta + S_y + n(\bar{y} - \theta)^2 = k\theta^2 - 2k\mu\theta + k\mu^2 + \eta + S_y + n\bar{y}^2 - 2n\theta\bar{y} + n\theta^2$$
$$= (k+n)\theta^2 + (-2k\mu - 2n\bar{y})\theta + k\mu^2 + \eta + S_y + n\bar{y}^2$$

$$ax^2 + bx + c = a(x - h)^2 + l$$

$h = -\frac{b}{2a}, \; l = c - ah^2$

$$(k+n)\theta^2 + (-2k\mu - 2n\bar{y})\theta + k\mu^2 + \eta + S_y + n\bar{y}^2 =$$
$$(k+n)\left(\theta - \frac{k\mu + n\bar{y}}{k+n}\right)^2 + k\mu^2 + \eta + S_y + n\bar{y}^2 - (k+n)\left(\frac{k\mu + n\bar{y}}{k+n}\right)^2 =$$
$$(k+n)\left(\theta - \frac{k\mu + n\bar{y}}{k+n}\right)^2 + \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y$$

$$p(\mathbf{y}|\theta, \omega) \propto \omega^{(d+n+1)/2-1}\exp\left[-\frac{\omega}{2}\left((k+n)\left(\theta - \frac{k\mu + n\bar{y}}{k+n}\right)^2 + \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y\right)\right]$$

$$\propto \omega^{(d+n+1)/2-1}\exp\left[-\frac{\omega}{2}\left((k+n)\left(\theta - \frac{k\mu + n\bar{y}}{k+n}\right)^2\right)\right]\exp\left[-\frac{\omega}{2}\left(\frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y\right)\right]$$

$$d^* = d + n$$
$$k^* = k + n$$
$$\mu^* = \frac{k\mu + n\bar{y}}{k+n}$$
$$\eta^* = \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y$$

(C) What is $p(\theta|\mathbf{y}, \omega)$?

$$p(\theta|\mathbf{y}, \omega) = \int_0^\infty p(\theta, \omega|\mathbf{y})d\omega$$

$$\propto \exp\left[-\frac{\omega}{2}\left((k+n)\left(\theta - \frac{k\mu + n\bar{y}}{k+n}\right)^2\right)\right]\int_0^\infty \omega^{(d+n+1)/2-1}\exp\left[-\frac{\omega}{2}\left(\frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y\right)\right]d\omega$$

$$(\theta|\mathbf{y}, \omega) \sim N\left(\frac{k\mu + n\bar{y}}{k+n}, (\omega(k+n))^{-1}\right)$$

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(D) What is $p(\omega|\mathbf{y})$?

$$p(\omega|\mathbf{y}) = \int_0^\infty p(\theta, \omega|\mathbf{y})d\theta$$

$$\propto \omega^{(d+n+1)/2-1}\exp\left[-\frac{\omega}{2}\left(\frac{nk(\mu-\bar{y})^2}{k+n} + \eta + S_y\right)\right]\int_{-\infty}^\infty \exp\left[-\frac{\omega}{2}\left((k+n)\left(\theta - \frac{k\mu+n\bar{y}}{k+n}\right)^2\right)\right]d\theta$$

$$\omega|\mathbf{y} \sim \text{Gamma}\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)$$

$d^* = d + n$, $\eta^* = \frac{nk(\mu-\bar{y})^2}{k+n} + \eta + S_y$

(E) From part A we had

$$p(\theta) \propto \left(1 + \frac{1}{d}\frac{(\theta-\mu)^2}{(d\eta/k)}\right)^{-\frac{d+1}{2}}$$

by construction

$$p(\theta|\mathbf{y}) \propto \left(1 + \frac{1}{d^*}\frac{(\theta-\mu^*)^2}{(d^*\eta^*/k^*)}\right)^{-\frac{d^*+1}{2}}$$

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