SDS 383D, Bayesian Inference in Simple Conjugate Families

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B: Change of variables for a gamma random variable

$$p(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) \tag{1}$$

$$x_1 \sim \text{Ga}(a_1, 1) \tag{2}$$
$$x_2 \sim \text{Ga}(a_2, 1) \tag{3}$$

$$f_{\mathbf{X}}(x_1, x_2) = \frac{x^{a_1-1}}{\Gamma(a_1)} \frac{x^{a_2-1}}{\Gamma(a_2)} x_1^{a_1-1} x_2^{a_2-1} e^{-x_1} e^{-x_2} \tag{4}$$

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}[\mathbf{h}(\mathbf{y})] \cdot |\det(\nabla \mathbf{h}(\mathbf{y}))| \tag{5}$$

$$y_1 = \frac{x_1}{(x_1 + x_2)} \tag{6}$$
$$y_2 = x_1 + x_2 \tag{7}$$

$$x_1 = y_1 y_2 \tag{8}$$
$$x_2 = y_2 - y_1 y_2 \tag{9}$$

$$\det(\nabla \mathbf{h}(\mathbf{y})) = \det \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = y_2 \tag{10}$$

$$f_{\mathbf{Y}}(y_1, y_2) = \frac{(y_1 y_2)^{a_1-1}}{\Gamma(a_1)} \frac{(y_2(1-y_1))^{a_2-1}}{\Gamma(a_2)} (y_1 y_2)^{a_1-1} (y_2(1-y_1))^{a_2-1} e^{-y_1 y_2} e^{-(y_2 - y_1 y_2)} y_2 \tag{11}$$

$$f_{\mathbf{Y}}(y_1, y_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1-1} (1-y_1)^{a_2-1} \cdot \frac{1}{\Gamma(a_1 + a_2)} y_2^{a_1+a_2-1} e^{-y_2} \tag{12}$$

$$y_1 \sim \text{Beta}(a_1, a_2) \tag{13}$$
$$y_2 \sim \text{Ga}(a_1 + a_2, 1) \tag{14}$$