# SDS 383D, Exercises 2: Bayes and the Gaussian Linear Model

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## A simple Gaussian location model

(A) We have

$$p(\theta) \propto \left(1 + \frac{1}{\nu}\frac{(x-m)^2}{s^2}\right)^{-\frac{\nu+1}{2}}$$

$$p(\theta, \omega) \propto \omega^{(d+1)/2-1} \exp\left(-\omega \frac{k(\theta-\mu)^2}{2}\right) \exp\left(-\omega\frac{\eta}{2}\right)$$

$$\begin{aligned} p(\theta) &\propto \int p(\theta,\omega) d\omega \\ &\propto \int \omega^{a-1} \exp(-\omega b) d\omega \end{aligned}$$

$a = \frac{d+1}{2}$, $b = \frac{k(\theta-\mu)^2}{2} + \frac{\eta}{2}$

$$\begin{aligned} p(\theta) &\propto \frac{\Gamma(a)}{b^a} \int \frac{b^a}{\Gamma(a)} \omega^{a-1} \exp(-\omega b) d\omega \\ &\propto \Gamma\left(\frac{d+1}{2}\right) \left[\frac{\eta}{2} + \frac{k(\theta-\mu)^2}{2}\right]^{-\frac{d+1}{2}} \\ &\propto \left[\frac{\eta}{2}\left(1 + \frac{kd(\theta-\mu)^2}{2d\eta/2}\right)\right]^{-\frac{d+1}{2}} \\ &\propto \left(1 + \frac{1}{d}\frac{(\theta-\mu)^2}{(d\eta/k)}\right)^{-\frac{d+1}{2}} \end{aligned}$$

$$\begin{aligned} \nu &= d \\ x &= \theta \\ s^2 &= \frac{d\eta}{k} \\ m &= \mu \end{aligned}$$

(B) Show that $p(\theta, \omega|\mathbf{y})$ has the form

$$p(\theta, \omega|\mathbf{y}) \propto \omega^{(d^*+1)/2-1} \exp\left[-\omega \frac{k^*(\theta-\mu^*)^2}{2}\right] \exp\left[-\omega\frac{\eta^*}{2}\right]$$

let

$$p(\mathbf{y}|\theta,\omega) \propto \omega^{n/2}\text{exp}\left[-\omega\left(\frac{S_y + n(\bar{y}-\theta)}{2}\right)\right]$$

$$\begin{aligned}
p(\theta,\omega|\mathbf{y}) &\propto \omega^{(d+1)/2-1}\text{exp}\left[-\omega\frac{(\theta-\mu)^2}{2}\right]\text{exp}\left[-\omega\frac{\eta}{2}\right]\omega^{n/2}\text{exp}\left[-\omega\left(\frac{S_y+n(\bar{y}-\theta)^2}{2}\right)\right] \\
&\propto \omega^{(d+n+1)/2-1}\text{exp}\left[-\frac{\omega}{2}\left(k(\theta-\mu)^2+\eta+S_y+n(\bar{y}-\theta)^2\right)\right]
\end{aligned}$$

$$\begin{aligned}
k(\theta-\mu)^2+\eta+S_y+n(\bar{y}-\theta)^2 &= k\theta^2-2k\mu\theta+k\mu^2+\eta+S_y+n\bar{y}^2-2n\theta\bar{y}+n\theta^2 \\
&= (k+n)\theta^2+(-2k\mu-2n\bar{y})\theta+k\mu^2+\eta+S_y+n\bar{y}^2
\end{aligned}$$

$$ax^2+bx+c = a(x-h)^2+l$$

$h = -\frac{b}{2a}$, $l = c - ah^2$

$$\begin{aligned}
&(k+n)\theta^2+(-2k\mu-2n\bar{y})\theta+k\mu^2+\eta+S_y+n\bar{y}^2 = \\
&(k+n)\left(\theta-\frac{k\mu+n\bar{y}}{k+n}\right)^2+k\mu^2+\eta+S_y+n\bar{y}^2-(k+n)\left(\frac{k\mu+n\bar{y}}{k+n}\right)^2 = \\
&(k+n)\left(\theta-\frac{k\mu+n\bar{y}}{k+n}\right)^2+\frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y
\end{aligned}$$

$$\begin{aligned}
p(\mathbf{y}|\theta,\omega) &\propto \omega^{(d+n+1)/2-1}\text{exp}\left[-\frac{\omega}{2}\left((k+n)\left(\theta-\frac{k\mu+n\bar{y}}{k+n}\right)^2+\frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y\right)\right] \\
&\propto \omega^{(d+n+1)/2-1}\text{exp}\left[-\frac{\omega}{2}\left((k+n)\left(\theta-\frac{k\mu+n\bar{y}}{k+n}\right)^2\right)\right]\text{exp}\left[-\frac{\omega}{2}\left(\frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y\right)\right]
\end{aligned}$$

$$\begin{aligned}
d^* &= d+n \\
k^* &= k+n \\
\mu^* &= \frac{k\mu+n\bar{y}}{k+n} \\
\eta^* &= \frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y
\end{aligned}$$

(C) What is $p(\theta|\mathbf{y},\omega)$?

$$\begin{aligned}
p(\theta|\mathbf{y},\omega) &= \int_0^\infty p(\theta,\omega|\mathbf{y})d\omega \\
&\propto \text{exp}\left[-\frac{\omega}{2}\left((k+n)\left(\theta-\frac{k\mu+n\bar{y}}{k+n}\right)^2\right)\right]\int_0^\infty \omega^{(d+n+1)/2-1}\text{exp}\left[-\frac{\omega}{2}\left(\frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y\right)\right]d\omega
\end{aligned}$$

$$(\theta|\mathbf{y},\omega) \sim \text{N}\left(\frac{k\mu+n\bar{y}}{k+n},(\omega(k+n))^{-1}\right)$$

(D) What is $p(\omega|\mathbf{y})$?

$$p(\omega|\mathbf{y}) = \int_0^\infty p(\theta, \omega|\mathbf{y})d\theta$$

$$\propto \omega^{(d+n+1)/2-1}\text{exp}\left[-\frac{\omega}{2}\left(\frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y\right)\right]\int_{-\infty}^{\infty}\text{exp}\left[-\frac{\omega}{2}\left((k+n)\left(\theta-\frac{k\mu+n\bar{y}}{k+n}\right)^2\right)\right]d\theta$$

$$\omega|\mathbf{y} \sim \text{Gamma}\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)$$

$d^* = d+n$, $\eta^* = \frac{nk(\mu-\bar{y})^2}{k+n}+\eta+S_y$

(E) From part A we had

$$p(\theta) \propto \left(1+\frac{1}{d}\frac{(\theta-\mu)^2}{(d\eta/k)}\right)^{-\frac{d+1}{2}}$$

by construction

$$p(\theta|\mathbf{y}) \propto \left(1+\frac{1}{d^*}\frac{(\theta-\mu^*)^2}{(d^*\eta^*/k^*)}\right)^{-\frac{d^*+1}{2}}$$