

SDS 383D, Exercises 2: Bayes and the Gaussian Linear Model

Jan-Michael Cabrera

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A simple Gaussian location model

(A) Show that the marginal prior, $p(\theta)$ takes on the following form:

$$p(\theta) \propto \left(1 + \frac{1}{\nu} \frac{(x - m)^2}{s^2}\right)^{-\frac{\nu+1}{2}}$$
$$p(\theta, \omega) \propto \omega^{(d+1)/2-1} \exp\left(-\omega \frac{k(\theta - \mu)^2}{2}\right) \exp\left(-\omega \frac{\eta}{2}\right)$$

$$p(\theta) \propto \int p(\theta, \omega) d\omega$$
$$\propto \int \omega^{a-1} \exp(-\omega b) d\omega$$

$$a = \frac{d+1}{2}, b = \frac{k(\theta - \mu)^2}{2} + \frac{\eta}{2}$$

$$p(\theta) \propto \frac{\Gamma(a)}{b^a} \int \frac{b^a}{\Gamma(a)} \omega^{a-1} \exp(-\omega b) d\omega$$
$$\propto \Gamma\left(\frac{d+1}{2}\right) \left[\frac{\eta}{2} + \frac{k(\theta - \mu)^2}{2}\right]^{-\frac{d+1}{2}}$$
$$\propto \left[\frac{\eta}{2} \left(1 + \frac{kd(\theta - \mu)^2}{2d\eta/2}\right)\right]^{-\frac{d+1}{2}}$$
$$\propto \left(1 + \frac{1}{d} \frac{(\theta - \mu)^2}{(d\eta/k)}\right)^{-\frac{d+1}{2}}$$

$$\nu = d$$
$$x = \theta$$
$$s^2 = \frac{d\eta}{k}$$
$$m = \mu$$

(B) Show that $p(\theta, \omega|\mathbf{y})$ has the form

$$p(\theta, \omega|\mathbf{y}) \propto \omega^{(d^*+1)/2-1} \exp\left[-\omega \frac{k^*(\theta - \mu^*)^2}{2}\right] \exp\left[-\omega \frac{\eta^*}{2}\right]$$

let

$$p(\mathbf{y}|\theta, \omega) \propto \omega^{n/2} \exp \left[-\omega \left(\frac{S_y + n(\bar{y} - \theta)^2}{2} \right) \right]$$

$$\begin{aligned} p(\theta, \omega|\mathbf{y}) &\propto \omega^{(d+1)/2-1} \exp \left[-\omega \frac{(\theta - \mu)^2}{2} \right] \exp \left[-\omega \frac{\eta}{2} \right] \omega^{n/2} \exp \left[-\omega \left(\frac{S_y + n(\bar{y} - \theta)^2}{2} \right) \right] \\ &\propto \omega^{(d+n+1)/2-1} \exp \left[-\frac{\omega}{2} (k(\theta - \mu)^2 + \eta + S_y + n(\bar{y} - \theta)^2) \right] \end{aligned}$$

$$\begin{aligned} k(\theta - \mu)^2 + \eta + S_y + n(\bar{y} - \theta)^2 &= k\theta^2 - 2k\mu\theta + k\mu^2 + \eta + S_y + n\bar{y}^2 - 2n\theta\bar{y} + n\theta^2 \\ &= (k+n)\theta^2 + (-2k\mu - 2n\bar{y})\theta + k\mu^2 + \eta + S_y + n\bar{y}^2 \end{aligned}$$

$$ax^2 + bx + c = a(x - h)^2 + l$$

$$h = -\frac{b}{2a}, l = c - ah^2$$

$$\begin{aligned} (k+n)\theta^2 + (-2k\mu - 2n\bar{y})\theta + k\mu^2 + \eta + S_y + n\bar{y}^2 &= \\ (k+n) \left(\theta - \frac{k\mu + n\bar{y}}{k+n} \right)^2 + k\mu^2 + \eta + S_y + n\bar{y}^2 - (k+n) \left(\frac{k\mu + n\bar{y}}{k+n} \right)^2 &= \\ (k+n) \left(\theta - \frac{k\mu + n\bar{y}}{k+n} \right)^2 + \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y & \end{aligned}$$

$$\begin{aligned} p(\mathbf{y}|\theta, \omega) &\propto \omega^{(d+n+1)/2-1} \exp \left[-\frac{\omega}{2} \left((k+n) \left(\theta - \frac{k\mu + n\bar{y}}{k+n} \right)^2 + \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y \right) \right] \\ &\propto \omega^{(d+n+1)/2-1} \exp \left[-\frac{\omega}{2} \left((k+n) \left(\theta - \frac{k\mu + n\bar{y}}{k+n} \right)^2 \right) \right] \exp \left[-\frac{\omega}{2} \left(\frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y \right) \right] \end{aligned}$$

$$d^* = d + n$$

$$k^* = k + n$$

$$\mu^* = \frac{k\mu + n\bar{y}}{k+n}$$

$$\eta^* = \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y$$

(C) From the joint posterior, what is the conditional posterior distribution $p(\theta|\mathbf{y}, \omega)$?

$$\begin{aligned} p(\theta|\mathbf{y}, \omega) &= \int_0^\infty p(\theta, \omega|\mathbf{y}) d\omega \\ &\propto \exp \left[-\frac{\omega}{2} \left((k+n) \left(\theta - \frac{k\mu + n\bar{y}}{k+n} \right)^2 \right) \right] \int_0^\infty \omega^{(d+n+1)/2-1} \exp \left[-\frac{\omega}{2} \left(\frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y \right) \right] d\omega \end{aligned}$$

$$(\theta|\mathbf{y}, \omega) \sim \mathcal{N} \left(\frac{k\mu + n\bar{y}}{k+n}, (\omega(k+n))^{-1} \right)$$

(D) From the joint posterior, what is the marginal posterior distribution $p(\omega|\mathbf{y})$?

$$\begin{aligned}
p(\omega|\mathbf{y}) &= \int_0^\infty p(\theta, \omega|\mathbf{y}) d\theta \\
&\propto \omega^{(d+n+1)/2-1} \exp \left[-\frac{\omega}{2} \left(\frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y \right) \right] \int_{-\infty}^\infty \exp \left[-\frac{\omega}{2} \left((k+n) \left(\theta - \frac{k\mu + n\bar{y}}{k+n} \right)^2 \right) \right] d\theta \\
\omega|\mathbf{y} &\sim \text{Gamma} \left(\frac{d^*}{2}, \frac{\eta^*}{2} \right)
\end{aligned}$$

$$d^* = d + n, \eta^* = \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y$$

(E) Show that the marginal posterior $p(\theta|\mathbf{y})$ takes the form of a centered, scaled t distribution and express the parameters in terms of the four parameters of the normal-gamma posterior for (θ, ω) .

$$p(\theta) \propto \left(1 + \frac{1}{d} \frac{(\theta - \mu)^2}{(d\eta/k)} \right)^{-\frac{d+1}{2}}$$

by construction

$$p(\theta|\mathbf{y}) \propto \left(1 + \frac{1}{d^*} \frac{(\theta - \mu^*)^2}{(d^*\eta^*/k^*)} \right)^{-\frac{d^*+1}{2}}$$

(F) True or false: in the limit as the prior parameters k , d , and η approach zero, the priors $p(\theta)$ and $p(\omega)$ are valid probability distributions.

$$\begin{aligned}
p(\theta) &\propto \left[1 + \frac{1}{d} \frac{(\theta - \mu)^2}{\eta/kd} \right]^{-\frac{d+1}{2}} \\
\lim_{k,d,\eta \rightarrow 0} \left[1 + \frac{1}{d} \frac{(\theta - \mu)^2}{\eta/kd} \right]^{-\frac{d+1}{2}} &= \lim_{k,d,\eta \rightarrow 0} \left[1 + \frac{k(\theta - \mu)^2}{\eta} \right]^{-\frac{d+1}{2}} \\
&= \left[1 + \frac{0(\theta - \mu)^2}{0} \right]^{-\frac{1}{2}} \\
p(\omega) &\propto \omega^{d/2-1} \exp \left(-\omega \frac{\eta}{2} \right) \\
\lim_{d,\eta \rightarrow 0} \omega^{d/2-1} \exp \left(-\omega \frac{\eta}{2} \right) &= \omega^{-1}
\end{aligned}$$

False

(G) True or false: in the limit as the prior parameters k , d , and η approach zero, the posteriors $p(\theta|\mathbf{y})$ and $p(\omega|\mathbf{y})$ are valid probability distributions.

$$\begin{aligned}
d^* = n + d &\xrightarrow{d \rightarrow 0} d^* = n \\
k^* = k + n &\xrightarrow{k \rightarrow 0} k^* = n \\
\mu^* = \frac{k\mu + n\bar{y}}{k+n} &\xrightarrow{k \rightarrow 0} \mu^* = \bar{y} \\
\eta^* = \frac{nk(\mu - \bar{y})^2}{k+n} + \eta + S_y &\xrightarrow{k,\eta \rightarrow 0} \eta^* = S_y
\end{aligned}$$

$$p(\theta|\mathbf{y}) \propto \left[1 + \frac{1}{n} \frac{(\theta - \bar{y})^2}{S_y/n^2} \right]^{-\frac{n+1}{2}}$$

$$p(\omega|\mathbf{y}) \propto \omega^{n/2-1} \exp\left(-\omega \frac{S_y}{2}\right)$$

- (H) True or false: In the limit as the prior parameters k , d , and η approach zero, the Bayesian credible interval for θ becomes identical to the classical (frequentist) confidence interval for θ at the same confidence level.

$$\theta \in m \pm t^* \cdot s$$

$$m = \mu^* \rightarrow \bar{y} \quad \text{for } k, d, \eta \rightarrow 0$$

$$s^{*2} = \frac{S_y}{n^2} \rightarrow s = \frac{1}{n} \left[\sum_{i=1}^n (y_i - \bar{y})^2 \right]^{1/2}$$

$$\theta \in \bar{y} \pm t^* \cdot \frac{1}{n} \left[\sum_{i=1}^n (y_i - \bar{y})^2 \right]^{1/2}$$

The Conjugate Gaussian Linear Model

- (A) Derive the conditional posterior $p(\beta|\mathbf{y}, \omega)$

$$\beta|\omega \sim N(m, (\omega K)^{-1})$$

$$\omega \sim \text{Gamma}\left(\frac{d}{2}, \frac{\eta}{2}\right)$$

$$\mathbf{y}|\beta, \omega \sim N(X\beta, (\omega \Lambda)^{-1})$$

first find

$$p(\beta, \omega|\mathbf{y}) \propto p(\omega)p(\beta|\omega)p(\mathbf{y}|\beta, \omega)$$

$$\propto \omega^{d/2-1} \exp\left(-\omega \frac{\eta}{2}\right) \omega^{p/2} \exp\left[-\frac{\omega}{2}(\beta - m)^T K(\beta - m)\right] \omega^{n/2} \exp\left[-\frac{\omega}{2}(y - X\beta)^T \Lambda(y - X\beta)\right]$$

$$\propto \omega^{\frac{d+p+n}{2}-1} \exp\left[-\frac{\omega}{2}((\beta - m)^T K(\beta - m) + (y - X\beta)^T \Lambda(y - X\beta) + \eta)\right]$$

$$\begin{aligned} & (\beta - m)^T K(\beta - m) + (y - X\beta)^T \Lambda(y - X\beta) + \eta \\ &= \beta^T K\beta - \beta^T K m - m^T K\beta + m^T K m + y^T \Lambda y - y^T \Lambda X\beta - \beta^T X^T \Lambda y + \beta^T X^T \Lambda X\beta + \eta \\ &= \beta^T K\beta - 2\beta^T K m + m^T K m + y^T \Lambda y - 2\beta^T X^T \Lambda y + \beta^T X^T \Lambda X\beta + \eta \\ &= \beta^T (K + X^T \Lambda X)\beta - 2\beta^T (K m + X^T \Lambda y) + m^T K m + y^T \Lambda y + \eta \end{aligned}$$

$$Q^T A Q + Q^T b + c = (Q - h)^T A (Q - h) + k, \text{ with } h = -\frac{1}{2} A^{-1} b \text{ and } k = c - \frac{1}{4} b^T A^{-1} b$$

$$\text{let } A = (K + X^T \Lambda X) = K^*, \quad b = -2(K m + X^T \Lambda y), \text{ and } c = m^T K m + y^T \Lambda y + \eta$$

$$\begin{aligned}
h &= -\frac{1}{2}(K + X^T \Lambda X)^{-1} \cdot (-2(Km + X^T \Lambda y)) \\
&= (K + X^T \Lambda X)^{-1}(Km + X^T \Lambda y) = m^*
\end{aligned}$$

$$\begin{aligned}
&\beta^T(K + X^T \Lambda X)\beta - 2\beta^T(Km + X^T \Lambda y) + m^T Km + y^T \Lambda y + \eta \\
&= (\beta - m^*)^T K^*(\beta - m) + m^T Km + y^T \Lambda y + \eta - (Km + X^T \Lambda y)^T K^{*-1}(Km + X^T \Lambda y)
\end{aligned}$$

$$\begin{aligned}
p(\beta, \omega | \mathbf{y}) &\propto \omega^{\frac{d+p+n}{2}-1} \exp \left[-\frac{\omega}{2} \left((\beta - m^*)^T K^*(\beta - m) + m^T Km + y^T \Lambda y + \eta - (Km + X^T \Lambda y)^T K^{*-1}(Km + X^T \Lambda y) \right) \right] \\
&\propto \omega^{\frac{d^*}{2}-1} \exp \left[-\omega \frac{\eta^*}{2} \right] \omega^{\frac{p}{2}} \exp \left[-\frac{\omega}{2} (\beta - m^*)^T K^*(\beta - m^*) \right]
\end{aligned}$$

$$\begin{aligned}
d^* &= d + n \\
K^* &= (K + X^T \Lambda X) \\
\eta^* &= m^T Km + y^T \Lambda y + \eta - (Km + X^T \Lambda y)^T K^{*-1}(Km + X^T \Lambda y) \\
m^* &= (K + X^T \Lambda X)^{-1}(Km + X^T \Lambda y)
\end{aligned}$$

$$\beta | \omega, \mathbf{y} \sim N(m^*, (\omega K^*)^{-1})$$

(B) Derive the marginal posterior $p(\omega | \mathbf{y})$

$$\begin{aligned}
p(\omega | \mathbf{y}) &\propto \int_{-\infty}^{\infty} p(\beta, \omega | \mathbf{y}) d\beta \\
&\propto \omega^{\frac{d^*}{2}-1} \exp \left[-\omega \frac{\eta^*}{2} \right] \int_{-\infty}^{\infty} \omega^{\frac{p}{2}} \exp \left[-\frac{\omega}{2} (\beta - m^*)^T K^*(\beta - m^*) \right] \\
&\propto \omega^{\frac{d^*}{2}-1} \exp \left[-\omega \frac{\eta^*}{2} \right]
\end{aligned}$$

$$\omega | \mathbf{y} \sim \text{Gamma} \left(\frac{d^*}{2}, \frac{\eta^*}{2} \right)$$

$$\begin{aligned}
d^* &= d + n \\
K^* &= (K + X^T \Lambda X) \\
\eta^* &= m^T Km + y^T \Lambda y + \eta - (Km + X^T \Lambda y)^T K^{*-1}(Km + X^T \Lambda y)
\end{aligned}$$

(C) Putting these together, derive the marginal posterior $p(\beta | \mathbf{y})$

$$\begin{aligned}
p(\beta|\mathbf{y}) &\propto \int_0^\infty p(\beta, \omega|\mathbf{y}) d\omega \\
&\propto \int_0^\infty \omega^{\frac{d^*}{2}-1} \exp\left[-\omega \frac{\eta^*}{2}\right] \omega^{\frac{p}{2}} \exp\left[-\frac{\omega}{2}(\beta - m^*)^T K^*(\beta - m^*)\right] d\omega \\
&\propto \int_0^\infty \omega^a \exp(-b\omega) d\omega
\end{aligned}$$

with $a = \frac{d^*+p}{2}$ and $b = \frac{1}{2}(\eta^* + (\beta - m^*)^T K^*(\beta - m^*))$

$$\begin{aligned}
p(\beta|\mathbf{y}) &\propto \frac{\Gamma(a)}{b^a} \int_0^\infty \frac{b^a}{\Gamma(a)} \omega^a \exp(-b\omega) d\omega \\
&\propto \Gamma(a) b^{-a} \\
&\propto [\eta^* + (\beta - m^*)^T K^*(\beta - m^*)]^{-\frac{d^*+p}{2}} \\
&\propto \left[1 + \frac{1}{d^*} \frac{(\beta - m^*)^T K^*(\beta - m^*)}{\eta^*/d^*}\right]^{-\frac{d^*+p}{2}}
\end{aligned}$$