SDS 383D, Bayesian Inference in Simple Conjugate Families

Jan-Michael Cabrera, JC7858

February 1, 2019

C: Normal prior and sampling distribution with unknown mean and known variance

$$p(\theta|x_1, ..., x_n) \propto p(\theta|m, v)p(x_1, ..., x_n|\theta) \tag{1}$$

$$p(\theta|m, v) = \left(\frac{1}{2\pi v}\right)^{1/2} \exp\left(-\frac{1}{2v}(\theta - m)^2\right) \tag{2}$$

$$p(x_i|\theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}(x_i - \theta)^2\right) \tag{3}$$

$$p(x_1, ..., x_n|\theta) = \prod_i^n p(x_i|\theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_i^n (x_i - \theta)^2\right) \tag{4}$$

$$p(\theta|x_1, ..., x_n) \propto \exp\left(-\frac{1}{2\sigma^2}(x_i - \theta)^2\right) \exp\left(-\frac{1}{2\sigma^2}\sum_i^n (x_i - \theta)^2\right) \tag{5}$$

$$\sum_i^n (x_i - \theta)^2 = \sum_i^n x_i^2 - 2\theta\sum_i^n x_i + \theta^2; \quad \bar{x} = \frac{1}{n}\sum_i^n x_i \tag{6}$$

$$p(\theta|x_1, ..., x_n) \propto \exp\left(-\frac{n}{2\sigma^2}(-2\theta\bar{x} + \theta^2) - \frac{1}{2v}(\theta^2 - 2\theta m)\right) \tag{7}$$

$$-\frac{1}{2}\left[\left(\frac{n}{\sigma^2} + \frac{1}{v}\right)\theta^2 - 2\left(\frac{n}{\sigma^2}\bar{x} + \frac{1}{v}m\right)\theta\right] \tag{8}$$

$$a = \frac{n}{\sigma^2} + \frac{1}{v}; \qquad b = \frac{n}{\sigma^2}\bar{x} + \frac{1}{v}m \tag{9}$$

$$-\frac{a}{2}\left[\theta^2 - 2\frac{b}{a}\theta\right] = -\frac{a}{2}\left[\theta^2 - 2\frac{b}{a}\theta + \frac{b^2}{a^2} - \frac{b^2}{a^2}\right] \tag{10}$$

$$p(\theta|x_1, ..., x_n) \propto \exp\left[-\frac{a}{2}\left(\theta - \frac{b}{a}\right)^2\right] \tag{11}$$

$$\theta|x_1, ..., x_n \sim \mathrm{N}\left(\frac{\frac{n}{\sigma^2}\bar{x} + \frac{1}{v}m}{\frac{n}{\sigma^2} + \frac{1}{v}}, \left[\frac{n}{\sigma^2} + \frac{1}{v}\right]^{-1}\right) \tag{12}$$