SDS 383D Conditionals and Marginals

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C: Conditional Multivariate Normal

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{\int p(x_1, x_2)dx_1} \propto p(x_1, x_2) \tag{1}$$

$$p(x_1, x_2) \propto \exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right] \tag{2}$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = \left[(x_1-\mu_1)^T, (x_2-\mu_2)^T\right]\begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{12}^T & \Omega_{22}\end{bmatrix}\begin{bmatrix}(x_1-\mu_1)\\(x_2-\mu_2)\end{bmatrix} \tag{3}$$

$$= \left[(x_1-\mu_1)^T\Omega_{11} + (x_2-\mu_2)^T\Omega_{12}^T, (x_1-\mu_1)^T\Omega_{12} + (x_2-\mu_2)^T\Omega_{22}\right]\begin{bmatrix}(x_1-\mu_1)\\(x_2-\mu_2)\end{bmatrix} \tag{4}$$

$$= ((x_1-\mu_1)^T\Omega_{11} + (x_2-\mu_2)^T\Omega_{12}^T)(x_1-\mu_1) + ((x_1-\mu_1)^T\Omega_{12} + (x_2-\mu_2)^T\Omega_{22})(x_2-\mu_2) \tag{5}$$

$$= (x_1-\mu_1)^T\Omega_{11}(x_1-\mu_1) + 2(x_1-\mu_1)^T\Omega_{12}^T(x_2-\mu_2) + ... \tag{6}$$

$$x^TAx + x^Tb = (x-h)^TA(x-h) + k \quad \text{where} \quad h = -\frac{1}{2}A^{-1}b \quad \text{and} \quad k = -\frac{1}{4}b^TA^{-1}b \tag{7}$$

$$(x_1-\mu_1)^TA(x_1-\mu_1) + (x_1-\mu_1)b = (x_1-\mu_1-h)^TA(x_1-\mu_1-h) + ... \tag{8}$$

$$\text{where} \quad A = \Sigma_{11} \quad \text{and} \quad b = 2\Sigma_{12}^T(x_2-\mu_2) \tag{9}$$

$$h = -\frac{1}{2}A^{-1}b = -\frac{1}{2}\Omega_{11}^{-1}2\Omega_{12}(x_2-\mu_2) \tag{10}$$

$$= -\Omega_{11}^{-1}\Omega_{12}^T(x_2-\mu_2) \quad \text{where} \quad \Omega_{12}^T = -\Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{11} \tag{11}$$

$$= \Sigma_{22}^{-1}\Sigma_{12}^T(x_2-\mu_2) \tag{12}$$

$$(x_1-\mu_1-h)^TA(x_1-\mu_1-h) = (x_1-\mu^*)^T\Sigma_{11}^*(x_1-\mu^*) \tag{13}$$

$$\mu^* = \mu_1 + \Sigma_{22}^{-1}\Sigma_{12}^T(x_2 - \mu_2) \tag{14}$$

$$\Sigma_{11}^* = \Omega_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T \tag{15}$$

$$x_1|x_2 \sim \mathrm{N}(\mu_1 + \Sigma_{22}^{-1}\Sigma_{12}^T(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T) \tag{16}$$