EE241 SPRING 2015: TUTORIAL #10

Friday, April 3, 2015

PROBLEM 1 (Linear independence): Which of the following sets are linearly independent?

(a) $\{0, 0, 0, 0, 0, 0\}$
(b) $\{0, 1, 2, 3, 4, 5\}$
(c) $\{[1, 1], [1, -1]\}$
(d) $\{[2, 5, 4], [3, 4, 5], [6, 4, 3], [1, 0, 0]\}$
(e) $\{[12, -11, 7, -8], [8, -12, 0, -10], [-4, -3, -9, -4]\}$

Solution. Recall the definition of linear independence: There is no non-trivial linear combination of the $\vec{v}_1 \ldots \vec{v}_k$ that yields the zero vector. In other words, there are *no* non-zero $c_1, \ldots, c_k$ such that

$$c_1\vec{v}_1 + \ldots c_k\vec{v}_k = 0$$

(a) This set is **linearly dependent** since *every* choice of linear combination yields 0.
(b) This set is **linearly dependent** since $1 \cdot 1 + (-1/2) \cdot 2 = 0$
(c) For only two vectors it is always each to check that no c_1, c_2 exist such that

$$c_1[1, 1] = -c_2[1, -1]$$

(d) For multiple and higher-dimensional vectors, it is best to cast the question as a matrix problem. Let $\vec{c} = [c_1, \ldots, c_k]$, let also A be the matrix formed by using $\vec{v}_1, \ldots, \vec{v}_k$ as its columns, then if there exists a non-zero $\vec{c}$ such that $A\vec{c} = \vec{0}$, the set is linearly dependent. In other words, the set is linearly dependent if
(e) $\{[12, -11, 7, -8], [8, -12, 0 - 10], [-4, -3, -9, -4]\}$

$\square$

PROBLEM 2 (From practice midterm 2): Consider the space P_3 of all polynomials of degree less than or equal to 3, and the following set of polynomials:

$$S = \left\{p_1(t) = t^3/6 + t^2/2 + t + 1, p_2(t) = t^2/2 + t + 1, p_3(t) = t + 1, p4(t) = 1\right\}.$$

Show that S is a basis for P_3.

Solution. For S to be a basis for P_3 it must satisfy two conditions: (a) that S be linearly independent and (b) that S span P_3. Let's start with (a). Consider representing the polynomials as vectors according to their coefficients, i.e.:

$$\begin{aligned}\vec{p}_1 &= [1/6, 1/2, 1, 1]\\ \vec{p}_2 &= [0, 1/2, 1, 1]\\ \vec{p}_3 &= [0, 0, 1, 1]\\ \vec{p}_4 &= [0, 0, 0, 1]\end{aligned}$$

We can now form the matrix with these vectors as its columns

$$A = \left[\vec{p}_1^T \mid \vec{p}_2^T \mid \vec{p}_3^T \mid \vec{p}_4^T\right] = \begin{bmatrix} 1/6 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly, the reduced row-echelon form of A is I_4. Since I_4 has no free-variables, and thus no nullspace, the columns of A are linearly independent and we can satisfy (a) that S **is** linearly independent.

The long way to go about showing (b) is to construct a new fixed but arbitrary polynomial $p(x) = ax^3 + bx^2 + cx + d$ and set

$$p(x) = \sum_{k=1}^{4} c_k p_k(x).$$

Then we could solve for c_1, c_2, c_3, c_4 in terms of a, b, c, d. However! there is a much easier way to show that (b) is satisfied using the vector representation above. Since A is full rank (i.e.: has 4 pivots and is equal to I_4 in RREF), we know that A^{-1} exists. Thus let the vector representation of $p(x)$ be $\vec{p} = [a, b, c, d]$ and we have that S spans P_3 if $A\vec{c} = \vec{p}$ has a solution. Since A^{-1} exists, then indeed we just set $\vec{c} = A^{-1}\vec{p}$ and we are done. Finally, S **is a basis for** P_3. □

PROBLEM 3 (Nullspace bound): Write a lower bound for the dimension of the nullspace of a $m \times n$ matrix where $m \leq n$.

Solution. For an $m \times n$ matrix, $rank(A)$ (the number of pivots in reduced row-echelon form) is at most $\min\{m, n\}$, since there cannot be more pivots than rows or columns. The dimension of the nullspace is given by the number of free variables in the reduced row-echelon form of the matrix. The number of free variables is $n - rank(A)$. Thus if $rank(A) \leq \min\{m, n\}$ then nullity$(A) \geq n - \min\{m, n\}$ and since $m \leq n$ then

$$\text{nullity}(A) \geq n - m$$

□