EE241 SPRING 2015: TUTORIAL #14

Friday, May 1, 2015

PROBLEM 1 (Symmetric matrices and orthogonal diagonalization): Diagonalize the following symmetric matrix

$$A = \begin{bmatrix} 3/2 & -1/2 & 0 & 0 \\ -1/2 & 3/2 & 0 & 0 \\ 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 3/2 & 1/2 \end{bmatrix}.$$

Solution. We begin with the characteristic equation

$$det\,(A - \lambda I_4) = 0.$$

We can use the following property for determinants of block-diagonal matrices (like the one above)

$$det\left(\begin{bmatrix} A & & & \\ & B & & \\ & & \ddots & \\ & & & Z \end{bmatrix}\right) = det\,(A)\,det\,(B)\dots det\,(Z)$$

where A, B, $\dots$ Z are square matrices of any size. In our case:

$$\begin{aligned}
0 &= det\,(A - \lambda I_4) \\
0 &= det\left(\begin{bmatrix} 3/2-\lambda & -1/2 \\ -1/2 & 3/2-\lambda \end{bmatrix}\right) det\left(\begin{bmatrix} 1/2-\lambda & 3/2 \\ 3/2 & 1/2-\lambda \end{bmatrix}\right) \\
0 &= \left(\left(\frac{3}{2}-\lambda\right)^2 - \frac{1}{4}\right)\left(\left(\frac{1}{2}-\lambda\right)^2 - \frac{9}{4}\right) \\
0 &= \left(2 - 3\lambda + \lambda^2\right)\left(-2 - \lambda + \lambda^2\right) \\
0 &= (2-\lambda)\,(1-\lambda)\,(-2+\lambda)\,(1+\lambda)
\end{aligned}$$

and $\lambda = 2, 1, 2, -1$. Let's now find the eigenvectors, first for $\lambda = -1$, find the one vector in the nullspace of the matrix $A - (-1)I_4$, or

$$\begin{aligned}
\vec{v}_{-1} \in \text{null}&\left(\begin{bmatrix} 5/2 & -1/2 & 0 & 0 \\ -1/2 & 5/2 & 0 & 0 \\ 0 & 0 & 3/2 & 3/2 \\ 0 & 0 & 3/2 & 3/2 \end{bmatrix}\right) \\
= \text{null}&\left(\begin{bmatrix} 0 & 12 & 0 & 0 \\ -1/2 & 5/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
= \text{null}&\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
= \text{null}&\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)
\end{aligned}$$

where in each line we've use some row operations to reduce the matrix. Thus, $\vec{v}_{-1} = \left[0, 0, 1/\sqrt{2}, -1/\sqrt{2}\right]$ (recall we must use the normalized vector). For the eigenvalue $\lambda = 1$ we find $\vec{v}_1 = \left[1/\sqrt{2}, 1/\sqrt{2}, 0, 0\right]$. Now, for eigenvalue $\lambda = 2$ we see something a little different,

$$\begin{aligned}
\vec{v}_2 \in \text{null}\left(\begin{bmatrix} -1/2 & -1/2 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & -3/2 & 3/2 \\ 0 & 0 & 3/2 & -3/2 \end{bmatrix}\right) \\
= \text{null}\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)
\end{aligned}$$

with two free variables, this nullspace is a two-dimensional space! Thus, we must choose two eigenvectors, the obvious choices are

$$\begin{aligned}
\vec{v}_2^{(1)} &= \left[1/\sqrt{2}, -1/\sqrt{2}, 0, 0\right], \\
\vec{v}_2^{(2)} &= \left[0, 0, 1/\sqrt{2}, 1/\sqrt{2}\right]
\end{aligned}$$

although any two vectors forming an orthonormal basis for the nullspace of $A - 2I_4$ would have worked. Now, pairing together the eigenvalues and their eigenvectors in the correct order we can write

$$A = \left[\begin{array}{c|c|c|c} \vec{v}_{-1} & \vec{v}_1 & \vec{v}_2^{(1)} & \vec{v}_2^{(2)} \end{array}\right] \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \left[\begin{array}{c|c|c|c} \vec{v}_{-1} & \vec{v}_1 & \vec{v}_2^{(1)} & \vec{v}_2^{(2)} \end{array}\right]^T$$

$$A = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

□

PROBLEM 2 (Orthogonal complements and projections into a subspace): Project the vector $\vec{v} = [1, 1, 1, 1]$ onto the orthogonal complement of the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 3 \\ -1 & -2 & -5 & -5 \end{bmatrix}.$$

Solution. We can go about solving this in two ways. We could (A) find a basis for the nullspace, use that to find a basis for the orthogonal complement, then project onto it or (B) notice that the *rowspace* is orthogonal to the nullspace and simply project onto that. We can find an orthonormal basis for the rowspace by using row operations on A, for example

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of this matrix are still not, however, orthonormal. Consider using one step of the Gram-Schmidt process:

$$\begin{aligned} \vec{v}_2 &= \vec{v}_2 - \text{proj}_{\vec{v}_1}\vec{v}_2 \\ &= [0,1,1,1] - \frac{2}{3}[1,0,1,1] \\ &= [-2/3, 1, 1/3, 1/3]\,. \end{aligned}$$

Now normalizing the two vectors

$$\begin{aligned} \hat{v}_1 &= \left[1/\sqrt{3}, 0, 1/\sqrt{3}, 1/\sqrt{3}\right] \\ \hat{v}_2 &= \left[-2\sqrt{3}/3\sqrt{5}, \sqrt{3/5}, 1/\sqrt{15}, 1/\sqrt{15}\right] \end{aligned}$$

The projection of $\vec{v}$ onto the orthogonal complement of the nullspace of A can now be written as

$$\text{proj}_{\text{null}(A)^\perp}\vec{v} = (\vec{v}\cdot\hat{v}_1)\,\hat{v}_1 + (\vec{v}\cdot\hat{v}_2)\,\hat{v}_2$$

which turns out to be

$$\text{proj}_{\text{null}(A)^\perp}\vec{v} = \sqrt{3}\hat{v}_1$$

□

PROBLEM 3 (Least squares): Find the parabola that is the least squares fit of the following four points $(-1,-1)$, $(0,-4)$, $(1,-4)$, $(2,-4)$.

Solution. Our model for the parabola will $y = ax^2 + bx + c$. This way, we can write a linear relationship between our parameters (a, b, and c) for every data point. The four points make 4 equations

$$\begin{cases} -1 &= a(-1)^2 + b(-1) + c(1) \\ -4 &= a(0)^2 + b(0) + c(1) \\ -4 &= a(1)^2 + b(1) + c(1) \\ -4 &= a(2)^2 + b(2) + c(1) \end{cases}$$

We can write this as a linear system

$$Y = X\vec{p} \quad \text{where} \quad Y = \begin{bmatrix} -1 \\ -4 \\ -4 \\ -4 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Of course, $Y = X\vec{p}$ has no solution (since our model does not fit our data exactly). However, the associated *normal equations* do. These are

$$X^T Y = X^T X \vec{p} \qquad \text{or} \qquad \begin{bmatrix} -21 \\ -11 \\ -13 \end{bmatrix} = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

which has the solution $a = 0.75$, $b = -1.65$, $c = -3.55$. □

PROBLEM 4 (Gram-Schmidt orthogonalization): Use the Gram-Schmidt method to construct an orthonormal basis for $\mathbb{R}^4$ starting with the following basis

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \right\}$$

Solution. Let us label the vectors in S as $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. We now follow the Gram-Schmidt to produce the orthonormal vectors $\hat{u}_i$ starting with $\vec{v}_1$,

$$\begin{aligned} \hat{u}_1 &= \frac{\vec{v}_1}{|\vec{v}_1|} \\ &= \left[1/\sqrt{2}, -1/\sqrt{2}, 0, 0\right] \end{aligned}$$

$$\begin{aligned} \vec{u}_2 &= \vec{v}_2 - (\hat{u}_1 \cdot \vec{v}_2)\, \hat{u}_1 \\ &= [0,1,2,0] - \left(-1/\sqrt{2}\right) \left[1/\sqrt{2}, -1/\sqrt{2}, 0, 0\right] \\ &= [0,1,2,0] + [1/2, -1/2, 0, 0] \\ &= [1/2, 1/2, 2, 0] \\ \hat{u}_2 &= \left[1/3\sqrt{2}, 1/3\sqrt{2}, 2\sqrt{2}/3, 0\right] \end{aligned}$$

$$\begin{aligned} \vec{u}_3 &= \vec{v}_3 - (\hat{u}_1 \cdot \vec{v}_3)\, \hat{u}_1 - (\hat{u}_2 \cdot \vec{v}_3)\, \hat{u}_2 \\ &= [0,0,1,-1] - \sqrt{8/9} \left[1/\sqrt{18}, 1/\sqrt{18}, \sqrt{8/9}, 0\right] \\ &= [0,0,1,-1] - [2/9, 2/9, 8/9, 0] \\ &= [-2/9, -2/9, 1/9, -1] \\ \hat{u}_3 &= \sqrt{3/10}\, [-2/9, -2/9, 1/9, -1] \end{aligned}$$

$$\begin{aligned} \vec{u}_4 &= \vec{v}_4 - (\hat{u}_1 \cdot \vec{v}_4)\, \hat{u}_1 - (\hat{u}_2 \cdot \vec{v}_4)\, \hat{u}_2 - (\hat{u}_3 \cdot \vec{v}_4)\, \hat{u}_3 \\ &= [2,0,0,1] - \sqrt{2} \left[1/\sqrt{2}, -1/\sqrt{2}, 0, 0\right] - \sqrt{2}/3 \left[1/3\sqrt{2}, 1/3\sqrt{2}, 2\sqrt{2}/3, 0\right] - (-13/30)\, [-2/9, -2/9, 1/9, -1] \\ &= [2,0,0,1] - [1,-1,0,0] - [1/9, 1/9, 4/9, 0] - [26/270, 26/270, 1/270, 13/30] \\ &= [107/135, 107/135, -121/270, 17/30] \\ \hat{u}_4 &= \sqrt{1543/2744}\, [107/135, 107/135, -121/270, 17/30] \end{aligned}$$

□

PROBLEM 5 (Eigenvalues and matrix exponentials): We can define the matrix exponential e^A via the Taylor series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

where $A^k = A \dots A$ (k-times) and $k! = k(k-1)\dots 1$. We also assume that $A^0 = I_n$. Using diagonalization of symmetric matrices, find e^A of

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution. First, let P be an orthonormal matrix and D be a diagonal matrix such that $A = PDP^T$ (i.e., A is diagonalized). Then we can write the matrix exponential above as

$$\begin{aligned}
e^A &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(PDP^T\right)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(PDP^T\right)\left(PDP^T\right)\dots\left(PDP^T\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} PDP^TPDP\dots PDP^T \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} PDI_nDI_n\dots I_nDP^T \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} PD^kP^T \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} P \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}^k P^T \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} P \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} P^T \\
&= P \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k \end{bmatrix} P^T \\
&= P \begin{bmatrix} e^{\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n} \end{bmatrix} P^T.
\end{aligned}$$

Thus, we can take the matrix exponent by simply exponentiating the eigenvalues in the diagonalized form. In our specific example we can use that

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -0.8507 & 0.5257 & 0 \\ 0.5257 & .8507 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2.2361 & 0 & 0 \\ 0 & 2.2361 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -0.8507 & 0.5257 & 0 \\ 0.5257 & .8507 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus e^A is given by taking the exponent of the each eigenvalue in the diagonal matrix

$$\begin{aligned} e^A &= \begin{bmatrix} -0.8507 & 0.5257 & 0 \\ 0.5257 & .8507 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1069 & 0 & 0 \\ 0 & 9.3565 & 0 \\ 0 & 0 & 0.3679 \end{bmatrix} \begin{bmatrix} -0.8507 & 0.5257 & 0 \\ 0.5257 & .8507 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2.6634 & 4.1365 & 0 \\ 4.1365 & 6.7999 & 0 \\ 0 & 0 & 0.3679 \end{bmatrix}. \end{aligned}$$

$\square$