EE241 SPRING 2015: TUTORIAL #8

Friday, March 13, 2015

PROBLEM 1: If $\vec{u}$ and $\vec{v}$ are related by angle θ, what is the angle between $\vec{u}/|\vec{u}|$ and $\vec{v}/|\vec{v}|$? What is the angle between $\vec{u}/|\vec{u}|$ and $\vec{v}$?

Solution. The angle between $\vec{u}/|\vec{u}|$ and $\vec{v}/|\vec{v}|$ is given by

$$\frac{\frac{\vec{u}}{|\vec{u}|}\cdot\frac{\vec{v}}{|\vec{v}|}}{\left|\frac{\vec{u}}{|\vec{u}|}\right|\left|\frac{\vec{v}}{|\vec{v}|}\right|} = \frac{\vec{u}}{|\vec{u}|}\cdot\frac{\vec{v}}{|\vec{v}|}\left(\frac{|\vec{u}|}{|\vec{u}|}\frac{|\vec{v}|}{|\vec{v}|}\right)^{-1} = \frac{\vec{u}}{|\vec{u}|}\cdot\frac{\vec{v}}{|\vec{v}|} = \cos\theta.$$

The angle between $\vec{u}/|\vec{u}|$ and $\vec{v}$ is given by

$$\frac{\frac{\vec{u}}{|\vec{u}|}\cdot\vec{v}}{\left|\frac{\vec{u}}{|\vec{u}|}\right||\vec{v}|} = \frac{\vec{u}}{|\vec{u}|}\cdot\frac{\vec{v}}{|\vec{v}|}\left(\frac{|\vec{u}|}{|\vec{u}|}\right)^{-1} = \frac{\vec{u}}{|\vec{u}|}\cdot\frac{\vec{v}}{|\vec{v}|} = \cos\theta.$$

Thus, normalizing either vector does not change the angles between them. □

PROBLEM 2: Find the set of all 3-dimensional vectors that are 30° away from $\vec{u} = [1,0,1]^T$ and orthogonal to $\vec{v} = [-3,1,0]^T$. **Hint:** Start by finding the set of all length-1 vectors (unit vectors) that satisfy these conditions.

Solution. Let $\vec{s} = [a,b,c]^T$ be any length-1 vector. Now apply the first condition

$$\frac{\vec{u}\cdot\vec{s}}{|\vec{u}||\vec{s}|} = \frac{a+c}{\sqrt{2}}$$

Note that $\cos(30^\circ) = \sqrt{3}/2$, thus

$$a + c = \sqrt{3/2}.$$

The second condition is that

$$\vec{v}\cdot\vec{s} = 0 \qquad \Longrightarrow \qquad -3a + b = 0.$$

Altogether

$$\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sqrt{3/2} \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sqrt{3/2} \\ 3\sqrt{3/2} \end{bmatrix}.$$

The solutions to this system are parametrized by t

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sqrt{3/2} \\ 3\sqrt{3/2} \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}.$$

Of course, the vector above is not length-1 for all values of t. However, we can now drop this requirement using arguments from Problem 1. The set of vectors that satisfy the conditions is just

$$S = \left\{\left[\sqrt{3/2}, 3\sqrt{3/2}, 0\right] + t\,[-1,-3,1] \;:\; t\in\mathbb{R}\right\}$$

□

PROBLEM 3: Find the approximate angle between the following two 100-dimensional vectors

$$\vec{u} = [1, -1, 1, -1, 1, -1, \ldots, 1, -1, 1, -1]$$
$$\vec{v} = \left[1, 1/2, 1/4, 1/8, 1/16, 1/32 \ldots, 2^{-96}, 2^{-97}, 2^{-98}, 2^{-99}\right].$$

What happens when the dimension of the above vectors grows (respecting the patterns above)?

Solution. First, note that

$$\begin{aligned} |\vec{u}| &= \sqrt{100} \\ &= 10 \\ |\vec{v}| &= \sqrt{\sum_{n=0}^{99} \left(\frac{1}{2^2}\right)^n} \\ &= \sqrt{\frac{1-(1/4)^{1000}}{1-1/4}} \\ &\approx 2/\sqrt{3} \end{aligned}$$

Their inner product yields

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \sum_{n=0}^{99} (-1/2)^n \\ &= \frac{1-(-1/2)^{100}}{1+1/2} \\ &\approx 2/3 \end{aligned}$$

Altogether

$$\cos\theta \approx \frac{2/3}{10 \cdot 2/\sqrt{3}} = \frac{1}{\sqrt{3} \cdot 10} \quad \Longrightarrow \quad \theta \approx 86.7^\circ$$

As the dimension of $\vec{u}$ and $\vec{v}$ grows, their internal angle approaches 90° and the vectors becomes nearly orthogonal. □

PROBLEM 4: Consider the following linear transformation

$$L\left(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} w+z \\ x+y \\ w+x+y+z \\ w-x-y+z \end{bmatrix}$$

What is the dimension of the range of L? How many equations define such a space? Give these equations.

Solution. The dimension of the range of L is equal to the number of pivots that the matrix associated with L contains. The matrix in question is

$$A_L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

By doing some quick row manipulations we can see that the reduced row echelon form of this matrix is

$$A_L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the range of L is a 2-dimensional space. Since we are in a 4-dimensional space, we will need two equations to restrict $\mathbb{R}^4$ to the range of L. In order to find a description of this space, consider using the Gauss-Jordan method on an augmented matrix

$$[A_L \mid I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right]$$

Let's identify the left and right parts of the above matrix as A'_L and B_L (Note that $A_L = B_L^{-1} A'_L$). If we let $[a, b, c, d]$ be a vector in the range of L then we have the relation

$$A'_L \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = B_L \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

This reduces to

$$\begin{bmatrix} w + z \\ x + y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ -a - b + c \\ -a + b + d \end{bmatrix}.$$

Thus, we can simply read off the last two equations since they do not contain w, x, y, or z. The equations that define our space are

$$\begin{aligned} 0 &= -a - b + c \\ 0 &= -a + b + d \end{aligned}$$

□

PROBLEM 5: Find the plane passing through the following three points $P_1 = [1, 1, 1]$, $P_2 = [1, 2, 3]$, and $P_3 = [2, 1, 1]$. Write it in a parametrized form and then again in terms of constrains.

Solution. Consider answering the question, "how do I get to a point in the plane from the origin?". First, you must traverse from the origin to one of the points, say P_1, and then walk in the direction of P_2 or P_3. Thus, any point in the plane can be written as

$$\vec{x} = \overrightarrow{P_1} + s\overrightarrow{P_1P_2} + t\overrightarrow{P_1P_3}$$

or

$$P = \left\{ \begin{bmatrix} 1 + t \\ 1 + s \\ 1 + 2s \end{bmatrix} \ : \ \forall s, t \in \mathbb{R} \right\}.$$

To write the constraint equation for this plane consider the following: take any point in the plane Q and draw a vector to P_1. This vector must be perpendicular to the normal vector of the plane. The normal vector of the plane is found by taking the cross product of $\overrightarrow{P_1P_2}$ with $\overrightarrow{P_1P_3}$,

$$\begin{aligned}\vec{n} &= \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} \\ &= [0, 2, -1]^T\end{aligned}$$

Now, if Q is at location $[x, y, z]^T$, then $\overrightarrow{P_1Q} = [x-1, y-1, z-1]^T$. In turn, this means that the condition for our plane is $\overrightarrow{P_1Q} \cdot \vec{n} = 0$, or

$$2y - z = 1$$

□

PROBLEM 6: A massive solar sail is deployed in space. The sail is shaped like a triangle and supported by three satellites. The satellites are reporting their positions relative to the international space station in metres. Satellite 1 is at $[-1000, 1500, 2000]$, satellite 2 is at $[1000, 1500, 0]$, satellite 3 is at $[500, -500, -500]$. What is the area of the solar sail? **Hint:** Consider a parallelogram-shaped solar sail first.

Solution. First, let's work in kilometers for easier notation, this means the satellites are at

$$\begin{aligned}S_1 &= [-1, 3/2, 2] \\ S_2 &= [1, 3/2, 0] \\ S_3 &= [1/2, -1/2, -1/2]\end{aligned}$$

If we consider a parallelogram-shaped sail, we can calculate it's area using the cross-product rule and the two vectors $\vec{u} = \overrightarrow{S_1S_2}$ and $\vec{v} = \overrightarrow{S_1S_3}$. First,

$$\begin{aligned}\vec{u} &= [2, 0, -2] \\ \vec{v} &= [3/2, -2, -5/2]\,.\end{aligned}$$

We can use the determinant method for calculating the cross product,

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 3/2 & -2 & -5/2 \end{vmatrix} &= \mathbf{i}\,(0 \cdot (-5/2) - (-2) \cdot (-2)) \\ &\quad - \mathbf{j}\,(2 \cdot (-5/2) - (3/2) \cdot (-2)) \\ &\quad + \mathbf{k}\,(2 \cdot (-2) - (3/2) \cdot 0) \\ &= [-4, -2, -4]\,.\end{aligned}$$

The area of a parallelogram sail is then $|\vec{u} \times \vec{v}| = 6\text{km}^2$. Since a triangle sail would be half of this area then the area of the solar sail is $\boxed{3\text{km}^2}$. □