EE241 Spring 2015: Tutorial #13

Friday, April 24, 2015

Problem 1 (Least squares): Find the least squares fit of the equation $y = ax + b$ for the points $(1, 2)$, $(2, 2)$, $(3, 2)$, and $(4, 3)$.

Solution. We can start with the exact fit equation of the form $X\vec{p} = Y$ where X and Y are our data and $\vec{p}$ are the parameters that fit them. The key is to treat the points enumerated above as the "constraints" and a and b as the variables we are solving for

$$X\vec{p} = Y \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

Notice that we've added the column of 1's that correspond to the coefficients on the variable b. Of course, the matrix equation we've written above has no solution. However! The associated *normal equations* do. We can get the normal equations by applying X^T to both sides of the equations above

$$X^T X\vec{p} = XY \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

This reduces to

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 24 \\ 9 \end{bmatrix}$$

which has the solution $a = 0.3$, $b = 1.5$. □

Problem 2 (Eigenvalues and eigenvectors practice): Find the eigenvalues of the following matrices

$$(a) \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Find the eigenvectors of matrix (a). **Aside:** In general, we can define eigen**functions** as well as eigenvectors. The derivative operator (d/dx) on real-valued functions $(f(x) : \mathbb{R} \to \mathbb{R})$ is a linear transformation. What functions are eigenfunctions of d/dx? What are their eigenvalues?

Solution. First, the eigenvalues

(a) The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 4 & -2 \\ 4 & 2-\lambda & 0 \\ -2 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (2-\lambda)^3 - 4\,(4(2-\lambda)) - 2\,(2(2-\lambda)) = 0$$

which means that

$$\begin{aligned} 0 &= (2-\lambda)\left((2-\lambda)^2 - 16 - 4\right) \\ 0 &= (2-\lambda)\left(4 - 4\lambda + \lambda^2 - 20\right) \\ 0 &= (2-\lambda)\left(\lambda^2 - 4\lambda - 16\right) \end{aligned}$$

Thus

$$\lambda = \begin{cases} 2, \\ 2 + \sqrt{20}, \\ 2 - \sqrt{20} \end{cases}$$

(b) In this case, the characteristic equation reduces much more quickly, consider

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 1 & 3-\lambda & 0 \\ 0 & 0 & 1 & 4-\lambda \end{vmatrix} \implies (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda) = 0$$

and so the eigenvalues are $\lambda = 1, 2, 3, 4$.

(c) In this case there is one eigenvalue repeated twice. The eigenvalues is 0.

To find the eigenvectors of matrix (a) We just need to solve for $\vec{v}$ in the three equations

$$\begin{aligned} (A - 2I_3)\,\vec{v}_1 &= \vec{0} \\ \left(A - (2+\sqrt{20})I_3\right)\vec{v}_2 &= \vec{0} \\ \left(A - (2-\sqrt{20})I_3\right)\vec{v}_3 &= \vec{0}. \end{aligned}$$

The three vectors (solved using MATLAB) are

$$\begin{aligned} \vec{v}_1 &= [0\ ,\ 0.4472\ ,\ 0.8944] \\ \vec{v}_2 &= [0.7071\ ,\ -0.6325\ ,\ 0.3162] \\ \vec{v}_3 &= [-0.7071\ ,\ -0.6325\ ,\ 0.3162]\,. \end{aligned}$$

As to the "aside" question on eigenfunctions, notice that

$$\frac{d}{dx} f(x) = \lambda f(x)$$

for $f(x) = e^{\lambda x}$. Thus the exponential function for *any* coefficient λ is an eigenfunction of the derivative operator with eigenvalue λ. □

PROBLEM 3 (Diagonalizing a matrix): Pick the diagonalizeable matrix below and find $A = PDP^T$ where D is diagonal and P is orthonormal.

$$(a)\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \quad (b)\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \quad (c)\begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution. In the above, only matrix (b) is symmetric and so can be diagonalized. To compute D we need the eigenvalues, to compute P we need the eigenvectors. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 3-\lambda & 3 \\ 2 & 3 & 3-\lambda \end{vmatrix} \implies (1-\lambda)\left((3-\lambda)^2 - 9\right) - 2\left(2(3-\lambda) - 6\right) + 2\left(6 - 2(3-\lambda)\right) = 0$$

Solving,

$$\begin{aligned} 0 &= (1-\lambda)\left((3-\lambda)^2 - 9\right) - 4\left(2(3-\lambda) - 6\right) \\ 0 &= (1-\lambda)\left((3-\lambda)^2 - 9\right) - 4\left(-2\lambda\right) \\ 0 &= (1-\lambda)\left(-6\lambda + \lambda^2\right) + 8\lambda \\ 0 &= \lambda\left((1-\lambda)(-6+\lambda) + 8\right) \\ 0 &= \lambda\left(-\lambda^2 + 7\lambda + 2\right) \end{aligned}$$

and the eigenvalues are $\lambda_1 = 0$ $\lambda_2 = (7+\sqrt{57})/2$ $\lambda_3 = (7-\sqrt{57})/2$. The eigenvectors come from solving the characteristic equation with each λ substituted in. That is,

$$\begin{aligned} A\vec{v}_1 &= \vec{0} \\ \left(A - (7+\sqrt{57})/2I_3\right)\vec{v}_2 &= \vec{0} \\ \left(A - (7-\sqrt{57})/2I_3\right)\vec{v}_3 &= \vec{0}. \end{aligned}$$

Note that $\vec{v}_1$ is simply the normalized vector that defines the nullspace of A. The three vectors are

$$\begin{aligned} \vec{v}_1 &= \left[0\ ,\ 1/\sqrt{2}\ ,\ -1/\sqrt{2}\right] \\ \vec{v}_2 &= [0.4109\ ,\ 0.6446\ ,\ 0.6446] \\ \vec{v}_3 &= [0.9117\ ,\ -0.2906\ ,\ -0.2906]\,. \end{aligned}$$

□

PROBLEM 4 (Adjacency matrices for social networks): Recall that for a graph G the adjacency matrix is given by

$$[A_G]_{ij} = \begin{cases} 1 & i,j \text{ connected} \\ 0 & \text{otherwise} \end{cases}$$

(assume no self-loops). We'll now also add to these definitions, the *degree matrix* D_G which is defined as

$$[D_G]_{ij} = \begin{cases} d_i & i = j \\ 0 & \text{otherwise} \end{cases}$$

where d_i is the number of connections at node i. Finally, if we take the difference of these two matrices we get the *graph Laplacian* L_G,

$$L_G = D_G - A_G$$

The number of times 0 appears as an eigenvalue in the Laplacian is the number of connected components in the graph. Connected components are like "islands" in social networks, it is impossible to be introduced to someone on another island via mutual friends. How many 0 eigenvalues are there in the following matrix?

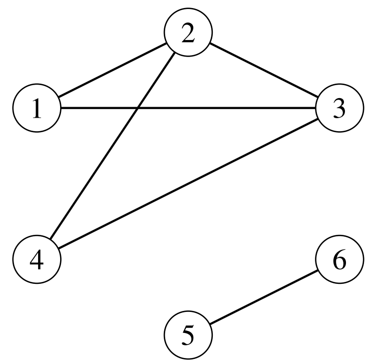
$$L_G = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Sketch the graph it represents. What fundamental matrix quantity does the number of 0 eigenvalues correspond to?

Solution. In the matrix L_G there are two connected components and therefore there are two 0 eigenvalues. We can check this fact by putting the matrix in MatLAB to get the eigenvalues (via the EIG function)

$$\lambda = 0, 0, 2, 2, 4, 4.$$

The graph associated with this matrix can be easily drawn by looking at the off-diagonal elements of L_G since they correspond to A_G,



The fundamental matrix quantity related to the number of 0 eigenvalues (and indeed, the number of connected components of the associated graph) is the *dimension of the nullspace*. Indeed, by using the NULL function in MatLab we get two linearly independent vectors

$$\vec{v}_1 = [1/2, 1/2, 1/2, 1/2, 0, 0] \quad \text{and} \quad \vec{v}_2 = \left[0, 0, 0, 0, 1/\sqrt{2}, 1/\sqrt{2}\right].$$

$\square$