EE241 SPRING 2015: TUTORIAL #5

Friday, Feb. 13, 2015

PROBLEM 1: Fill in the following table with True ($\checkmark$), False (leave blank), or Possible (P).

System	RREF	Pivots	Solutions	$m < n$	$m = n$	$m > n$
$A\vec{x} = \vec{b}$	A has row(s) of 0 in RREF	$[A, b]$ has a pivot in the last column	No solution	$\checkmark$	$\checkmark$	$\checkmark$
			1 solution			
			∞ solutions			
		$[A, b]$ has no pivots in the last column	No solution			
			1 solution			P
			∞ solutions	$\checkmark$	$\checkmark$	P
	A no row(s) of 0 in RREF	$[A, b]$ has a pivot in the last column	No solution	$\checkmark$	$\checkmark$	$\checkmark$
			1 solution			
			∞ solutions			
		$[A, b]$ has no pivots in the last column	No solution			
			1 solution		$\checkmark$	
			∞ solutions	$\checkmark$		
$A\vec{x} = \vec{0}$	A has row(s) of 0 in RREF		No solution			
			1 solution			P
			∞ solutions	$\checkmark$	$\checkmark$	P
	A no row(s) of 0 in RREF		No solution			
			1 solution		$\checkmark$	
			∞ solutions	$\checkmark$		

PROBLEM 2 (Fitting): The first three *physicists' Hermite polynomials* are the following

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2. \end{aligned}$$

Find a linear combination of these polynomials such that the resulting function passes through $(0, 0)$ $(1, 0)$, and the first derivative at $x = 1$ is -1.

Solution. First, let $f(x) = b_0 H_0(x) + b_1 H_1(x) + b_2 H_2(x)$. The three conditions we have are that $f(0) = 0$, $f(1) = 0$, and the $df/dx|_1 = -1$. Consider the first derivative of $f(x)$,

$$\begin{aligned} \frac{df}{dx} &= b_0 \frac{dH_0}{dx} + b_1 \frac{dH_1}{dx} + b_2 \frac{dH_2}{dx} \\ &= b_0 \cdot 0 + b_1 \cdot 2 + b_2 \cdot (8x) \\ &= 2b_1 H_0(x) + 4b_2 H_1(x) \end{aligned}$$

In the last line, I've rewritten the polynomial in terms of the original $H_i(x)$ to show that the derivative operation doesn't produce functions that cannot be written as a linear combination of $H_i(x)$. Now our three conditions are

$$\begin{cases} 0 = b_0 - 2b_2 & \text{from} \quad f(0) = 0 \\ 0 = b_0 + 2b_1 + 2b_2 & \text{from} \quad f(1) = 0 \\ -1 = 2b_1 + 8b_2 & \text{from} \quad df/dx|_1 = -1 \end{cases}$$

The linear system for $\vec{b} = [b_0, b_1, b_2]$ can be solved by row reduction on

$$\left[A|\vec{f}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 2 & 8 & -1 \end{array}\right] \quad \longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/4 \end{array}\right]$$

Thus $\vec{b} = [-1/2, 1/2, -1/4]$ and

$$\boxed{f(x) = -1/2 H_0(x) + 1/2 H_1(x) - 1/4 H_2(x) = x - x^2.}$$

□

PROBLEM 3 (Permutations): Group the following permutations into even and odd,

$$\{1, 2, 4, 3\} \ , \ \{1, 4, 2, 3\} \ , \ \{4, 1, 2, 3\} \ , \ \{4, 1, 3, 2\} \ , \ \{4, 3, 1, 2\} \ , \ \{4, 3, 2, 1\}$$

Solution. The first permutation is odd (swap 3 and 4) and every other permutation in the list results from one swap on the previous permutation. Thus, the groups are

$$\begin{aligned} \text{even} &= \{1, 4, 2, 3\} \ , \ \{4, 1, 3, 2\} \ , \ \{4, 3, 2, 1\} \\ \text{odd} &= \{1, 2, 4, 3\} \ , \ \{4, 1, 2, 3\} \ , \ \{4, 3, 1, 2\} \,. \end{aligned}$$

□

PROBLEM 4 (Vector lengths): Which is longer: a n-dimensional vector of 1's? Or a $2n$-dimensional vector of $1/2$?

Solution. A n-dimensional vector of 1's has length

$$\sqrt{\sum_{i=1}^{n} (1)^2} = \sqrt{n}.$$

A $2n$-dimensional vector of $1/2$ has length

$$\sqrt{\sum_{i=1}^{2n}(1/2)^2} = 1/2 \cdot \sqrt{2n} = \sqrt{n/2}.$$

Thus, the vector of 1's is longer. □

PROBLEM 5 (Determinants): Find the determinant of the following matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Solution. First, reorder the rows to get as close as possible to upper-triangular form,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now subtract the first row and add the third row to the fifth row to get

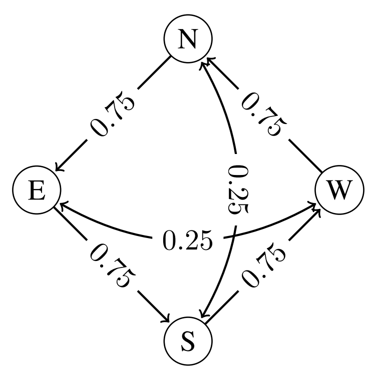
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now subtract the fourth row from the fifth to get

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there is a row of 0's, the determinant itself is also 0. □

PROBLEM 6: The graph below represents the following process: You are steering a ship probabilistically. Your first mate flips two fair coins every half-an-hour. If he gets two heads, you turn the ship around. Otherwise, you turn the ship 90° "port" (left). This graph contains no self-loops, i.e.: a "stay" is not a valid move.



(a) What is the matrix M representing this Markov process? This matrix should transform the vector $\vec{p} = [p_N, p_E, p_S, p_W]$ to new probabilities $\vec{p}'$ according to the rules of the coin flips. (To check your answer, you can apply the matrix to $\vec{p} = [1,0,0,0], [0,1,0,0], \dots$).
(b) You have no idea which direction you initially set sail in ($\vec{p}$). However, after an hour and a half, you estimate the following probabilities for your direction $\vec{p}' = [9/64, 27/64, 27/64, 1/64]$. Which direction did you set sail in?

Solution. (1) The (i,j) entry of the matrix M are the probability of transitioning from direction j (column, "input") to direction i (row, "output"). Thus

$$M = \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 1/4 & 0 & 0 & 3/4 \\ 3/4 & 1/4 & 0 & 0 \end{bmatrix}.$$

(2) After an hour and a half we've made 3 transitions, so we need to calculate M^3. To help with our mental math, let's calculate N^3 instead where $N = 4M$ (we won't need to worry about denominators).

$$\begin{aligned} N^3 &= \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 9 & 6 \\ 6 & 1 & 0 & 9 \\ 9 & 6 & 1 & 0 \\ 0 & 9 & 6 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 9 & 1 & 27 \\ 27 & 27 & 9 & 1 \\ 1 & 27 & 27 & 9 \\ 9 & 1 & 27 & 27 \end{bmatrix} \end{aligned}$$

Now $M^3 = N^3/4^3$ so

$$M^3 = \begin{bmatrix} 27/64 & 9/64 & 1/64 & 27/64 \\ 27/64 & 27/64 & 9/64 & 1/64 \\ 1/64 & 27/64 & 27/64 & 9/64 \\ 9/64 & 1/64 & 27/64 & 27/64 \end{bmatrix}$$

Our original direction and our new estimated direction are related by $\vec{p}' = M^3\vec{p}$. The natural thing to do would be to find $(M^3)^{-1}$ and solve $\vec{p} = (M^3)^{-1}\vec{p}'$. However, we can use a special fact about $\vec{p}$ to simplify our work. Since $\vec{p}$ represents only one of four directions, we can test the four associated vectors to check which of them solves $\vec{p}' = M^3\vec{p}$. The four directions are

$$\vec{p} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Each of these vectos, in order, returns the first, second, third, and fourth column of M^3 when we calculate $M^3\vec{p}$. Thus, we just need to find *which* column of M^3 is identical to $\vec{p}'$. This happens to be the second column, and so we conclude that we must have initially been heading East.

□