EE241 SPRING 2015: TUTORIAL #11

Friday, April 10, 2015

PROBLEM 1 (Complementary basis): Find a set S of vectors in $\mathbb{R}^4$ such that $W = S$ is a 3-dimensional subspace of $\mathbb{R}^4$ that does not include the vector $\vec{v}_0 = [1, 1, 1, 1]$. Then, find a basis for the vector space of all vectors orthogonal to $\vec{v}_0$

Solution. For the first part, we can simply choose

$$\begin{aligned} S &= \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} \\ &= \{[1, 0, 0, 0] \ , \ [0, 1, 0, 0] \ , \ [0, 0, 1, 0]\} \,. \end{aligned}$$

Note that $W \subset \mathbb{R}^4$ but $\vec{v}_0 W$, which can be seen by just checking the last component. However, W clearly contains vectors that are non orthogonal to $\vec{v}_0$. For example, none of the $\vec{u}_i$ even have inner-product 0 with it. One way to make the new set S' such that $W' \perp \vec{v}_0$ is to simply subtract from each $\vec{u}_i$ the projection of $\vec{v}_0$, i.e.:

$$\vec{u}_i' = \vec{u}_i - \frac{\vec{u}_i \cdot \vec{v}_0}{|\vec{u}_i||\vec{v}_0|} \frac{\vec{v}_0}{|\vec{v}_0|}$$

Or

$$\begin{aligned} \vec{u}_1' &= [1, 0, 0, 0] - \frac{1}{4} [1, 1, 1, 1] \\ &= \left[\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right] \\ \vec{u}_2' &= \left[-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right] \\ \vec{u}_3' &= \left[-\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}\right] \end{aligned}$$

Finally, to show that for any $\vec{w} \in W'$ we have that $\vec{w} \cdot \vec{v}_0 = 0$, we can simply write $\vec{w}$ as a linear combination of the new basis S',

$$\begin{aligned} \vec{w} \cdot \vec{v}_0 &= \left(w_1 \vec{u}_1' + w_2 \vec{u}_2' + w_3 \vec{u}_3'\right) \cdot \vec{v}_0 \\ &= w_1 \vec{u}_1' \cdot \vec{v}_0 + w_2 \vec{u}_2' \cdot \vec{v}_0 + w_3 \vec{u}_3' \cdot \vec{v}_0 \\ &= 0 + 0 + 0 \\ &= 0. \end{aligned}$$

□

PROBLEM 2 (Change of basis): Write the vector $\vec{v} = [1, -2, 1]$ in the basis $S = \{[1, 1, 0], [1, 0, 1], [0, 1, 1]\}$

Solution. To write the vector $\vec{v}$ in the new basis S, we must solve for coefficients v_1, v_2, v_3 in the equation

$$v_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Equivalently, we can solve

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} v_1 v_2 v_3 [] = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

From MATLAB (or a simple application of Gauss-Jordan) we find that

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Thus $[\vec{v}]_S = A^{-1}\vec{v} = [-1, 2, -1]$. □

PROBLEM 3 (Checking linear independence with functions): Is the set $\{2, 4\sin^2(x), \cos^2(x)\}$ linearly dependent or independent?

Solution. To check for linear independence, we attempt to solve the following equation for non-zero c_1, c_2, c_3,

$$c_1 \cdot 2 + c_2 \cdot 4\sin^2(x) + c_3 \cdot \cos^2(x) = 0 \quad \forall x \in \mathbb{R}.$$

Recall that $\cos^2(x) = 1 - \sin^2(x)$, so

$$\begin{aligned} c_1 \cdot 2 + c_2 \cdot 4\sin^2(x) + c_3 \cdot \left(1 - \sin^2(x)\right) &= 0 && \forall x \in \mathbb{R} \\ (2c_1 + c_3) \cdot 1 + (4c_2 - c_3) \cdot \sin^2(x) &= 0 && \forall x \in \mathbb{R} \end{aligned}$$

Thus, if we set $c_1 = -2$, $c_2 = 1$, and $c_3 = 4$ we see that the set of functions is indeed linearly dependent. □

PROBLEM 4 (Matrix bases): Find a basis for 2×2 symmetric matrices where each element in the set has determinant 1 or -1. Then, write the the following matrix as a linear combination of the elments of this basis

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

Solution. First, let's write out the conditions for a 2×2 matrix to have determinant ± 1. Consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then we must have that $ad - bc = \pm 1$. Furthermore, since we are dealing with symmetric matrices, we also need that $b = c$ so really we have that $ad - b^2 = \pm 1$. Our strategy for choosing basis elements should be to keep as many of the terms 0 as possible. For example, if we choose a to be 0 then we must have that $b^2 = \mp 1$ which means that $b = 1$. Our first element can be

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now we can try the case where $a = 1$ and we are guaranteed to generate a matrix that is linearly independent from the one above. If $a = 1$ then $d - b^2 = \pm 1$. Following our strategy, we can set $b = 0$ this time and let $d = \pm 1$. Thus we have two new matrices

$$M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad M_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Finally, we need to check that this set does indeed span the vector space of 2×2 symmetric matrices. To do so, consider an arbitrary symetric matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

Can we find coefficients m_i such that $m_1 M_1 + m_2 M_2 + m_3 M_3 = A$ for any choice of a, b, and c? If we can write m_1, m_2, m_3 in terms of a, b, and c, then we can answer this positively. Consider

$$m_1 M_1 + m_2 M_2 + m_3 M_3 = \begin{bmatrix} m_2 + m_3 & m_1 \\ m_1 & m_2 - m_3 \end{bmatrix} \implies \begin{cases} m_2 + m_3 = a \\ m_1 = b \\ m_2 - m_3 = c \end{cases}$$

Thus $m_1 = b$, $m_2 = (a + c)/2$, and $m_3 = (a - c)/2$ and our set spans the vector space of 2×2 symmetric matrices. The matrix in the question can be written with $m_1 = 2$, $m_2 = 1$, and $m_3 = 1$. □