EE241 SPRING 2015: TUTORIAL #12

Friday, April 17, 2015

PROBLEM 1 (Change of basis for vectors): Haar wavelets can be thought of as basis vectors for discrete signals. The are useful for representing "edges" in a signal. The 4-dimensional Haar wavelet basis is

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad H_4^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & -1/2 & 0 \\ 1/4 & -1/4 & 0 & 1/2 \\ 1/4 & -1/4 & 0 & -1/2 \end{bmatrix}.$$

Important note: this is the standard representation you find in literature but it lists the basis vectors in the *rows* of the matrix, not the columns (as we're used to doing), so remember that as you read the solution below. (I've included H_4^{-1} as a hint.) I give you a signal in the Haar basis $[\vec{x}]_H = [10, 1, 0, 0]$ and you have a signal in the time-basis (standard basis) $\vec{y} = [1, -1, 1, -1]$. I ask you to modulate (i.e.: add together) the two signals and return the answer to me in the Haar basis.

Solution. There are two ways to solve this. The simplest way is to bring $\vec{y}$ into the Haar basis and add it to $[\vec{x}]_H$. To do this we can just write

$$[\vec{y}]_H = \left(H_4^T\right)\vec{y} = [0, 0, 1, 1]$$

and our answer is

$$[\vec{z}]_H = [\vec{x}]_H + [\vec{y}]_H = [10, 1, 2, 2]\,.$$

Our other option was to move $[\vec{x}]_H$ to the standard basis, perform the addition there and then transform the result *back* into the Haar basis, to do this we would do

$$[\vec{z}]_H = H_4^T\left(\left(H_4^T\right)^{-1}[\vec{x}]_H + \vec{y}\right)$$

and this would give us exactly the same answer. ☐

PROBLEM 2 (Ortho- gonal/normal matrices and bases):

(a) If $\vec{u} \perp \vec{v}$, is $(\alpha\vec{u}) \perp \vec{v}$ for any $\alpha \in \mathbb{R}$?
(b) Starting with the row-ortho<u>gonal</u> matrix H_4 from problem (1), find the related ortho<u>normal</u> matrix.
(c) Given an orthonormal matrix A, prove that $AA^T = I_n$. Furthermore, prove that if B is also orthonormal, then $C = AB$ is orthonormal. **Hint:** Use that the $(i,j)^{\text{th}}$ entry of AB is $\vec{a}_i \cdot \vec{b}_j$ where $\vec{a}_i$ is the i^{th} row of A and $\vec{b}_j$ is the j^{th} column of B.

Solution.

(a) This is always the case since if $\vec{u} \perp \vec{v}$ then $\vec{u} \cdot \vec{v} = 0$ and multiplying both sides by α we see that

$$\alpha\left(\vec{u} \cdot \vec{v}\right) = \alpha \cdot 0 \qquad \Longrightarrow \qquad (\alpha\vec{u}) \cdot \vec{v} = 0.$$

(b) Here we can use the fact from part (a) that rescaling vectors does not change their orthogonality and we can simply scale each <u>row</u> of H_4 such that the row vectors are normal.

$$|\,[1,1,1,1]\,| = 2|\,[1,1,-1,-1]\,| = 2|\,[1,-1,0,0]\,| = \sqrt{2}|\,[0,0,1,-1]\,| = \sqrt{2}$$

Thus the new matrix H_4' is

$$H_4 = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Note that now the rows AND the columns are orthonormal.

(c) Using the hint we can more succinctly write the product AA^T

$$\begin{aligned}\left[AA^T\right]_{i,j} &= \vec{a}_i \cdot \vec{a}_j \\ &= \begin{cases} |\vec{a}_i|^2 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases} \\ &= \begin{cases} 1 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases}\end{aligned}$$

Thus $AA^T = I_n$. Now to show that $C = AB$ is orthonormal, we should note that all of the things we need to check about C we can check by verifying that $CC^T = I_n$. Each of the 1's on the diagonal prove that the columns of C are normal and each of the 0's off of the diagonal prove that the columns are mutually orthogonal. Thus, since A and B are orthonormal then $AA^T = I_n$ and $BB^T = I_n$ and

$$\begin{aligned}CC^T &= (AB)\left(AB\left(^T\right.\right. \\ &= ABB^TA^T \\ &= AI_nA^T \\ &= AA^T \\ &= I_n.\end{aligned}$$

Thus C is orthonormal as well.

□

PROBLEM 3 (Change of basis for matrices): Consider the following linear transformation on vectors in $\mathbb{R}^3$

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ y-z \\ 2x \end{pmatrix}$$

and the following basis for $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

What is the matrix representing L in the S basis?

Solution. The matrix representing L in the standard basis can be easily read out from the definition of the linear transformation

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}.$$

To find $[L]_S$ we consider what happens to a vector $[\vec{x}]_S$ at the input to this transformation. First, we must take $[\vec{x}]_S$ into the standard basis thus we must apply the change of basis

$$\vec{x} = S\,[\vec{x}]_S$$

where S is the matrix formed from the basis vectors as columns. Now we can apply L in the standard basis to get the output of the transformation, again, in the standard basis

$$\vec{y} = LS\,[\vec{x}]_S\,.$$

Finally, to get back to the S basis, we apply S^{-1} as given below:

$$S^{-1} = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}$$

and
$$[\vec{y}]_S = S^{-1}LS\,[\vec{x}]_S\,.$$
Thus, the matrix that represents the action of L in the basis S is what is left in this equation, i.e.:
$$[L]_S = S^{-1}LS$$
or
$$[L]_S = \begin{bmatrix} 3/2 & 1/2 & 1 \\ -3/2 & -1/2 & 1 \\ 1/2 & 3/2 & 1 \end{bmatrix}.$$

$\square$