EE241 SPRING 2015: TUTORIAL #4

Friday, Feb. 6, 2015

PROBLEM 1: Invert the following matrix using the Gauss-Jordan method and record the elementary row operation matrices that are being applied. How can you write A using these elementary matrices?

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 0 \\ -1 & 2 & 2 \end{bmatrix}$$

Solution. To use the Gauss-Jordan method we begin by augmenting the matrix A with I_3 and proceed to reduced row echelon form.

$$[A|I_3] = \left[\begin{array}{ccc|ccc} -1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \qquad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 \end{array}\right] \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_4E_3E_2E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_5E_4E_3E_2E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \qquad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E_6E_5E_4E_3E_2E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_7E_6E_5E_4E_3E_2E_1\,[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 3/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \qquad E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus

$$A^{-1} = E_7E_6E_5E_4E_3E_2E_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1/2 & 3/2 \\ 1 & 0 & -1 \end{bmatrix}.$$

We can also check that

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}.$$

Note that these last two equations automatically imply that $A^{-1}A = AA^{-1} = I_3$. □

PROBLEM 2: LU decomposition is a technique that speeds up the process of repeatedly solving $A\vec{x} = \vec{b}$ for multiple instances of $\vec{b}$. The key to LU decomposition is to find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$. These matrices actually appear quite naturally during the Gauss-Jordan process. Consider

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{bmatrix}.$$

(a) Perform the Gauss-Jordan method on $[A|I_3]$ but stop at *row echelon form* instead of *reduced row echelon form*. Use the matrices you've found to write down L and U.
(b) Use the LU factors to solve $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{c}$, where

$$\vec{b} = \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

(c) Use the LU factors to determine A^{-1}.

Solution.

(a) We'll use elementary matrices to keep track of the operations we perform. At first, we begin as with Gaussian elimination and reduce the lower-left entries of A,

$$E_1 A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 3 & 16 & 30 \end{bmatrix} \qquad \text{where} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, the third row,

$$E_2 E_1 A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 4 & 15 \end{bmatrix} \qquad \text{where} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

And the third row second column,

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{where} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

In this way we've found the U part of the LU decomposition to be $E_3 E_2 E_1 A$. Note that

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_3 E_2 E_1 A = \left(E_3^{-1} E_2^{-1} E_3^{-1}\right) U.$$

Note two things: First, the inverse of each lower-triangular matrix $E_{1,2,3}$ is itself lower-triangular. Second, if $\left(E_1^{-1} E_2^{-1} E_3^{-1}\right)$ is the product of lower-triangular matrices, then it is itself lower-triangular as well and therefore serves as our L matrix. Finally,

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) In solving $A\vec{x} = \vec{b}$ we break up the solution into two parts, namely $U\vec{x} = L^{-1}\vec{b}$. Computing $L\vec{b}$ is simple

$$L^{-1}\vec{b} = E_3 E_2 E_1 \vec{b} = \begin{bmatrix} 6 \\ -24 \\ 24 \end{bmatrix}$$

since we just perform on b the same operations we performed on A originally. The $U\vec{x}$ part is now just back-substitution,

$$U\vec{x} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 6 \\ -24 \\ 24 \end{bmatrix} \qquad \Longrightarrow \qquad \vec{x} = \begin{bmatrix} 110 \\ -36 \\ 8 \end{bmatrix}$$

Similarly for $\vec{c}$ we find

$$L^{-1}\vec{c} = \begin{bmatrix} 6 \\ -18 \\ 30 \end{bmatrix} \qquad \text{and} \qquad \vec{x} = U^{-1}\left(L^{-1}\vec{c}\right) = \begin{bmatrix} 112 \\ -39 \\ 10 \end{bmatrix}.$$

(c) First, note that $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$ and we already have L^{-1}. Now we must only find U^{-1} and we can easily compute A^{-1}. This can be easily done by finding solutions to the three equations

$$U\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad U\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad U\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

the solutions will each yield the third, second, and first columns of U. In the first case, $\vec{x} = [7/3, -1, 1/3]^T$ by back-substitution. In the second case $\vec{x} = [-2, 1/2, 0]^T$, and in the last case $\vec{x} = [1, 0, 0]^T$ and we have found all of the columns of U^{-1}. Now computing A^{-1} we find

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 7/3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 62/3 & -20/3 & 7/3 \\ -7 & 5/2 & -1 \\ 5/3 & -2/3 & 1/3 \end{bmatrix}$$

□

PROBLEM 3 (Optional): Use Gauss-Jordan method to find the 4×2 matrix B satisfying

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} \cdot B = \begin{bmatrix} 50 & 60 \\ 38 & 48 \\ 21 & 28 \\ 7 & 10 \end{bmatrix}$$

Solution. To use Gauss-Jordan method, we set up our augmented matrix (augmented by 2 columns instead of the usual 1),

$$\left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 50 & 60 \\ 2 & 4 & 2 & 2 & 38 & 48 \\ 3 & 2 & 1 & 1 & 21 & 28 \\ 1 & 2 & 0 & 0 & 7 & 10 \end{array}\right] \begin{array}{c} \xrightarrow{r_2-2r_1} \\ \xrightarrow{r_3-3r_1} \\ \xrightarrow{r_4-r_1} \end{array} \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 50 & 60 \\ 0 & 0 & -4 & -6 & -62 & -72 \\ 0 & -4 & -8 & -11 & -129 & -152 \\ 0 & 0 & -3 & -4 & -43 & -50 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{r_2\leftrightarrow r_3} \\ \xrightarrow{r_3\leftrightarrow r_2} \end{array} \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 50 & 60 \\ 0 & -4 & -8 & -11 & -129 & -152 \\ 0 & 0 & -4 & -6 & -62 & -72 \\ 0 & 0 & -3 & -4 & -43 & -50 \end{array}\right] \begin{array}{c} \xrightarrow{r_2/(-4)} \\ \xrightarrow{r_3/(-4)} \end{array} \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 50 & 60 \\ 0 & 1 & 2 & 2.75 & 32.25 & 38 \\ 0 & 0 & 1 & 1.5 & 15.5 & 18 \\ 0 & 0 & -3 & -4 & -43 & -50 \end{array}\right]$$

$$\xrightarrow{r_4+3r_3} \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 50 & 60 \\ 0 & 1 & 2 & 2.75 & 32.25 & 38 \\ 0 & 0 & 1 & 1.5 & 15.5 & 18 \\ 0 & 0 & 0 & 0.5 & 3.5 & 4 \end{array}\right] \begin{array}{c} \xrightarrow{r_1-8r_4} \\ \xrightarrow{r_2-5.5r_4} \\ \xrightarrow{r_3-3r_4} \\ \xrightarrow{2r_4} \end{array} \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 0 & 22 & 28 \\ 0 & 1 & 2 & 0 & 13 & 16 \\ 0 & 0 & 1 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 7 & 8 \end{array}\right]$$

$$\xrightarrow{r_1-3r_3} \atop \xrightarrow{r_2-2r_3} \left[\begin{array}{cccc|cc} 1 & 2 & 0 & 0 & 7 & 10 \\ 0 & 1 & 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 7 & 8 \end{array}\right] \xrightarrow{r_1-2r_2} \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 7 & 8 \end{array}\right]$$

Thus

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$\square$