EE241 SPRING 2015: TUTORIAL #7

Friday, March 6, 2015

PROBLEM 1: Solve the following system using Cramer's rule

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

Solution. Recall Cramer's rule, i.e.: that $\vec{x}$ has components

$$x_i = \frac{det(A_i)}{det(A)}$$

where A_i is the A matrix with the i^{th} column replaced by $\vec{b}$. For us this means (expanding up from the bottom right entry each time)

$$\begin{aligned} det(A_1) &= \begin{vmatrix} 4 & 3 & 2 \\ 6 & 3 & 2 \\ 4 & 2 & 0 \end{vmatrix} \\ &= 8, \\ det(A_2) &= \begin{vmatrix} 3 & 4 & 2 \\ 1 & 6 & 2 \\ -1 & 4 & 0 \end{vmatrix} \\ &= -12, \\ det(A_3) &= \begin{vmatrix} 3 & 3 & 4 \\ 1 & 3 & 6 \\ -1 & 2 & 4 \end{vmatrix} \\ &= -10. \end{aligned}$$

We also have that

$$\begin{aligned} det(A) &= \begin{vmatrix} 3 & 3 & 2 \\ 1 & 3 & 2 \\ -1 & 2 & 0 \end{vmatrix} \\ &= 0 \cdot \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 3 \\ -1 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} \\ &= -8. \end{aligned}$$

Thus, altogether,

$$\vec{x} = \begin{bmatrix} -1 \\ 3/2 \\ 5/4 \end{bmatrix}$$

$\square$

PROBLEM 2: Let $\vec{u} = [1, 2, 3]$ and $\vec{v} = [3, 4, 6]$.

(a) What is $|\vec{u}|$?
(b) What is $|\vec{v}|$?
(c) What is the distance between the points associated with $\vec{u}$ and $\vec{v}$?
(d) Consider a matrix A such that $A^T A = 2I$. What is the distance between $\vec{x} = A\vec{u}$ and $\vec{y} = A\vec{v}$?

Solution.

(a) $|\vec{u}| = \sqrt{14}$?
(b) $|\vec{v}| = \sqrt{61}$?
(c) Simply find the length of the vector $\vec{w} = \vec{v} - \vec{u} = [2, 2, 3]$. The distance is $|\vec{w}| = \sqrt{17}$.
(d) Note that there are a few ways to calculate distance,

$$\begin{aligned}
|\vec{x} - \vec{y}| &= |A\vec{u} - A\vec{v}| \\
&= |A\left(\vec{u} - \vec{v}\right)| \\
&= \sqrt{\left(A\left(\vec{u} - \vec{v}\right)\right)^T \left(A\left(\vec{u} - \vec{v}\right)\right)} \\
&= \sqrt{\left(\vec{u} - \vec{v}\right)^T A^T A \left(\vec{u} - \vec{v}\right)} \\
&= \sqrt{\left(\vec{u} - \vec{v}\right)^T 2I \left(\vec{u} - \vec{v}\right)} \\
&= \sqrt{2}\sqrt{\left(\vec{u} - \vec{v}\right)^T \left(\vec{u} - \vec{v}\right)} \\
&= \sqrt{2}|\vec{u} - \vec{v}| \\
&= \sqrt{34}.
\end{aligned}$$

$\square$