Week 5: Brute Force Algorithms and Asymptotic Analysis

Class presentation (= reduced version of full class notes)

Brute Force Algorithms

What is a brute force algorithm?

A straight forward approach to solving a problem

usually directly based on the problem statement and definitions of the concepts involved.

Intuitive

Direct

Relies on sheer computing power

Could involve checking all possible solutions

Advantages:

Guaranteed to find a correct solution

Good for smaller problems

Simple strategy

Provides a baseline for algorithms that solve the problem in question

Disadvantages:

Slow

Inefficient

Much work

Six examples of brute force algorithms from the class notes:

Definition (Sorting Problem).
Input: A sequence of n numbers $A = [a_1, a_2, \ldots, a_n]$
Output: A permutation $A' = [a'_1, a'_2, \ldots, a'_n]$ of the input such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$

This is ascending order.
We can also sort into descending order.

Example 1: Selection Sort

Here is one simple method:
Find the smallest element in the sequence; swap it into slot 1.
Find the 2nd smallest element in the sequence; swap it into slot 2.
Find the 3rd smallest element in the sequence; swap it into slot 3.
etc.

Let's look at an example of the idea in action. Let the original sequence be

33; 77; 55; 22; 11; 44.

In what follows bold numbers will denote the most recently placed number.

33; 77; 55; 22; 11; 44 This is the original array A.

11; 77; 55; 22; 33; 44
11; ***22*****;** 55; 77; 33; 44
11; 22**; 33;** 77; 55; 44
11; 22; 33; **44**; 55; 77
11; 22; 33; 44; **55**; 77 No further swaps are done. This is the result A'.

This method of sorting is called the Selection Sort. The selection sort algorithm is given as:

```
1   # Input: An array A and indices i and j.
2   # Output: An array where A[i] and A[j] have been swapped.
3   def Swap(A, i, j):
4       temp = A[i]
5       A[i] = A[j]
6       A[j] = temp
7
8   # Input: An array A of integers.
9   # Output: Array A sorted in increasing order.
10  def SelectionSort(A):
11      for i in range(len(A)-1):
12          m = i
13
14      # Find the smallest element in A[i .. n-1]
15          for j in range(i+1, len(A)):
16              if A[j] < A[m]:
17                  m = j
18          Swap(A, i, m)
```

Asymptotic Analysis of Selection Sort

What is our input size?
Answer: n.

What's the worst-case running time?

To formally compute the worst-case running time, we create a chart to track each line of code.

Line	Cost	Count
11	c_1	n-1
12	c_2	n-1
15	c_3	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
16	c_4	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
17	c_5	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
4	c_6	n-1
5	c_7	n-1
6	c_8	n-1

We then total the cost multiplied by the count for each line.

$T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1)n/2 + c_4(n-1)n/2 + c_5(n-1)n/2 + c_6(n-1) + c_7(n-1) + c_8(n-1)$
$= (c_3 + c_4 + c_5)n^2 + (c_1 + c_2 - .5c_3 - .5c_4 - .5c_5 + c_6 + c_7 + c_8)n + (-c_1 - c_2 - c_6 - c_7 - c_8)1$
$= c_9n^2 + c_{10}n + c_{11}1$
$\leq (c_9 + c_{10} + c_{11})n^2$
$= c_{12}n^2$

Since $T(n) \leq$ some constant multiple of n^2, we say
$T(n)$ is in the asymptotic class $O(n^2)$
so $T(n) = O(n^2)$.

Informally, we can identify the basic operation to be the comparison in line 16. Since this is inside of a pair of nested loops with the outer loop running n-1 times and the inner loop running (n-1)+(n-2)+(n-3)+…+3+2+1 = $\frac{(n-1)n}{2}$ times, the work is $O(n^2)$. Done.

example 2 - Bubble Sort

Another sort that people think about when talking about brute force sorts is bubble sort.
The intuition behind the bubble sort is:

> Walk through the array comparing adjacent elements in the list,
> exchanging them if they are out of order.

The first pass will cause the largest element to "bubble up" and end up in the last slot in the list.
Pass i bubbles up the i-th largest element in the same way.
After pass i, at least the i largest elements will be located in the last i slots of the array, so do not do any comparisons in this sorted tail portion of the array.

Let's look at an example of the bubble sort idea in action. Let the original sequence be

33; 77; 55; 22; 11; 44.

In what follows bold numbers will denote the most recently compared numbers.

33; 77; 55; 22; 11; 44 This is the original array A.

33; 77; 55; 22; 11; 44 compare 33 and 77
33; **77; 55;** 22; 11; 44 compare 77 and 55, and swap them
33; 55; **77; 22**; 11; 44 compare 77 and 22, and swap them
33; 55; 22; **77; 11**; 44 compare 77 and 11, and swap them
33; 55; 22; 11; **77; 44** compare 77 and 44, and swap them
33; 55; 22; 11; 44; 77

finished 1st pass through array, the last item is in its correct place

33; 55; 22; 11; 44; 77 compare 33 and 55
33; **55; 22**; 11; 44; 77 compare 55 and 22, and swap them
33; 22; **55; 11;** 44; 77 compare 55 and 11, and swap them
33; 22; 11; **55; 44**; 77 compare 55 and 44, and swap them
33; 22; 11; 44; 55; 77

finished 2nd pass through array, the last 2 items are in their correct places

33; 22; 11; 44; 55; 77 compare 33 and 22, and swap them
22; **33; 11**; 44; 55; 77 compare 33 and 11, and swap them
22; 11; **33; 44**; 55; 77 compare 33 and 44
22; 11; 33; 44; 55; 77

finished 3rd pass through array, the last 3 items are in their correct places

22; 11; 33; 44; 55; 77 compare 22 and 11, and swap them
11; **22; 33**; 44; 55; 77 compare 22 and 33
11; 22; 33; 44; 55; 77

finished 4th pass through array, the last 4 items are in their correct places

11; 22; 33; 44; 55; 77 compare 11 and 22
11; 22; 33; 44; 55; 77

finished 5th pass through array, the last 5 items are in their correct places
No further swaps are done. This is the result A'.

The Bubble sort algorithm is given as:

```
1       # Input: An array A of integers.
2       # Output: An array A sorted in increasing order.
3       def BubbleSort(A):
4               for i in range(len(A)-1) :
5                       for j in range(len(A)-i-1) :
6                               if A[j+1] < A[j] :
7                                       Swap(A, j+1, j)
```

Asymptotic Analysis of Bubble Sort

What is our input size? Answer: n.

What's the worst-case running time?

Line	Cost	Count
4	c_1	n-1
5	c_2	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
6	c_3	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
7	c_4	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2 for each of the three lines of Swap

We then total the cost multiplied by the count for each line.

$T(n) = c_1(n-1) + c_2(n-1)n/2 + c_3(n-1)n/2 + c_4(n-1)n/2$
$= (c_2 + c_3 + c_4)n^2 + (c_1 - .5c_2 - .5c_3 - .5c_4)n + (-c_1)1$
$= c_5n^2 + c_6n + c_71$
$\leq (c_5 + c_6 + c_7)n^2$
$= c_8n^2$

Since T(n) ≤ some constant multiple of n^2, we say
T(n) is in the asymptotic class $O(n^2)$
so T(n) = $O(n^2)$.

Informally, we can identify the basic operation to be the comparison in line 6. Since this is inside of a pair of nested loops with the outer loop running n-1 times and the inner loop running (n-1)+(n-2)+(n-3)+…+3+2+1 = $\frac{(n-1)n}{2}$ times, the work is $O(n^2)$. Done.

example 3 - String Matching

Another classic problem that has a brute force solution is the string matching problem.

Definition (String Matching Problem).
Input: An n symbol string $\sigma = \sigma_1\sigma_2\sigma_3 \ldots \sigma_n$ and an m≤n symbol search string $s = s_1s_2s_3 \ldots s_m$
Output: If there exists an index i such that $\sigma_i = s_1, \sigma_{i+1} = s_2, \sigma_{i+2} = s_3, \sigma_{i+m-1} = s_m$ then output i; otherwise, output None.

Alternatively, we can call these the text (what we are looking in) and the pattern (what we are looking for).
Or we can call them the haystack and the needle.

How might you go about solving this problem using brute force?
Here is one way:

> Continually shift the search-string/pattern/needle one character right until it lines up. Once you have less than m-1 characters to the right of the start of the match attempt you can safely stop.

For example, let us trace the algorithm on the symbol string/text/haystack σ= "ababacdabc" with search string/pattern/needle s = "abc".

a	**b**	**a**	**b**	**a**	**c**	**d**	**a**	**b**	**c**
a	b	c							
	a	b	c						
		a	b	c					
			a	b	c				
				a	b	c			
					a	b	c		
						a	b	c	
							a	b	c

In pseudo-code we have the following string match algorithm:

```
1       # Input: An array A of n symbols and an array S of m < n
2       #        symbols.
3       # Output: The index i of the start of the pattern in A or
4       #         None, otherwise.
5       def StringMatch(A, S):
6               for i in range(len(A)-m):
7                       j = 0
8                       while j < len(S) and A[i+j] == S[j]:
9                               j = j+1
10                      if j == len(S):
11                              return i
12              return None
```

What is our input size?
Answer: (n, m). n for the length of the haystack and m for the length of the needle

What's the worst-case running time?

Line	Cost	Count
6	c_1	n-1
7	c_2	n-1
8	c_3	m·(n-m+1)
9	c_4	m·(n-m+1)
10	c_5	n-1
11	c_6	1 at most
12	c_7	1 at most

$T(n, m) = c_1(n-1) + c_2(n-1) + c_3\, m\cdot(n-m+1) + c_4\, m\cdot(n-m+1) + c_5(n-1) + \max(c_6, c_7)$
$= (c_3 + c_4 + c_5)nm + (c_3 + c_4 + c_5)m^2 + (c_1 + c_2 - c_3 - c_4 + c_5)n + (-c_1 - c_2 + c_3 + c_4 - c_5 + \max(c_6, c_7))1$
$= c_8 mn + c_9 m^2 + c_{10} n + c_{11} 1$
$\leq (c_8 + c_9 + c_{10} + c_{11})nm$
$= c_{12} nm$

Since T(n, m) ≤ some constant multiple of nm,
T(n, m) is in the asymptotic class O(nm)
so T(n) = $O(nm)$.

We will consider two very fundamental problems in computational geometry:
1. The closest-pair problem
2. The convex-hull problem

example 4 - Closest Pair

Definition (Closest Pair Problem).

Input: A set $P = \{p_1, p_2, p_3, \dots, p_n\}$ of n points in d-dimensional space and a metric
$d: P \times P \to R$
Output: The distance between the two closest points p_i and p_j according to the metric d.

For our purposes we will only consider the 2-D Euclidean plane and with the metric

$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

What is the brute force approach here?

For every pair of points, compute the distance between the two points and find the pair with the smallest distance.

Formally, we have the algorithm:

```
1   # Input: Two points A and B in the XY plane.
2   # Output: The Euclidean distance between the two points.
3   def Distance(A, B) :
4       return math.sqrt ((A[0]-B[0])**2 + (A[1]-B[1])**2)
5
6   # Input: P a list of n>2 points in the XY plans.
7   # Output: The distance between the closest pair of points.
8   def ClosestPair(P) :
9       d = 1e12  # a large number, might as well be infinity
10      for i in range(len(P)-1) :
11          for j in range(i+1, len(P)) :
12              d = min (d, Distance(P[i], P[j]))
13      return d
```

Note: math.sqrt() might require you to import math.

Asymptotic analysis

What is our input size?
Answer: n = number of points

What's the worst-case running time?

Line	Cost	Count
9	c_1	1
10	c_2	n-1
11	c_3	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
12	c_4	(n-1)+(n-2)+(n-3)+…+3+2+1 = (n-1)n/2
13	c_5	1

$T(n) = c_1(1) + c_2(n-1) + c_3 (n-1)n/2 + c_4 (n-1)n/2 + c_5 (1)$
$= (c_2 + c_3 + c_4)n^2 + (-.5c_2 – .5c_3 – .5c_4)n + (c_1 - c_2 + c_5)1$
$= c_6n^2 + c_7n + c_81$
$\leq (c_6 + c_7 + c_8)n^2$
$= c_9n^2$

Since $T(n) \leq$ some constant multiple of n^2,
$T(n)$ is in the asymptotic class $O(n^2)$
so $T(n) = O(n^2)$.

example 5 - Convex Hull

We now turn our attention to what is considered the most important problem in computational geometry. The convex hull problem. Applications include:

- Collision detection in video games.
- Path planning
- Processing satellite maps for accessibility
- Detecting statistical outliers
- Computing the diameter of a set of points (i.e. the distance between the two farthest points).

First let's define what it means for a set to be convex.
Definition (Convex Set). A set of points (finite or infinite) in the plane is called convex if for any two points p and q in the set, the entire line segment pq belongs to the set.
Examples of convex sets are the set of points that make up a square, rectangle, parallelepiped, etc. In fact, the vertices of any convex polygon form a convex set.

The convex hull of a set of points is the boundary of the region in the plane that includes the line segments joining all pairs of points in the set.

"Same Side Test": A line segment connecting two points pi and pj of a set of n points is part of the convex hull's boundary if all other points of the set lie on the same side of the straight line through these two points.

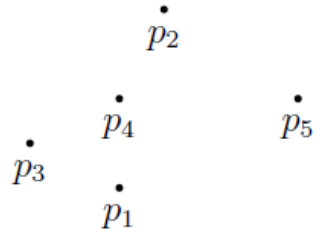
Given a set of n points in the plane we can think of the convex hull as the smallest convex polygon such that the points in the set are either inside or on the boundary of the polygon.

Exams of convex hulls are:
If S is a single point, S is its own convex hull.
If S is 2 points the convex hull is the line segment joining the two points.
If S is three points not on the same line then the convex hull is a triangle.

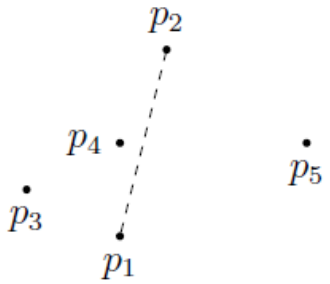
We are now equipped to define the convex hull problem.
Definition 8 (Convex Hull Problem).
Input: A set S of n points
Output: The list of points that form the vertices of the convex polygon

Our brute force algorithm involves repeating the "Same Side Test" for every pair of points in the set S, and compiling a list of points whose line segments pass the test.

We will not provide pseudocode for this algorithm, but we will work through a few steps on the following example set of points:

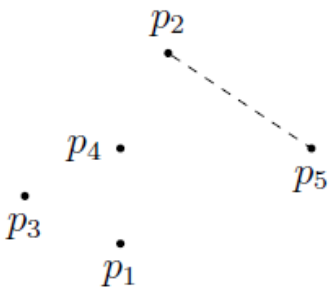


Bad segment example: Test the line between points p1 and p2.



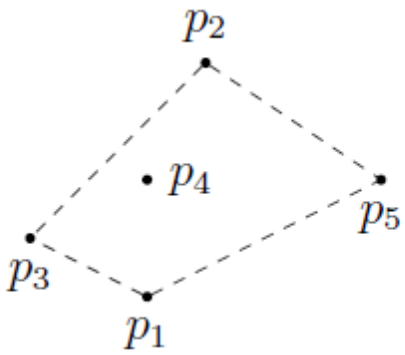
This line segment "splits the plane" in such a way that there are points on both sides of the line segment. Therefore, this segment cannot be part of the convex hull.

Good segment example: Test the line between points p2 and p5.



This line segment "splits the plane" in such a way that all remaining points lie on the same side of the line segment. Therefore, this segment is part of the convex hull.

The convex hull for the entire set is



so, our algorithm would return the subset of points {p1, p2, p3, p5} that lie on the convex hull.

This is a very inefficient way of solving the problem (as most brute force algorithms are). For every pair of points, we need to consider the line segment joining these two points and check to see if all remaining points lie on the same side of the line segment.

For a set of n points, how many line segments are there? There is one line segment for each pair of points, so there are "n choose 2" = $nC2 = \binom{n}{2} = \frac{n(n-1)}{2}$ segments. For each line segment, we need to determine on which side of the line does each of the remaining n-2 points lie. How we do this is quite complicated. But clearly, this is the essential unit of work that we need to count. The counting is simple in the worst case:

$$\frac{n(n-1)}{2} \cdot (n-2) = \frac{(n^2-n)(n-2)}{2} = \frac{n^3-3n^2+2n}{2} < n^3$$

so, this brute force algorithm for the convex hull of a set of n points is $O(n^3)$.

Searching in Graphs

We can use exhaustive search to process all vertices and edges in a graph.
There are two such algorithms we will investigate to perform this type of search
1. Depth-First Search (DFS)
2. Breadth-First Search (BFS)
Both types of graph searches are extremely frequent in Computer Science. We will implement both algorithms using the adjacency list representation of a graph.

BFS is a brute-force algorithm that uses a queue to organize the work to be done next. Algorithm DFS is a recursive algorithm, we will cover it here because it is logically related to BFS. We could have implemented DFS as a brute-force algorithm using a stack, but that is exactly what is taking place behind the scenes in recursion.

example 6 - Breadth First Search (BFS)

The basic idea of Breadth First Search is to start at a chosen vertex, visits all of its not-yet-visited neighbors, then all of their not-yet-visited neighbors, ... If the graph is connected (if there is a path from the source vertex to every other vertex), all vertices will eventually be visited. The algorithm gets its name from its attempt to go as wide as possible before proceeding to vertices that are another level away from the source.

A high-level algorithm using a queue is as follows:

We use a queue to maintain the list of discovered vertices.
Queue initially contains the source vertex.
At each iteration of the BFS algorithm:

1. Enqueue all vertices adjacent to the vertex that is at the front of the queue.
2. Mark each newly inserted vertex visited.
3. Dequeue the vertex at the front of the queue.

If we finish our search from the source vertex, and not all nodes have been visited we must start BFS again from an arbitrary unvisited vertex.
When a visited vertex considers one of its visited neighbors, we insert a new “cross edge” that goes from the vertex to its previously visited neighbor.
Like DFS the pseudo-code for BFS is divided into two algorithms BFS and BFSVisit.

The BFS algorithm is as follows:

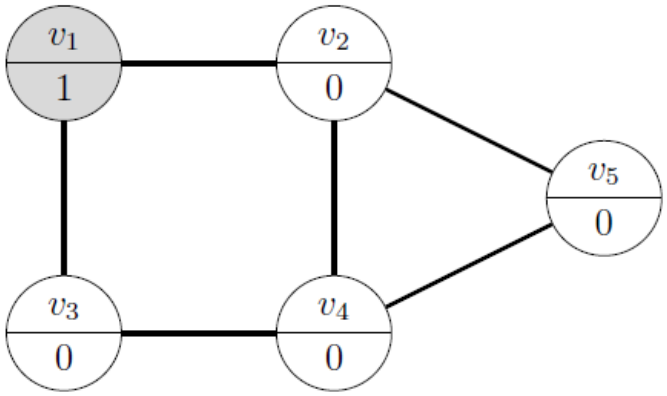
```
1       # Input: A graph G = (V, E) and a global variable, count.
2       # Output: A graph G = (V,' E') where E' = E and V' = V
3       #       except that the nodes of V' are marked with integers
4       #       that indicate in what order they were visited.
5       def BFS(G):
6               global count
7               count = 0
8       # Mark every vertex in V with 0 to indicate not yet visited.
9       for v in G:
10              v.visited = 0
11      # Start BFSVisit algorithm for the first unvisited node.
12      # Restart as needed (if graph is not connected).
13      for v in G:
14              if v.visited == 0:
15                      BFSVisit(v)
```

The BFSVisit algorithm is as follows:

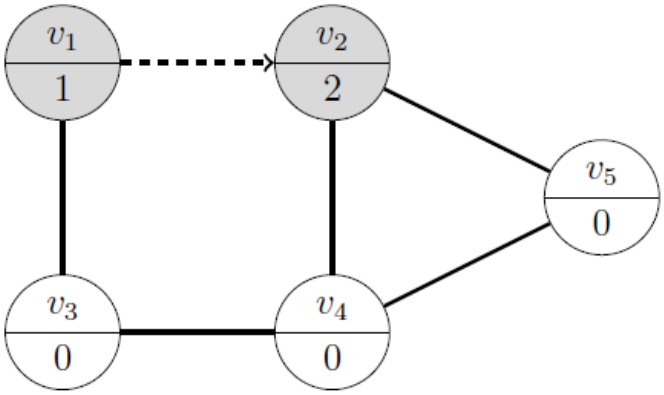
```
1       # Input: v is an unvisited vertex.
2       # Output: All descendants of v have been visited.
3       def BFSVisit(v):
4               count = count + 1
5               v.visited = count
6       # Create a queue, enqueueing the vertex.
7               Q = [v]
8               while Q is not empty:
9                       w = dequeue(Q)
10              # for each unvisited neighbor of w, visit it, mark it as visited, and enqueue it
11                      for u in w.getAdjacent():
12                              if u.visited == 0;
13                                      count = count + 1
14                                      u.visited = count
15                                      Q.append(u)
```

Let's look at an example trace of BFS on a small graph. In what follows, the numbers represent the visit counter value, the tree edges are denoted by dashed lines with arrows, and the cross edges are denoted by dotted lines with arrows.

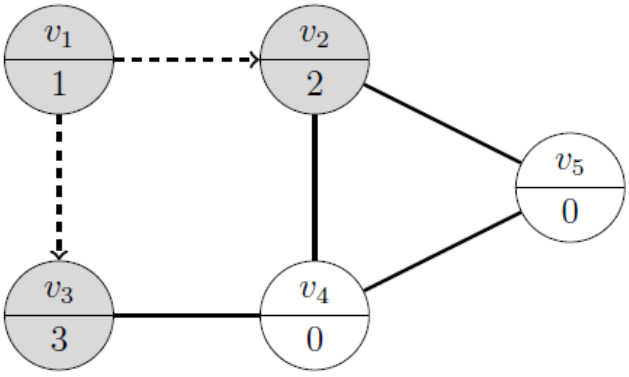
1. The source vertex is v1 and Q = [v1]. Mark v1 as the 1st visited vertex.



2. We visit the first descendant of v1 and Q = [v1, v2]. Mark v2 as the 2nd visited vertex. Make edge (v1, v2) a tree edge.

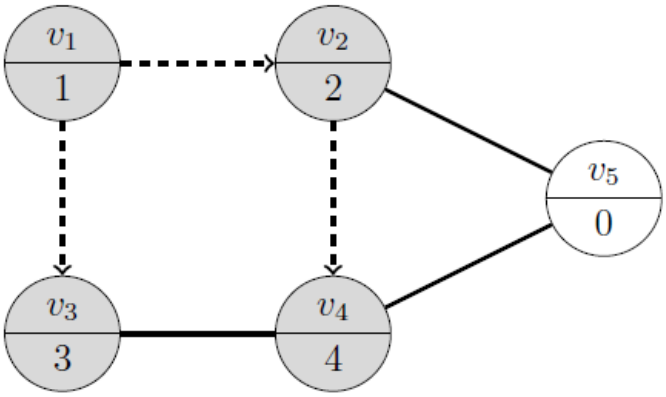


3. We visit the next descendant of v1 and Q = [v1, v2, v3]. Mark v3 as the 3rd visited vertex. Make edge (v1, v3) a tree edge.

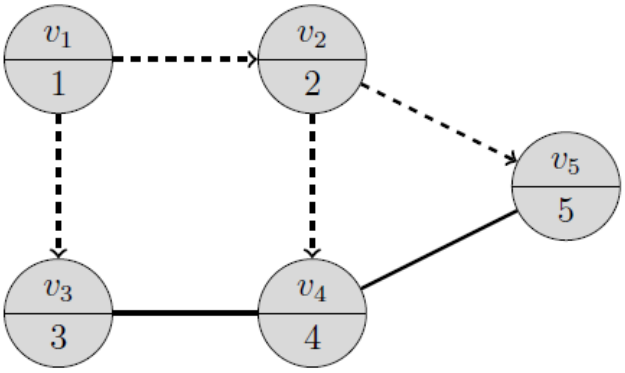


4. Vertex v1 has no more descendants, it is removed from the queue. The queue is now Q = [v2, v3].

5. We visit the first descendant of v2. Q = [v2, v3, v4]. Mark v4 as the 4th visited vertex. Make edge (v2, v4) a tree edge.

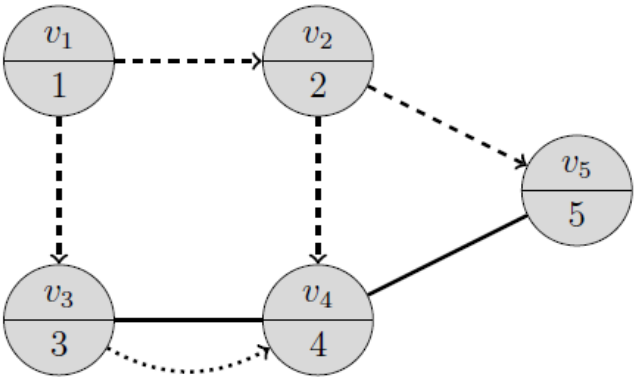


6. We visit the second descendant of v2. Q = [v2, v3, v4, v5]. Mark v5 as the 5th visited vertex. Make edge (v2, v5) a tree edge.

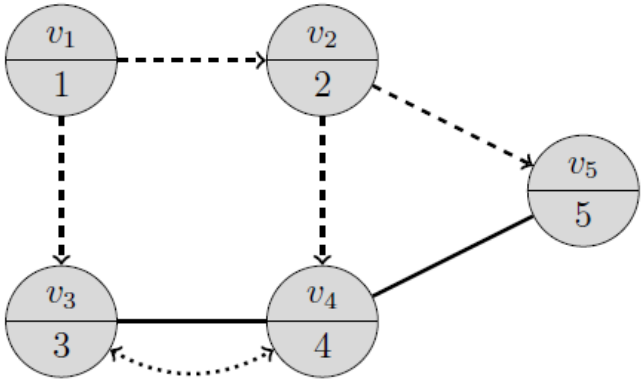


7. Vertex v2 has no more descendants, it is removed from the queue. The queue is now Q = [v3, v4, v5].

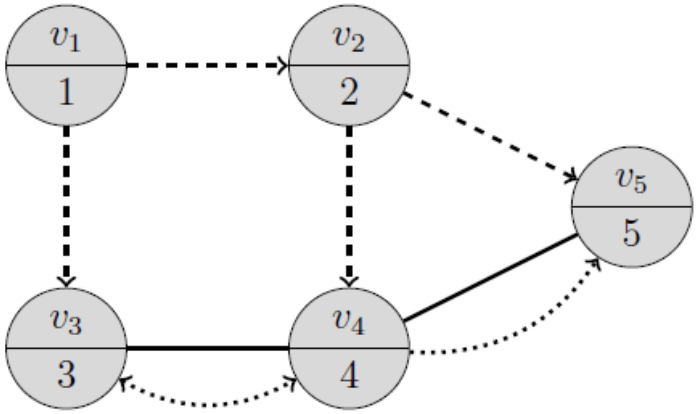
8. Vertex v3's only descendant is v4 which has already been visited. This means (v3, v4) is a cross edge. Vertex v3 is removed from the queue so Q = [v4, v5].



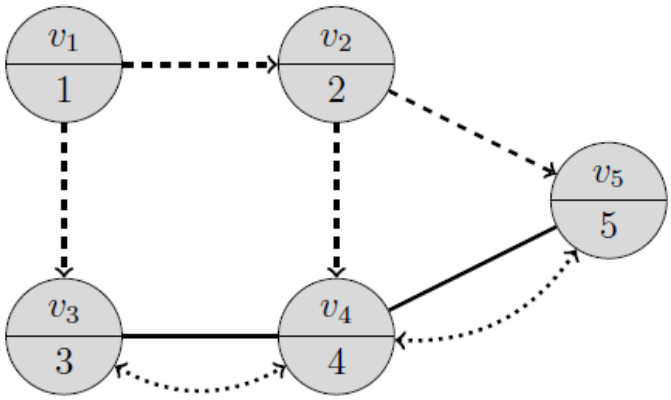
9. Vertex v4's first descendant is v3 which has already been visited this means (v4, v3) is a cross edge.



10. Vertex v4's second descendant is v5 which has already been visited. This means (v4, v5) is a cross edge. Since, there are no additional descendants v4 is removed from the queue. This results in Q = [v5].



11. Vertex v5's only descendant is v4 which has already been visited. This means (v5, v4) is a cross edge. V5 is removed from the queue so Q = [ ] and our BFS has finished since there are no unvisited vertices in G.



What is the work done by the BFS algorithm in the worst case?

We will not do this formally with a chart. Instead, we identify the crucial unit of work to be considering whether or not to enqueue a vertex.

In BFS(), after marking each vertex as unvisited, the algorithm calls BFSVisit() at most once for each vertex (lines 13-15) and in BSFVisit, the vertex is added to the queue (line 7). So, this bound is $|V|$, the number of vertices in the graph.

In BFSVisit(), lines 11-15 examine every edge from this vertex to decide whether or not to enqueue each of its neighbors. Since each edge appears twice in the graph's adjacency structure, this bound is $2|E|$ or twice the number of edges in the graph.

Therefore, the number of "enqueings" that are considered is $\leq |V| + 2|E| < 2(|V| + |E|)$, so the BFS algorithm's asymptotic class is $O(|V| + |E|)$.