

# Databases Autumn 2025

## Hand-In Exercise 5

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| <b>Total Points</b> |  |
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## Task 1 Relational Algebra and Operator Tree

We are given the relations

$$U(a, b, c), \quad V(c, d, e), \quad W(e, f, g)$$

and the following SQL query:

```
SELECT b, f
FROM (
  SELECT V.c, V.d, W.e, W.f, W.g
  FROM V, W
  WHERE V.e = W.e
        AND f = 2
        AND g = 10
)
JOIN U ON U.c = V.c
WHERE a LIKE 'Ben';
```

### a. Relational algebra and algebraic operator tree

A translation of the query into relational algebra is:

$$\pi_{b,f} \left( \sigma_{a \text{ LIKE 'Ben'}} \left( \left( \pi_{V.c, V.d, W.e, W.f, W.g} \left( \sigma_{V.e=W.e \wedge W.f=2 \wedge W.g=10} (V \times W) \right) \right) \bowtie_{U.c=V.c} U \right) \right).$$

The corresponding algebraic operator tree is shown in Figure 1.

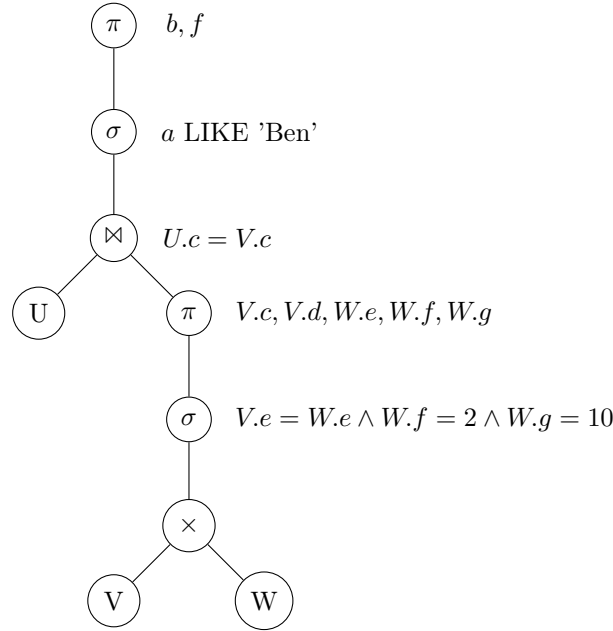


Figure 1: Algebraic operator tree for the given SQL query.

## b. Query optimizer

We transform the canonical algebraic expression step by step using equivalence rules of the relational algebra. The goal is to push selections and projections as far down as possible and to replace Cartesian products plus join conditions by joins, in order to reduce the size of intermediate results.

Rule 10 – Split selection with conjunction

$$\sigma_{P_1 \wedge P_2 \wedge P_3}(R) = \sigma_{P_1}(\sigma_{P_2}(\sigma_{P_3}(R))).$$

This rule is used to turn the single selection

$$\sigma_{V.e=W.e \wedge W.f=2 \wedge W.g=10}(V \times W)$$

into a cascade of three selections, so that the conditions can later be pushed down independently.

Rule 15 – Swap selection and Cartesian product

$$\sigma_P(R \times S) = \sigma_P(R) \times S \quad \text{if } P \text{ uses only attributes of } R.$$

This rule is used to push the selections  $\sigma_{W.f=2}$  and  $\sigma_{W.g=10}$  from  $\sigma_{W.f=2}(\sigma_{W.g=10}(V \times W))$  down onto  $W$ , since both conditions refer only to attributes of  $W$ .

Rule 22 – Combine selection and product into join

$$\sigma_{R.A=S.B}(R \times S) = R \bowtie_{R.A=S.B} S.$$

This rule is used to replace

$$\sigma_{V.e=W.e}(V \times W')$$

by the join

$$V \bowtie_{V.e=W.e} W',$$

where  $W' = \sigma_{W.f=2}(\sigma_{W.g=10}(W))$ . This eliminates the large Cartesian product in favor of a join with already filtered  $W$ .

Rule 14 – Swap selection and join

$$\sigma_P(R \bowtie S) = \sigma_P(R) \bowtie S \quad \text{if } P \text{ uses only attributes of } R.$$

This rule is used to pull the selection  $\sigma_{a \text{ LIKE 'Ben'}}$  down onto  $U$  in

$$\sigma_{a \text{ LIKE 'Ben'}}(R \bowtie_{U.c=V.c} U) \equiv R \bowtie_{U.c=V.c} \sigma_{a \text{ LIKE 'Ben'}}(U),$$

since the predicate involves only attribute  $a$  of relation  $U$ .

Rule 16 – Swap projection and join (projection push-down)

$$\pi_Q(R \bowtie_P S) = \pi_Q(\pi_{Q_1}(R) \bowtie_P \pi_{Q_2}(S)),$$

where  $Q_1$  and  $Q_2$  contain the attributes from  $Q$  and from the join predicate  $P$  that belong to each side.

This rule is used to introduce projections on  $U$ ,  $V$ , and  $W$  so that each relation only carries the attributes that are actually needed:

- result attributes:  $b$  (from  $U$ ) and  $f$  (from  $W$ ),
- join attributes:  $c$  on  $U$  and  $V$ ,  $e$  on  $V$  and  $W$ ,
- selection attributes:  $a$  on  $U$ ;  $f$  and  $g$  on  $W$ .

Thus we choose the minimal attribute sets

$$U : \{a, b, c\}, \quad V : \{c, e\}, \quad W : \{e, f, g\},$$

and obtain the optimized algebraic expression

$$Q_{\text{opt}} = \pi_{b,f} \left( \left( \pi_{c,e}(V) \bowtie_{V.e=W.e} \sigma_{W.f=2 \wedge W.g=10}(\pi_{e,f,g}(W)) \right) \bowtie_{U.c=V.c} \sigma_{a \text{ LIKE 'Ben'}}(\pi_{a,b,c}(U)) \right).$$

In this form, all selections and projections are pushed down to the leaves, the Cartesian product is replaced by joins, and intermediate results are reduced.

The corresponding optimized operator tree is:

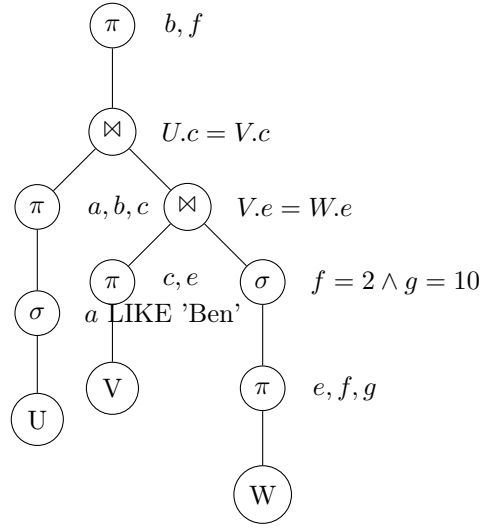


Figure 2: Optimized algebraic operator tree for the given SQL query.

### c. Cardinality estimation

We assume that the attributes  $a$ ,  $f$  and  $g$  are independently and uniformly distributed over 10 distinct values. Hence, a single equality condition such as  $f = 2$  has selectivity  $1/10$ , and the conjunction  $f = 2 \wedge g = 10$  has selectivity  $1/10 \cdot 1/10 = 1/100$ . The relation cardinalities are:

$$\text{Card}(U) = 10, \quad \text{Card}(V) = 100000, \quad \text{Card}(W) = 1000000.$$

**Non-optimized operator tree** In the canonical (non-optimized) tree, the operations on  $V$  and  $W$  start with the Cartesian product:

$$V \times W, \quad \text{Card}(V \times W) = \text{Card}(V) \cdot \text{Card}(W) = 100000 \cdot 1000000 = 10^{11}.$$

Then the selection

$$\sigma_{V.e=W.e \wedge f=2 \wedge g=10}(V \times W)$$

is applied. We can reason about its effect as follows:

- The conditions  $f = 2$  and  $g = 10$  refer only to  $W$  and are independent and uniform, so together they have selectivity  $1/100$ . Applying them to  $V \times W$  reduces the size from  $10^{11}$  to

$$10^{11} \cdot \frac{1}{100} = 10^9.$$

- The remaining condition  $V.e = W.e$  is an equality join on a uniformly distributed key. With uniformity, the probability that  $V.e$  and  $W.e$  match for a random pair is approximately  $1/\text{Card}(W)$ , so we obtain

$$10^9 \cdot \frac{1}{1000000} = 1000$$

tuples after this selection.

The subsequent projection  $\pi_{V.c,V.d,W.e,W.f,W.g}$  does not change the number of tuples, so the intermediate result before joining with  $U$  still has about 1000 tuples. The join with  $U$  and the selection on  $a$  produce a very small final result compared to the sizes above.

The *dominant* intermediate result in the non-optimized tree is therefore the Cartesian product  $V \times W$  with

$$\text{Card}_{\max}^{(\text{non-opt})} \approx 10^{11} \text{ tuples.}$$

**Optimized operator tree** In the optimized tree from the previous subtask, selections and projections are pushed down:

- On  $U$ : first  $\pi_{a,b,c}(U)$  keeps 10 tuples, then  $\sigma_a \text{ LIKE 'Ben'}$  selects one out of 10 values of  $a$ :

$$\text{Card}(\sigma_a \text{ LIKE 'Ben'}(\pi_{a,b,c}(U))) = 10 \cdot \frac{1}{10} = 1.$$

- On  $W$ : first  $\pi_{e,f,g}(W)$  keeps 1000000 tuples, then  $\sigma_{f=2 \wedge g=10}$  has selectivity  $1/100$ :

$$\text{Card}(\sigma_{f=2 \wedge g=10}(\pi_{e,f,g}(W))) = 1000000 \cdot \frac{1}{100} = 10000.$$

- On  $V$ :  $\pi_{c,e}(V)$  keeps all 100000 tuples (only the attributes change, not the number of rows).

We then join the reduced  $V$  and  $W$ :

$$\pi_{c,e}(V) \bowtie_{V.e=W.e} \sigma_{f=2 \wedge g=10}(\pi_{e,f,g}(W)).$$

Under the uniform foreign-key assumption, the fraction of  $W$ 's key values that survive the selection is

$$\frac{\text{Card}(\sigma_{f=2 \wedge g=10}(W))}{\text{Card}(W)} = \frac{10000}{1000000} = \frac{1}{100}.$$

Each tuple in  $V$  therefore matches with probability  $1/100$ , so the join result has expected size

$$\text{Card}(V \bowtie W') \approx \text{Card}(V) \cdot \frac{1}{100} = 100000 \cdot \frac{1}{100} = 1000,$$

where  $W' = \sigma_{f=2 \wedge g=10}(W)$ .

Joining this intermediate result (about 1000 tuples) with  $\sigma_a \text{ LIKE 'Ben'}(\pi_{a,b,c}(U))$  (one tuple) on  $c$  yields a very small final result. The largest intermediate results in the optimized plan are thus the base relation  $W$  with  $10^6$  tuples and the filtered subset of  $W$  with only

$$\text{Card}(\sigma_{f=2 \wedge g=10}(W)) = 10000$$

tuples before the join.

Compared to the non-optimized tree, the *largest* intermediate result in the optimized tree is:

$$\text{Card}_{\max}^{(\text{opt})} \approx 10000 \text{ tuples (on the filtered } W),$$

instead of the huge Cartesian product with  $\text{Card}_{\max}^{(\text{non-opt})} \approx 10^{11}$  tuples. This shows why pushing selections down and avoiding Cartesian products is crucial for query performance.

## Task 2

i Parameters given in the sheet:

| Parameter                 | Symbol             | Value  |
|---------------------------|--------------------|--------|
| Attribute size            | $s_a$              | 6 B    |
| Tuple size                | $s_t$              | 12 B   |
| Cardinality of R          | $\text{Card}(R)$   | 50 000 |
| Page size                 | page               | 8192 B |
| Fill degree (data pages)  | $f_{\text{data}}$  | 0.9    |
| Fill degree (index pages) | $f_{\text{index}}$ | 0.7    |
| Page header               | head               | 48 B   |
| Pointer size              | $s_p$              | 6 B    |

ii **FF(A)**

Attribute A is uniformly distributed in [0..999], therefore **FF(A = a) = 1/1000**.

iii **FF(B)**

Attribute B is not uniformly distributed:

- 40 000 tuples are uniformly distributed over the values [0..99] (100 values)

The tuples are spread uniformly across 100 values so each value occurs  $\frac{40\,000}{100} = 400$  times.

Therefore **FF(B = 10) =  $\frac{400}{50\,000}$** .

- 10 000 tuples are uniformly distributed over the values [100..999] (900 values)

Here the tuples are spread uniformly across 900 different values so each value occurs  $\frac{10\,000}{900} \approx 11.11$  times.

Therefore **FF(B = 500) =  $\frac{10\,000/900}{50\,000}$** .

### 1. No index (Layout R)

Page-layout:  $\frac{(\text{page-head}) \cdot f_{\text{data}}}{r_{\text{avg}} + p_{\text{slot}}} \Rightarrow x = 407$

(with  $r_{\text{avg}} = s_t$  and  $p_{\text{slot}} = s_p$ )

Number of data pages:  $\lceil \text{Card}(R)/x \rceil \Rightarrow \underline{N\text{Pages}(R) = 123}$

Since there is no index:  $\Rightarrow \mathbf{C(A = 10) = C(A = 500) = C(B = 10) = C(B = 500) = 123}$

### 2. Indirect B+ tree on A (Layout RA)

Leaf capacity:  $\left\lceil \frac{(\text{page-head} - 2 \cdot p_{\text{leaf}}) \cdot f_{\text{index}}}{k + r_k \cdot p_{\text{rec}}} \right\rceil \Rightarrow \underline{t_{\text{leaf}} = 18}$

(with  $k = s_a$ ,  $r_k = 50$ ,  $p_{\text{leaf}} = p_{\text{rec}} = s_p$ )

Number of leaf pages:  $\lceil \frac{n_k}{t_{\text{leaf}}} \rceil \Rightarrow \underline{n_{\text{leaf}} = 56}$

(with  $n_k = 1000$ )

Inner node capacity:  $\left\lceil \frac{((\text{page-head}) \cdot f_{\text{index}}) - p}{k + p} \right\rceil \Rightarrow \underline{e_i = 474}$

(with  $p = s_p$ )

Height:  $\lceil \log_{e_i+1}(n_{\text{leaf}}) \rceil + 1 \Rightarrow \mathbf{h = 2}$

Index-only selection cost:  $(h - 1) + \lceil FF(A = a) \cdot n_{\text{leaf}} \rceil \Rightarrow \mathbf{C(A = 10) = C(A = 500) = 2}$

Queries on B require table scan:  $\Rightarrow \mathbf{C(B = 10) = C(B = 500) = 123}$

### 3. Indirect B+ tree on B (Layout RB)

The index structure parameters are the same as RA:

$$\Rightarrow t_{\text{leaf}} = 18, n_{\text{leaf}} = 56, e_i = 474,$$

$$\text{Height: } \lceil \log_{e_i+1}(n_{\text{leaf}}) \rceil + 1 \Rightarrow \mathbf{h} = 2$$

$$\text{Queries on A require table scan: } \Rightarrow \mathbf{C(A = 10) = C(A = 500) = 123}$$

$$\text{Index-only selection cost: } (h-1) + \lceil FF(B=b) \cdot n_{\text{leaf}} \rceil \Rightarrow \mathbf{C(B = 10) = C(B = 500) = 2}$$

### 4. Two indirect indexes on A and B (Layout RAB)

Both RA and RB exist.

Heights and Cost identical (to calculation of indexes):

$$\Rightarrow \mathbf{h} = 2$$

$$\Rightarrow \mathbf{C(A = 10) = C(A = 500) = 2}$$

$$\Rightarrow \mathbf{C(B = 10) = C(B = 500) = 2}$$

### 5. Clustered, direct index on A (Layout RA\$)

Means physically sorted by A  $\rightarrow$  Leaf pages = table pages.

The parameters are the same as before:

$$\Rightarrow x = 407, n_{\text{leaf}} \rightarrow NPages(R) = 123, e_i = 474,$$

$$\text{Height: } \lceil \log_{e_i+1}(n_{\text{leaf}}) \rceil + 1 \Rightarrow \mathbf{h} = 2$$

$$\text{Cost for A: } (h-1) + \lceil FF(A=a) \cdot n_{\text{leaf}} \rceil \Rightarrow \mathbf{C(A = 10) = C(A = 500) = 2}$$

$$\text{B has no usable index: } \Rightarrow \mathbf{C(B = 10) = C(B = 500) = 123}$$

### 6. Combined B+ tree on (A,B) (Layout RC)

Combined search key:  $s_A + s_B \Rightarrow k = 12$ .

$$\text{Number of distinct key pairs: } 1000 \cdot 1000 \Rightarrow n_k = 1\,000\,000.$$

$$\text{Average number of tuples per key pair: } \frac{Card(R)}{n_k} \Rightarrow r_k = 0.05.$$

$$\text{Leaf capacity (page-layout indirect index): } \left\lceil \frac{(\text{page-head} - 2 \cdot p_{\text{leaf}}) \cdot f_{\text{index}}}{k + r_k \cdot p_{\text{rec}}} \right\rceil \Rightarrow t_{\text{leaf}} = 462.$$

$$\text{Number of leaf pages: } \left\lceil \frac{n_k}{t_{\text{leaf}}} \right\rceil \Rightarrow n_{\text{leaf}} = 2165$$

$$\text{Inner node capacity: } \left\lceil \frac{((\text{page-head}) \cdot f_{\text{index}}) - p}{k + p} \right\rceil \Rightarrow e_i = 316.$$

$$\text{Height of the index: } \lceil \log_{e_i+1}(n_{\text{leaf}}) \rceil + 1 \Rightarrow \mathbf{h} = 3$$

$$\text{Cost for queries on A: } (h-1) + \lceil FF(A=a) \cdot n_{\text{leaf}} \rceil \Rightarrow \mathbf{C(A = 10) = C(A = 500) = 5}.$$

$$\text{B alone cannot use the combined index (no leading A): } \Rightarrow \mathbf{C(B = 10) = C(B = 500) = 123}.$$

Summary:

| Layout | A=10 | B=10 | A=500 | B=500 | Height   |
|--------|------|------|-------|-------|----------|
| R      | 123  | 123  | 123   | 123   | no index |
| RA     | 2    | 123  | 2     | 123   | 2        |
| RB     | 123  | 2    | 123   | 2     | 2        |
| RAB    | 2    | 2    | 2     | 2     | 2        |
| RA\$   | 2    | 123  | 2     | 123   | 2        |
| RC     | 5    | 123  | 5     | 123   | 3        |



## Task 3