

# Estimation and Inference in Boundary Discontinuity Designs\*

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April 22, 2025

## Abstract

Boundary Discontinuity Designs are used to learn about treatment effects along a continuous boundary that splits units into control and treatment groups according to a bivariate score variable. These research designs are also called Multi-Score Regression Discontinuity Designs, a leading special case being Geographic Regression Discontinuity Designs. We study the statistical properties of commonly used local polynomial treatment effects estimators along the continuous treatment assignment boundary. We consider two distinct approaches: one based explicitly on the bivariate score variable for each unit, and the other based on their univariate distance to the boundary. For each approach, we present pointwise and uniform estimation and inference methods for the treatment effect function over the assignment boundary. Notably, we show that methods based on univariate distance to the boundary exhibit an irreducible large misspecification bias when the assignment boundary has kinks or other irregularities, making the distance-based approach unsuitable for empirical work in those settings. In contrast, methods based on the bivariate score variable do not suffer from that drawback. We illustrate our methods with an empirical application. Companion general-purpose software is provided.

*Keywords:* regression discontinuity, uniform estimation and inference, causal inference.

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\*We thank Eric Gautier, Boris Hanin, Jason Klusowski, Juliana Londoño-Vélez, Xinwei Ma, Boris Shigida, Mykhaylo Shkolnikov, and Jeff Wooldridge for comments. Cattaneo and Titiunik gratefully acknowledge financial support from the National Science Foundation (SES-2019432, DMS-2210561, and SES-2241575).

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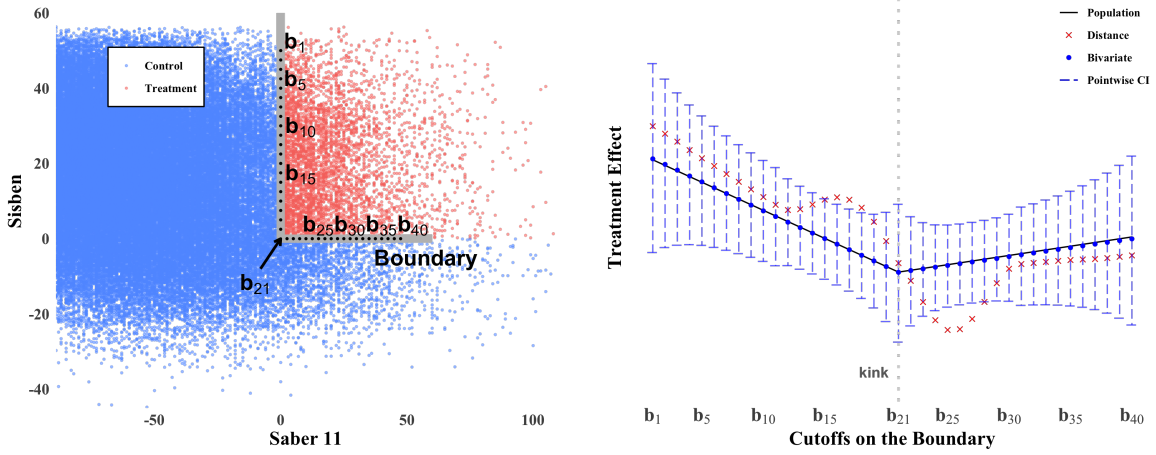
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# 1 Introduction

We study estimation and inference in Boundary Discontinuity Designs (Black, 1999; Dell, 2010; Jardim et al., 2024), where the goal is to learn about causal treatment effects along a continuous boundary that splits units into control and treatment groups according to the value of a bivariate location score variable. This setup is also known as a Multi-Score Regression Discontinuity (RD) Design (Papay et al., 2011; Reardon and Robinson, 2012), a leading special case being the Geographic RD Design and variations thereof (Keele and Titiunik, 2015, 2016; Keele et al., 2017; Galiani et al., 2017; Diaz and Zubizarreta, 2023). See Cattaneo and Titiunik (2022, Section 2.3) for an overview of the literature on Multi-Dimensional RD Designs, and Cattaneo et al. (2024, Section 5) for a practical introduction.

To describe the setup formally, suppose that  $(Y_i(0), Y_i(1), \mathbf{X}_i^\top)^\top$ ,  $i = 1, 2, \dots, n$ , is a random sample, where  $Y_i(0)$  and  $Y_i(1)$  denote the scalar potential outcomes for unit  $i$  under control and treatment assignment, respectively, and the score  $\mathbf{X}_i = (X_{1i}, X_{2i})^\top$  is a continuous bivariate vector with support  $\mathcal{X} \subseteq \mathbb{R}^2$ . Units are assigned to either the control group or treatment group according to their location  $\mathbf{X}_i$  relative to a known one-dimensional boundary curve  $\mathcal{B}$  splitting the support  $\mathcal{X}$  in two disjoint regions:  $\mathcal{X} = \mathcal{A}_0 \cup \mathcal{A}_1$  with  $\mathcal{A}_0$  and  $\mathcal{A}_1$  the control and treatment disjoint (connected) regions, respectively, and  $\mathcal{B} = \text{bd}(\mathcal{A}_0) \cap \text{bd}(\mathcal{A}_1)$ , where  $\text{bd}(\mathcal{A}_t)$  denotes the boundary of the set  $\mathcal{A}_t$ . The observed response variable is  $Y_i = \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_0) \cdot Y_i(0) + \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_1) \cdot Y_i(1)$ . Without loss of generality, we assume that the boundary belongs to the treatment group, that is,  $\text{bd}(\mathcal{A}_1) \subset \mathcal{A}_1$  and  $\mathcal{B} \cap \mathcal{A}_0 = \emptyset$ .

Boundary discontinuity designs are commonly used in quantitative social, behavioral, and biomedical sciences. For example, consider the substantive application in Londoño-Vélez, Rodríguez and Sánchez (2020). The authors studied the effects of a governmental subsidy for post-secondary education in Colombia, called *Ser Pilo Paga* (SPP), an anti-poverty policy providing tuition support for four-year or five-year undergraduate college students in any government-certified higher education institution with high-quality status. Eligibility to the program SSP was based on both merit and economic need: in order to qualify for the program, students had to obtain a high grade in Colombia’s national standardized high school exit exam, SABER 11, and they also had to come from economically disadvantaged families, as measured by the survey-based wealth index



(a) Scatterplot and Boundary (SPP data).

(b) Estimation and Inference (Simulations).

Figure 1: Scatterplot, Treatment Boundary, and Estimation and Inference Methods.

Note: Panel (a) presents a scatterplot of the bivariate score  $\mathbf{X}_i$  using the SPP data, and also plots the treatment boundary  $\mathcal{B}$  with 40 marked grid points. Panel (b) presents estimation and inference results over the 40 boundary grid points depicted in Panel (a) based on one simulated dataset calibrated using the real SPP data, where the population treatment effect curve  $\tau(\mathbf{x})$  is assumed linear to remove smoothing bias. Point estimates and confidence intervals/bands end points are averaged over 200 Monte Carlo simulations. Implementation employ the R software package `rd2d`, and omitted details are available in the replication files.

SISBEN. Eligibility followed a deterministic rule with a fixed bivariate cutoff: students had to obtain a SABER 11 score in the top 9 percent of scores or better, and they had to come from a household with SISBEN index below a region-specific threshold. Formally, each student was assigned a bivariate score  $\mathbf{X}_i = (\text{SABER11}_i, \text{SISBEN}_i)^\top$ , where  $X_{1i} = \text{SABER11}_i$  recorded the SABER11 score and  $X_{2i} = \text{SISBEN}_i$  recorded the SISBEN wealth score. After recentering each variable at its corresponding threshold, the treatment assignment boundary becomes  $\mathcal{B} = \{(\text{SABER11}, \text{SISBEN}) : (\text{SABER11}, \text{SISBEN}) \in \{\text{SABER11} \geq 0 \text{ and } \text{SISBEN} = 0\} \cup \{\text{SABER11} = 0 \text{ and } \text{SISBEN} \geq 0\}\}$ . Figure 1a presents a scatterplot of the bivariate score of the data of students in the 2014 cohort ( $n = 363,096$  observations), and also plots the bivariate assignment boundary  $\mathcal{B}$  together with 40 evenly-spaced cutoff points along the boundary.

Section 2 presents the core assumption underlying the causal inference framework, which generalizes the standard univariate RD design to boundary discontinuity designs. Given the standard (continuity and finite-moments) conditions in Assumption 1, the goal is to conduct estimation and

inference for the *average treatment effect curve along the boundary*:

$$\tau(\mathbf{x}) = \mathbb{E}[Y_i(1) - Y_i(0) | \mathbf{X}_i = \mathbf{x}], \quad \mathbf{x} \in \mathcal{B},$$

both pointwise for each  $\mathbf{x} \in \mathcal{B}$ , and uniformly over  $\mathcal{B}$ . For example, in the SPP application, the outcome variable is  $Y_i = 1$  if student  $i$  attended college or  $Y_i = 0$  otherwise, and thus the causal parameter  $\tau(\mathbf{x})$  captures the treatment effect of SPP on the probability of college education for students at the margin of program eligibility, as determined by their bivariate score  $\mathbf{X}_i = (\text{SABER11}_i, \text{SISBEN}_i)^\top \in \mathcal{B}$ . As a result, the parameter  $\tau(\mathbf{x})$  captures policy-relevant heterogeneous treatment effects along the boundary  $\mathcal{B}$ : for example, in Figure 1a,  $\tau(\mathbf{b}_1)$  corresponds to a (local) average treatment effect for students with high SISBEN score (wealth) and low SABER11 score (academic), while  $\tau(\mathbf{b}_{40})$  corresponds to a (local) average treatment effect for students with low SISBEN score and higher SABER11 score. Identification of these boundary treatment effects parallels standard continuity-based univariate RD design arguments (Hahn et al., 2001): treatment assignment changes abruptly along the boundary  $\mathcal{B}$ , which implies that conditional expectations on each side of the assignment boundary can be used to identify  $\tau(\mathbf{x})$  whenever there is no systematic “sorting” of units across the boundary, that is, whenever  $\mathbb{E}[Y_i(0) | \mathbf{X}_i = \mathbf{x}]$  and  $\mathbb{E}[Y_i(1) | \mathbf{X}_i = \mathbf{x}]$  are continuous for all  $\mathbf{x} \in \mathcal{B}$  (Assumption 1 below). For more discussion, see Papay et al. (2011), Reardon and Robinson (2012), Keele and Titiunik (2015), and references therein.

Motivated by the local to the assignment boundary identifiability of  $\tau(\mathbf{x})$ , researchers employ flexible regression methods using only observations with their score near the boundary  $\mathcal{B}$ . Local polynomial methods are the preferred choice for estimation and inference because they are simple (weighted) linear regression estimates that intuitively incorporate localization to the assignment boundary, while also retaining most of the familiar features of least squares regression methods. Two distinct implementations can be considered in this setting:

- (i) regression analysis based on the *univariate distance* to the boundary  $\mathcal{B}$ , or
- (ii) regression analysis based on the *bivariate location* relative to the boundary  $\mathcal{B}$ .

The first approach is more commonly used in practice because it is perceived as simpler, while the second approach is rarely encountered in applications. Despite their widespread use, however, there is no foundational understanding of their statistical properties and relative merits. This paper fills

the gap in the literature by providing a comprehensive set of large sample results for each of the two approaches, which we then use to offer specific practical recommendations for the analysis and interpretation of boundary discontinuity designs. In particular, we provide pointwise and uniform (over  $\mathcal{B}$ ) estimation and inference methods for both local polynomial regression approaches, and demonstrate theoretical and practical advantages of the approach based on bivariate location over the approach based on distance (albeit both are shown to be valid under appropriate assumptions).

Using simulated data calibrated with the real SPP data, Figure 1b illustrates graphically the two local polynomial approaches for estimation of  $\tau(\mathbf{x})$ . The solid black line corresponds to the population treatment effect curve  $\tau(\mathbf{x})$ , taken to be linear in order to eliminate smoothing bias issues. The red crosses and blue dots correspond to local polynomial estimators based on, respectively, univariate distance to the boundary and bivariate location relative to the boundary, for the 40 grid points depicted on the assignment boundary  $\mathcal{B}$  in Figure 1a. The estimator based on distance exhibits a higher bias near the boundary kink point  $\mathbf{x} = \mathbf{b}_{21}$ , relative to the bivariate local polynomial estimator based on the bivariate score. The figure also reports robust bias corrected confidence intervals based on the latter estimator.

Section 3 presents pointwise and uniform estimation and inference methods for local polynomial regression methods based on the univariate distance to the boundary  $\mathcal{B}$ . We begin by providing interpretable sufficient conditions (Assumption 2 below) for identification of  $\tau(\mathbf{x})$ , which restrict the distance function in conjunction with the univariate kernel function used and the shape of the boundary  $\mathcal{B}$ . We then present an important negative result: near kinks of the boundary  $\mathcal{B}$ , a  $p$ th order distance-based local polynomial estimator exhibits an irreducible bias of order  $h$ , the bandwidth used for implementation, no matter the polynomial order  $p$  used. This drawback is due to the fact that the underlying population regression function is at most Lipschitz continuous near kinks of the boundary  $\mathcal{B}$ . Figure 1b illustrated the phenomenon numerically, and our paper provides a theoretical explanation for the large bias of the distance-based treatment effect estimator. In contrast, when the boundary  $\mathcal{B}$  is smooth, we show that the  $p$ th order distance-based local polynomial estimator exhibits the usual bias of order  $h^{p+1}$ . Thus, our results show that the standard distance-based treatment effect estimator, while consistent for  $\tau(\mathbf{x})$ , both pointwise and uniformly over  $\mathcal{B}$ , it can exhibit a large bias affecting bandwidth selection, point estimation, and statistical inference in applications, whenever  $\mathcal{B}$  has kinks or other irregularities. For uncertainty

quantification, Section 3 presents pointwise and uniform (over  $\mathcal{B}$ ) large sample distribution theory, which is used to propose both confidence intervals for  $\tau(\mathbf{x})$  and confidence bands for the entire treatment effect curve ( $\tau(\mathbf{x}) : \mathbf{x} \in \mathcal{B}$ ). The last part of Section 3 discusses implementation, and also compares our results with related methods for standard univariate RD designs.

The analysis based on univariate distance to the boundary is not as straight forward as it may seem, and can in fact lead to invalid empirical findings in some applications. More precisely, our results show that standard methods in the literature for univariate RD estimation and inference will not be valid when deployed directly to a bivariate RD design, upon constructing distance to the boundary, because the specific features of the assignment boundary can lead to a large misspecification bias near kinks or other irregularities of  $\mathcal{B}$ . The large misspecification bias we highlight when the assignment boundary is non-smooth is particularly worrisome in Geographic RD designs (Keele and Titiunik, 2015), and similar settings, where the boundary naturally exhibits many kinks or other irregularities.

An alternative is to employ the bivariate location score directly. Section 4 studies this approach, and offers pointwise and uniform over  $\mathcal{B}$  estimation and inference methods. The pointwise results follow directly from the literature, provided an additional regularity condition on the bivariate kernel function and boundary  $\mathcal{B}$  hold (Assumption 3 below). On the other hand, the uniform inference results require some additional technical care, and are new to the literature. The main potential issue is that a uniform distributional approximation is established over the lower-dimensional manifold  $\mathcal{B}$ , and thus its shape can affect the validity of the results. Section 4 also discusses new bandwidth selection methods based on mean square error (MSE) expansions, robust bias-corrected inference, and related implementation details. Our results provide natural generalizations of well-established results for univariate RD designs; see Calonico et al. (2020) for bandwidth selection, and Calonico et al. (2014, 2018, 2022) for robust bias correction.

Section 5 deploys our theoretical and methodological results to the SPP data, revising the main results reported in Londoño-Vélez, Rodríguez and Sánchez (2020). In addition to providing further empirical evidence in favor of their empirical findings, we also find some evidence of treatment effect heterogeneity along the assignment boundary  $\mathcal{B}$ . All empirical results are obtained using the companion R package `rd2d`, and we provide complete replication files.

Section 6 concludes with specific recommendations for practice. The supplemental appendix

presents generalizations of our theoretical results, reports their proofs, and gives other theoretical and methodological results that may be of independent interest. In particular, a new strong approximation result for empirical process with bounded moments is presented, extending recent work by Cattaneo and Yu (2025). The companion software article (Cattaneo et al., 2025) discusses the general-purpose software R package `rd2d` (<https://rdpackages.github.io/rd2d>) implementing the methods developed in this paper.

## 2 Setup

We maintain throughout the paper the following basic conditions on the underlying data generating process.

**Assumption 1** (Data Generating Process). *Let  $t \in \{0, 1\}$ .*

- (i)  $(Y_1(t), \mathbf{X}_1)^\top, \dots, (Y_n(t), \mathbf{X}_n)^\top$  are independent and identically distributed random vectors.
- (ii) The distribution of  $\mathbf{X}_i$  has a Lebesgue density  $f_X(\mathbf{x})$  that is continuous and bounded away from zero on its support  $\mathcal{X} = [a_1, b_1] \times [a_2, b_2]$ , for  $-\infty < a_l < b_l < \infty$  with  $l \in \{1, 2\}$ .
- (iii)  $\mu_t(\mathbf{x}) = \mathbb{E}[Y_i(t) | \mathbf{X}_i = \mathbf{x}]$  is  $(p+1)$ -times continuously differentiable on  $\mathcal{X}$ .
- (iv)  $\sigma_t^2(\mathbf{x}) = \mathbb{V}[Y_i(t) | \mathbf{X}_i = \mathbf{x}]$  is bounded away from zero and continuous on  $\mathcal{X}$ .
- (v)  $\sup_{\mathbf{x} \in \mathcal{X}} \mathbb{E}[|Y_i(t)|^{2+v} | \mathbf{X}_i = \mathbf{x}] < \infty$  for some  $v \geq 2$ .

This assumption is on par with the usual assumptions encountered in the classical RD literature with an univariate score. In particular, part (ii) goes beyond the usual compact support restriction and further assumes a tensor product structure on  $\mathcal{X}$  to avoid technicalities underlying our strong approximation results used for uniform distribution theory: the assumption is not practically restrictive because all methods considered in this paper localize to the boundary. Part (iii) imposes standard smoothness conditions on the bivariate conditional expectations of interest, which will play an important role in misspecification (smoothing bias) reduction in our upcoming results. Identification of  $\tau(\mathbf{x})$  follows directly from Assumption 1: see Papay et al. (2011), Reardon and Robinson (2012), Keele and Titiunik (2015), and references therein.

## 2.1 Notation

We employ standard concepts and notations from empirical process theory (van der Vaart and Wellner, 1996; Giné and Nickl, 2016) and geometric measure theory (Simon et al., 1984; Federer, 2014). For a random variable  $\mathbf{V}_i$ , we write  $\mathbb{E}_n[g(\mathbf{V}_i)] = n^{-1} \sum_{i=1}^n g(\mathbf{V}_i)$ . For a vector  $\mathbf{v} \in \mathbb{R}^k$ , the Euclidean norm is  $\|\mathbf{v}\| = (\sum_{i=1}^k \mathbf{v}_i^2)^{1/2}$ . For a matrix  $A \in \mathbb{R}^{m \times n}$ , the operator norm is  $\|A\| = \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ .  $C^k(\mathcal{X}, \mathcal{Y})$  denotes the class of  $k$ -times continuously differentiable functions from  $\mathcal{X}$  to  $\mathcal{Y}$ , and  $C^k(\mathcal{X})$  is a shorthand for  $C^k(\mathcal{X}, \mathbb{R})$ . For a Borel set  $\mathcal{S} \subseteq \mathcal{X}$ , the De Giorgi perimeter of  $\mathcal{S}$  is  $\text{perim}(\mathcal{S}) = \sup_{g \in \mathcal{D}_2(\mathcal{X})} \int_{\mathbb{R}^2} \mathbf{1}(\mathbf{x} \in \mathcal{S}) \text{div } g(\mathbf{x}) d\mathbf{x} / \|g\|_\infty$ , where  $\text{div}$  is the divergence operator, and  $\mathcal{D}_2(\mathcal{X})$  denotes the space of  $C^\infty$  functions with values in  $\mathbb{R}^2$  and with compact support included in  $\mathcal{X}$ . When  $\mathcal{S}$  is connected, and the boundary  $\text{bd}(\mathcal{S})$  is a smooth simple closed curve,  $\mathcal{S}$  simplifies to the curve length of  $\text{bd}(\mathcal{S})$ . A curve  $\mathcal{B} \subseteq \mathbb{R}^2$  is a *rectifiable curve* if there exists a Lipschitz continuous function  $\gamma : [0, 1] \mapsto \mathbb{R}^2$  such that  $\mathcal{B} = \gamma([0, 1])$ . For a function  $f : \mathbb{R}^2 \mapsto \mathbb{R}$ ,  $\text{Supp}(f)$  denotes closure of the set  $\{\mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) \neq 0\}$ . For reals sequences  $a_n = o(b_n)$  if  $\limsup_{n \rightarrow \infty} \frac{|a_n|}{|b_n|} = 0$ ,  $|a_n| \lesssim |b_n|$  if there exists some constant  $C$  and  $N > 0$  such that  $n > N$  implies  $|a_n| \leq C|b_n|$ . For sequences of random variables  $a_n = o_{\mathbb{P}}(b_n)$  if  $\text{plim}_{n \rightarrow \infty} \frac{|a_n|}{|b_n|} = 0$ ,  $|a_n| \lesssim_{\mathbb{P}} |b_n|$  if  $\limsup_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}[|\frac{a_n}{b_n}| \geq M] = 0$ . Finally,  $\Phi(x)$  denotes the standard Gaussian cumulative distribution function.

## 3 Analysis based on Univariate Distance

For each unit  $i = 1, \dots, n$ , define their scalar distance-based score  $D_i(\mathbf{x}) = \mathcal{d}(\mathbf{X}_i, \mathbf{x})(\mathbf{1}(\mathbf{X}_i \in \mathcal{A}_1) - \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_0))$  to the point  $\mathbf{x} \in \mathcal{B}$ , where  $\mathcal{d}(\cdot, \cdot)$  denotes a distance function. It is customary to use the Euclidean distance  $\mathcal{d}(\mathbf{X}_i, \mathbf{x}) = \|\mathbf{X}_i - \mathbf{x}\| = \sqrt{(X_{1i} - x_1)^2 + (X_{2i} - x_2)^2}$  for  $\mathbf{x} = (x_1, x_2)^\top \in \mathcal{B}$  in applications, but other choices are sometimes encountered. For each  $\mathbf{x} \in \mathcal{B}$ , the setup thus reduces to a standard univariate RD design with  $D_i(\mathbf{x}) \in \mathbb{R}$  the score variable and  $c = 0$  the cutoff, where  $D_i(\mathbf{x}) \geq 0$  if unit  $i$  is assigned to treatment status and  $D_i(\mathbf{x}) < 0$  if unit  $i$  is assigned to control status. The local polynomial treatment effect curve estimator based on distance is

$$\hat{\tau}_{\text{dis}}(\mathbf{x}) = \mathbf{e}_1^\top \hat{\gamma}_1(\mathbf{x}) - \mathbf{e}_1^\top \hat{\gamma}_0(\mathbf{x}), \quad \mathbf{x} \in \mathcal{B},$$



where, for  $t \in \{0, 1\}$ ,

$$\hat{\gamma}_t(\mathbf{x}) = \arg \min_{\gamma \in \mathbb{R}^{p+1}} \mathbb{E}_n \left[ (Y_i - \mathbf{r}_p(D_i(\mathbf{x}))^\top \gamma)^2 k_h(D_i(\mathbf{x})) \mathbf{1}(D_i(\mathbf{x}) \in \mathcal{J}_t) \right],$$

with  $\mathbf{r}_p(u) = (1, u, u^2, \dots, u^p)^\top$  the usual univariate polynomial basis,  $k_h(u) = k(u/h)/h^2$  for univariate kernel function  $k(\cdot)$  and bandwidth parameter  $h$ , and  $\mathcal{J}_0 = (-\infty, 0)$  and  $\mathcal{J}_1 = [0, \infty)$ . See [Cattaneo and Titiunik \(2022\)](#) for a literature review of RD designs, and [Cattaneo et al. \(2020, 2024\)](#) for practical introductions. The univariate kernel function typically down-weights observations as the distance to  $\mathbf{x} \in \mathcal{B}$  increases, while the bandwidth determines the level of localization to each point on the boundary  $\mathcal{B}$ . In this case, observations contribute relative to their distance  $\mathcal{d}(\mathbf{X}_i, \mathbf{x})$  to the point  $\mathbf{x} \in \mathcal{B}$ .

We impose the following conditions on the underlying features of the distance-based local polynomial estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$ .

**Assumption 2** (Univariate Distance-Based Kernel). *Let  $t \in \{0, 1\}$ .*

- (i)  $\mathcal{d} : \mathbb{R}^2 \mapsto [0, \infty)$  satisfies  $\|\mathbf{x}_1 - \mathbf{x}_2\| \lesssim \mathcal{d}(\mathbf{x}_1, \mathbf{x}_2) \lesssim \|\mathbf{x}_1 - \mathbf{x}_2\|$  for all  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ .
- (ii)  $k : \mathbb{R} \rightarrow [0, \infty)$  is compact supported and Lipschitz continuous, or  $k(u) = \mathbf{1}(u \in [-1, 1])$ .
- (iii)  $\liminf_{h \downarrow 0} \inf_{\mathbf{x} \in \mathcal{B}} \int_{\mathcal{A}_t} k_h(\mathcal{d}(\mathbf{u}, \mathbf{x})) d\mathbf{u} \gtrsim 1$ .

Part (i) of this assumption requires the distance function be equivalent (up to constants) to the Euclidean distance, while Assumption 2(ii) imposes standard conditions on the (univariate) kernel function. The last part of this assumption is novel to the literature: it implicitly restricts the geometry of the boundary  $\mathcal{B}$  relative to the kernel and distance functions. More precisely, it rules out settings where highly irregular boundary shapes will lead to regions with too “few” data: a necessary condition for the estimator  $\hat{\gamma}_t(\mathbf{x})$  to be well-defined in large samples is that  $\mathbb{E}[k_h(D_i(\mathbf{x})) \mathbf{1}(D_i(\mathbf{x}) \in \mathcal{J}_t)] = \int_{\mathcal{A}_t} k_h(\mathcal{d}(\mathbf{u}, \mathbf{x})) f_X(\mathbf{u}) d\mathbf{u} > 0$ . Our theoretical results show that Assumption 2(iii) is a simple sufficient condition, due to Assumption 1(ii). Commonly encountered treatment assignment boundaries and distance functions typically satisfy Assumption 2; a potential exception being highly irregular geographic boundaries with low density of observations.

### 3.1 Identification and Interpretation

For each treatment group  $t \in \{0, 1\}$ , boundary point  $\mathbf{x} \in \mathcal{B}$ , and distance score  $D_i(\mathbf{x})$ , the univariate distance-based local polynomial estimator  $\hat{\gamma}_t(\mathbf{x})$  is the sample analog of the coefficients associated with the best (weighted, local) mean square approximation of the conditional expectation  $\mathbb{E}[Y_i|D_i(\mathbf{x})]$  based on  $\mathbf{r}_p(D_i(\mathbf{x}))$ :

$$\gamma_t^*(\mathbf{x}) = \arg \min_{\gamma \in \mathbb{R}^{p+1}} \mathbb{E} \left[ (Y_i - \mathbf{r}_p(D_i(\mathbf{x}))^\top \gamma)^2 k_h(D_i(\mathbf{x})) \mathbf{1}(D_i(\mathbf{x}) \in \mathcal{J}_t) \right].$$

Therefore, letting  $\hat{\theta}_{t,\mathbf{x}}(0) = \mathbf{e}_1^\top \hat{\gamma}_t(\mathbf{x})$ ,  $\theta_{t,\mathbf{x}}^*(0) = \mathbf{e}_1^\top \gamma_t^*(\mathbf{x})$ , and  $\theta_{t,\mathbf{x}}(r) = \mathbb{E}[Y_i|D_i(\mathbf{x}) = r, D_i(\mathbf{x}) \in \mathcal{J}_t]$  for  $r \in \mathbb{R}$ , we have the standard least squares (best linear prediction) decomposition

$$\hat{\theta}_{t,\mathbf{x}}(0) - \theta_{t,\mathbf{x}}(0) = \mathbf{e}_1^\top \Psi_{t,\mathbf{x}}^{-1} \mathbf{O}_{t,\mathbf{x}} + [\theta_{t,\mathbf{x}}^*(0) - \theta_{t,\mathbf{x}}(0)] + [\mathbf{e}_1^\top (\hat{\Psi}_{t,\mathbf{x}}^{-1} - \Psi_{t,\mathbf{x}}^{-1}) \mathbf{O}_{t,\mathbf{x}}] \quad (1)$$

for each group  $t \in \{0, 1\}$ , and where

$$\begin{aligned} \Psi_{t,\mathbf{x}} &= \mathbb{E} \left[ \mathbf{r}_p \left( \frac{D_i(\mathbf{x})}{h} \right) \mathbf{r}_p \left( \frac{D_i(\mathbf{x})}{h} \right)^\top k_h(D_i(\mathbf{x})) \mathbf{1}(D_i(\mathbf{x}) \in \mathcal{J}_t) \right], \\ \hat{\Psi}_{t,\mathbf{x}} &= \mathbb{E}_n \left[ \mathbf{r}_p \left( \frac{D_i(\mathbf{x})}{h} \right) \mathbf{r}_p \left( \frac{D_i(\mathbf{x})}{h} \right)^\top k_h(D_i(\mathbf{x})) \mathbf{1}(D_i(\mathbf{x}) \in \mathcal{J}_t) \right], \quad \text{and} \\ \mathbf{O}_{t,\mathbf{x}} &= \mathbb{E}_n \left[ \mathbf{r}_p \left( \frac{D_i(\mathbf{x})}{h} \right) k_h(D_i(\mathbf{x})) (Y_i - \theta_{t,\mathbf{x}}^*(D_i(\mathbf{x}))) \mathbf{1}(D_i(\mathbf{x}) \in \mathcal{J}_t) \right]. \end{aligned}$$

In the decomposition (1), the first term is the stochastic linear representation of the centered estimator,  $\hat{\theta}_{t,\mathbf{x}}(0) - \theta_{t,\mathbf{x}}(0)$ , since it is an average of (unconditional) mean-zero random variables, the second term is the mean square approximation bias, and the third term is a non-linearity error arising from the convergence of the Gram matrix associated with the (weighted) least squares estimator.

Since  $\hat{\tau}_{\text{dis}}(\mathbf{x}) = \hat{\theta}_{1,\mathbf{x}}(0) - \hat{\theta}_{0,\mathbf{x}}(0)$ , the following lemma characterizes the target estimand of the distance-based local polynomial estimator.

**Lemma 1** (Identification). *Suppose Assumptions 1(i)–(iii) and 2(i) hold. Then,  $\tau(\mathbf{x}) = \lim_{r \downarrow 0} \theta_{1,\mathbf{x}}(r) - \lim_{r \uparrow 0} \theta_{0,\mathbf{x}}(r)$  for all  $\mathbf{x} \in \mathcal{B}$ .*

This lemma is established by noting that, for each group  $t \in \{0, 1\}$  and  $r \in \mathbb{R}$ ,

$$\begin{aligned}\theta_{t,\mathbf{x}}(r) &= \mathbb{E}[Y_i | D_i(\mathbf{x}) = r, D_i(\mathbf{x}) \in \mathcal{J}_t] \\ &= \mathbb{E}[Y_i | \mathcal{d}(\mathbf{X}_i, \mathbf{x}) = |r|, \mathbf{X}_i \in \mathcal{A}_t] = \mathbb{E}[Y_i(t) | \mathcal{d}(\mathbf{X}_i, \mathbf{x}) = |r|],\end{aligned}$$

and then verifying that  $\lim_{u \downarrow 0} \mathbb{E}[Y_i(t) | \mathcal{d}(\mathbf{X}_i, \mathbf{x}) = u] = \mu_t(\mathbf{x})$  under the conditions imposed. Without restricting the data generating process, as well as the assignment boundary and distance function,  $\theta_{t,\mathbf{x}}(0)$  and  $\mu_t(\mathbf{x})$  need not agree. Employing Lemma 1 and the decomposition (1), we obtain:

$$\widehat{\tau}_{\text{dis}}(\mathbf{x}) - \tau(\mathbf{x}) = \mathfrak{L}_n(\mathbf{x}) + \mathfrak{B}_n(\mathbf{x}) + \mathfrak{Q}_n(\mathbf{x}), \quad \mathbf{x} \in \mathcal{B},$$

where  $\mathfrak{L}_n(\mathbf{x}) = \mathbf{e}_1^\top \Psi_{1,\mathbf{x}}^{-1} \mathbf{O}_{1,\mathbf{x}} - \mathbf{e}_1^\top \Psi_{0,\mathbf{x}}^{-1} \mathbf{O}_{0,\mathbf{x}}$  is a mean-zero linear statistic,  $\mathfrak{B}_n(\mathbf{x}) = \theta_{1,\mathbf{x}}^*(0) - \theta_{0,\mathbf{x}}^*(0) - \tau(\mathbf{x})$  is the bias of the estimator, and  $\mathfrak{Q}_n(\mathbf{x}) = \mathbf{e}_1^\top (\widehat{\Psi}_{1,\mathbf{x}}^{-1} - \Psi_{1,\mathbf{x}}^{-1}) \mathbf{O}_{1,\mathbf{x}} - \mathbf{e}_1^\top (\widehat{\Psi}_{0,\mathbf{x}}^{-1} - \Psi_{0,\mathbf{x}}^{-1}) \mathbf{O}_{0,\mathbf{x}}$  is the higher-order linearization error.

In standard local polynomial regression settings,  $\mathfrak{L}_n(\mathbf{x})$  is approximately Gaussian,  $\mathfrak{B}_n(\mathbf{x})$  is of order  $h^{p+1}$ , and  $\mathfrak{Q}_n(\mathbf{x})$  is neglected. However, because estimation is conducted along the boundary  $\mathcal{B}$ , not all of those standard results remain valid in the context of boundary discontinuity designs.

### 3.2 Bias Along the Boundary

Unlike the case of standard univariate local polynomial estimation, in boundary discontinuity designs the smoothness of  $\theta_{t,\mathbf{x}}(r) = \mathbb{E}[Y_i | D_i(\mathbf{x}) = r, D_i(\mathbf{x}) \in \mathcal{J}_t]$  depends on smoothness of the boundary  $\mathcal{B}$ . More specifically, the bias of the distance-based local polynomial estimator can be affected by the shape of the boundary  $\mathcal{B}$ , regardless of the polynomial order  $p$  used when constructing the estimator. Figure 2 demonstrates the problem graphically: for a point  $\mathbf{x} \in \mathcal{B}$  that is close enough to a kink point  $(0, 0)$  on the boundary, the conditional expectation  $\theta_{1,\mathbf{x}}(r)$  is not differentiable for all  $r \geq 0$ . This problem arises because, given the distance function  $\mathcal{d}(\cdot, \cdot)$ , for any point  $\mathbf{x} \in \mathcal{B}$  near a kink, a “small”  $r$  gives a complete arc  $\{\mathbf{x} \in \mathcal{A}_1 : \mathcal{d}(\mathbf{X}_i, \mathbf{x}) = r\}$ , while for a “large”  $r$  the arc is truncated by the boundary. As a result, for the example in Figure 2,  $\theta_{1,\mathbf{x}}(r)$  is smooth for all  $r \leq r_3$  and  $r > r_3$ , but the function is not differentiable at  $r = r_3$ . Furthermore, at  $r = r_3$ , the left derivative is constant, but the right derivative is equal to infinity. The supplemental

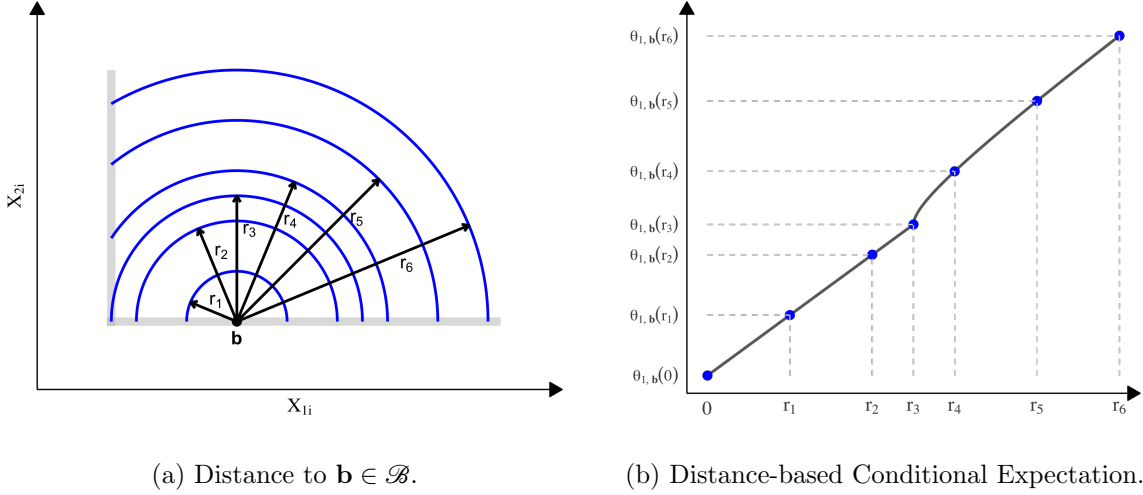


Figure 2: Lack of Smoothness of Distance-Based Conditional Expectation near a Kink.  
Note: Analytic example of  $\theta_{1,\mathbf{b}}(r) = \mathbb{E}[Y(1)|D_i(\mathbf{b}) = r]$ ,  $r \geq 0$ , for distance transformation  $D_i(\mathbf{b}) = \mathcal{d}(\mathbf{X}_i, \mathbf{b}) = \|\mathbf{X}_i - \mathbf{b}\|$  to point  $\mathbf{b} \in \mathcal{B}$  near a kink point on the boundary, based on location  $\mathbf{X}_i = (X_{1i}, X_{2i})^\top$ . The induced univariate conditional expectation  $r \mapsto \theta_{1,\mathbf{b}}(r)$  is continuous but not differentiable at  $r = r_3$ .

appendix gives details on this analytic example.

Although smoothness of the boundary  $\mathcal{B}$  can affect the smoothness of  $\theta_{t,\mathbf{x}}(r)$ , the fact that the distance-based estimator is “local” means that the *approximation error* will be no greater than that of a local constant estimator, regardless of the choice of polynomial order  $p$ . The following lemma formalizes this result, and also shows that this bias order cannot be improved by increasing  $p \geq 0$ .

**Lemma 2** (Approximation Error: Minimal Guarantee). *For some  $L > 0$ , let  $\mathcal{P}$  be the class of data generating processes satisfying Assumptions 1(i)-(iii) and 2 with  $\mathcal{X} \subseteq [-L, L]^2$ , and the following three additional conditions:*

- (i)  $L^{-1} \leq \inf_{\mathbf{x} \in \mathcal{X}} f_X(\mathbf{x}) \leq \sup_{\mathbf{x} \in \mathcal{X}} f_X(\mathbf{x}) \leq L$ ,
- (ii)  $\max_{0 \leq |\nu| \leq p} \sup_{\mathbf{x} \in \mathcal{X}} |\partial^\nu \mu(\mathbf{x})| + \max_{0 \leq |\nu| \leq p} \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{X}} \frac{|\partial^\nu \mu(\mathbf{x}) - \partial^\nu \mu(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|} \leq L$ , and
- (iii)  $\liminf_{h \downarrow 0} \inf_{\mathbf{x} \in \mathcal{B}} \int_{\mathcal{A}_t} k_h(\mathcal{d}(\mathbf{u}, \mathbf{x})) d\mathbf{u} \geq L^{-1}$ .

For any  $p \geq 1$ , if  $nh^2 \rightarrow \infty$ , then

$$1 \lesssim \liminf_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{P}} \sup_{\mathbf{x} \in \mathcal{B}} \frac{\mathfrak{B}_n(\mathbf{x})}{h} \leq \limsup_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{P}} \sup_{\mathbf{x} \in \mathcal{B}} \frac{\mathfrak{B}_n(\mathbf{x})}{h} \lesssim 1.$$

This lemma characterizes precisely the uniform over  $\mathcal{B}$  (and data generating processes) bias of

the distance-based local polynomial estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$ . The upper bound in Lemma 2 is established uniformly over the class of data generating processes because we can show that  $|\theta_t(0) - \theta_t(r)| \lesssim r$  for  $t \in \{0, 1\}$  in general. The lower bound is shown using the following example. Suppose  $\mathbf{X}_i \sim \text{Uniform}([-2, 2]^2)$ ,  $\mu_0(x_1, x_2) = 0$ ,  $\mu_1(x_1, x_2) = x_2$  for all  $(x_1, x_2) \in [-2, 2]^2$ , and  $Y_i(0)|\mathbf{X}_i \sim \text{Normal}(\mu_0(\mathbf{X}_i), 1)$  and  $Y_i(1)|\mathbf{X}_i \sim \text{Normal}(\mu_1(\mathbf{X}_i), 1)$ . Let  $\mathcal{d}(\cdot, \cdot)$  be the Euclidean distance, and suppose that the control and treatment regions are  $\mathcal{A}_1 = \{(x, y) \in \mathbb{R}^2 : x \leq 0, y \geq 0\}$  and  $\mathcal{A}_0 = \mathbb{R}^2 / \mathcal{A}_1$ , respectively, and hence  $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 0, y = 0 \text{ or } x = 0, 0 \leq y \leq 2\}$ . This boundary is as in Figure 2a, having a  $90^\circ$  kink at  $\mathbf{x} = (0, 0)$ . It follows that the conditions of Lemma 2 hold, and hence this is an allowed data generating process. The supplemental appendix establishes the lower bound by careful analysis of the resulting approximation bias.

As a point of contrast, note that  $\mathfrak{B}_n(\mathbf{x}) \lesssim_{\mathbb{P}} h^{p+1}$  pointwise in  $\mathbf{x} \in \mathcal{B}$ , for small enough  $h$ , provided that the kinks on the boundary  $\mathcal{B}$  are sufficiently far apart of each other relative to the bandwidth. However, Lemma 2 demonstrates that, no matter how large the sample size is (i.e., how small the bandwidth is), there will always be a region near a kink of the boundary  $\mathcal{B}$  where the misspecification bias of the distance-based local polynomial estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  is at most of order  $h$  (i.e., not of order  $h^{p+1}$  as expected), regardless of the polynomial order  $p$  employed. The problem arises when the boundary  $\mathcal{B}$  changes non-smoothly, leading to a non-differentiable induced conditional expectation  $\theta_{t,\mathbf{x}}(r) = \mathbb{E}[Y_i | D_i(\mathbf{x}) = r, D_i(\mathbf{x}) \in \mathcal{J}_t]$ ,  $t \in \{0, 1\}$ , as illustrated in Figure 2b. On the other hand, if the boundary  $\mathcal{B}$  is smooth enough, then a better smoothing bias can be established.

**Lemma 3** (Approximation Error: Smooth Boundary). *Suppose Assumptions 1(i)-(iii) and 2 hold, with  $\mathcal{d}(\cdot, \cdot)$  the Euclidean distance.*

- (i) *For  $\mathbf{x} \in \mathcal{B}$ , and for some  $\delta, \varepsilon > 0$ , suppose that  $\mathcal{B} \cap \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| \leq \varepsilon\} = \gamma([- \delta, \delta])$ , where  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  is a one-to-one function in  $C^{\kappa+2}([- \delta, \delta], \mathbb{R}^2)$ . Then,  $\theta_{0,\mathbf{x}}(\cdot)$  and  $\theta_{1,\mathbf{x}}(\cdot)$  are  $(\kappa \wedge (p+1))$ -times continuously differentiable on  $[0, \varepsilon]$ . Therefore, there exists a positive constant  $C$  such that  $|\mathfrak{B}_n(\mathbf{x})| \leq Ch^{\kappa \wedge (p+1)}$  for all  $h \in [0, \varepsilon]$  and  $\mathbf{x} \in \mathcal{B}$ .*
- (ii) *Suppose  $\mathcal{B} = \gamma([0, L])$  where  $\gamma$  is a one-to-one function in  $C^{\iota+2}([0, L], \mathbb{R}^2)$  for some  $L > 0$ . Suppose there exists  $\delta, \varepsilon > 0$  such that for all  $\mathbf{x} \in \mathcal{B}^\circ = \gamma([\delta, L - \delta])$ ,  $r \in [0, \varepsilon]$ , and  $t = 0, 1$ , the set  $\{\mathbf{u} \in \mathbb{R}^2 : \mathcal{d}(\mathbf{u}, \mathbf{x}) = r\}$  intersects with  $\text{bd}(\mathcal{A}_t)$  at only two points in  $\mathcal{B}$ . Then, for all*

$\mathbf{x} \in \gamma([\delta, L - \delta])$ ,  $\theta_{0,\mathbf{x}}(\cdot)$  and  $\theta_{1,\mathbf{x}}(\cdot)$  are  $(\iota \wedge (p+1))$ -times continuously differentiable on  $[0, \varepsilon]$ , and  $\lim_{r \downarrow 0} \frac{d^v}{dr^v} \theta_{t,\mathbf{x}}(r)$  exists and is finite for all  $0 \leq v \leq p+1$  and  $t \in \{0, 1\}$ . Therefore, there exists a positive constant  $C$  such that  $\sup_{\mathbf{x} \in \mathcal{B}^o} |\mathfrak{B}_n(\mathbf{x})| \leq Ch^{\iota \wedge (p+1)}$  for all  $h \in [0, r]$ .

This lemma gives sufficient conditions in terms of smoothness of the boundary  $\mathcal{B}$  to achieve a smaller misspecification bias of  $\hat{\tau}_{\text{dis}}(\mathbf{x})$ , when compared to the minimal guarantee given by Lemma 2. In particular, for a uniform bound on misspecification bias, the sufficient condition in Lemma 3 requires the boundary  $\mathcal{B}$  be *uniformly smooth* in that it can be parameterized by *one* smooth function. The sufficient condition ensuring a smooth boundary everywhere is crucial because, as shown in Lemma 2, even a piecewise smooth  $\mathcal{B}$  with only one kink will have detrimental effects on the convergence rate of the bias (Figure 2).

In Appendix A, we also show that if we consider the class of rectifiable boundaries, then no estimator based on univariate distance, not just local polynomial estimators, can achieve a uniform convergence rate better than  $n^{-1/4}$ , that is, the non-parametric mean square minimax rate for estimating bivariate Lipschitz functions (Stone, 1982). Since we show below that the local polynomial estimator based on univariate distance has a uniform convergence no larger than  $(n/\log n)^{-1/4}$ , it is *nearly* minimax optimal (up to the  $\log^{1/4} n$  term). This minimax result means that no matter which univariate distance-based estimator is used, it is always possible to find a rectifiable boundary and a data generating process such that the estimation error is no better than  $n^{-1/4}$ .

### 3.3 Treatment Effect Estimation and Inference

Using technical results established in the supplemental appendix, we obtain the following convergence rates for the univariate distance-based local polynomial treatment effect estimator.

**Theorem 1** (Rates of Convergence). *Suppose Assumptions 1 and 2 hold. If  $nh^2/\log(1/h) \rightarrow \infty$ , then*

- (i)  $|\hat{\tau}_{\text{dis}}(\mathbf{x}) - \tau(\mathbf{x})| \lesssim_{\mathbb{P}} \frac{1}{\sqrt{nh^2}} + \frac{1}{n^{\frac{1+v}{2+v}} h^2} + |\mathfrak{B}_n(\mathbf{x})|$  for  $\mathbf{x} \in \mathcal{B}$ , and
- (ii)  $\sup_{\mathbf{x} \in \mathcal{B}} |\hat{\tau}_{\text{dis}}(\mathbf{x}) - \tau(\mathbf{x})| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^2}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}} h^2} + \sup_{\mathbf{x} \in \mathcal{B}} |\mathfrak{B}_n(\mathbf{x})|.$

This theorem establishes the pointwise and uniform (over  $\mathcal{B}$ ) convergence rates for the distance-based treatment effect estimator. By Lemma 2,  $\hat{\tau}_{\text{dis}}(\mathbf{x}) \rightarrow_{\mathbb{P}} \tau(\mathbf{x})$ , both pointwise and uniformly,

if  $h \rightarrow 0$ . However,  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  has a variance convergence rate of order  $n^{-1}h^{-2}$  despite being a “univariate” local polynomial estimator, which would have naïvely suggested a variance convergence of order  $n^{-1}h^{-1}$  instead. In addition to exhibiting a bivariate curse of dimensionality, the convergence rate of the treatment effect estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  along the boundary  $\mathcal{B}$  is determined by the smoothness of the  $\mathcal{B}$ , as discussed in the previous subsection. Finally, it is not difficult to establish valid pointwise, or integrated over  $\mathcal{B}$ , MSE convergence rates matching those in Theorem 1(i). See the supplemental appendix for omitted details. Section 3.4 discusses the implications of these results for implementation, leveraging standard methods from univariate RD designs.

To develop companion pointwise and uniform inference procedures along the treatment assignment boundary  $\mathcal{B}$ , we consider the feasible t-statistic at each boundary point  $\mathbf{x} \in \mathcal{B}$  (for given a bandwidth choice):

$$\hat{T}_{\text{dis}}(\mathbf{x}) = \frac{\hat{\tau}_{\text{dis}}(\mathbf{x}) - \tau(\mathbf{x})}{\sqrt{\hat{\Xi}_{\mathbf{x},\mathbf{x}}}},$$

where, using standard least squares algebra, for all  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{B}$  and  $t \in \{0, 1\}$ , we define

$$\hat{\Xi}_{\mathbf{x}_1, \mathbf{x}_2} = \hat{\Xi}_{0, \mathbf{x}_1, \mathbf{x}_2} + \hat{\Xi}_{1, \mathbf{x}_1, \mathbf{x}_2}, \quad \hat{\Xi}_{t, \mathbf{x}_1, \mathbf{x}_2} = \frac{1}{nh^2} \mathbf{e}_1^\top \hat{\Psi}_{t, \mathbf{x}_1}^{-1} \hat{\Upsilon}_{t, \mathbf{x}_1, \mathbf{x}_2} \hat{\Psi}_{t, \mathbf{x}_2}^{-1} \mathbf{e}_1,$$

and

$$\begin{aligned} \hat{\Upsilon}_{t, \mathbf{x}_1, \mathbf{x}_2} = h^2 \mathbb{E}_n \Big[ & \mathbf{r}_p \left( \frac{D_i(\mathbf{x}_1)}{h} \right) k_h(D_i(\mathbf{x}_1)) (Y_i - \mathbf{r}_p(D_i(\mathbf{x}_1))^\top \hat{\gamma}_t(\mathbf{x}_1)) \mathbf{1}(D_i(\mathbf{x}_1) \in \mathcal{J}_t) \\ & \times \mathbf{r}_p \left( \frac{D_i(\mathbf{x}_2)}{h} \right)^\top k_h(D_i(\mathbf{x}_2)) (Y_i - \mathbf{r}_p(D_i(\mathbf{x}_2))^\top \hat{\gamma}_t(\mathbf{x}_2)) \mathbf{1}(D_i(\mathbf{x}_2) \in \mathcal{J}_t) \Big]. \end{aligned}$$

Thus, feasible confidence intervals and confidence bands over  $\mathcal{B}$  take the form:

$$\hat{\mathbf{I}}_{\text{dis}}(\mathbf{x}; \alpha) = \left[ \hat{\tau}_{\text{dis}}(\mathbf{x}) - \varphi_\alpha \sqrt{\hat{\Xi}_{\mathbf{x}, \mathbf{x}}}, \hat{\tau}_{\text{dis}}(\mathbf{x}) + \varphi_\alpha \sqrt{\hat{\Xi}_{\mathbf{x}, \mathbf{x}}} \right], \quad \mathbf{x} \in \mathcal{B},$$

for any  $\alpha \in [0, 1]$ , and where  $\varphi_\alpha$  denotes the appropriate quantile depending on the desired inference procedure.

For pointwise inference, it is straightforward to show that  $\sup_{t \in \mathbb{R}} |\mathbb{P}[\hat{T}_{\text{dis}}(\mathbf{x}) \leq t] - \Phi(t)| \rightarrow 0$  for each  $\mathbf{x} \in \mathcal{B}$ , under standard regularity conditions, and provided that  $\sqrt{nh^2} |\mathfrak{B}_n(\mathbf{x})| \rightarrow 0$ . Thus,

in this case,  $q_\alpha = \Phi^{-1}(1 - \alpha/2)$  is an asymptotically valid choice. For uniform inference, we first establish a novel strong approximation for the entire stochastic process  $(\widehat{T}_{\text{dis}}(\mathbf{x}) : \mathbf{x} \in \mathcal{B})$ , assuming that  $\sqrt{nh^2} \sup_{\mathbf{x} \in \mathcal{B}} |\mathfrak{B}_n(\mathbf{x})| \rightarrow 0$ . With some technical work, we can then choose the appropriate  $q_\alpha$  for uniform inference because

$$\mathbb{P}[\tau(\mathbf{x}) \in \widehat{I}_{\text{dis}}(\mathbf{x}; \alpha), \text{ for all } \mathbf{x} \in \mathcal{B}] = \mathbb{P}\left[\sup_{\mathbf{x} \in \mathcal{B}} |\widehat{T}_{\text{dis}}(\mathbf{x})| \leq q_\alpha\right],$$

and the distribution of  $\sup_{\mathbf{x} \in \mathcal{B}} |\widehat{T}_{\text{dis}}(\mathbf{x})|$  can be deduced from the strong approximation of the stochastic process  $(\widehat{T}_{\text{dis}}(\mathbf{x}) : \mathbf{x} \in \mathcal{B})$ .

**Theorem 2** (Statistical Inference). *Suppose Assumptions 1 and 2 hold. Let  $\mathcal{U}_n$  be the  $\sigma$ -algebra generated by  $((Y_i, (D_i(\mathbf{x}) : \mathbf{x} \in \mathcal{B})) : 1 \leq i \leq n)$ .*

(i) *For all  $\mathbf{x} \in \mathcal{B}$ , if  $n^{\frac{v}{2+v}} h^2 \rightarrow \infty$  and  $nh^2 \mathfrak{B}_n^2(\mathbf{x}) \rightarrow 0$ , then*

$$\mathbb{P}[\tau(\mathbf{x}) \in \widehat{I}_{\text{dis}}(\mathbf{x}; \alpha)] \rightarrow 1 - \alpha,$$

*for  $q_\alpha = \Phi^{-1}(1 - \alpha/2)$ .*

(ii) *If  $n^{\frac{v}{2+v}} h^2 / \log n \rightarrow \infty$ ,  $\liminf_{n \rightarrow \infty} \frac{\log h}{\log n} > -\infty$ ,  $nh^2 \sup_{\mathbf{x} \in \mathcal{B}} \mathfrak{B}_n^2(\mathbf{x}) \rightarrow 0$ , and  $\text{perim}(\{\mathbf{y} \in \mathcal{A}_t : \mathcal{d}(\mathbf{y}, \mathbf{x})/h \in \text{Supp}(k)\}) \lesssim h$  for all  $\mathbf{x} \in \mathcal{B}$  and  $t \in \{0, 1\}$ , then*

$$\mathbb{P}[\tau(\mathbf{x}) \in \widehat{I}_{\text{dis}}(\mathbf{x}; \alpha), \text{ for all } \mathbf{x} \in \mathcal{B}] \rightarrow 1 - \alpha,$$

*for  $q_\alpha = \inf\{c > 0 : \mathbb{P}[\sup_{\mathbf{x} \in \mathcal{B}} |\widehat{Z}_n(\mathbf{x})| \geq c | \mathcal{U}_n] \leq \alpha\}$ , where  $(\widehat{Z}_n : \mathbf{x} \in \mathcal{B})$  is a Gaussian process conditional on  $\mathcal{U}_n$ , with  $\mathbb{E}[\widehat{Z}_n(\mathbf{x}_1) | \mathcal{U}_n] = 0$  and  $\mathbb{E}[\widehat{Z}_n(\mathbf{x}_1) \widehat{Z}_n(\mathbf{x}_2) | \mathcal{U}_n] = \widehat{\Xi}_{\mathbf{x}_1, \mathbf{x}_2} / \sqrt{\widehat{\Xi}_{\mathbf{x}_1, \mathbf{x}_1} \widehat{\Xi}_{\mathbf{x}_2, \mathbf{x}_2}}$ , for all  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{B}$ .*

This theorem establishes asymptotically valid inference procedures using the distance-based local polynomial treatment effect estimator  $\widehat{\tau}_{\text{dis}}(\mathbf{x})$ . For uniform inference, an additional mild restriction on the assignment boundary  $\mathcal{B}$  is imposed: the De-Giorgi perimeter condition can be verified when the boundary of  $\{\mathbf{y} \in \mathcal{A}_t : \mathcal{d}(\mathbf{y}, \mathbf{x})/h \in \text{Supp}(k)\}$  is a curve of length no greater than  $h$  up to a constant; since the set is contained in a  $h$ -ball centered at  $\mathbf{x}$ , the curve length condition holds as long as the curve  $\mathcal{B} \cap \{\mathbf{y} \in \mathbb{R}^2 : \mathcal{d}(\mathbf{y}, \mathbf{x}) \leq h\}$  is not “too wiggly”.



### 3.4 Discussion and Implementation

Although the distance-based estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  looks like an univariate local polynomial estimation procedure, based on the scalar score variable  $D_i(\mathbf{x})$ , Theorem 1 shows that its pointwise and uniform variance convergence rates are equal to that of a bivariate nonparametric estimator (which are unimprovable). Theorem 2 shows that inference results derived using univariate local polynomial regression methods can be deployed directly in distance-based settings, provided the side conditions are satisfied. This result follows from the fact that  $\hat{T}_{\text{dis}}(\mathbf{x})$  is constructed as a self-normalizing statistic, and therefore it is *adaptive* to the fact that the univariate covariate  $D_i(\mathbf{x})$  is actually based on the bivariate covariate  $\mathbf{X}_i$ . This finding documents another advantage of employing pre-asymptotic variance estimators and self-normalizing statistics for distributional approximation and inference (Calonico et al., 2018). It follows that standard estimation and inference methods from the univariate RD design literature could be deployed to boundary discontinuity designs employing univariate distance to the boundary, provided an appropriate bandwidth is chosen to ensure that the bias due to the shape of the assignment boundary  $\mathcal{B}$  (documented in Section 3.2) is small.

For implementation, consider first the case that  $\mathfrak{B}_n(\mathbf{x}) \lesssim h^{p+1}$ , that is, the assignment boundary  $\mathcal{B}$  is smooth (in the sense of Lemma 3). Establishing a precise MSE expansion for  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  is cumbersome due to the added complexity introduced by the distance transformation, but the convergence rates can be deduced from Theorem 1. The *incorrect* univariate MSE-optimal bandwidth is  $h_{1d} \asymp n^{-1/(3+2p)}$ , while the *correct* MSE-optimal bandwidth is  $h_{\text{dis}} \asymp n^{-1/(4+2p)}$ , implying that taking the distance variable as the univariate covariate, and thus ignoring its intrinsic bivariate dimension, implies that the resulting bandwidth choice will be smaller  $n^{-1/(3+2p)} < n^{-1/(4+2p)}$ , thereby undersmoothing the point estimator (relative to the *correct* optimal MSE bandwidth choice). As a consequence, the point estimator will not be MSE-optimal but rather exhibit more variance and less bias, and the associated inference procedures will be more conservative. An obvious solution is to rescale the *incorrect* univariate bandwidth,  $\frac{n^{1/(3+2p)}}{n^{1/(4+2p)}} h_{1d}$ , but this may not be necessary if the empirical implementation of  $h_{1d}$  employs a pre-asymptotic variance estimator, as in the software package `rdrobust` (<https://rdpackages.github.io/rdrobust/>). See Calonico et al. (2020), and references therein, for details on bandwidth selection for standard univariate RD designs.

When the assignment boundary  $\mathcal{B}$  exhibits kinks (or other irregularities), no matter how large

the sample size or the polynomial order, there will always be a region near each kink where the bias will be large, that is, of irreducible order  $h$ , as shown by Lemma 2. In this case, near each kink the *correct* MSE-optimal bandwidth is  $h_{\text{dis}} \asymp n^{-1/4}$ . Away from each kink, the MSE-optimal bandwidths are as discussed in the preceeding paragraph. This phenomenon is generated by the lack of smoothness of  $\mathcal{B}$ , and leads to different MSE-optimal bandwidths for each  $\mathbf{x} \in \mathcal{B}$ , thus making automatic implementation more difficult. A simple solution to this problem is to set  $h_{\text{dis}} \asymp n^{-1/4}$  for all  $\mathbf{x} \in \mathcal{B}$ , which is generically suboptimal but always no larger than the pointwise MSE-optimal bandwidth, thus delivering a more variable (than optimal) point estimator and more conservative (than possible) associated inference procedure.

Given a choice of bandwidth  $h$ , valid statistical inference can be developed by controlling the remaining misspecification bias. When the boundary  $\mathcal{B}$  is smooth, robust bias correction for standard univariate RD designs continues to be valid in the context of distance-based estimation (Calonico et al., 2014, 2018, 2022). On the other hand, when  $\mathcal{B}$  exhibits kinks, undersmoothing relative to the MSE-optimal bandwidth for  $p = 0$  is needed due to the fact that increasing  $p$  does not necessarily imply a reduction in misspecification bias, and thus bias correction techniques are ineffective (uniformly over  $\mathbf{x} \in \mathcal{B}$ ).

It remains to explain how uniform inference is implemented based on Theorem 5(ii). In practice, we discretize along the boundary with point of evaluations  $\mathbf{b}_1, \dots, \mathbf{b}_M \in \mathcal{B}$ , as in Figure 1a, and hence the feasible (conditional) Gaussian process  $(\hat{Z}_n(\mathbf{x}) : \mathbf{x} \in \mathcal{B})$  is reduced to the  $M$ -dimensional (conditional) Gaussian random vector  $\hat{\mathbf{Z}}_n = (\hat{Z}_n(\mathbf{b}_1), \dots, \hat{Z}_n(\mathbf{b}_M))$  with covariance matrix having a typical element  $\mathbb{E}[\hat{Z}_n(\mathbf{b}_1)\hat{Z}_n(\mathbf{b}_2)|\mathcal{U}]$ . Finding  $q_\alpha$  reduces to finding the  $\alpha$ -quantile of the distribution of  $\max_{1 \leq l \leq M} |\hat{Z}_n(\mathbf{b}_l)|$ , which can be easily simulated.

In the companion software package `rd2d`, we employ a simple rule-of-thumb bandwidth selector targeting  $h_{\text{dis}} \asymp n^{-1/4}$  to implement the univariate distance-based estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  as default for all  $\mathbf{x} \in \mathcal{B}$ , which is valid for estimation whether  $\mathcal{B}$  is smooth or not. For inference, following Calonico et al. (2018), the package employs the undersmoothed choice of order  $n^{-1/3}$ . If  $\mathcal{B}$  is known to be smooth, then the package implements a rule-of-thumb bandwidth selector targeting  $h_{\text{dis}} \asymp n^{-1/(4+2p)}$  for point estimation, and then employs robust bias corrected inference based on that bandwidth choice. Section 5 illustrates the performance of these methods with the SPP empirical application, and also considers the software package `rdrobust` for bandwidth selection

with  $p = 0$  as another simple rule-of-thumb bandwidth implementation. Further implementation details and simulation evidence are given in [Cattaneo et al. \(2025\)](#).

## 4 Analysis based on Bivariate Location

The previous section demonstrated the potential detrimental aspects of employing distance-based local polynomial regression to analyze boundary discontinuity designs. When the boundary has kinks, as it is common in many applications based on sharp or geographic assignment rules, univariate methods based on distance to the boundary can lead to large biases near kinks or other irregularities on the assignment boundary. This section shows that an easy way to avoid the distance-based bias is to employ bivariate local polynomial regression methods.

The location-based treatment effect curve estimator of  $\tau(\mathbf{x})$  is

$$\hat{\tau}(\mathbf{x}) = \mathbf{e}_1^\top \hat{\boldsymbol{\beta}}_1(\mathbf{x}) - \mathbf{e}_1^\top \hat{\boldsymbol{\beta}}_0(\mathbf{x}), \quad \mathbf{x} \in \mathcal{B},$$

where, for  $t \in \{0, 1\}$ ,

$$\hat{\boldsymbol{\beta}}_t(\mathbf{x}) = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^{p_p+1}} \mathbb{E}_n \left[ (Y_i - \mathbf{R}_p(\mathbf{X}_i - \mathbf{x})^\top \boldsymbol{\beta})^2 K_h(\mathbf{X}_i - \mathbf{x}) \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_t) \right],$$

with  $p_p = (2 + p)(1 + p)/2 - 1$ ,  $\mathbf{R}_p(\mathbf{u}) = (1, u_1, u_2, u_1^2, u_2^2, u_1 u_2, \dots, u_1^p, u_2^p)^\top$  denotes the  $p$ th order polynomial expansion of the bivariate vector  $\mathbf{u} = (u_1, u_2)^\top$ ,  $K_h(\mathbf{u}) = K(u_1/h, u_2/h)/h^2$  for a bivariate kernel function  $K(\cdot)$  and a bandwidth parameter  $h$ . We employ the same bandwidth for both dimensions of  $\mathbf{X}_i$  only for simplicity, and because it is common practice to first standardize each dimension of the bivariate score.

We impose the following assumption on the bivariate kernel function and assignment boundary.

**Assumption 3** (Kernel Function). *Let  $t \in \{0, 1\}$ .*

- (i)  $K : \mathbb{R}^2 \rightarrow [0, \infty)$  is compact supported and Lipschitz continuous, or  $K(\mathbf{u}) = \mathbf{1}(\mathbf{u} \in [-1, 1]^2)$ .
- (ii)  $\liminf_{h \downarrow 0} \inf_{\mathbf{x} \in \mathcal{B}} \int_{\mathcal{A}_t} K_h(\mathbf{u} - \mathbf{x}) d\mathbf{u} \gtrsim 1$ .

The first condition is standard in the literature, while the second condition is new but rather minimal. Similar to the case of distance-based estimation (Assumption 2(iii)), the goal of Assump-

tion 3(ii) is to ensure enough data is available in large samples for each point  $\mathbf{x} \in \mathcal{B}$  because  $\mathbb{E}[K_h(\mathbf{X}_i - \mathbf{x})\mathbf{1}(\mathbf{X}_i \in \mathcal{A}_t)] \gtrsim \int_{\mathcal{A}_t} K_h(\mathbf{u} - \mathbf{x})d\mathbf{u}$  under Assumption 1. Given these conditions, pointwise and uniform estimation results, as well as valid MSE expansions, follow from standard local polynomial calculations and empirical process theory. On the other hand, uniform distribution theory needs to be established with some extra technical care: the supplemental appendix establishes a key new strong approximation result that takes into account the specific features of the estimator and manifold  $\mathcal{B}$ .

#### 4.1 Treatment Effect Estimation

Using standard concentration techniques from empirical process theory, we obtain the pointwise and uniform convergence rate of  $\hat{\tau}(\mathbf{x})$ .

**Theorem 3** (Rate of Convergence). *Suppose Assumptions 1 and 3 hold. If  $nh^2/\log(n) \rightarrow \infty$ , then*

- (i)  $|\hat{\tau}(\mathbf{x}) - \tau(\mathbf{x})| \lesssim_{\mathbb{P}} \frac{1}{\sqrt{nh^2}} + \frac{1}{n^{\frac{1+v}{2+v}}h^2} + h^{p+1}$  for  $\mathbf{x} \in \mathcal{B}$ , and
- (ii)  $\sup_{\mathbf{x} \in \mathcal{B}} |\hat{\tau}(\mathbf{x}) - \tau(\mathbf{x})| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^2}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^2} + h^{p+1}.$

This theorem immediately establishes consistency of the treatment effect estimator based on the bivariate location score, provided that  $h \rightarrow 0$ . More importantly, the theorem shows that the bias is of order  $h^{p+1}$  regardless of whether there are kinks or other irregularities in  $\mathcal{B}$ , under the assumptions imposed. Comparing Theorems 1 and 3, it follows that both estimators can achieve the same (optimal) convergence rate when the boundary  $\mathcal{B}$  is smooth (c.f., Lemma 3), but otherwise  $\hat{\tau}(\mathbf{x})$  can achieve a faster (indeed, optimal) convergence rate, while  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  cannot because of the large order- $h$  bias when  $\mathcal{B}$  is not smooth documented in Lemma 2.

Furthermore, it is interesting to note that the radial (univariate) kernel  $K(\mathbf{u}) = k(\|\mathbf{u}\|)$  satisfies Assumption 3(b). Therefore, the large (order- $h$ ) bias of the distance-based estimator  $\hat{\tau}_{\text{dis}}(\mathbf{x})$  due to a non-smooth assignment boundary, as demonstrated in Lemma 2, are not necessarily due to the choice of distance-based kernel smoothing, but rather due to the choice of distance-based (univariate) local polynomial approximation:  $\mathbf{r}_p(\|\mathbf{u}\|)$  (in  $\hat{\tau}(\mathbf{x})$ ) vs.  $\mathbf{R}_p(\mathbf{u})$  (in  $\hat{\tau}_{\text{dis}}(\mathbf{x})$ ). In other words, Theorem 3 demonstrates that a bivariate local polynomial approximation does not suffer from the bias documented in Lemma 2 even when the radial kernel is used for localization.

Given its more standard structure, it is possible to establish pointwise and integrated (conditional) MSE expansions for the estimator  $\hat{\tau}(\mathbf{x})$ . Using standard multi-index notation, define the leading conditional bias  $B_{\mathbf{x}} = B_{1,\mathbf{x}} - B_{0,\mathbf{x}}$  with

$$B_{t,\mathbf{x}} = \mathbf{e}_1^\top \hat{\Gamma}_{t,\mathbf{x}}^{-1} \sum_{|\mathbf{k}|=p+1} \frac{\mu_t^{(\mathbf{k})}(\mathbf{x})}{\mathbf{k}!} \mathbb{E}_n \left[ \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}}{h} \right) \left( \frac{\mathbf{X}_i - \mathbf{x}}{h} \right)^{\mathbf{k}} K_h(\mathbf{X}_i - \mathbf{x}) \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_t) \right]$$

and  $\hat{\Gamma}_{t,\mathbf{x}} = \mathbb{E}_n \left[ \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}}{h} \right) \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}}{h} \right)^\top K_h(\mathbf{X}_i - \mathbf{x}) \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_t) \right]$ , for  $t \in \{0, 1\}$ . Similarly, the leading conditional variance is  $V_{\mathbf{x}} = V_{1,\mathbf{x}} + V_{0,\mathbf{x}}$  with  $V_{t,\mathbf{x}} = \mathbf{e}_1^\top \hat{\Gamma}_{t,\mathbf{x}}^{-1} \Sigma_{t,\mathbf{x},\mathbf{x}} \hat{\Gamma}_{t,\mathbf{x}}^{-1} \mathbf{e}_1$  and

$$\Sigma_{t,\mathbf{x},\mathbf{x}} = h^2 \mathbb{E}_n \left[ \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}}{h} \right) \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}}{h} \right)^\top K_h(\mathbf{X}_i - \mathbf{x})^2 \sigma_t^2(\mathbf{X}_i) \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_t) \right],$$

for  $t \in \{0, 1\}$ . The following theorem gives the MSE expansions.

**Theorem 4** (MSE Expansions). *Suppose Assumptions 1 and 3 hold, and let  $w(\mathbf{x})$  be a non-negative continuous function on  $\mathcal{B}$  such that  $\int_{\mathcal{B}} w(\mathbf{x}) d\mathbf{x} < \infty$ . If  $nh^2 / \log(n) \rightarrow \infty$ , then*

- (i)  $\mathbb{E}[(\hat{\tau}(\mathbf{x}) - \tau(\mathbf{x}))^2 | \mathbf{X}] = h^{2(p+1)} B_{\mathbf{x}}^2 + \frac{1}{nh^2} V_{\mathbf{x}} + o_{\mathbb{P}}(r_n)$ , and
- (ii)  $\int_{\mathcal{B}} \mathbb{E}[(\hat{\tau}(\mathbf{x}) - \tau(\mathbf{x}))^2 | \mathbf{X}] w(\mathbf{x}) d\mathbf{x} = h^{2(p+1)} \int_{\mathcal{B}} B_{\mathbf{x}}^2 dw(\mathbf{x}) + \frac{1}{nh^2} \int_{\mathcal{B}} V_{\mathbf{x}} w(\mathbf{x}) d\mathbf{x} + o_{\mathbb{P}}(r_n)$ ,

with  $r_n = h^{2p+2} + n^{-1}h^{-2} + n^{-\frac{2(1+v)}{2+v}}h^{-4}$ .

Ignoring the asymptotically constant and higher-order terms, approximate MSE-optimal and IMSE-optimal bandwidth choices are

$$h_{\text{MSE},\mathbf{x}} = \left( \frac{2V_{\mathbf{x}}}{(2p+2)B_{\mathbf{x}}^2} \frac{1}{n} \right)^{1/(2p+4)} \quad \text{and} \quad h_{\text{IMSE}} = \left( \frac{2 \int_{\mathcal{B}} V_{\mathbf{x}} w(\mathbf{x}) d\mathbf{x}}{(2p+2) \int_{\mathcal{B}} B_{\mathbf{x}}^2 w(\mathbf{x}) d\mathbf{x}} \frac{1}{n} \right)^{1/(2p+4)},$$

provided that  $B_{\mathbf{x}} \neq 0$  and  $\int_{\mathcal{B}} B_{\mathbf{x}}^2 w(\mathbf{x}) d\mathbf{x} \neq 0$ , respectively. These choices are infeasible because a preliminary bandwidth, as well as estimates of the conditional variances and higher-order derivatives of the conditional mean, are needed. We discuss implementation below and in our companion software article [Cattaneo et al. \(2025\)](#).

## 4.2 Uncertainty Quantification

Given a bandwidth choice, the feasible t-statistic at each boundary point  $\mathbf{x} \in \mathcal{B}$  is

$$\hat{T}(\mathbf{x}) = \frac{\hat{\tau}(\mathbf{x}) - \tau(\mathbf{x})}{\sqrt{\hat{\Omega}_{\mathbf{x},\mathbf{x}}}},$$

where, using standard least squares algebra, for all  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{B}$  and  $t \in \{0, 1\}$ ,

$$\hat{\Omega}_{\mathbf{x}_1, \mathbf{x}_2} = \hat{\Omega}_{0, \mathbf{x}_1, \mathbf{x}_2} + \hat{\Omega}_{1, \mathbf{x}_1, \mathbf{x}_2}, \quad \hat{\Omega}_{t, \mathbf{x}_1, \mathbf{x}_2} = \frac{1}{nh^2} \mathbf{e}_1^\top \hat{\Gamma}_{t, \mathbf{x}_1}^{-1} \hat{\Sigma}_{t, \mathbf{x}_1, \mathbf{x}_2} \hat{\Gamma}_{t, \mathbf{x}_2}^{-1} \mathbf{e}_1$$

with

$$\hat{\Sigma}_{t, \mathbf{x}_1, \mathbf{x}_2} = h^2 \mathbb{E}_n \left[ \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}_1}{h} \right) \mathbf{r}_p \left( \frac{\mathbf{X}_i - \mathbf{x}_2}{h} \right)^\top K_h(\mathbf{X}_i - \mathbf{x}_1) K_h(\mathbf{X}_i - \mathbf{x}_2) \varepsilon_i(\mathbf{x}_1) \varepsilon_i(\mathbf{x}_2) \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_t) \right]$$

and  $\varepsilon_i(\mathbf{x}) = Y_i - \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_0) \mathbf{R}_p(\mathbf{X}_i - \mathbf{x})^\top \hat{\beta}_0(\mathbf{x}) - \mathbf{1}(\mathbf{X}_i \in \mathcal{A}_1) \mathbf{R}_p(\mathbf{X}_i - \mathbf{x})^\top \hat{\beta}_1(\mathbf{x})$ .

Feasible confidence intervals and confidence bands over the treatment boundary  $\mathcal{B}$  take the form:

$$\hat{I}(\mathbf{x}; \alpha) = \left[ \hat{\tau}(\mathbf{x}) - \varphi_\alpha \sqrt{\hat{\Omega}_{\mathbf{x}, \mathbf{x}}}, \hat{\tau}(\mathbf{x}) + \varphi_\alpha \sqrt{\hat{\Omega}_{\mathbf{x}, \mathbf{x}}} \right], \quad \mathbf{x} \in \mathcal{B},$$

for any  $\alpha \in [0, 1]$ , and where  $\varphi_\alpha$  denotes the appropriate quantile depending on the targeted inference procedure. For pointwise inference, as in the previous section, it is a textbook exercise to show that  $\sup_{t \in \mathbb{R}} |\mathbb{P}[\hat{T}(\mathbf{x}) \leq t] - \Phi(t)| \rightarrow 0$  for each  $\mathbf{x} \in \mathcal{B}$ , under standard regularity conditions, and provided that  $nh^{2p+4} \rightarrow 0$ . For uniform inference, as in the previous section, we approximate the distribution of the entire stochastic process  $(\hat{T}(\mathbf{x}) : \mathbf{x} \in \mathcal{B})$  first, and then deduce an approximation for  $\sup_{\mathbf{x} \in \mathcal{B}} |\hat{T}(\mathbf{x})|$ . This approach enable us to construct asymptotically valid confidence bands because, as noted previously,

$$\mathbb{P}[\tau(\mathbf{x}) \in \hat{I}(\mathbf{x}; \alpha), \text{ for all } \mathbf{x} \in \mathcal{B}] = \mathbb{P}\left[\sup_{\mathbf{x} \in \mathcal{B}} |\hat{T}(\mathbf{x})| \leq \varphi_\alpha\right].$$

See the supplemental appendix for omitted technical details.

**Theorem 5** (Inferences). *Suppose Assumptions 1 and 3 hold. Let  $\mathbf{W}_n = ((Y_1, \mathbf{X}_1^\top)^\top, \dots, (Y_n, \mathbf{X}_n^\top)^\top)$ .*

(i) For all  $\mathbf{x} \in \mathcal{B}$ , if  $n^{\frac{v}{2+v}} h^2 \rightarrow \infty$  and  $nh^{2p+4} \rightarrow 0$ , then

$$\mathbb{P}[\tau(\mathbf{x}) \in \hat{\mathbf{I}}(\mathbf{x}; \alpha)] \rightarrow 1 - \alpha,$$

for  $q_\alpha = \Phi^{-1}(1 - \alpha/2)$ .

(ii) If  $n^{\frac{v}{2+v}} h^2 / \log n \rightarrow \infty$ ,  $\liminf_{n \rightarrow \infty} \frac{\log h}{\log n} > -\infty$ ,  $nh^{2p+4} \rightarrow 0$ , and  $\text{perim}(\{\mathbf{y} \in \mathcal{A}_t : (\mathbf{y} - \mathbf{x})/h \in \text{Supp}(K)\}) \lesssim h$  for all  $\mathbf{x} \in \mathcal{B}$  and  $t \in \{0, 1\}$ , then

$$\mathbb{P}[\tau(\mathbf{x}) \in \hat{\mathbf{I}}(\mathbf{x}; \alpha), \text{ for all } \mathbf{x} \in \mathcal{B}] \rightarrow 1 - \alpha,$$

for  $q_\alpha = \inf\{c > 0 : \mathbb{P}[\sup_{\mathbf{x} \in \mathcal{B}} |\hat{Z}_n(\mathbf{x})| \geq c | \mathbf{W}_n] \leq \alpha\}$ , where  $(\hat{Z}_n : \mathbf{x} \in \mathcal{B})$  is a Gaussian process conditional on  $\mathbf{W}_n$  with  $\mathbb{E}[\hat{Z}_n(\mathbf{x}_1) | \mathbf{W}_n] = 0$  and  $\mathbb{E}[\hat{Z}_n(\mathbf{x}_1) \hat{Z}_n(\mathbf{x}_2) | \mathbf{W}_n] = \hat{\Omega}_{\mathbf{x}_1, \mathbf{x}_2} / \sqrt{\hat{\Omega}_{\mathbf{x}_1, \mathbf{x}_1} \hat{\Omega}_{\mathbf{x}_2, \mathbf{x}_2}}$ , for all  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{B}$ .

This theorem gives valid pointwise and inference procedures for the boundary treatment effect  $\tau(\mathbf{x})$  based on the bivariate local polynomial estimator. Unlike the case of distance-based estimation, the bias condition is the same for  $\mathbf{x} \in \mathcal{B}$ , making implementation substantially easier. However, as in the case of Theorem 2, and additional technical restriction on  $\mathcal{B}$  is needed in order to avoid overly “wiggly” boundary designs that would lead to invalid statistical inference.

### 4.3 Implementation

The bivariate local polynomial estimator  $\hat{\tau}(\mathbf{x})$  and associated t-statistic  $\hat{\mathbf{T}}(\mathbf{x})$  is fully adaptive to kinks and other irregularities of the boundary  $\mathcal{B}$ . Therefore, it is straightforward to implement local and global bandwidth selectors based on Theorem 4 and the discussion given above. In particular, replacing the (asymptotic) bias and variance constants,  $B_{\mathbf{x}}$  and  $V_{\mathbf{x}}$ , with preliminary estimators, we obtain the feasible plug-in bandwidth selectors

$$\hat{h}_{\text{MSE}} = \left( \frac{2\hat{V}_{\mathbf{x}}}{(2p+2)\hat{B}_{\mathbf{x}}^2} \frac{1}{n} \right)^{1/(2p+4)} \quad \text{and} \quad \hat{h}_{\text{IMSE}} = \left( \frac{2 \int_{\mathcal{B}} \hat{V}_{\mathbf{x}} w(\mathbf{x}) d\mathbf{x}}{(2p+2) \int_{\mathcal{B}} \hat{B}_{\mathbf{x}}^2 w(\mathbf{x}) d\mathbf{x}} \frac{1}{n} \right)^{1/(2p+4)},$$

where  $\hat{B}_{\mathbf{x}}$  is an appropriate estimator of  $B_{\mathbf{x}}$ , and  $\hat{V}_{\mathbf{x}} = nh^2 \hat{\Omega}_{\mathbf{x}, \mathbf{x}}$ . Omitted details are given in the supplemental appendix. These bandwidth choices can now be used to implement (I)MSE-optimal

$\hat{\tau}(\mathbf{x})$  point treatment effect estimators pointwise and uniformly along the boundary  $\mathcal{B}$ . Furthermore, leveraging the results in Theorem 5, a simple application of robust bias-corrected inference proceeds by employing the same (I)MSE-optimal bandwidth (for  $p$ th order point estimation), but then constructing the t-statistic  $\hat{T}(\mathbf{x})$  with a choice of  $p + 1$  (instead of  $p$ ). This inference approach has several theoretical advantages (Calonico et al., 2014, 2018, 2022), and has been validated empirically (Hyytinen et al., 2018; De Magalhães et al., 2025). Our companion software, `rd2d`, implements the procedures described above. See Cattaneo et al. (2025) for more details.

## 5 Empirical Applications

We illustrate our proposed methodology with the SPP application. We consider both univariate distance-based and bivariate location-based estimation and inference. The data set has  $n = 363,096$  complete observations for the first cohort of the program (2014). Each observation corresponds to one student, and the bivariate score  $\mathbf{X}_i = (X_{1i}, X_{2i})^\top = (\text{SABER11}_i, \text{SISBEN}_i)^\top$  is composed by the student’s SABER11 test score (ranging from  $-310$  to  $172$ ) and SISBEN wealth index (ranging from  $-103.41$  to  $127.21$ ). As discussed in the introduction, and without loss of generality, the score is recentered at the corresponding cutoff for program eligibility, so that the treatment assignment boundary is as shown in Figure 1a. Furthermore, also without loss of generality, each dimension of  $\mathbf{X}_i$  is standardized in order to more naturally accommodate a common bandwidth  $h$ . Recall that  $Y_i$  is an indicator variable equal to one if the student enrolled in a higher education institution after receiving the SPP subsidy, or equal to zero otherwise.

Figure 3 presents the empirical results using our proposed methods, both pointwise across the 40 boundary points depicted in 1a and uniformly over  $\mathcal{B}$ . Figure 3a compares the two point estimation approaches, and demonstrates empirically the presence of some bias associated with the distance-based approach: the blue dots correspond to  $\hat{\tau}(\mathbf{x})$  implemented using the data-driven MSE-optimal bandwidth; the red crosses correspond to  $\hat{\tau}_{\text{dis}}$  implemented a rule-of-thumb bandwidth ignoring the presence of the kink at  $\mathbf{x} = (0, 0) \in \mathcal{B}$ ; and the gray circles correspond to  $\hat{\tau}(\mathbf{x})$  implemented using a rule-of-thumb bandwidth of order  $n^{-1/4}$ . Figure 3b demonstrates empirically the pointwise and uniform inference results from Section 4 based on  $\hat{\tau}(\mathbf{x})$  implemented using the data-driven MSE-optimal bandwidth. The treatment effects are highly statistically significant along the assignment



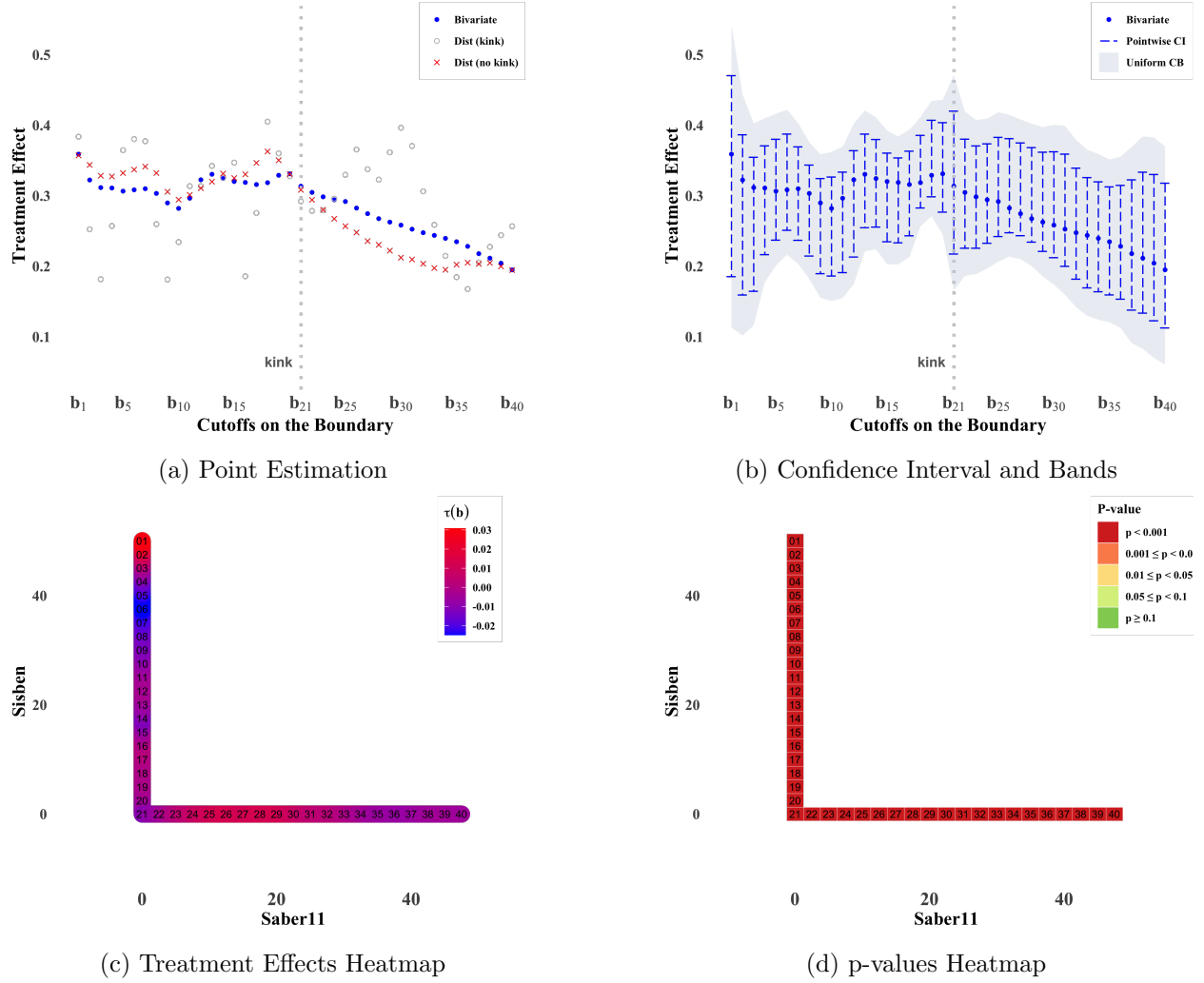
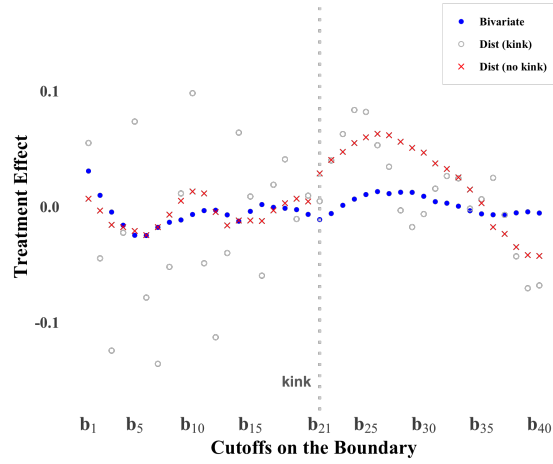


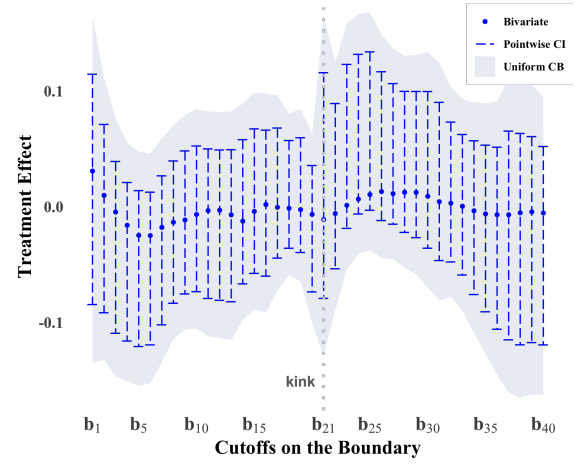
Figure 3: Boundary Treatment Effect Estimation and Inference (SPP application). Outcome: College enrollment.

boundary, indicating mostly homogeneous treatment effects along poverty ( $\mathbf{b}_1 - -\mathbf{b}_{21}$ ) but some degree of heterogeneity in treatment effects along academic performance ( $\mathbf{b}_{21} - -\mathbf{b}_{40}$ ). Specifically, college attendance remains constant as students become wealthier ( $\hat{\tau}(\mathbf{x}) \approx 0.3$  for  $\mathbf{x} \in \{\mathbf{b}_1, \mathbf{b}_{21}\}$ ), but the treatment effects appear to decrease as students exhibit higher academic performance (from  $\hat{\tau}(\mathbf{b}_{21}) \approx 0.3$  to  $\hat{\tau}(\mathbf{b}_{40}) \approx 0.2$ ). Finally, Figures 3c and 3d offer “heat maps” for the point estimates and associated RBC p-values along the assignment boundary  $\mathcal{B}$ .

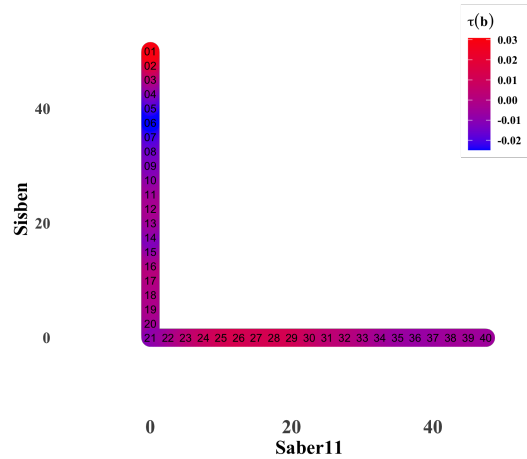
To demonstrate the credibility of the boundary discontinuity design, we repeat the empirical analysis using a pre-intervention covariate: mother’s education of the student. This corresponds to a standard “placebo” analysis, where the treatment effect is expected to be statistically indistin-



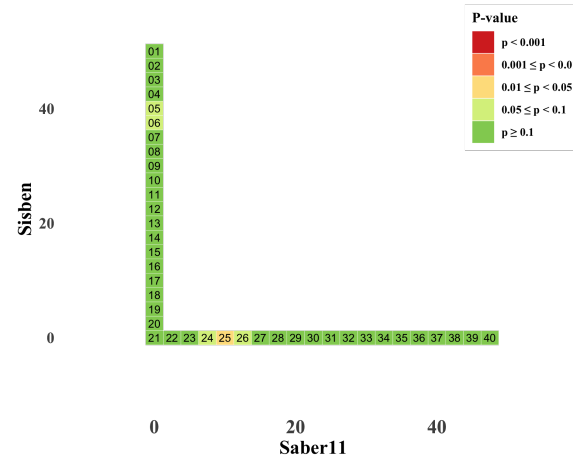
(a) Point Estimation



(b) Confidence Interval and Bands



(c) Heatmap of Treatment Effect



(d) Heatmap of P-value

Figure 4: Boundary Treatment Effect Estimation and Inference (SPP application). Outcome: Mother's education (placebo).

guishable from zero. Figure 4 shows that this is indeed the case.

## 6 Conclusion

We studied the pointwise and uniform statistical properties of the two most popular local polynomial methods for treatment effect estimation and inference in boundary discontinuity designs. Our theoretical and numerical results demonstrated that methods based on the univariate distance to the assignment boundary can exhibit a large bias in the presence of kinks or other irregularities of  $\mathcal{B}$ . In contrast, methods based on the bivariate score do not suffer from those problems, and thus can perform better in applications. We thus recommend employing bivariate local polynomial analysis whenever possible in boundary discontinuity designs. Our companion software package `rd2d` (<https://rdpackages.github.io/rd2d>) offers general-purpose implementations of all the estimation and inference methods developed.

Building on the framework and results presented, there are (at least) two important directions for future research. First, it would be of interest to incorporate predetermined covariates for either efficiency improvements in estimation and inference, or for treatment effect heterogeneity analysis. Second, it would be of interest to consider functionals of the average treatment effect curve along the boundary,  $\tau(\mathbf{x})$ , which can help summarize empirical findings. For example, interesting scalar parameters include the *weighted average treatment effect along the boundary*  $\tau_p = \int_{\mathcal{R}} \tau(\mathbf{x})w(\mathbf{x})d\mathbf{x}$ , and the *largest average treatment effect along the boundary*  $\tau_\infty = \sup_{\mathbf{x} \in \mathcal{R}} \tau(\mathbf{x})$ , for some region  $\mathcal{R} \subseteq \mathcal{B}$ . In research underway we are investigating how our (pointwise and) uniform estimation and inference results can be extended to study the properties of covariate-adjusted estimators of  $\tau(\mathbf{x})$ , and of the plug-in estimators  $\hat{\tau}_p = \int_{\mathcal{R}} \hat{\tau}(\mathbf{x})w(\mathbf{x})d\mathbf{x}$  and  $\hat{\tau}_\infty = \sup_{\mathbf{x} \in \mathcal{R}} \hat{\tau}(\mathbf{x})$ . (For the latter estimator, the results in the supplemental appendix apply directly.)

## A Minimax Convergence Rate

The following theorem presents a minimax uniform convergence rate result for estimation of bivariate compact supported nonparametric functions employing a class of estimators based on distance to the evaluation point.

**Theorem 6** (Distance-based Minimax Convergence Rate). *For a constants  $q \geq 1$  and  $L > 0$ , let  $\mathcal{P}_{\text{NP}} = \mathcal{P}_{\text{NP}}(L, q)$  be the class of (joint) probability laws  $\mathbb{P}$  of  $(Y_1, \mathbf{X}_1), \dots, (Y_n, \mathbf{X}_n)$  satisfying the following:*

- (i)  *$((Y_i, \mathbf{X}_i) : 1 \leq i \leq n)$  are i.i.d. taking values in  $\mathbb{R} \times \mathbb{R}^2$ .*
- (ii)  *$\mathbf{X}_i$  admits a Lebesgue density  $f$  that is continuous on its compact support  $\mathcal{X} \subseteq [-L, L]^2$ , with  $L^{-1} \leq \inf_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \leq \sup_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \leq L$ , and  $\mathcal{B} = \text{bd}(\mathcal{X})$  is a rectifiable curve.*
- (iii)  *$\mu(\mathbf{x}) = \mathbb{E}[Y_i | \mathbf{X}_i = \mathbf{x}]$  is  $q$ -times continuously differentiable on  $\mathcal{X}$  with*

$$\max_{0 \leq |\nu| \leq \lfloor q \rfloor} \sup_{\mathbf{x} \in \mathcal{X}} |\partial_{\nu} \mu(\mathbf{x})| + \max_{|\nu| = q} \sup_{\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}} \frac{|\partial_{\nu} \mu(\mathbf{x}_1) - \partial_{\nu} \mu(\mathbf{x}_2)|}{\|\mathbf{x}_1 - \mathbf{x}_2\|^{q - \lfloor q \rfloor}} \leq L.$$

- (iv)  *$\sigma^2(\mathbf{x}) = \mathbb{V}[Y_i | \mathbf{X}_i = \mathbf{x}]$  is continuous on  $\mathcal{X}$  with  $L^{-1} \leq \inf_{\mathbf{x} \in \mathcal{X}} \sigma^2(\mathbf{x}) \leq \sup_{\mathbf{x} \in \mathcal{X}} \sigma^2(\mathbf{x}) \leq L$ .*

*In addition, let  $\mathcal{T}$  be the class of all distance-based estimators  $T_n(\mathbf{U}_n(\mathbf{x}))$  with  $\mathbf{U}_n(\mathbf{x}) = ((Y_i, \|\mathbf{X}_i - \mathbf{x}\|))^\top : 1 \leq i \leq n)$  for each  $\mathbf{x} \in \mathcal{X}$ . Then,*

$$\liminf_{n \rightarrow \infty} n^{1/4} \inf_{T_n \in \mathcal{T}} \sup_{\mathbb{P} \in \mathcal{P}_{\text{NP}}} \mathbb{E}_{\mathbb{P}} \left[ \sup_{\mathbf{x} \in \mathcal{B}} |T_n(\mathbf{U}_n(\mathbf{x})) - \mu(\mathbf{x})| \right] \gtrsim 1,$$

where  $\mathbb{E}_{\mathbb{P}}[\cdot]$  denotes an expectation taken under the data generating process  $\mathbb{P}$ .

In contrast to Theorem 6, under the assumptions imposed, a classical result of [Stone \(1982\)](#) shows that the uniform minimax convergence rate is

$$\liminf_{n \rightarrow \infty} \left( \frac{n}{\log n} \right)^{\frac{q}{2q+2}} \inf_{S_n \in \mathcal{S}} \sup_{\mathbb{P} \in \mathcal{P}_{\text{NP}}} \mathbb{E}_{\mathbb{P}} \left[ \sup_{\mathbf{x} \in \mathcal{B}} |S_n(\mathbf{x}; \mathbf{W}_n) - \mu(\mathbf{x})| \right] \gtrsim 1,$$

where  $\mathcal{S}$  is the (unrestricted class) of all estimators based on  $(\mathbf{W}_n = (Y_i, \mathbf{X}_i^\top)^\top : 1 \leq i \leq n)$ . Theorem 6 shows that if we restrict the class of estimators to only using the distance  $\|\mathbf{X}_i - \mathbf{x}\|$  to the evaluation point  $\mathbf{x}$  (as opposed to using  $\mathbf{X}_i$  directly) for estimation of  $\mu(\mathbf{x})$ , then for any such estimator there is a data generating process such that the largest estimation error uniformly along the boundary is at least of order  $n^{-1/4}$ , when the boundary can have possibly countably many kinks. Notably, Theorem 6 shows that increasing smoothness of the underlying bivariate regression function  $\mu(\mathbf{x})$  does not improve the uniform estimation accuracy of estimators in  $\mathcal{T}$  due to the possible lack of smoothness introduced by a non-smooth boundary  $\mathcal{B}$  of the support  $\mathcal{X}$ .

By Theorem 1 and Lemma 2, the univariate distance-based  $p$ -th order local polynomial nonparametric regression estimator

$$\hat{\mu}_{\text{dis}}(\mathbf{x}) = \mathbf{e}_1^\top \hat{\boldsymbol{\gamma}}(\mathbf{x}), \quad \hat{\boldsymbol{\gamma}}(\mathbf{x}) = \arg \min_{\boldsymbol{\gamma} \in \mathbb{R}^{p+1}} \mathbb{E}_n \left[ (Y_i - \mathbf{r}_p(D_i(\mathbf{x}))^\top \boldsymbol{\gamma})^2 k_h(D_i(\mathbf{x})) \right], \quad \mathbf{x} \in \mathcal{B},$$

satisfies

$$\limsup_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P} \left[ \left( \frac{n}{\log n} \right)^{1/4} \sup_{\mathbf{x} \in \mathcal{B}} |\hat{\mu}_{\text{dis}}(\mathbf{x}) - \mu(\mathbf{x})| \geq M \right] = 0.$$

This shows that the distance-based local polynomial regression estimator is minimax optimal in the sense of Theorem 6, up to the  $\log^{1/4} n$  factor, which we conjecture is unimprovable (i.e., we conjecture the lower bound in Theorem 6 is loose by the factor  $\log^{1/4} n$ ).

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