Please determine **analytically** (i.e. handwritten proof) and by graphing in Matlab the convexity of the following sets:

✓ 1) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, x^2 + y^2 \leq 4\}$
✗ 2) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 = 4\}$
✓ 3) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, x \geq 0, y \geq 0, 2x + 4y \leq 12, x + y \leq 4\}$
✓ 4) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, \sqrt{x} \geq y, y \leq 2, 0 \leq x \leq 4\}$
✗ 5) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, x^2 \geq y, y \leq 1\}$
✗ 6) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, x^2 \geq 1 - y^2\}$
✓ 7) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, \log_{10} x \geq y, x \geq 1, y \leq 100, y \geq 0\}$
✓ 8) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, \sin(x) \leq y, y \geq \sqrt{2}/2, -\pi/4 \leq x \leq \pi/4\}$
✓ 9) $x, y \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, |x| \leq y, -|x| + 5 \geq y\}$
✓ 10) $x, y, z \in \mathcal{S}, \mathcal{S} = \{x, y \in \mathbb{R}, x + y = 11 - z\}$

(1) $x, y \in S,\ S = \{x, y \in \mathbb{R},\ x^2+y^2 \leq 4\}$

$$= \left\{ \underline{v} = \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, \|\underline{v}\|_2^2 \leq 4 \right\}$$

$$= \left\{ \underline{v} \in \mathbb{R}^2 \,\middle|\, \|\underline{v}\|_2 \leq 2 \right\}$$

let $\underline{V_1}, \underline{V_2} \in S$, then $\|\underline{V_1}\|_2 \leq 2$, $\|\underline{V_2}\|_2 \leq 2$. $\theta \in [0,1]$

$\Rightarrow \|\theta \underline{V_1} + (1-\theta)\underline{V_2}\|_2 \leq \|\theta \underline{V_1}\|_2 + \|(1-\theta)\underline{V_2}\|_2$ (Triangle inequality)

$$= \theta \|\underline{V_1}\|_2 + (1-\theta)\|\underline{V_2}\|_2 \leq 2$$

$\Rightarrow \theta \underline{V_1} + (1-\theta)\underline{V_2} \in S$ for $\theta \in [0,1]$

$\Rightarrow S$ is a convex set

(2) $x, y, z \in S,\ S = \{x, y, z \in \mathbb{R},\ x^2+y^2+z^2 = 4\}$

$$= \left\{ \underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, \|\underline{v}\|_2^2 = 4 \right\}$$

$$= \left\{ \underline{v} \in \mathbb{R}^3 \,\middle|\, \|\underline{v}\|_2 = 2 \right\}$$

Counterexample: $\underline{V_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\underline{V_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $\|\underline{V_1}\|_2 = 2$, $\|\underline{V_2}\|_2 = 2$

$\underline{V_3} = \frac{1}{2}\underline{V_1} + \frac{1}{2}\underline{V_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\|\underline{V_3}\|_2 = \sqrt{2} \notin S$

$\Rightarrow S$ is not a convex set

(3) $x, y \in S$, $S = \{x, y \in \mathbb{R},\ x \geq 0,\ y \geq 0,\ 2x+4y \leq 12,\ x+y \leq 4\}$

$$= \left\{ \underline{v} = \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ \underline{v} \succcurlyeq \underline{0},\ \underline{a}^T \underline{v} \leq 12,\ \underline{b}^T \underline{v} \leq 4,\ \underline{a} = \begin{bmatrix} 2 \\ 4 \end{bmatrix},\ \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

let $\underline{v}_1, \underline{v}_2 \in S$, then $\underline{v}_1 \succcurlyeq \underline{0},\ \underline{a}^T \underline{v}_1 \leq 12,\ \underline{b}^T \underline{v}_2 \leq 4$

$\underline{v}_2 \succcurlyeq \underline{0},\ \underline{a}^T \underline{v}_2 \leq 12,\ \underline{b}^T \underline{v}_2 \leq 4$, $\theta \in [0,1]$

$\Rightarrow \theta \underline{v}_1 + (1-\theta) \underline{v}_2 \succcurlyeq 0$

$\underline{a}^T(\theta \underline{v}_1 + (1-\theta)\underline{v}_2) = \theta \underline{a}^T \underline{v}_1 + (1-\theta) \underline{a}^T \underline{v}_2 \leq 12$

$\underline{b}^T(\theta \underline{v}_1 + (1-\theta)\underline{v}_2) = \theta \underline{b}^T \underline{v}_1 + (1-\theta) \underline{b}^T \underline{v}_2 \leq 4$

$\Rightarrow \theta \underline{v}_1 + (1-\theta) \underline{v}_2 \in S$ for $\theta \in [0,1] \Rightarrow S$ is a convext set

(4) $x, y \in S$, $S = \{x, y \in \mathbb{R},\ \sqrt{x} \geq y,\ y \leq 2,\ 0 \leq x \leq 4\}$

$$= \left\{ \underline{v} = \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ \sqrt{x} \geq y,\ y \leq 2,\ 0 \leq x \leq 4 \right\}$$

let $\underline{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in S$, $\underline{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$, then $\sqrt{x_1} \geq y_1,\ y_1 \leq 2,\ 0 \leq x_1 \leq 4$

$\sqrt{x_2} \geq y_2,\ y_2 \leq 2,\ 0 \leq x_2 \leq 4$

$\theta \in [0,1]$

$\Rightarrow \theta \underline{v}_1 + (1-\theta) v_2 = \theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix}$

$x_1 \geq y_1^2,\ x_2 \geq y_2^2$, check whether $\theta x_1 + (1-\theta) x_2 \geq [\theta y_1 + (1-\theta) y_2]^2$

pf: $\theta x_1 + (1-\theta) x_2 \geq \theta y_1^2 + (1-\theta) y_2^2 \cdots$ ①

$\theta y_1^2 + (1-\theta) y_2^2 - [\theta y_1 + (1-\theta) y_2]^2$

$= \theta y_1^2 - \theta^2 y_1^2 + (1-\theta) y_2^2 - (1-\theta)^2 y_2^2 - 2\theta(1-\theta) y_1 y_2$

$= \theta(1-\theta) y_1^2 + (1-\theta)\theta y_2^2 - 2\theta(1-\theta) y_1 y_2$

$= \theta(1-\theta)[y_1^2 + y_2^2 - 2y_1y_2]$

$= \theta(1-\theta)[y_1 - y_2]^2 \geq 0 \Rightarrow \theta y_1^2 + (1-\theta)y_2^2 \geq [\theta y_1 + (1-\theta)y_2]^2 \cdots ②$

According to ①.②, $\theta x_1 + (1-\theta)x_2 \geq [\theta y_1 + (1-\theta)y_2]^2$ for $\theta \in [0,1]$ $\cdots ③$

$y_1 \leq 2$, $y_2 \leq 2$, check whether $\theta y_1 + (1-\theta)y_2 \leq 2$

pf: $\because y_1 \leq 2$, $y_2 \leq 2$, $\therefore \theta y_1 \leq 2\theta$, $(1-\theta)y_2 \leq (1-\theta)2$

$\Rightarrow \theta y_1 + (1-\theta)y_2 \leq 2\theta + (1-\theta) \times 2 = 2$ for $\theta \in [0,1] \cdots ④$

$0 \leq x_1 \leq 4$, $0 \leq x_2 \leq 4$, check whether $\theta x_1 + (1-\theta)x_2 \leq 4$

pf: $\because 0 \leq x_1 \leq 4$, $0 \leq x_2 \leq 4$ $\therefore 0 \leq \theta x_1 \leq 4\theta$, $0 \leq (1-\theta)x_2 \leq (1-\theta)4$

$\Rightarrow 0 \leq \theta x_1 + (1-\theta)x_2 \leq 4\theta + (1-\theta)4 = 4$ for $\theta \in [0,1] \cdots ⑤$

According to ③④⑤, $\theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$ for $\theta \in [0,1]$

$\Rightarrow S$ is a convex set

(5) $x, y \in S$, $S = \{x, y \in \mathbb{R}, x^2 \geq y, y \leq 1\}$

Counterexample: $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix} \in S$, $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{5} \end{bmatrix} \in S$

$\Rightarrow \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} \notin S \Rightarrow S$ is not a convex set

(6) $x, y \in S$, $S = \{x, y \in \mathbb{R},\ x^2 \geq 1 - y^2\}$
$= \{x, y \in \mathbb{R},\ x^2 + y^2 \geq 1\}$

Counterexample: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S$, $\begin{bmatrix} -1 \\ -1 \end{bmatrix} \in S$

$\Rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$

$\Rightarrow S$ is not a convex set

(7) $x, y \in S$, $S = \{x, y \in \mathbb{R},\ \log_{10} x \geq y,\ x \geq 1,\ 0 \leq y \leq 100\}$.

let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in S$, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S \Rightarrow$ $\log_{10} x_1 \geq y_1,\ x_1 \geq 1,\ 0 \leq y_1 \leq 100$
$\log_{10} x_2 \geq y_2,\ x_2 \geq 1,\ 0 \leq y_2 \leq 100$

$\theta\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta x_1 + (1-\theta)x_2 \\ \theta y_1 + (1-\theta)y_2 \end{bmatrix}$, $\theta \in [0,1]$

$\because x_1 \geq 1,\ x_2 \geq 1$ $\therefore \theta x_1 \geq \theta,\ (1-\theta)x_2 \geq 1-\theta$

$\Rightarrow \theta x_1 + (1-\theta)x_2 \geq \theta + (1-\theta) = 1$ ...... ①

$\because 0 \leq y_1 \leq 100,\ 0 \leq y_2 \leq 100$. $\therefore 0 \leq \theta y_1 \leq 100\theta,\ 0 \leq (1-\theta)y_2 \leq 100(1-\theta)$

$\Rightarrow 0 \leq \theta y_1 + (1-\theta)y_2 \leq 100\theta + 100(1-\theta) = 100$ ...... ②

$\because \log_{10} x_1 \geq y_1,\ \log_{10} x_2 \geq y_2$ $\therefore x_1 \geq 10^{y_1},\ x_2 \geq 10^{y_2}$

$\Rightarrow \theta x_1 \geq \theta 10^{y_1},\ (1-\theta)x_2 \geq (1-\theta)10^{y_2}$

$\Rightarrow \theta x_1 + (1-\theta)x_2 \geq \theta 10^{y_1} + (1-\theta)10^{y_2}$

$\because 10^y$ is a convex function and $\theta \in [0,1]$

$\therefore$ By Jensen's inequality, $\theta 10^{y_1} + (1-\theta)10^{y_2} \geq 10^{\theta y_1 + (1-\theta)y_2}$

$\Rightarrow \theta x_1 + (1-\theta)x_2 \geq 10^{\theta y_1 + (1-\theta)y_2}$

$\Rightarrow \log(\theta x_1 + (1-\theta)x_2) \geq \theta y_1 + (1-\theta)y_2$ ...... ③

According to ① ② ③, $\theta\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$ for $\theta \in [0,1]$

$\Rightarrow S$ is a convex set

(8) $x, y \in S$, $S = \{x, y \in \mathbb{R}, \sin(x) \leq y, y \geq \frac{\sqrt{2}}{2}, -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$

let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in S$, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$, then $\begin{matrix} \sin(x_1) \leq y_1, & y_1 \geq \frac{\sqrt{2}}{2}, & -\frac{\pi}{4} \leq x_1 \leq \frac{\pi}{4} \\ \sin(x_2) \leq y_2, & y_2 \geq \frac{\sqrt{2}}{2}, & -\frac{\pi}{4} \leq x_2 \leq \frac{\pi}{4} \end{matrix}$

$\theta\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta x_1 + (1-\theta)x_2 \\ \theta y_1 + (1-\theta)y_2 \end{bmatrix}$, $\theta \in [0, 1]$

$\because -\frac{\pi}{4} \leq x_1 \leq \frac{\pi}{4}$, $-\frac{\pi}{4} \leq x_2 \leq \frac{\pi}{4}$ $\therefore -\frac{\pi}{4}\theta \leq \theta x_1 \leq \frac{\pi}{4}\theta$

$-\frac{\pi}{4}(1-\theta) \leq (1-\theta)x_2 \leq \frac{\pi}{4}(1-\theta)$

$\Rightarrow -\frac{\pi}{4}\theta - \frac{\pi}{4}(1-\theta) \leq \theta x_1 + (1-\theta)x_2 \leq \frac{\pi}{4}\theta + \frac{\pi}{4}(1-\theta)$

$\Rightarrow -\frac{\pi}{4} \leq \theta x_1 + (1-\theta)x_2 \leq \frac{\pi}{4}$ ... ①

$\because y_1 \geq \frac{\sqrt{2}}{2}$, $y_2 \geq \frac{\sqrt{2}}{2}$ $\therefore \theta y_1 \geq \frac{\sqrt{2}}{2}\theta$, $(1-\theta)y_2 \geq \frac{\sqrt{2}}{2}(1-\theta)$

$\Rightarrow \theta y_1 + (1-\theta)y_2 \geq \frac{\sqrt{2}}{2}\theta + \frac{\sqrt{2}}{2}(1-\theta) = \frac{\sqrt{2}}{2}$ ... ②

$\because -\frac{\pi}{4} \leq \theta x_1 + (1-\theta)x_2 \leq \frac{\pi}{4}$ $\therefore -\frac{\sqrt{2}}{2} \leq \sin(\theta x_1 + (1-\theta)x_2) \leq \frac{\sqrt{2}}{2}$

$\because \theta y_1 + (1-\theta)y_2 \geq \frac{\sqrt{2}}{2}$ $\therefore \sin(\theta x_1 + (1-\theta)x_2) \leq \theta y_1 + (1-\theta)y_2$ .... ③

According to ① ② ③, $\theta\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$ for $\theta \in [0,1]$

$\Rightarrow S$ is a convex set.

(9) $x, y \in S$, $S = \{x, y \in \mathbb{R}, |x| \le y, -|x| + 5 \ge y\}$

let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in S$, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$. then $\begin{matrix} |x_1| \le y_1 & -|x_1| + 5 \ge y_1 \\ |x_2| \le y_2 & -|x_2| + 5 \ge y_2 \end{matrix}$

$$\theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix}, \quad \theta \in [0,1]$$

$\because |x_1| \le y_1$, $|x_2| \le y_2$ $\therefore |\theta x_1| \le \theta y_1$, $|(1-\theta)x_2| \le (1-\theta) y_2$

$\Rightarrow |\theta x_1| + |(1-\theta)x_2| \le \theta y_1 + (1-\theta) y_2$ and $|\theta x_1 + (1-\theta)x_2| \le |\theta x_1| + |(1-\theta)x_2|$

$\Rightarrow |\theta x_1 + (1-\theta)x_2| \le \theta y_1 + (1-\theta) y_2 \quad \cdots ①$

$\because -|x_1| + 5 \ge y_1$, $-|x_2| + 5 \ge y_2$ $\therefore -|\theta x_1| + 5\theta \ge \theta y_1$

$-|(1-\theta)x_1| + 5(1-\theta) \ge (1-\theta) y_2$

$\Rightarrow -|\theta x_1| - |(1-\theta)x_2| + 5 \ge \theta y_1 + (1-\theta) y_2$

$\Rightarrow -\left[|\theta x_1| + |(1-\theta)x_2|\right] + 5 \ge \theta y_1 + (1-\theta) y_2$ and $|\theta x_1| + |(1-\theta)x_2| \ge |\theta x_1 + (1-\theta)x_2|$

$\Rightarrow -|\theta x_1 + (1-\theta)x_2| + 5 \ge \theta y_1 + (1-\theta) y_2 \quad \cdots ②$

According to ① ②, $\theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$ for $\theta \in [0,1]$

$\Rightarrow$ S is a convex set

10. $x, y, z \in S$. $S = \{x, y \in \mathbb{R}, x+y = 11-z\}$

$$= \left\{ \underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3, x+y+z = 11 \right\}$$

$$= \left\{ \underline{v} \in \mathbb{R}^3, \underline{a}^T \underline{v} = 11, \underline{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Let $\underline{v}_1, \underline{v}_2 \in S$, then $\underline{a}^T \underline{v}_1 = 11$, $\underline{a}^T \underline{v}_2 = 11$, $\theta \in [0,1]$

$$\Rightarrow \underline{a}^T (\theta \underline{v}_1 + (1-\theta) \underline{v}_2) = \theta \underline{a}^T \underline{v}_1 + (1-\theta) \underline{a}^T \underline{v}_2 = 11\theta + 11(1-\theta) = 11$$

$\Rightarrow \theta \underline{v}_1 + (1-\theta) \underline{v}_2 \in S \Rightarrow S$ is a convex set