let $0 \le \theta \le 1$

312510197 羅兆傑

① if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta)x_2 \\ \theta y_1 + (1-\theta)y_2 \end{bmatrix},$$

$$[\theta x_1 + (1-\theta)x_2]^2 + [\theta y_1 + (1-\theta)y_2]^2$$

$$= \theta^2 x_1^2 + 2\theta(1-\theta)x_1x_2 + (1-\theta)^2 x_2^2$$

$$+ \theta^2 y_1^2 + 2\theta(1-\theta)y_1y_2 + (1-\theta)^2 y_2^2$$

$$= \theta^2(x_1^2+y_1^2) + (1-\theta)^2(x_2^2+y_2^2) + 2\theta(1-\theta)(x_1x_2+y_1y_2)$$

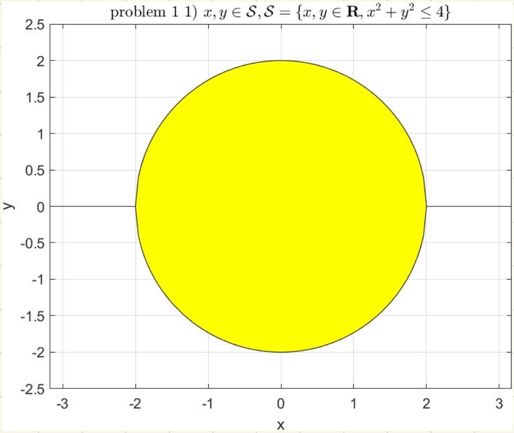
$$(\because x_1x_2 + y_1y_2 = \underline{x}_1^T \underline{x}_2 \le 4)$$

$$\le \theta^2 \cdot 4 + (1-\theta)^2 \cdot 4 + 2\theta(1-\theta) \cdot 4$$

$$= 4[\theta^2 + 2\theta(1-\theta) + (1-\theta)^2] = 4[\theta + (1-\theta)]^2 = 4$$

$$\Rightarrow \|\theta \underline{x}_1 + (1-\theta)\underline{x}_2\|_2^2 \le 4 \Rightarrow \theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in S$$

$\Rightarrow S$ is a convex set



from the figure it's indeed a convex set.

② if we let $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\underline{x}_1, \underline{x}_2 \in S$

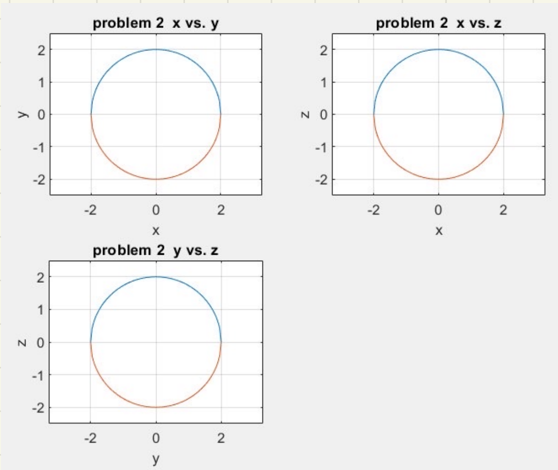
$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \\ \theta z_1 + (1-\theta) z_2 \end{bmatrix}, \text{ then}$$

counterexample : let $\underline{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $\theta = 0.5$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ then}$$

$$\| \theta \underline{x}_1 + (1-\theta)\underline{x}_2 \|_2^2 = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 4$$

$\Rightarrow$ S is not a convex set



I split the view into 3 perspectives, we can see that randomly pick two points and their convex combination that $0<\theta<1$ is not in the original set

③ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$

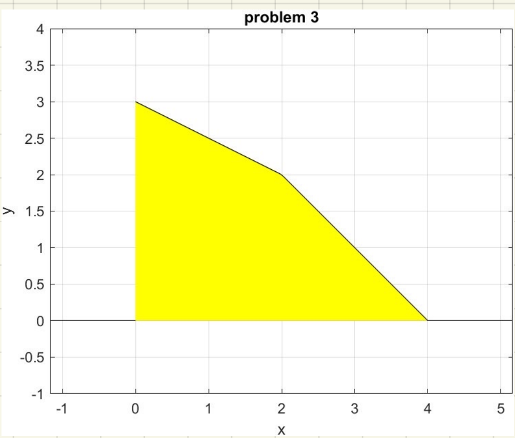
(1)
$$\theta, (1-\theta), x_1, x_2 \geq 0 \Rightarrow \theta x_1 + (1-\theta) x_2 \geq 0$$

(2)
$$\theta, (1-\theta), y_1, y_2 \geq 0 \Rightarrow \theta y_1 + (1-\theta) y_2 \geq 0$$

(3)
$$2[\theta x_1 + (1-\theta) x_2] + 4[\theta y_1 + (1-\theta) y_2]$$
$$= \theta(2x_1 + 4y_1) + (1-\theta)(2x_2 + 4y_2)$$
$$\leq \theta \cdot 12 + (1-\theta) \cdot 12 = 12$$

(4)
$$[\theta x_1 + (1-\theta) x_2] + [\theta y_1 + (1-\theta) y_2]$$
$$= \theta(x_1 + y_1) + (1-\theta)(x_2 + y_2)$$
$$\leq \theta \cdot 4 + (1-\theta) \cdot 4 = 4$$

$$\Rightarrow \theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in S \Rightarrow S \text{ is a convex set}$$



$\Leftarrow$ from this figure we know that it's indeed a convex set.

④ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$

(1)

$$\theta y_1^2 + (1-\theta) y_2^2 - (\theta y_1 + (1-\theta) y_2)^2$$

$$= \theta y_1^2 + (1-\theta) y_2^2 - \theta^2 y_1^2 + 2\theta(1-\theta) y_1 y_2 + (1-\theta)^2 y_2^2$$

$$= \theta(1-\theta) y_1^2 + (1-\theta)\theta y_2^2 + 2\theta(1-\theta) y_1 y_2$$

$$= \theta(1-\theta)(y_1 - y_2)^2 \geq 0$$

$$\left(\sqrt{\theta x_1 + (1-\theta) x_2}\right)^2 = \theta x_1 + (1-\theta) x_2 \geq \theta y_1^2 + (1-\theta) y_2^2$$

$$\geq (\theta y_1 + (1-\theta) y_2)^2$$

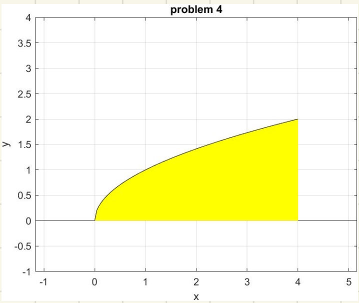
$$\Rightarrow \sqrt{\theta x_1 + (1-\theta) x_2} \geq \theta y_1 + (1-\theta) y_2$$

(2) $\theta y_1 + (1-\theta) y_2 \leq \theta \cdot 2 + (1+\theta) \cdot 2 = 2$

(3) $\theta x_1 + (1-\theta) x_2 \geq \theta \cdot 0 + (1-\theta) \cdot 0 = 0$

$\theta x_1 + (1-\theta) x_2 \leq \theta \cdot 4 + (1-\theta) \cdot 4 = 4$

$$\Rightarrow \theta \underline{x}_1 + (1-\theta) \underline{x}_2 \in S \Rightarrow S \text{ is a convex set}$$



$\Leftarrow$ it's indeed a convex set.

⑤ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

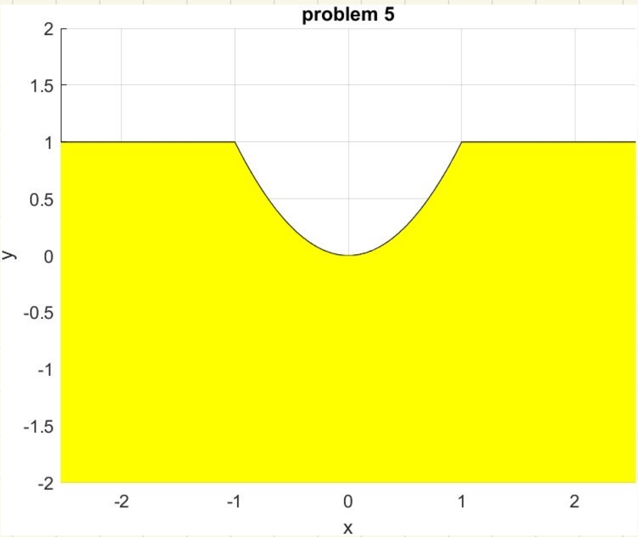
$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$

Counterexample : let $\underline{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

let $\theta = 0.5$, $\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

but $(0.5)^2 < 0.5$ violate $x^2 \geq y$

$\Rightarrow$ S is not a convex set



$\Uparrow$

we can clearly see there's a concave region

so it's not a convex set.

⑥ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

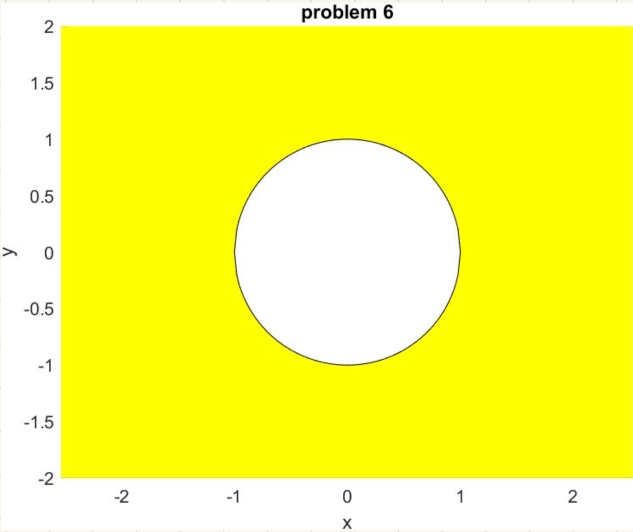
$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$

Counterexample: let $\underline{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

let $\theta = 0.5$, $\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

but $(0.5)^2 < 1 - (0.5)^2$

$\Rightarrow$ S is not a convex set



$\Uparrow$

it's clearly that it's not a convex set

⑦ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$

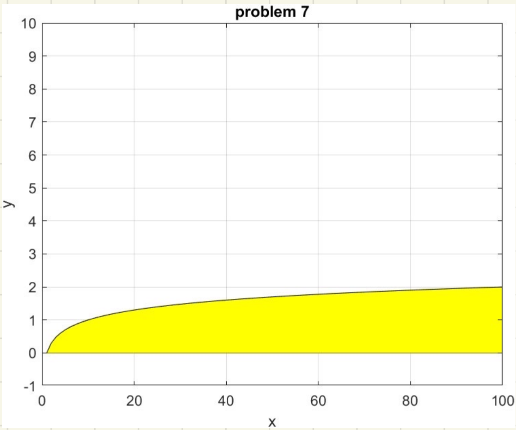
(1)
$$\theta x_1 + (1-\theta) x_2 \geq \theta\, 10^{y_1} + (1-\theta)\, 10^{y_2}$$
$$\geq 10^{\theta y_1 + (1-\theta) y_2}$$ (according to Jensen's ineq.)

(2) $\theta x_1 + (1-\theta) x_2 \geq \theta \cdot 1 + (1+\theta) \cdot 1 = 1$

(3) $\theta y_1 + (1-\theta) y_2 \leq \theta\, 100 + (1-\theta)\, 100 = 100$

(4) $\theta y_1 + (1-\theta) y_2 \geq \theta\, 0 + (1-\theta)\, 0 = 0$

$\Rightarrow$ S is a convex set

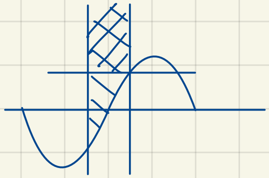


$\Uparrow$

I didn't draw the part $y \leq 100$, but since it's a affine set, so we know that it's still a convex set.

⑧ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$



$\Rightarrow S = \{ x, y \in \mathbb{R},\ y \geq \frac{\sqrt{2}}{2},\ \frac{-\pi}{4} \leq x \leq \frac{\pi}{4} \}$

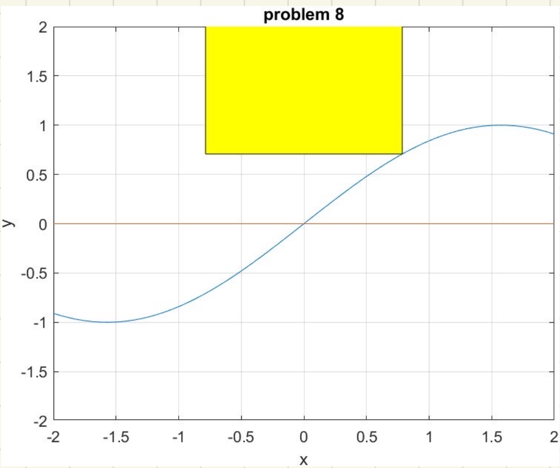
(1) $\theta x_1 + (1-\theta) x_2 \geq \theta \cdot \frac{-\pi}{4} + (1+\theta) \cdot \frac{-\pi}{4} = \frac{-\pi}{4}$

(2) $\theta x_1 + (1-\theta) x_2 \leq \theta \cdot \frac{\pi}{4} + (1-\theta) \frac{\pi}{4} = \frac{\pi}{4}$

(3) $\theta y_1 + (1-\theta) y_2 \geq \theta \frac{\sqrt{2}}{2} + (1-\theta) \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

intersection of convex sets is still a convex set

$\Rightarrow$ S is a convex set



Obviously it's a convex set.

⑨ if we let $\underline{x}$ to be $\begin{bmatrix} x \\ y \end{bmatrix}$, assume $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta) \underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix},$$

$$S = \{ x, y \in \mathbb{R} \mid y \geq x,\ y \geq -x,\ y \leq -x+5,\ y \leq x+5 \}$$

(1) $\theta y_1 + (1-\theta) y_2 \geq \theta x_1 + (1-\theta) x_2$

(2) $\theta y_1 + (1-\theta) y_2 \geq \theta(-x_1) + (1-\theta)(-x_2)$

$$= -(\theta x_1 + (1-\theta) x_2)$$

(3) $\theta y_1 + (1-\theta) y_2 \geq \theta(-x_1+5) + (1-\theta)(-x_2+5)$

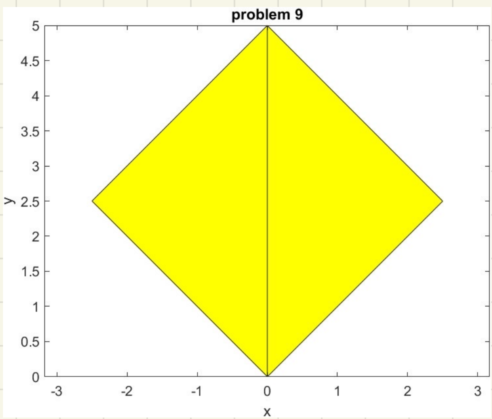
$$= -(\theta x_1 + (1-\theta) x_2) + 5$$

(4) $\theta y_1 + (1-\theta) y_2 \geq \theta(x_1+5) + (1-\theta)(x_2+5)$

$$= (\theta x_1 + (1-\theta) x_2) + 5$$

intersection of convex sets is still a convex set

$\Rightarrow$ S is a convex set



$\Leftarrow$ it's enclosed by four straight lines so it's obvious that it's a cvx set

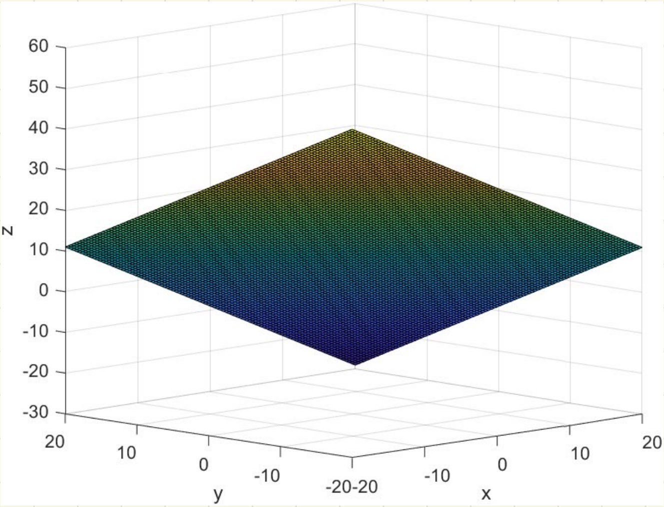
⑩ if we let $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\underline{x}_1, \underline{x}_2 \in S$

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 = \begin{bmatrix} \theta x_1 + (1-\theta)x_2 \\ \theta y_1 + (1-\theta)y_2 \\ \theta z_1 + (1-\theta)z_2 \end{bmatrix}, \text{ then}$$

$$(\theta x_1 + (1-\theta)x_2) + (\theta y_1 + (1-\theta)y_2)$$

$$= \theta(11 - z_1) + (1-\theta)(11 - z_2) = 11 - (\theta z_1 + (1-\theta)z_2)$$

$$\Rightarrow \theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in S \Rightarrow S \text{ is a convex set}$$



$\Uparrow$

it's a plane in $\mathbb{R}^3$ so it's a convex set