[MAT 3] - Z8 - Wektory i wartości własne

Zad. 1.

wielomian charakterystyczny

$$\chi_A(x) = \det(A - x\,\mathrm{Id})$$

1. Znaleźć wielomian charakterystyczny macierzy:

(a) $\begin{pmatrix} 2+i & 1 \\ 2 & 2-i \end{pmatrix}$,

$$x\,\mathrm{Id} = x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

$$A - x\,\mathrm{Id} = \begin{bmatrix} 2+i-x & 1 \\ 2 & 2-i-x \end{bmatrix}$$

$$\chi_A(x) = \begin{vmatrix} 2+i-x & 1 \\ 2 & 2-i-x \end{vmatrix} = (2+i-x)(2-i-x) - 2$$

$$\chi_A(x) = (2-x)^2 - i^2 - 2 = x^2 - 4x + 4 + 1 - 2 = x^2 - 4x + 3$$

$$\underbrace{\chi_A(x) = (x-3)(x-1)}$$

(b) $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.

$$\chi_A(x) = \det(A - x\,\mathrm{Id})$$

$$\chi_A(x) = \begin{vmatrix} 2-x & -1 & 2 \\ 5 & -3-x & 3 \\ -1 & 0 & -2-x \end{vmatrix} = (-1) \cdot (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ -3-x & 3 \end{vmatrix} +$$

$$+ (-2-x) \cdot (-1)^{3+3} \begin{vmatrix} 2-x & -1 \\ 5 & -3-x \end{vmatrix} = (-1) \cdot (-3+6x) - (x+2)\Big[(2-x)(-3-x) + 5\Big]$$

$$= -2x - 3 - (x+2)(x^2 + x - 6 + 5) = -2x - 3 - (x+2)(x^2 + x - 1) = -x^3 - x^2 + x - 2x^2 -$$

$$-2x + 2 - 2x - 3 = -(x^3 + 3x^2 + 3x + 1) = \underbrace{-(x+1)^3}$$

Zad. 2

2. Znaleźć wartości własne i wektory własne podanych liniowych przekształceń przestrzeni liniowych. Dla każdej wartości własnej λ znaleźć podprzestrzeń N_λ wektorów własnych.

(a) $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F([x_1, x_2]) = [x_1, x_1 + x_2]$,

baza standardowa: $[1,0], [0,1]$

$F([1,0]) = [1,1]$

$F([0,1]) = [0,1]$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\chi_A(x) = \det(A - x\,Id) = \begin{vmatrix} 1-x & 0 \\ 1 & 1-x \end{vmatrix} = \underset{x=1}{(1-x)^2}.$$

$X = 1$ → wartość własna

wektory własne: $(A - x\,Id) \cdot v = 0$

$$\begin{bmatrix} 1-1 & 0 \\ 1 & 1-1 \end{bmatrix} \cdot v = 0$$

$V = [x_1, x_2]$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{\begin{bmatrix} 0 \\ x_1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$ $\quad x_2 \in R$

$$v = [0, x_2] = x_2 [0, 1]$$

$\boxed{N_\lambda :}$ dla $\lambda = 1$:

$$\underline{\text{Baza } N_1 = \mathcal{L}([0,1])}$$

(b) $F : \mathbb{R}^3 \to \mathbb{R}^3$, $F([x_1, x_2, x_3]) = [x_1 - x_3, 2x_2, x_1 + x_3]$.

baza standardowa: $[1,0,0], [0,1,0], [0,0,1]$

$F([1,0,0]) = [1,0,1]$

$F([0,1,0]) = [0,2,0]$ $\qquad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$F([0,0,1]) = [-1,0,1]$

$$\chi_A(x) = \det(A - x\,Id) = \begin{vmatrix} 1-x & 0 & -1 \\ 0 & 2-x & 0 \\ 1 & 0 & 1-x \end{vmatrix} = (2-x)\cdot(-1)^{2+2}\cdot\begin{vmatrix} 1-x & -1 \\ 1 & 1-x \end{vmatrix} =$$

$$= (2-x)\cdot\left[(1-x)^2 + 1\right] = (2-x)\,(x^2 - 2x + 2)$$

$$\Delta < 0$$

$x = 2$ ← wartość własna

$(A - x\,Id)\cdot v = 0 \qquad v = [x_1, x_2, x_3]$

$$\begin{bmatrix} 1-x & 0 & -1 \\ 0 & 2-x & 0 \\ 1 & 0 & 1-x \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -x_1 - x_3 \\ 0 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases} \quad / -$$

$$x_1 = 0$$
$$x_3 = 0$$
$$x_2 \in R$$

$$v = [0, x_2, 0] = \overset{t \in R}{\overset{//}{x_2}} [0, 1, 0]$$

dla $\lambda = 2$

$$V_2 = \mathcal{L}([0,1,0])$$

(c) $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F([x_1, x_2]) = [-x_2, x_1]$,

baza standardowa: $[1,0], [0,1]$

$$F([1,0]) = [0,1]$$
$$F([0,1]) = [-1,0]$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\chi_A(\lambda) = \det(A - \lambda \, Id) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = \lambda^2 - i^2 = $$
$$= (\lambda - i)(\lambda + i)$$

dla $\lambda = i$

$$\chi_A(i)\cdot \underset{\begin{bmatrix}x_1\\x_2\end{bmatrix}}{\underset{\parallel}{v}} = 0$$

$$\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\begin{bmatrix}-ix_1 - x_2\\ x_1 - ix_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}-i & -1 & 0\\ 1 & -i & 0\end{array}\right] \xrightarrow{r_2 - ir_1} \left[\begin{array}{cc|c}\boxed{-i} & \overset{t}{-1} & 0\\ 0 & 0 & 0\end{array}\right]$$

$-ix_1 - t = 0$

$x_1 = t\cdot i$

$x_2 = t$

$t \in \mathbb{C}$

$v = t[i,1]$

$V_i = \mathcal{L}([i,1])$

dla $\lambda = -i$

$$\chi_A(-i)\cdot \underset{\begin{bmatrix}x_1\\x_2\end{bmatrix}}{\underset{\parallel}{u}} = 0$$

$$\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}\begin{bmatrix}ix_1 - x_2\\ x_1 + ix_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \xrightarrow{r_2 + i r_1} \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$i x_1 - t = 0$

$x_1 = -it$

$x_2 = t$

$t \in \mathbb{C}$

$u = t[-i, 1]$

$N_{-i} = \mathcal{L}([-i, 1])$

(d) $F : \mathbb{C}^3 \to \mathbb{C}^3$, $F([x_1, x_2, x_3]) = [x_1 - x_3, 2x_2, x_1 + x_3]$.

baza standardowa: $[1,0,0], [0,1,0], [0,0,1]$

$F([1,0,0]) = [1,0,1]$

$F([0,1,0]) = [0,2,0]$

$F([0,0,1]) = [-1,0,1]$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\chi_A(\lambda) = \det(A - \lambda \, Id) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (2-\lambda) \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)\left[(1-\lambda)^2 + 1\right] =$$

$$= (2-\lambda)(\lambda^2 - 2\lambda + 2)$$

$\Delta = 4 - 8 = -4 = i^2 2^2$ $\quad \sqrt{\Delta} = 2i$

$\lambda_1 = \frac{2 - 2i}{2} = 1 - i \quad \wedge \quad \lambda_2 = 1 + i \quad \wedge \quad \lambda_3 = 2$

dla $\lambda_1 = 1-i$

$$\begin{bmatrix} i & 0 & -1 \\ 0 & 1+i & 0 \\ 1 & 0 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} i & 0 & -1 & 0 \\ 0 & 1+i & 0 & 0 \\ 1 & 0 & i & 0 \end{array}\right] \xrightarrow{r_3 + i r_1} \left[\begin{array}{ccc|c} i & 0 & \overset{t}{-1} & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(1+i)\, x_2 = 0$

$x_2 = 0$

$x_3 = t$

$i x_1 = t$

$x_1 = \frac{t}{i} = -ti$

$V = [-it, 0, t] = t[-i, 0, 1]$

$N_{1-i} = \mathcal{L}\,[-i, 0, 1]$

dla $\lambda_2 = 1+i$

$$\begin{bmatrix} -i & 0 & -1 \\ 0 & 1-i & 0 \\ 1 & 0 & -i \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 0 & 1-i & 0 & 0 \\ 1 & 0 & -i & 0 \end{array}\right] \xrightarrow{r_3 - i r_1} \left[\begin{array}{ccc|c} -i & 0 & \overset{t}{-1} & 0 \\ 0 & 1-i & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & 0 \end{array}\right]$$

$x_3 = t$

$x_2 = 0$

$-i x_1 = t$

$x_1 = \frac{t}{-i} = \frac{t i^2}{i} = t i$

$u = [i, 0, 1] \cdot t$

$N_{1+i} = \mathcal{L}([i, 0, 1])$

dla $\lambda = 2$ tak jak w podpunkcie b)

Zad. 3

3. Niech $F\colon V \to V$ będzie przekształceniem liniowym o podanej macierzy $A = M_{\mathcal{B}}^{\mathcal{B}}(F)$ w bazie standardowej $\mathcal{B}$ przestrzeni $V(\mathbb{C})$. Sprawdzić, czy istnieje baza $\mathcal{C}$ odpowiedniej przestrzeni wektorowej, w której $M_{\mathcal{C}}^{\mathcal{C}}(F)$ jest macierzą diagonalną. Jeśli tak, znaleźć macierze zmiany bazy: $M_{\mathcal{C}}^{\mathcal{B}}(Id_V)$ oraz $M_{\mathcal{B}}^{\mathcal{C}}(Id_V)$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda Id) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

$\lambda = -1$

$$(A - \lambda Id) \cdot v = 0$$

dla $\lambda = -1$:

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} & t & \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + t = 0$

$x_1 = -t$

$v = t[-1, 1]$

$N_{-1} = \mathcal{L}([-1, 1])$

$\lambda = 2$

$$\left[\begin{array}{cc|c} & t & \\ -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 = 2t$

$u = t[2, 1]$

$N_2 = \mathcal{L}([2, 1])$

baza $C = \{v_1, v_2\}$ – baza złożona z wektorów własnych

istnieje

$$C = \{[2,1], [-1,1]\}$$

$$M_C^C(F) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{r_2 - \frac{1}{2}r_1} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \xrightarrow{r_1 + \frac{1}{3}r_2}$$

$$\to \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \xrightarrow{r_2 \cdot (-\frac{1}{3})} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\dim N_{-1} + \dim N_2 = 1+1 = 2 = n$$

$\Rightarrow$ F jest diagonalna

$M_C^B(Id_V) = ?$ $\qquad M_B^C(Id_V) = ?$

$$M_C^B(Id\,V) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

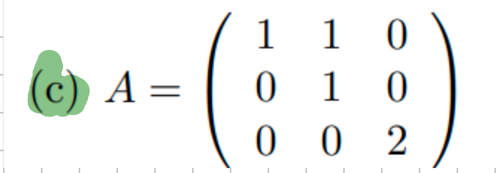
$$M_C^C(Id\,V) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M_B^C(Id\,V) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - \frac{1}{2} r_1} \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{r_1 + \frac{2}{3} r_2}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 2 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow[r_3 \cdot \frac{2}{3}]{r_1 \cdot \frac{1}{2}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$M^C_B(\mathrm{Id}\, V) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$



(c) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda \mathrm{Id}) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\,(-1)^{2+2} \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(2-\lambda)\right] =$$

$$= (1-\lambda)^2\,(2-\lambda)$$

$$\lambda = 1 \ \wedge \ \lambda = 2$$

$$(A - \lambda \mathrm{Id}) \cdot v = 0$$

dla $\lambda = 1$

$$\left[\begin{array}{ccc|c} \overset{t}{0} & \boxed{1} & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

$x_2 = 0$

$x_3 = 0$

$x_1 = t \quad , \quad t \in \mathbb{C}$

$v = [t, 0, 0] = t[1, 0, 0]$

$N_1 = \mathcal{L}([1, 0, 0])$

dla $\lambda = 2$

$$\left[\begin{array}{ccc|c} \boxed{-1} & 1 & \overset{t}{0} & 0 \\ 0 & \boxed{-1} & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

$-x_2 = 0 \Rightarrow x_2 = 0$

$-x_1 = 0 \Rightarrow x_1 = 0$

$x_3 = t \quad , \quad t \in \mathbb{C}$

$u = [0, 0, t] = t[0, 0, 1]$

$N_2 = \mathcal{L}([0, 0, 1])$

$C = \{[1, 0, 0], [0, 0, 1]\}$ $\quad n = 3$

$\dim N_1 + \dim N_2 = 2 \neq n$