

JANO & Natalia

AI tools have been used to review parts of this solution.

```
In [38]: import matplotlib.pyplot as plt
import numpy as np
```

1. Analytical solutions to ODEs

(a): $\dot{x} = -x$

$$\begin{aligned}\frac{dx}{dt} &= -x \\ \frac{1}{x} dx &= -dt \\ \int \frac{1}{x} dx &= \int -dt \\ \ln|x| &= -t + C \\ x &= C \exp(-t)\end{aligned}$$

Initial condition:

$$x(0) = C \exp(-0) \stackrel{!}{=} 1 \Rightarrow C = 1$$

Solution:

$$x(t) = \exp(-t)$$

(b): $\dot{x} = -\frac{1}{x}$

$$\begin{aligned}x dx &= -dt \\ \int x dx &= \int -dt \\ \frac{x^2}{2} &= -t + C \\ x &= \sqrt{2(C-t)}\end{aligned}$$

Initial condition:

$$x(0) = \sqrt{2(C-0)} \stackrel{!}{=} 1 \Rightarrow C = \frac{1}{2}$$

Solution:

$$x(t) = \sqrt{1 - 2t}$$

(c): $\dot{x} = 1 - x$

$$\begin{aligned}\frac{dx}{dt} &= 1 - x \\ \frac{1}{1-x} dx &= dt \\ \int \frac{1}{1-x} dx &= \int dt \\ -\ln|1-x| &= t + C \\ x &= 1 - C \exp(-t)\end{aligned}$$

Initial condition:

$$x(0) = 1 - C \exp(-0) \stackrel{!}{=} 0 \Rightarrow C = 1$$

Solution:

$$x(t) = 1 - \exp(-t)$$

(d): $\dot{x} = x(1 - x)$

Partial fractions:

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

Integrate:

$$\begin{aligned}\int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx &= \int dt \\ \ln \left| \frac{x}{1-x} \right| &= t + C\end{aligned}$$

Solve for x :

$$\begin{aligned}\frac{x}{1-x} &= C \exp(t) \\ x &= \frac{C \exp(t)}{1 + C \exp(t)}\end{aligned}$$

Initial condition:

$$x(0) = \frac{C \exp(0)}{1 + C \exp(0)} \stackrel{!}{=} \frac{1}{2} \Rightarrow C = 1$$

Solution:

$$x(t) = \frac{\exp(t)}{1 + \exp(t)}$$

```
In [59]: # defining the solutions for later usage
```

```
def sol_1(t):
    return np.exp(-t)

def sol_2(t):
    return np.sqrt(2*t + 1)

def sol_3(t):
    return 1 - np.exp(-t)

def sol_4(t):
    return np.exp(t) / (1 + np.exp(t))

solutions = [sol_1, sol_2, sol_3, sol_4]
```

2. Numerical solutions to ODEs

a) & b)

```
In [40]: # defining the ODEs
```

```
def func1(t, x):
    return -x

def func2(t, x):
    return 1/x

def func3(t, x):
    return 1-x

def func4(t, x):
    return x*(1-x)

functions = [func1, func2, func3, func4]
```

```
In [41]: # implementing the euler scheme
```

```
def euler(f_func, x_0, t_max, dt, *args, **kwargs):
    t = 0
    x = x_0

    x_t = [x_0]
    ts = [t]

    while t < t_max:
        t += dt
        ts.append(t)

        x = x + f_func(t, x, *args, **kwargs) * dt # calculate
```

```
x_t.append(x)

return np.array(ts), np.array(x_t)
```

```
In [42]: # calculate the numerical solutions for the given ODEs using the euler sc

times = []
numeric_solutions = []
for func, ic in zip([func1, func2, func3, func4], [1, 1, 0, 1/2]):
    ts, x_t = euler(f_func=func, x_0=ic, t_max=10, dt=0.01)
    times.append(np.array(ts))
    numeric_solutions.append(x_t)
```

```
In [ ]: # plotting the solutions that have been obtained in the previous cell

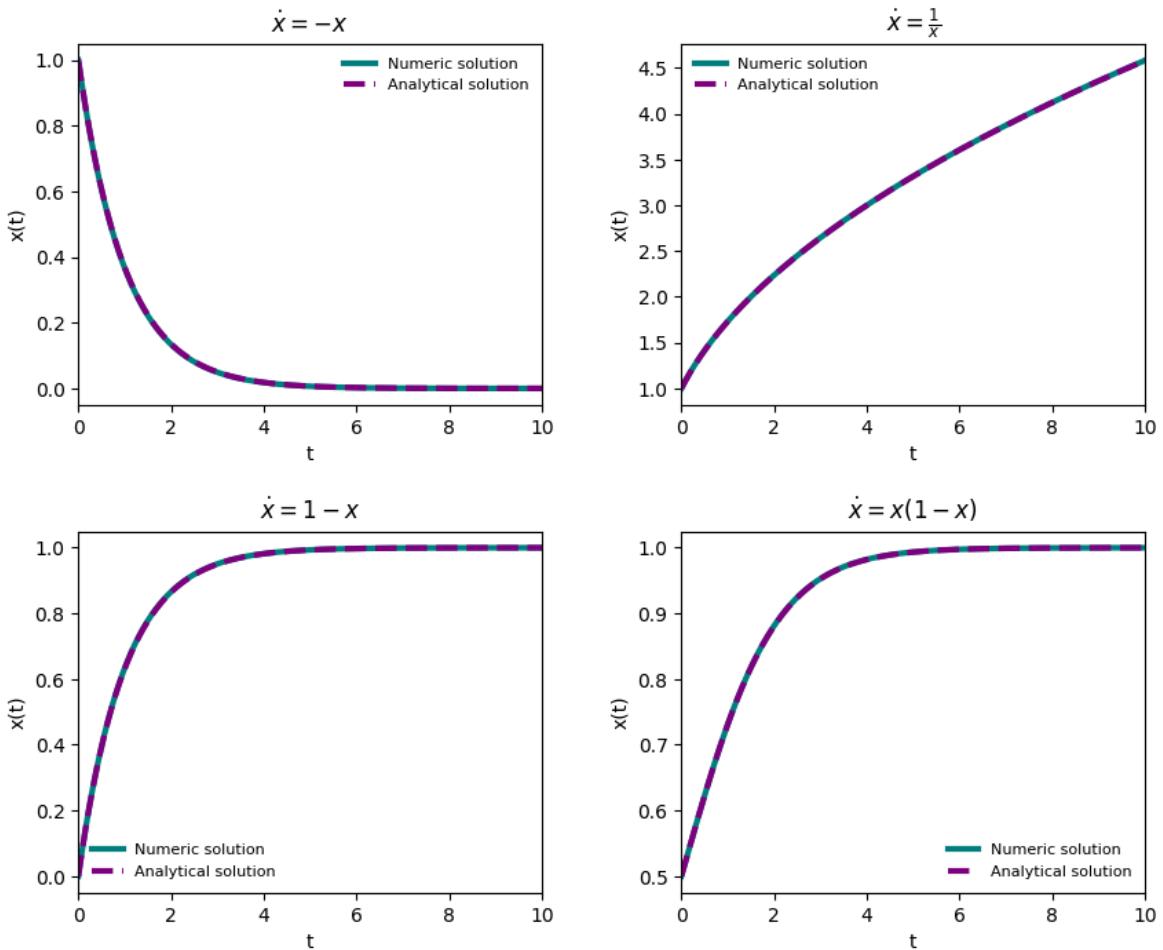
titles = ["$\\dot{x} = -x", "$\\dot{x} = \\frac{1}{x}$", "$\\dot{x} = 1-x$"]
fig, ((ax1, ax2), (ax3, ax4)) = plt.subplots(2, 2, figsize=(10, 8))
plt.subplots_adjust(hspace=0.35, wspace=0.3)

for i, (ax, sol) in enumerate(zip([ax1, ax2, ax3, ax4], solutions)):

    ax.plot(times[i], numeric_solutions[i], label="Numeric solution", color='blue')
    ax.plot(times[i], sol(times[i]), linestyle="--", label="Analytical solution")
    ax.set_title(titles[i])

    ax.set_xlim([0, 10])
    ax.set_xlabel("t")
    ax.set_ylabel("x(t)")
    ax.legend(fontsize=8, loc="best", frameon=False)

plt.show()
```



Compare (discuss) the analytical and numeric solutions:

The numerical and analytical solutions seem to be indistinguishable only by looking at the plots.

C)

```
In [44]: from scipy.integrate import solve_ivp
```

```
In [45]: # defining the squared error as the element wise difference (squared) between the numerical and analytical solution

def squared_error(array_a, array_b):
    s_err = [(b - a)**2 for a, b in zip(array_a, array_b)]
    return s_err
```

```
In [46]: # defining a function to compare the numerical and analytical solutions |

def compare_solutions(func, dt, x0, t_min, t_max, sol):
    ts = np.arange(t_min, t_max, dt)

    ana_solution = sol(ts)
    scipy_solution = solve_ivp(func, (t_min, t_max), [x0], t_eval=ts).y[0]
    numeric_solution = euler(f_func=func, x_0=x0, t_max=t_max, dt=dt)[1]

    qe_num = squared_error(ana_solution, numeric_solution)
    qe_sci = squared_error(ana_solution, scipy_solution)
```

```
    return ts, qe_num, qe_sci
```

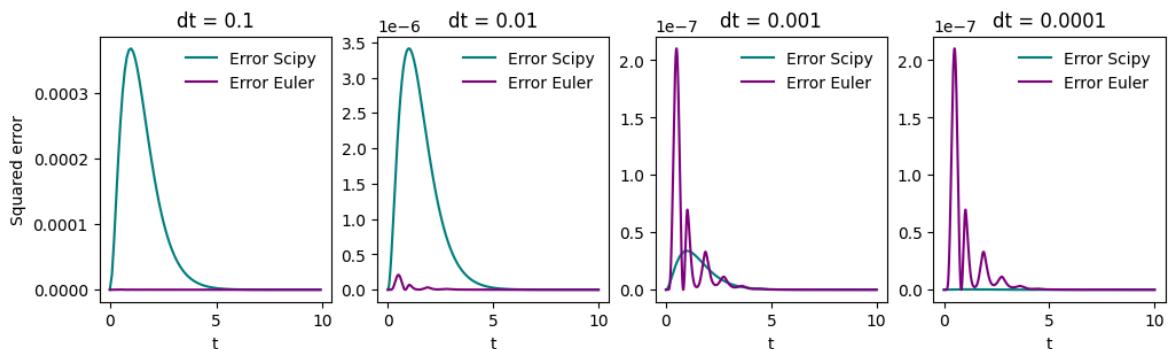
```
In [47]: # plotting the squared error of the euler scheme and scipy.solve_ivp for

fig, (ax1, ax2, ax3, ax4) = plt.subplots(1, 4, figsize=(12, 3))

for dt, ax in zip([0.1, 0.01, 0.001, 0.0001], [ax1, ax2, ax3, ax4]):
    ts, qe_num, qe_sci = compare_solutions(func=func1, dt=dt, x0=1, t_min

    ax.plot(ts, qe_num, label="Error Scipy", color="teal")
    ax.plot(ts, qe_sci, label="Error Euler", color="purple")
    ax.set_title(f"dt = {dt}")
    ax.legend(frameon=False)
    ax.set_xlabel("t")
    if ax == ax1:
        ax.set_ylabel("Squared error")

plt.show()
```



Method comparison:

While both numerical methods find solutions that deviate relatively little from the analytical solution, the results become even better when decreasing the step size. It is also interesting that the Euler method seems to yield a better approximation of the analytical solution for a bigger step size, while the scipy method starts to shine as the step size decreases.

3. Passive membrane

a)

```
In [48]: # define the ODE describing the passive membrane

def dVdt(t, V, tau_m, E_m, R_m, I):
    return (-V + E_m + R_m*I[int(t/dt)]) / tau_m
```

```
In [49]: # Params
simtime = 100 * 10**-3 # 100 ms
dt = 0.1 * 10**-3 # 0.1 ms simulation time step
R_m = 1 * 10**7 # 10**7 Ohm
I_0 = 1 * 10**-9 # 1 nA
```

```
tau_m = 10 * 10**-3                                # 10 ms
E_m = -80 * 10**-3                                # 80 mV

t_array = np.arange(0, simtime + dt, dt)
I_ext = I_0 * np.ones_like(t_array)
```

In [50]:

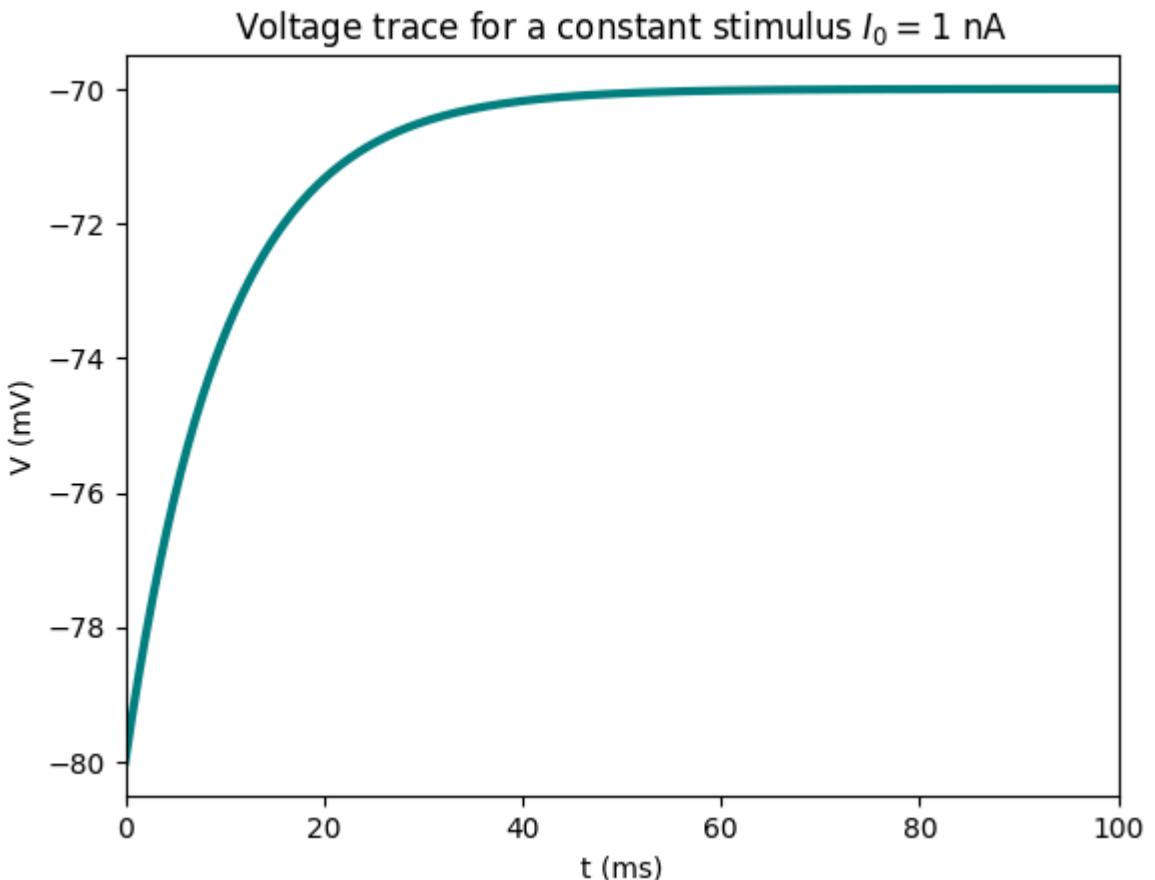
```
# grabbing the solution and converting it into sensible units

ts, V_steady = euler(dVdt, x_0=E_m, tau_m=tau_m, E_m=E_m, R_m=R_m, I=I_ex
ts_ms = ts/10**-3
V_steady_mV = V_steady/10**-3
```

In [51]:

```
# plotting the solution

plt.plot(ts_ms, V_steady_mV, color="teal", linewidth=3)
plt.ylabel("V (mV)")
plt.xlabel("t (ms)")
plt.xlim([0, 100])
plt.title("Voltage trace for a constant stimulus $I_0 = 1$ nA")
plt.show()
```



b)

In [52]:

```
from matplotlib.gridspec import GridSpec
```

In [53]:

```
# plotting the voltage trace and the rescaled sinusoidal stimulus for dif

fig = plt.figure(figsize=(16, 4))
gs = GridSpec(1, 3, wspace=0.4)
```

```

for i, (f, c) in enumerate(zip([1, 10, 30], ["teal", "purple", "orange"]))

    T = 1 / f
    t_max = T * 10
    t_array = np.arange(0, t_max+dt, dt)
    stim = I_0 * np.sin(2*np.pi*f*t_array)

    ts, V_t = euler(dVdt, x_0=E_m, tau_m=tau_m, E_m=E_m, R_m=R_m, I=stim,
                     ts_ms = ts
                     V_t_mV = V_t / 10**-3
                     input_mV = (E_m + R_m*stim) / 10**-3

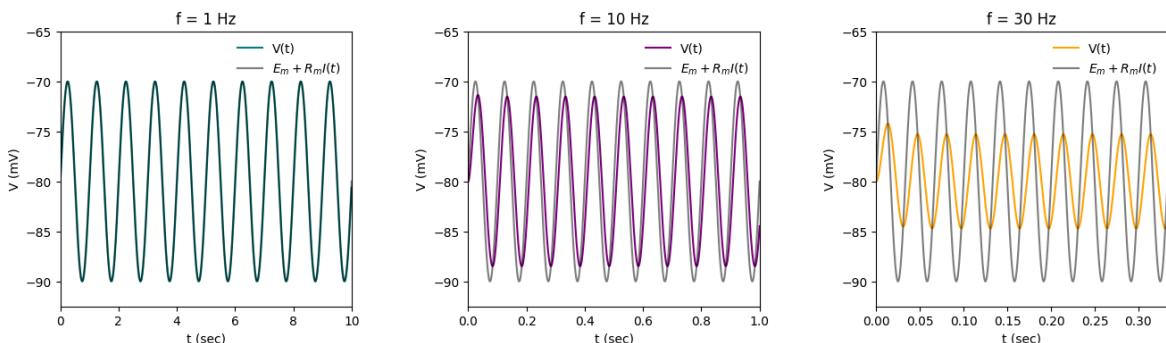
    ax1 = fig.add_subplot(gs[0, i])
    ax1.plot(ts, V_t_mV, color=c, label="V(t)")
    ax1.plot(t_array, input_mV, color="black", alpha=0.5, label="$E_m + R_m I(t)$")

    ax1.set_title(f"f = {f} Hz")
    ax1.legend(loc="upper right", frameon=False)

    ax1.set_xlim([-92.5, -65])
    ax1.set_xlabel("t (sec)")
    ax1.set_ylabel("V (mV)")

plt.show()

```



Comment on the results:

Plotting the voltage response to sinusoidal stimuli of different frequency is a great way of illustrating the passive membranes low pass filtering properties. If the stimulus frequency is too low, the stimulus will not pass all its energy onto the change in membrane potential, resulting in a smaller amplitude of the voltage oscillations and phase delay.

C)

```
In [54]: # defining the two functions
```

```
def A(nu):
    return I_0*R_m*np.sqrt(1 / (1 + (2*np.pi*nu*tau_m)**2))

def dphi(nu):
    return np.arctan(2*np.pi*nu*tau_m)
```

```
In [55]: # calculating the output for frequencies between 1 and 100 Hz with 1 Hz f
```

```
nu = np.arange(1, 100, 1)
A_calc = A(nu) / 10**-3
dphi_calc = dphi(nu)
```

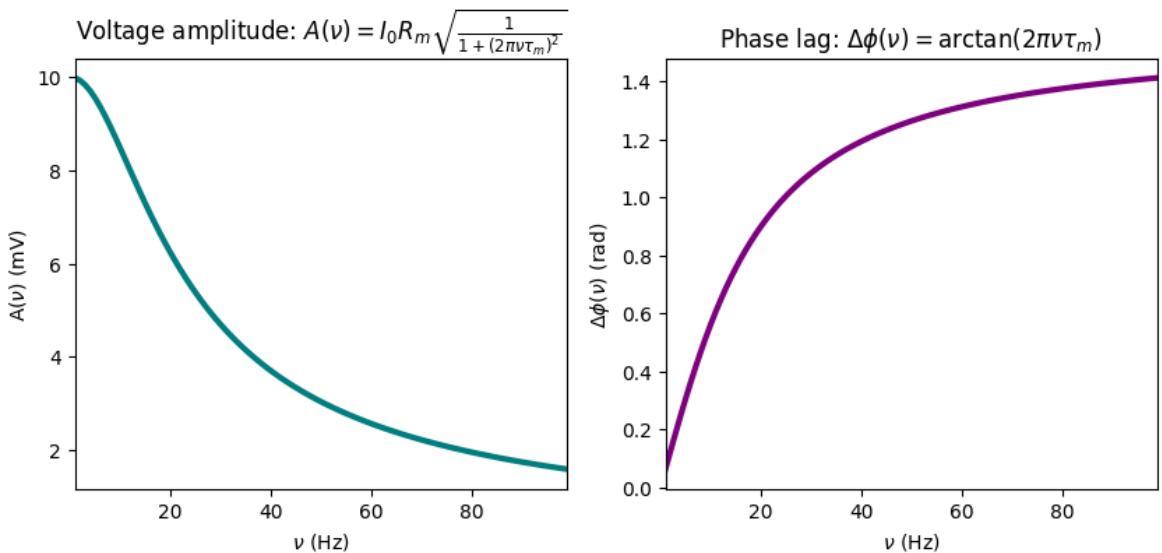
```
In [56]: # plotting A(\nu) and \Delta\phi(\nu)
```

```
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(10, 4))

ax1.plot(nu, A_calc, color="teal", linewidth=3)
ax1.set_ylabel("A($\nu$) (mV)")
ax1.set_xlabel("$\nu$ (Hz)")
ax1.set_xlim([min(nu), max(nu)])
ax1.set_title("Voltage amplitude: $A(\nu) = I_0 R_m \sqrt{\frac{1}{1+(2\pi\nu\tau_m)^2}}$")

ax2.plot(nu, dphi_calc, color="purple", linewidth=3)
ax2.set_ylabel("\Delta\phi(\nu) (rad)")
ax2.set_xlabel("$\nu$ (Hz)")
ax2.set_xlim([min(nu), max(nu)])
ax2.set_title("Phase lag: $\Delta\phi(\nu) = \arctan(2\pi\nu\tau_m)$")

plt.show()
```



Comment on the result:

Plotting $A(\nu)$ and $\Delta\phi(\nu)$ clearly shows the relationship between the stimulus frequency and the response of the passive membrane. The higher the input frequency, the smaller the amplitude of the resulting voltage oscillations. Increasing the stimulus frequency also introduces a phase delay to the voltage response.

d)

```
In [57]: # Approximating A(\nu) and \Delta\phi(\nu)

A_approx = np.empty(len(nu))
dphi_approx = np.empty(len(nu))

for i, f in enumerate(nu):

    T = 1 / f
    t_max = T * 30 # simulating for
    t_array = np.arange(0, t_max+dt, dt)
    stim = I_0 * np.sin(2*np.pi*f*t_array)

    ts, V_t = euler(dVdt, x_0=E_m, tau_m=tau_m, E_m=E_m, R_m=R_m, I=stim,
                     V_t_stable = V_t[len(V_t) - int(len(V_t)/5):] / 10**-3 # ass
    A_approx[i] = (np.max(V_t_stable) - np.mean(V_t_stable)) # Cal

    n_per = int(np.round(T / dt)) # cal
    V_last = V_t[-n_per:] # che
    I_last = stim[-n_per:] # che

    idx_I = np.argmax(I_last) + (len(stim) - n_per) # obt
    idx_V = np.argmax(V_last) + (len(V_t) - n_per) # obt

    t_peak_I = ts[idx_I] # rei
    t_peak_V = ts[idx_V] # rei

    dphi_approx[i] = 2.0 * np.pi * f * (t_peak_V - t_peak_I) # cal
```

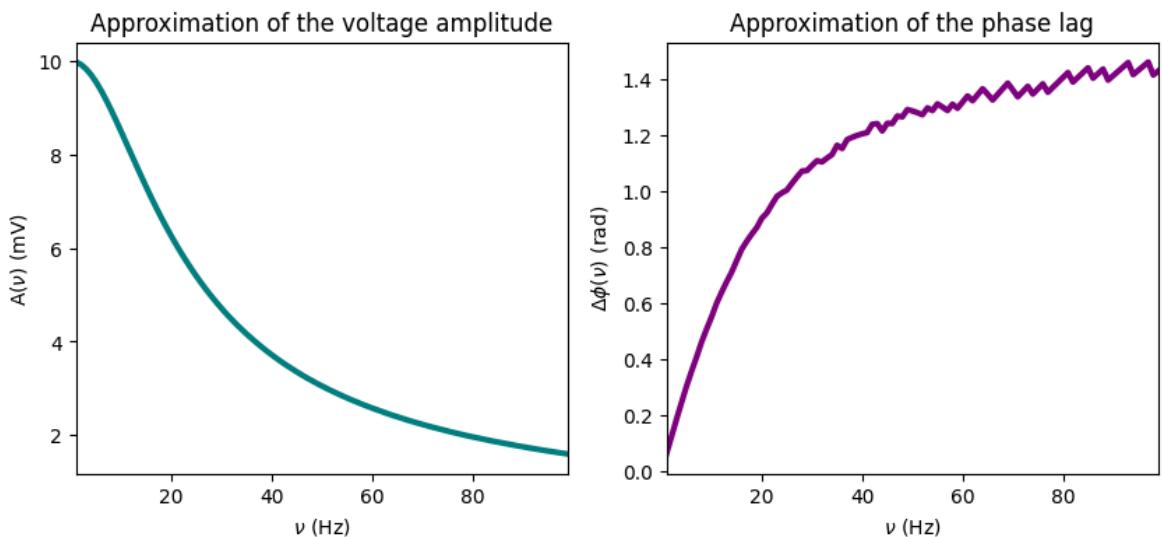
```
In [58]: # plotting the results of the approximation

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(10, 4))

ax1.plot(nu, A_approx, color="teal", linewidth=3)
ax1.set_ylabel("A($\nu$) (mV)")
ax1.set_xlabel("$\nu$ (Hz)")
ax1.set_xlim([min(nu), max(nu)])
ax1.set_title("Approximation of the voltage amplitude")

ax2.plot(nu, dphi_approx, color="purple", linewidth=3)
ax2.set_ylabel("\Delta \phi (\nu) (rad)")
ax2.set_xlabel("$\nu$ (Hz)")
ax2.set_xlim([min(nu), max(nu)])
ax2.set_title("Approximation of the phase lag")

plt.show()
```



sorry for not inserting them into the plots of the analytical solutions