Expander Graphs

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1 Definitions

1.1 Expansion

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Definition 1.1.1. Let $P = (V, E)$ be a finite graph.

(1) For any disjoint subsets of vertives $V_1, V_2 \subseteq V$, we denote by $\mathcal{E}(V_1, V_2)$ the set of edges of Γ with one extremity in V_1 and the other in V_2. We denote $\mathcal{E}(V_1) := \mathcal{E}(V_1, V \setminus V_1)$.

(2) The ***Cheeger constant*** or ***expansion constant*** of Γ is

$$h(\Gamma) := \min\left\{ \frac{|\mathcal{E}(W)|}{|W|} \;\middle|\; \emptyset \neq W \subseteq V \wedge |W| \leq \frac{1}{2}|V| \right\}$$

with the convention that $h(\Gamma) = +\infty$ if Γ has at most one vertex.

The larger $h(P)$ is, the more difficult it is to disconnect a large subset of V from the rest of the graph. This will be our way of measuring "high connectivity".

Lemma 1.1.2. Let Γ be a finite graph with at least two vertices (so that $h(\Gamma) < \infty$). We have $h(\Gamma) > 0$ if and only if Γ is connected.

Proof. The condition $h(\Gamma) = 0$ means that there exists some nonempty $W \subseteq V$, $|W| \leq \frac{1}{2}|V|$, such that $|\mathcal{E}|(W) = 0$. Since $W \neq V$, there is no path between W and $V \setminus W$, so Γ is disconnected.

Conversely, if Γ is disjoint, there is some componenet, say W, of size at most $\frac{1}{2}|V|$. Hence, $\mathcal{E}(W) = \emptyset$ and $h(\Gamma) = 0$. $\square$

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Example 1.1.3. Let K_m be the complete graph on m vertieces. We have

$$h(K_m) = \min_{1 \leq j \leq \frac{m}{2}} \frac{|\mathcal{E}(\{1, 2, \ldots, j\})|}{j} = \min_{1 \leq j \leq \frac{m}{2}} \frac{j(m-j)}{j} = m - \left\lfloor \frac{m}{2} \right\rfloor.$$

This goes do infinity, as m goes to infinity. This unsurprisingly tells us that complete graphs are very well connected.

Example 1.1.4. Let C_m be the m-cycle. We have

$$h(\Gamma) = \frac{2}{\left\lfloor \frac{m}{2} \right\rfloor}.$$

This goes to zero.

Example 1.1.5. Let $T_{d,k}$ be the d-regular tree of depth k, where $d \geq 2$. Let W be one of the subtrees hanging from the root. We have

$$h(T_{d,k}) \leq \frac{|\mathcal{E}(W)|}{|W|} = \frac{1}{\frac{|T|-1}{d}} = \frac{d}{|T|-1}.$$

This is a small upper bound.

Bounding

Remark 1.1.6. It is much easier to give upper bounds for h than lower bounds.

Let's calculate some trivial bounds. Let Γ be connected. Then

$$\frac{2}{|\Gamma|} \leq h(\Gamma) \leq \min_{x \in V} \operatorname{var}(x)$$

For the first inequality we use $\frac{|\mathcal{E}(W)|}{|W|} \geq \frac{1}{|V|/2}$, and for the second, we take $W = \{x\}$.

Our goal will be to find graphs with large $h(\Gamma)$ and few edges (unlike K_m).

Conncetivity with diameter

Proposition 1.1.7. Let Γ be connected. Let $v = \max_{x \in V} \operatorname{var}(x)$. Then

$$\operatorname{diam} \Gamma \leq 2 \cdot \frac{\log \frac{|\Gamma|}{2}}{\log(1 + \frac{h(\Gamma)}{V})} + 3.$$

Proof. Let $x \in V$. For $n \in \mathbb{N}$, consider the ball

$$B_x(v) = \{y \in V \mid d(x, y) \leq n\}.$$

Claim:

$$|B_x(n)| \geq \min \left\{ \frac{|\Gamma|}{2}, \left(1 + \frac{h(\Gamma)}{v}\right)^n \right\}$$

Proof. We use induction on n. Suppose $|B_x(n)| < \frac{|\Gamma|}{2}$ and take $W = B_x(n)$. Note

$$\begin{aligned} |B_x(n+1) \setminus B_x(n)| &\geq \frac{|\mathcal{E}(B_x(n))|}{v} \\ &\geq \frac{h(\Gamma)\,|B_x(n)|}{v}. \end{aligned}$$

Hence,

$$|B_x(n+1)| \geq \left(1 + \frac{h(\Gamma)}{v}\right) |B_x(n)|. \qquad \square$$

Let's use the claim to prove the proposition. Let $x, y \in V$. Let n be smallest such that

$$\left(1 + \frac{h(\Gamma)}{v}\right)^n \geq \frac{|\Gamma|}{2}.$$

This means

$$|B_x(n)| \geq \frac{|\Gamma|}{2}, \qquad |B_y(n)| \geq \frac{|\Gamma|}{2}.$$

Hence,

$$|B_x(n+1) \cap B_y(n)| > 0.$$

Hence, $d(x, y) \leq 2n + 1$. $\square$

Expander graphs

Definition 1.1.8. A family $(\Gamma_i)_{i\in\mathbb{N}}$ of graphs is an ***expander family*** if there are constants $v \geq 1$, $h > 0$ such that

1. $|V_i| \to \infty$
2. $\forall i.\ \max_{x\in V} \operatorname{var}(x) \leq v$
3. $\forall i.\ h(\Gamma_i) \geq h$.

Remark 1.1.9. The second condition implies $|E_i| \leq v\,|V_i|$, so there are only linearly many edges in Γ_i. This gives us a low cost of construction (in applications).

Diameters of expanders

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be an expander family. Then

$$\operatorname{diam}\Gamma_i \ll \log|V_i|\,,$$

where the implicit constant depends only on (v, h), so it is absolute in the family.

This fact follows immediatly from the previous proposition.

Remark 1.1.10. Having log-diameter is not the same as being an expander family.

Example 1.1.11. The tree $T_{d,k}$ has log-diameter and it does have bounded valency (for fixed d), but the Cheeger constant goes to 0 as k goes to infinity.

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