

Graph Theory

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1 Independence, matching, covers

1.1 Definitions

Let $G = (V, E)$ be a graph.

Definition 1.1.1. A set $S \subseteq V$ is an *independent set* if $G[S]$ contains no edges. The *independence number* $\alpha(G)$ is the maximum possible cardinality of an independent set.

Definition 1.1.2. A set $T \subseteq V$ is a *vertex cover* if

$$\forall e \in E. T \cap e \neq \emptyset.$$

The *vertex cover number* $\beta(G)$ is the minimum possible cardinality of a vertex cover.

Definition 1.1.3. A set $M \subseteq E$ is a *matching* if

$$\forall e, f \in M. e \neq f \Rightarrow e \cap f = \emptyset.$$

The *matching number* $\alpha'(G)$ is the maximum possible cardinality of a matching.

Definition 1.1.4. A set $C \subseteq E$ is an *edge cover* if every vertex of G is covered by at least one edge from C . If $\delta(G) \geq 1$, we define the *edge cover number* $\beta'(G)$ as the minimum possible cardinality of an edge cover.

Lemma 1.1.5. Let G be a graph. The following holds:

- $\alpha(G) + \beta(G) = n(G)$.
- $\alpha'(G) \leq \beta(G)$.
- $\alpha(G) \leq \beta'(G)$.
- If $\delta(G) \geq 1$, then $\alpha'(G) \leq \frac{n}{2} \leq \beta(G)$.

Theorem 1.1.6 (Gallai). If $\delta(G) \geq 1$, then $\alpha'(G) + \beta'(G) = n(G)$.

1.2 Matchings

Definition 1.2.1. Let M be a matching. A path is an *M -alternating path* if the edges along the path alternate between M and $\overline{M} = E \setminus M$.

Definition 1.2.2. An M -alternating path is called an *M -augmenting path* if both ends of the path are uncovered by M .

Proposition 1.2.3. Let G be a graph and M a matching. If there exists an M -augmenting path P , then M is not a maximum matching.

Proof. We can construct a bigger matching $M' = M \triangle E(P)$, where \triangle is the disjunctive union. \square

Theorem 1.2.4 (König). Let G be a bipartite graph. Then the following holds:

(a) $\alpha'(G) = \beta(G)$.

(a) If M is a matching with no M -augmenting path, then M is a maximum matching.

Remark 1.2.5. There also exist graphs with $\alpha'(G) = \beta(G)$ that are not bipartite.

Corollary 1.2.6. If G is a bipartite graph, then $\alpha(G) = \beta'(G)$

Proof. We have

$$\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G) = \beta'(G),$$

where the latter two equalities are due to König's and Gallai's theorem respectively. \square

Definition 1.2.7. Let G be a bipartite graph with partite classes A and B . **Hall's condition** (HC) holds for A , if

$$\forall S \subseteq A. |S| \leq |N(S)|, \quad (\text{HC})$$

where $N(S)$ is the neighbourhood of S .

Theorem 1.2.8 (Hall). Let $G = (A \cup B, E)$ be a bipartite graph. A matching that covers A exists if and only if Hall's condition holds for A .

Definition 1.2.9. A matching M is a **perfect matching** if it covers all vertices.

Corollary 1.2.10. Let G be a bipartite graph. There exists a perfect matching of G if and only if $|A| = |B|$ and A satisfies Hall's condition.

Definition 1.2.11. Let $S \subseteq A$. The **deficiency** of S is defined as $\text{def}(S) = |S| - |N(S)|$.

Theorem 1.2.12. Let $G = (A \cup B, E)$ be a bipartite graph and M a matching. Then at most

$$|A| - \max_{S \subseteq A} (\text{def}(S))$$

vertices of A are covered.

Theorem 1.2.13. If G is a regular bipartite graph, then G has a perfect matching.

Theorem 1.2.14. Let M be a matching in G . There exists an M -augmenting path in G if and only if M is not maximum.

The **Blossom algorithm** is a known algorithm in $O(n\sqrt{n})$ that finds M -augmenting paths (in polynomial time). It also provides a maximum matching, and it allows us to find $\alpha'(G)$ and $\beta'(G)$ in polynomial time.

Let $o(G)$ be the number of odd components ($|V(C)| \equiv 1 \pmod{2}$) in G .

Theorem 1.2.15. Tutte A graph G has a perfect matching if and only if

$$\forall S \subseteq V(G). |S| \geq o(G \setminus S). \quad (\text{TC})$$

The condition (TC) is called **Tutte's condition**.

Remark 1.2.16. In bipartite graphs (TC) implies (HC).

Theorem 1.2.17 (Berge-Tutte formula). A maximum matching in a graph G leaves exactly

$$\max_{S \subseteq V(G)} (o(G \setminus S) - |S|).$$

vertices uncovered. Equivalently,

$$\alpha'(G) = \frac{1}{2} \left(n - \max_{S \subseteq V(G)} \{o(G \setminus S) - |S|\} \right).$$

1.3 Factors

Definition 1.3.1. A **factor** is a spanning subgraph (subgraph that contains all vertices). A **k -factor** is a k -regular spanning subgraph.

Theorem 1.3.2 (Petersen). If G is a cubic graph with at most one bridge, then G has a 1-factor.

Theorem 1.3.3 (Petersen). If G is a cubic graph and all cut edges lie on the same path then G has a 1-factor.

Proof. Omitted. □

Example 1.3.4. [This](#) is the smallest example of a cubic graph with no 1-factor.

Theorem 1.3.5 (Petersen). Every bridgeless cubic graph decomposes into a 1-factor and 2-factor.

Theorem 1.3.6. If G is a k -regular graph and k is even, then G has a 2-factor.

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2 Connectivity

2.1 k -connectivity

Definition 2.1.1. The **connectivity number** $\kappa(G)$ is the minimum number of vertices in $S \subseteq V(G)$ such that $G - S$ is disconnected or contains only one vertex.

Remark 2.1.2. The latter conditions handles the case of complete graphs.

Definition 2.1.3. A graph G is k -connected if $\kappa(G) \geq k$.

Alternatively: G is k -connected if the removal of at most $k - 1$ vertices always results in a connected graph with at most two vertices.

Proposition 2.1.4. 1. $\kappa(G) \leq \delta(G)$

2. $\kappa(G) \leq \beta(G)$

3. $\kappa(G) \leq n(G) - 2$

Theorem 2.1.5. The minimum number of edges in a k -connected graph of order $n > k > 2$ is $\lceil nk/2 \rceil$.

Proof. Using **Harary graphs** $H_{n,k}$. □

Definition 2.1.6. A set $F \subseteq E(G)$ is a **disconnecting set** if $E - F$ is disconnected

Definition 2.1.7. Let $\emptyset \neq A \subsetneq V(G)$ and let $E(A, \overline{A})$ be the set of edges between A and \overline{A} . Then $E(A, \overline{A})$ is an **edge cut**.

Clearly, an edge cut is a disconnecting set. Similarly, a minimal disconnecting set is an edge cut.

Definition 2.1.8. The **edge-connectivity number** of G , $\kappa'(G)$ is the minimum number of edges in a disconnecting set.

Definition 2.1.9. A graph is **k -edge-connected** if the removal of less than k edges always leaves a connected graph.

Equivalently, a graph G is k -edge-connected if $k \leq \kappa'(G)$.

Theorem 2.1.10. Let G be a simple graph with $n(G) \geq 2$. Then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Corollary 2.1.11. Let G be a graph.

- If G is k -connected, then G is k -edge-connected.
- The minimum number of edges in a k -edge-connected graph on $n > k \geq 2$ vertices is $\lceil kn/2 \rceil$.

Theorem 2.1.12 (Whitney). A graph G is 2-connected if and only if for every pair of distinct vertices $u, v \in V(G)$ there exist two internally disjoint u, v -paths.

Lemma 2.1.13 (Expansion). If G is a k -connected and we add a new vertex v and k incident edges to the graph, we obtain a k -connected graph.

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Theorem 2.1.14. If G is a graph with $n(G) \geq 3$, then the following statements are equivalent.

- (a) G is 2-connected.
- (b) G is connected and there is no cut-vertex.
- (c) For every pair of distinct vertices $u, v \in V(G)$ there exist two internally disjoint u, v -paths.
- (d) Every pair of distinct vertices lies on a common cycle.
- (e) Every pair of edges lies on a common cycle and $\gamma(G) \geq 1$.

Lemma 2.1.15 (Subdivision). Let G' be a graph obtained from G by subdividing an edge. Then G is 2-connected if and only if G' is 2-connected

Definition 2.1.16. In a graph G , a path P is an *open ear* if all internal vertices of P are of degree 2 and the end vertices are of degree at least 3.

Definition 2.1.17. An *open ear decomposition* of G' is a sequence P_0, P_1, \dots, P_k such that P_0 is a cycle and for every $i \in [k]$ P_i is an open ear in the graph $G_i = P_0 \cup P_1 \cup \dots \cup P_i$ and $G_k = G$.

Theorem 2.1.18. A graph G is 2-connected if and only if it has an open ear decomposition.

Proposition 2.1.19. A graph G is 2-edge-connected if and only if it is connected and every edge of G lies in a cycle.

Definition 2.1.20. A *closed ear* in G is such a cycle that only one vertex has degree at least 4.

Definition 2.1.21. A *closed ear decomposition* of G is P_0, P_1, \dots, P_k such that P_0 is a cycle and for every $i \in [k]$ P_i is an open/closed ear in the graph $G_i = P_0 \cup P_1 \cup \dots \cup P_i$.

Theorem 2.1.22. A graph G is 2-edge-connected if and only if it has a closed ear decomposition.

A Exercises

Exercise sheet 1

1. For each of the following graphs G , determine $\alpha(G)$, $\alpha'(G)$, $\beta(G)$, and $\beta'(G)$:

(a) $G = P_n$

(b) $G = C_n$

2. Prove that $\frac{\beta(G)}{2} \leq \alpha'(G) \leq \beta(G)$ for any graph G .

3. (From lectures) Let G be a bipartite graph with bipartition $\{A, B\}$. The deficiency of the subset $U \subseteq A$ is defined as:

$$\text{def}_G(U) := |U| - |N_G(U)|,$$

and the deficiency of G is defined as:

$$\text{def}(G) := \max_{U \subseteq A} \text{def}_G(U).$$

Prove that $\alpha'(G) = |A| - \text{def}(G)$.

4. Let $F = \{F_1, \dots, F_n\}$ be a family of subsets of a set Y . Prove there are distinct elements a_1, \dots, a_n such that $a_i \in F_i$ if and only if $|F| \leq |\cup_{S \in F} S|$ for all $F \subseteq S$. (Such a set is called a system of distinct representatives for F .)
5. (Extra) Let Γ be a finite group and $H \leq \Gamma$ with index n . Let $L = \{L_i\}_{i=1}^n$ be the left cosets of H and $R = \{R_i\}_{i=1}^n$ be the right cosets of H . Prove that there is a subset $\{h_1, \dots, h_n\} \subseteq G$ such that $L = \{h_i H\}_{i=1}^n$ and $R = \{H h_i\}_{i=1}^n$.

Exercise sheet 2

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1. For each of the following graphs G , determine the number of maximum matchings:
 - (a) $G = K_n$
 - (b) $G = K_{a,b}, a \leq b$
2. Let G be a graph such that $\delta(G) \geq 1$, with maximal matching M and minimal edge cover C . Prove the following equivalences:
 - (a) M is a maximum matching if and only if M is contained in a minimum edge cover.
 - (b) C is a minimum edge cover if and only if C contains a maximum matching.
3. Use deficiency (see sheet 1) to prove the König-Egerváry theorem. (Hint: find a matching and a vertex cover of the same size using a subset of maximum deficiency).
4. (From lectures) Let G be a bipartite graph with bipartition $\{A, B\}$ such that $|A| = |B|$. Prove that for A , Hall's condition holds if and only if Tutte's condition holds.
5. Prove Theorem 1.2.17 (Berge-Tutte formula).

B Homework

1. Use König's Theorem to prove that every bipartite graph G has a matching of size $e(G)/\Delta(G)$. Use this to show that any subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k .
2. Let G be a graph with $\delta(G) \geq k$ and $n(G) \geq 2k$. Prove that $\alpha'(G) \geq k$ (Hint: you can use the Berge-Tutte formula).
3. Let G be an r -connected graph of even order, and which contains no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor.
4. A trail in a graph G is a list of vertices and edges $v_0, e_1, v_1, e_2, \dots, e_{k-1}, v_k$ such that $e_i = v_{i-1}v_i$ and the edges e_i are pairwise distinct.

Let G be a graph with $n(G) \geq 3$. Prove the following are equivalent:

- (a) G is 2-edge-connected.
 - (b) Every edge appears in a cycle.
 - (c) G has a closed trail containing any specified pair of edges.
 - (d) G has a closed trail containing any specified pair of vertices.
5. Let G be a simple graph. Prove that $\chi(G) + \chi(G^c) \leq n(G) + 1$.
 6. A graph G with $\chi(G) = k$ is called colour-critical, or k -critical, if $\chi(H) < k$ for all proper subgraphs H of G .

Prove the following:

- (a) Every k -chromatic graph has a k -critical subgraph.
- (b) A graph G with no isolated vertices is colour-critical if and only if $\chi(G - e) < \chi(G)$ for all $e \in E(G)$.
- (c) Let G and H be k -critical graphs sharing exactly 1 vertex v , and with edges $vu \in E(G)$, $vw \in E(H)$. Prove that $(G - uv) \cup (H - vw) \cup uw$ is k -critical.
- (d) For all $k \geq 3$, use (c) to obtain a k -critical graph other than K_k .
- (e) For $n \geq 4$, $n \neq 5$, construct a 4-critical graph with n vertices.

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