Weighted Finite Automata and Noncommutative Rational Series

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1 Weighted finite automata, linear representations, rational series

1.1 Weighted finite automata

Definition 1.1.1. Let K be a semiring and A an alphabet.

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(1) A ***weighted (finite) automaton*** (WFA) with weights in K is a tuple (Q, I, E, T) consisting of a finite set Q of ***states***, and maps $I\colon Q \to K$ (***initial weights***), $E\colon Q \times A \times Q \to K$ (***transition function***), $T\colon Q \to K$ (***terminal weights***).

(2) A triple (p, a, q) with $E(p, a, q) \neq 0$ is an ***edge/transition*** with ***label*** a, ***starting state*** p, ***ending state*** q and ***weight*** $E(p, a, q) \in K$.

(3) A ***path/run*** is a sequence of edges $c = (q, a_1, q_1)(q_1, a_2, p_2) \ldots (q_{n-1}, a, q_n)$. Its ***weight*** is

$$E(q, a_1, q_1) \cdot E(q_1, a_2, p_2) \cdots E(q_{n-1}, a_n, q_n)$$

and its ***label*** is $w = a_1 a_2 \ldots a_n \in A^*$.

(4) The ***behaviour*** of $\mathcal{A}$ is the series $[[\mathcal{A}]] \in K\langle\langle A\rangle\rangle$ defined by

$$([[\mathcal{A}]], w) = \sum_{q_0, \ldots, q_n \in Q} I(q_0) E(q, a_1, q_1) E(q_1, a_2, p_2) \cdots E(q_{n-1}, a_n, q_n) T(q_n),$$

where $w = a_1 \cdots a_n$, $a_i \in A$.

Definition 1.1.2. Terminology:

1. A state q is ***initial*** if $I(q) \neq 0$ and ***terminal*** if $T(q) \neq 0$.
2. A ***successful run/accepting run*** is a run from an initial state to a terminal state.

1.2 Linear representation, recognizable series

We wish to represent the data in a weighted finite automaton using adjacency matrices.

Definition 1.2.1. Let K be a semiring and A an alphabet.

(1) A series $S \in K\langle\langle A\rangle\rangle$ is ***(K-)recognizable*** if there exist $n \geq 0$, $\lambda \in K^{1\times b}$, $\gamma \in K^{n\times 1}$, and a monoid morphism $\mu\colon A^* \to K^{d\times d}$ such that for every $w \in A^*$ we have

$$(S, w) = \lambda \mu(w) \gamma.$$

(2) The triple (λ, μ, γ) is a ***linear representation*** of S with ***dimension*** n.

Proposition 1.2.2. A series $S \in K\langle\langle A\rangle\rangle$ is recognizable if and only if there exists a WFA $\mathcal{A}$ such that $S = [[\mathcal{A}]]$.

Proof. Suppose $S = [[\mathcal{A}]]$ with $\mathcal{A} = (Q, I, E, T)$. Without loss of generality let $Q = \{1, \ldots, n\}$. Let

$$\lambda := \begin{bmatrix} I(1) & \ldots & I(n) \end{bmatrix}, \quad \gamma := \begin{bmatrix} T(1) & \vdots & T(n) \end{bmatrix}^\top,$$

and for every $a \in A$ let

$$\mu(a) := \begin{bmatrix} E(1,a,1) & \ldots & E(1,a,n) \\ \vdots & & \vdots \\ E(1,a,n) & \ldots & E(n,a,n) \end{bmatrix}.$$

This extends to a morphism

$$\mu \colon A^* \to K^{d \times d},$$
$$\mu(a_1 \cdots a_n) = \mu(a_1)\mu(a_2)\cdots\mu(a_n).$$

Then, for $p, q \in Q$, $w = a_1 \cdots a_n$, we have

$$\begin{aligned} u(w)_{p,q} &= [\mu(a_1)\cdots\mu(a_n)]_{p,q} \\ &= \sum_{p_1,\ldots,p_{m-1}=1}^{n} \mu(a_1)_{p,p_1}\mu(a_2)_{p_1,p_2}\cdots\mu(a_m)_{p_{m-1},q} \end{aligned}$$

and

$$\begin{aligned} \lambda\mu(w)\gamma &= \sum_{p,p_1,\ldots,p_{m-1},q=1}^{n} \lambda_p\mu(a_1)_{p,p_1}\mu(a_2)_{p_1,p_2}\cdots\mu(a_m)_{p_{m-1},q}\gamma_q \\ &= \sum_{p,p_1,\ldots,p_{m-1},q=1}^{n} I(p)E(p,a_1,p_1)E(p_1,a_2,p_2)\cdots E(p_{m-1},a_m,q)T(q) \\ &= ([[\mathcal{A}]], w). \end{aligned}$$

Conversely, let (λ, μ, γ) be a linear representation recognizing S. Let $Q := \{1, \ldots, n\}$, $I(p) := \lambda_p$, $T(q) := \gamma_q$, $E(p, a, q) := \mu(a)_p q$. The computation above shows that S is the behaviour of (Q, I, E, T). □

1.3 Model-theoretic characterization

Lemma 1.3.1. Let K be a semiring and A an alphabet.

(1) For $x \in A^*$, the map $S \mapsto x^{-1}X$ is a K-module morphism.

(2) For every $x, y \in A^*$ and for every $S \in K\langle\langle A \rangle\rangle$ we have $(xy)^{-1}S = y^{-1}(x^{-1}S)$.

Proof. Left as an exercise. □

Definition 1.3.2. A submodule $M \subseteq K\langle\langle A \rangle\rangle$ is ***stable*** if for every $S \in M$ and every $x \in A^*$ we have $x^{-1}S \subseteq M$ (equivalently for every $a \in A$ we have $a^{-1}M \subseteq M$).

Theorem 1.3.3. A series $S \in K\langle\langle A \rangle\rangle$ is recognizable if and only if there exists a stable finitely generated left K-submodule $M \subseteq K\langle\langle A \rangle\rangle$ such that $S \in M$.

Proof. Next lecture. □

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