Expander Graphs

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1 Definitions

1.1 Expansion

Definition 1.1.1. Let $P = (V, E)$ be a finite graph.

(1) For any disjoint subsets of vertives $V_1, V_2 \subseteq V$, we denote by $\mathcal{E}(V_1, V_2)$ the set of edges of Γ with one extremity in V_1 and the other in V_2. We denote $\mathcal{E}(V_1) := \mathcal{E}(V_1, V \setminus V_1)$.

(2) The ***Cheeger constant*** or ***expansion constant*** of Γ is

$$h(\Gamma) := \min \left\{ \frac{|\mathcal{E}(W)|}{|W|} \;\middle|\; \emptyset \neq W \subseteq V \wedge |W| \leq \frac{1}{2} |V| \right\}$$

with the convention that $h(\Gamma) = +\infty$ if Γ has at most one vertex.

The larger $h(P)$ is, the more difficult it is to disconnect a large subset of V from the rest of the graph. This will be our way of measuring "high connectivity".

Lemma 1.1.2. Let Γ be a finite graph with at least two vertices (so that $h(\Gamma) < \infty$). We have $h(\Gamma) > 0$ if and only if Γ is connected.

Proof. The condition $h(\Gamma) = 0$ means that there exists some nonempty $W \subseteq V$, $|W| \leq \frac{1}{2} |V|$, such that $|\mathcal{E}| (W) = 0$. Since $W \neq V$, there is no path between W and $V \setminus W$, so Γ is disconnected.

Conversely, if Γ is disjoint, there is some componenet, say W, of size at most $\frac{1}{2} |V|$. Hence, $\mathcal{E}(W) = \emptyset$ and $h(\Gamma) = 0$. $\square$

Index