Graph Theory

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1 Independence, matching, covers

1.1 Definitions

Let $G = (V, E)$ be a graph.

Definition 1.1.1. A set $S \subseteq V$ is an ***independent set*** if $G[S]$ contains no edges. The ***independence number*** $\alpha(G)$ is the maximum possible cardinality of an independent set.

Definition 1.1.2. A set $T \subseteq V$ is a ***vertex cover*** if

$$\forall e \in E.\ T \cap e \neq \emptyset.$$

The ***vertex cover number*** $\beta(G)$ is the minimum possible cardinality of a vertex cover.

Definition 1.1.3. A set $M \subseteq E$ is a ***matching*** if

$$\forall e, f \in M.\ e \neq f \Rightarrow e \cap f = \emptyset.$$

The ***matching number*** $\alpha'(G)$ is the maximum possible cardinality of a matching.

Definition 1.1.4. A set $C \subseteq E$ is an edge cover if every vertex of G is covered by at least one edge from C. If $\delta(G) \geq 1$, we define the ***edge cover number*** $\beta'(G)$ as the minimum possible cardinality of an edge cover.

Lemma 1.1.5. Let G be a graph. The following holds:

- $\alpha(G) + \beta(G) = n(G)$.
- $\alpha'(G) \leq \beta(G)$.
- $\alpha(G) \leq \beta'(G)$.
- If $\delta(G) \geq 1$, then $\alpha'(G) \leq \frac{n}{2} \leq \beta(G)$.

Proof. TO DO □

Theorem 1.1.6 (Gallai)**.** If $\delta(G) \geq 1$, then $\alpha'(G) + \beta'(G) = n(G)$.

Proof. TO DO □

Definition 1.1.7. Let M be a matching. A path is an M***-alternating path*** if the edges along the path alternate between M and $\overline{M} = E \setminus M$.

Definition 1.1.8. An M-alternating path is called an M***-augmenting path*** if both ends of the path are uncovered by M.

Proposition 1.1.9. Let G be a graph and M a matching. If there exists an M-augmenting path P, then M is not a maximum matching.

Proof. We can construct a bigger matching $M' = M \triangle E(P)$, where $\triangle$ is the disjunctive union. □

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Theorem 1.1.10 (König)**.** Let G be a bipartite graph. Then the following holds:

(a) $\alpha'(G) = \beta(G)$.

(a) If M is a matching witn no M-augmenting path, then M is a maximum matching.

Proof. TO DO □

Remark 1.1.10.1. There also exist graphs with $\alpha'(G) = \beta(G)$ that are not bipartite.

Corollary 1.1.11. If G is a bipartite graph, then $\alpha(G) = \beta'(G)$

Proof. We have

$$\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G) = \beta'(G),$$

where the latter two equalities are due to König's and Gallai's theorem respectively. □

Definition 1.1.12. Let G be a bipartite graph with partite classes A and B. ***Hall's condition*** (HC) holds for A, if

$$\forall S \subseteq A.\ |S| \leq |N(S)|\,,$$

where $N(S)$ is the neighbourhood of S.

Theorem 1.1.13 (Hall)**.** Let $G = (A \cup B, E)$ be a bipartite graph. A matching that covers A exists if and only if Hall's condition holds for A.

Proof. TO DO □

Definition 1.1.14. A matching M is a ***perfect matching*** if it covers all vertices.

Corollary 1.1.15. Let G be a bipartite graph. There exists a perfect matching of G if and only if $|A| = |B|$ and A satisfies Hall's condition.

Definition 1.1.16. Let $S \subseteq A$. The ***deficiency*** of S is defined as $\mathrm{def}(S) = |S| - |N(S)|$.

Theorem 1.1.17. Let $G = (A \cup B, E)$ be a bipartite graph and M a matching. Then at most

$$|A| - \max_{S \subseteq A}(\mathrm{def}(S))$$

vertices of A are covered.

Proof. Left as an exercise. □

Theorem 1.1.18. If G is a regular bipartite graph, then G has a perfect matching.

Proof. TO DO □

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