Weighted Finite Automata and Noncommutative Rational Series

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1 WFA, Linear Representations, Rational Series

1.1 Weighted Finite Automata

Definition 1.1.1. Let K be a semiring and A an alphabet.

(1) A ***weighted (finite) automaton*** (WFA) with weights in K is a tuple (Q, I, E, T) consisting of a finite set Q of ***states***, and maps $I\colon Q \to K$ (***initial weights***), $E\colon Q \times A \times Q \to K$ (***transition function***), $T\colon Q \to K$ (***terminal weights***).

(2) A triple (p, a, q) with $E(p, a, q) \neq 0$ is an ***edge/transition*** with ***label*** a, ***starting state*** p, ***ending state*** q and ***weight*** $E(p, a, q) \in K$.

(3) A ***path/run*** is a sequence of edges $c = (q, a_1, q_1)(q_1, a_2, p_2) \dots (q_{n-1}, a, q_n)$. Its ***weight*** is

$$E(q, a_1, q_1) \cdot E(q_1, a_2, p_2) \cdots E(q_{n-1}, a_n, q_n)$$

and its ***label*** is $w = a_1 a_2 \dots a_n \in A^*$.

(4) The ***behaviour*** of $\mathcal{A}$ is the series $[[\mathcal{A}]] \in K\left\langle\langle A \rangle\right\rangle$ defined by

$$([[\mathcal{A}]], w) = \sum_{q_0, \dots, q_n \in Q} I(q_0) E(q, a_1, q_1) E(q_1, a_2, p_2) \cdots E(q_{n-1}, a_n, q_n) T(q_n),$$

where $w = a_1 \cdots a_n$, $a_i \in A$.

Definition 1.1.2. Terminology:

1. A state q is ***initial*** if $I(q) \neq 0$ and ***terminal*** if $T(q) \neq 0$.

2. A ***successful run/accepting run*** is a run from an initial state to a terminal state.

1.2 Linear representation, recognizable series

Definition 1.2.1. Let K be a semiring and A an alphabet.

(1) A series $S \in K\left\langle\langle A \rangle\right\rangle$ is ***(K-)recognizable*** if there exist $n \geq 0$, $\lambda \in K^{1 \times b}$, $\gamma \in K^{n \times 1}$, and a monoid morphism $\mu\colon A^* \to K^{d \times d}$ such that for every $w \in A^*$ we have

$$(S, w) = \lambda \mu(w) \gamma.$$

(2) The triple (λ, μ, γ) is a ***linear representation*** of S with ***dimension*** n.

Proposition 1.2.2. A series $S \in K\left\langle\langle A \rangle\right\rangle$ is recognizable if and only if there exists a WFA $\mathcal{A}$ such that $S = [[\mathcal{A}]]$.

Proof. □

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1.3 Model-theoretic Characterization

Lemma 1.3.1. Let K be a semiring and A an alphabet.

(1) For $x \in A^*$, the map $S \mapsto x^{-1}X$ is a K-module morphism.

(2) For every $x, y \in A^*$ and for every $S \in K\langle\langle A \rangle\rangle$ we have $(xy)^{-1}S = y^{-1}(x^{-1}S)$.

Proof. Left as an exercise. □

Definition 1.3.2. A submodule $M \subseteq K\langle\langle A \rangle\rangle$ is ***stable*** if for every $S \in M$ and every $x \in A^*$ we have $x^{-1}S \subseteq M$ (equivalently for every $a \in A$ we have $a^{-1}M \subseteq M$).

Theorem 1.3.3. A series $S \in K\langle\langle A \rangle\rangle$ is recognizable if and only if there exists a stable finitely generated left submodule $M \subseteq K\langle\langle A \rangle\rangle$ such that $S \in M$.

Proof. □

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