

# Graph Theory

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# 1 Independence, matching, covers

## 1.1 Definitions

Let  $G = (V, E)$  be a graph.

**Definition 1.1.1.** A set  $S \subseteq V$  is an *independent set* if  $G[S]$  contains no edges. The *independence number*  $\alpha(G)$  is the maximum possible cardinality of an independent set.

**Definition 1.1.2.** A set  $T \subseteq V$  is a *vertex cover* if

$$\forall e \in E. T \cap e \neq \emptyset.$$

The *vertex cover number*  $\beta(G)$  is the minimum possible cardinality of a vertex cover.

**Definition 1.1.3.** A set  $M \subseteq E$  is a *matching* if

$$\forall e, f \in M. e \neq f \Rightarrow e \cap f = \emptyset.$$

The *matching number*  $\alpha'(G)$  is the maximum possible cardinality of a matching.

**Definition 1.1.4.** A set  $C \subseteq E$  is an edge cover if every vertex of  $G$  is covered by at least one edge from  $C$ . If  $\delta(G) \geq 1$ , we define the *edge cover number*  $\beta'(G)$  as the minimum possible cardinality of an edge cover.

**Lemma 1.1.5.** Let  $G$  be a graph. The following holds:

- $\alpha(G) + \beta(G) = n(G)$ .
- $\alpha'(G) \leq \beta(G)$ .
- $\alpha(G) \leq \beta'(G)$ .
- If  $\delta(G) \geq 1$ , then  $\alpha'(G) \leq \frac{n}{2} \leq \beta(G)$ .

**Theorem 1.1.6** (Gallai). If  $\delta(G) \geq 1$ , then  $\alpha'(G) + \beta'(G) = n(G)$ .

**Definition 1.1.7.** Let  $M$  be a matching. A path is an  *$M$ -alternating path* if the edges along the path alternate between  $M$  and  $\overline{M} = E \setminus M$ .

**Definition 1.1.8.** An  $M$ -alternating path is called an  *$M$ -augmenting path* if both ends of the path are uncovered by  $M$ .

**Proposition 1.1.9.** Let  $G$  be a graph and  $M$  a matching. If there exists an  $M$ -augmenting path  $P$ , then  $M$  is not a maximum matching.

*Proof.* We can construct a bigger matching  $M' = M \triangle E(P)$ , where  $\triangle$  is the disjunctive union.  $\square$

**Theorem 1.1.10** (König). Let  $G$  be a bipartite graph. Then the following holds:

- (a)  $\alpha'(G) = \beta(G)$ .
- (a) If  $M$  is a matching with no  $M$ -augmenting path, then  $M$  is a maximum matching.

**Remark 1.1.11.** There also exist graphs with  $\alpha'(G) = \beta(G)$  that are not bipartite.

**Corollary 1.1.12.** If  $G$  is a bipartite graph, then  $\alpha(G) = \beta'(G)$

*Proof.* We have

$$\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G) = \beta'(G),$$

where the latter two equalities are due to König's and Gallai's theorem respectively.  $\square$

**Definition 1.1.13.** Let  $G$  be a bipartite graph with partite classes  $A$  and  $B$ . **Hall's condition** (HC) holds for  $A$ , if

$$\forall S \subseteq A. |S| \leq |N(S)|, \quad (\text{HC})$$

where  $N(S)$  is the neighbourhood of  $S$ .

**Theorem 1.1.14** (Hall). Let  $G = (A \cup B, E)$  be a bipartite graph. A matching that covers  $A$  exists if and only if Hall's condition holds for  $A$ .

**Definition 1.1.15.** A matching  $M$  is a **perfect matching** if it covers all vertices.

**Corollary 1.1.16.** Let  $G$  be a bipartite graph. There exists a perfect matching of  $G$  if and only if  $|A| = |B|$  and  $A$  satisfies Hall's condition.

**Definition 1.1.17.** Let  $S \subseteq A$ . The **deficiency** of  $S$  is defined as  $\text{def}(S) = |S| - |N(S)|$ .

**Theorem 1.1.18.** Let  $G = (A \cup B, E)$  be a bipartite graph and  $M$  a matching. Then at most

$$|A| - \max_{S \subseteq A} (\text{def}(S))$$

vertices of  $A$  are covered.

**Theorem 1.1.19.** If  $G$  is a regular bipartite graph, then  $G$  has a perfect matching.

**Theorem 1.1.20.** Let  $M$  be a matching in  $G$ . There exists an  $M$ -augmenting path in  $G$  if and only if  $M$  is not maximum.

The **Blossom algorithm** is a known algorithm in  $O(n\sqrt{n})$  that finds  $M$ -augmenting paths (in polynomial time). It also provides a maximum matching, and it allows us to find  $\alpha'(G)$  and  $\beta'(G)$  in polynomial time.

Let  $o(G)$  be the number of odd components ( $|V(C)| \equiv 1 \pmod{2}$ ) in  $G$ .

**Theorem 1.1.21.** Tutte A graph  $G$  has a perfect matching if and only if

$$\forall S \subseteq V(G). |S| \geq o(G \setminus S). \quad (\text{TC})$$

The condition (TC) is called **Tutte's condition**.

**Remark 1.1.22.** In bipartite graphs (TC) implies (HC).

**Theorem 1.1.23** (Berge-Tutte formula). A maximum matching in a graph  $G$  leaves exactly

$$\max_{S \subseteq V(G)} (o(G \setminus S) - |S|).$$



# A Exercises

## Exercise sheet 1

1. For each of the following graphs  $G$ , determine  $\alpha(G)$ ,  $\alpha'(G)$ ,  $\beta(G)$ , and  $\beta'(G)$ :

(a)  $G = P_n$

(b)  $G = C_n$

2. Prove that  $\frac{\beta(G)}{2} \leq \alpha'(G) \leq \beta(G)$  for any graph  $G$ .

3. (From lectures) Let  $G$  be a bipartite graph with bipartition  $\{A, B\}$ . The deficiency of the subset  $U \subseteq A$  is defined as:

$$\text{def}_G(U) := |U| - |N_G(U)|,$$

and the deficiency of  $G$  is defined as:

$$\text{def}(G) := \max_{U \subseteq A} \text{def}_G(U).$$

Prove that  $\alpha'(G) = |A| - \text{def}(G)$ .

4. Let  $F = \{F_1, \dots, F_n\}$  be a family of subsets of a set  $Y$ . Prove there are distinct elements  $a_1, \dots, a_n$  such that  $a_i \in F_i$  if and only if  $|F| \leq |\cup_{S \in F} S|$  for all  $F \subseteq S$ . (Such a set is called a system of distinct representatives for  $F$ .)
5. (Extra) Let  $\Gamma$  be a finite group and  $H \leq \Gamma$  with index  $n$ . Let  $L = \{L_i\}_{i=1}^n$  be the left cosets of  $H$  and  $R = \{R_i\}_{i=1}^n$  be the right cosets of  $H$ . Prove that there is a subset  $\{h_1, \dots, h_n\} \subseteq G$  such that  $L = \{h_i H\}_{i=1}^n$  and  $R = \{H h_i\}_{i=1}^n$ .

## Exercise sheet 2

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1. For each of the following graphs  $G$ , determine the number of maximum matchings:

(a)  $G = K_n$

(b)  $G = K_{a,b}, a \leq b$

2. Let  $G$  be a graph such that  $\delta(G) \geq 1$ , with maximal matching  $M$  and minimal edge cover  $C$ . Prove the following equivalences:

(a)  $M$  is a maximum matching if and only if  $M$  is contained in a minimum edge cover.

(b)  $C$  is a minimum edge cover if and only if  $C$  contains a maximum matching.

3. Use deficiency (see sheet 1) to prove the König-Egerváry theorem. (Hint: find a matching and a vertex cover of the same size using a subset of maximum deficiency).

4. (From lectures) Let  $G$  be a bipartite graph with bipartition  $\{A, B\}$  such that  $|A| = |B|$ . Prove that for  $A$ , Hall's condition holds if and only if Tutte's condition holds.

5. (From lectures, the Berge-Tutte formula) Prove that a maximum matching in a graph  $G$  leaves exactly

$$\max_{S \subseteq V(G)} (o(G \setminus S) - |S|).$$

vertices uncovered. Equivalently,

$$\alpha'(G) = \frac{1}{2} \left( n - \max_{S \subseteq V(G)} \{o(G \setminus S) - |S|\} \right).$$

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