Expander Graphs

List of exercises

All the exercises are taken from An Introduction to Expander Graphs by E. Kowalski (the reference to the book is in brackets).

2 Graphs

Exercise 2.1 (Exercise 2.1.3) Show that if $\Gamma = (V, E, \epsilon)$ is a finite graph, then

$$\sum_{x \in V} \mathrm{val}(x) = 2|E_2| + |E_1|$$

where $E_i = \{\alpha \in E \mid \alpha \text{ has } i \text{ extremities}\}$, i.e., $|E_1|$ is the number of loops and $|E_2|$ is the number of edges joining distinct vertices.

Exercise 2.2 (Exercise 2.1.7) Write the adjacency matrices of the following graphs:

(i) Cycles with 4 and 5 vertices.

(ii) Paths with 4 and 5 vertices.

(iii) Complete graphs with 3, 4, and 5 vertices.

(iv) Cayley graph of the symmetric group S_3 with the generating sets $\{(12), (13)\}$ and $\{(12), (13), (23)\}$.

Exercise 2.3 (Exercise 2.1.19)

(i) Let Γ be a finite d-regular graph with girth $g \geq 3$. Prove that

$$|\Gamma| \geq d(d-1)^{\lfloor \frac{g-3}{2} \rfloor}.$$

(ii) Show that the girth of a finite d-regular graph Γ with $d \geq 3$ is $\ll \log(|\Gamma|)$, where the implied constant depends only on d.

Exercise 2.4 (Exercise 2.1.19) Show that the number of vertices and edges of the finite tree $T_{d,k}$ are given by

$$|T_{d,k}| = d\frac{(d-1)^k - 1}{d-2} + 1, \quad |E_{d,k}| = |T_{d,k}| - 1 = d\frac{(d-1)^k - 1}{d-2}$$

if $d \geq 3$, and $|T_{2,k}| = 2k + 1$, $|E_{2,k}| = 2k$.

Exercise 2.5 (Forests and Trees (Exercise 2.2.7)) Recall that forests are graphs of infinite girth, and trees are connected forests.

(i) Show that the diameter of a finite tree $T_{d,k}$ with $d \geq 2$ and $k \geq 0$ is equal to $2k$, and is achieved by the distance between any two distinct vertices labeled with words $(s_1, \ldots, s_k)$ and $(s'_1, \ldots, s'_k)$ of (maximal) length with $s_1 \neq s'_1$.

(ii) Show that if T is a tree, then for any two vertices x and y, there exists a unique geodesic on T with endpoints x and y (the image of all paths of length $d_T(x, y)$ between two vertices x and y in T is the same).

(iii) If $T = T_{d,k}$ with the "root" vertex $x_0 = \emptyset$ and $0 \leq j \leq k$, show that

$$V' = \{x \in V_T \mid d_T(x_0, x) \leq j\}$$

induces a full subgraph isomorphic to $T_{d,j}$.

(iv) If $T = T_{d,k}$ with root x_0 and $x \in T$ is any vertex, show that

$$V'' = \{y \in V_T \mid d_T(y, x_0) \geq d_T(y, x)\}$$

induces a full subgraph T'' of T which is also a tree.

(v) Let Γ be any graph with girth $\ell \geq 1$, and let $x_0 \in V$. Show that the subgraph of Γ induced by

$$V' = \{x \in V \mid d_\Gamma(x_0, x) < \frac{\ell}{2}\}$$

is a tree.

Exercise 2.6 (Exercise 2.2.9) Let $k \geq 2$ be an integer, and let $G = S_k$ be the symmetric group on k elements. Suppose we are given a subgroup H of G such that:

- H acts transitively on $\{1, \ldots, k\}$;
- H contains at least one transposition;
- H contains a cycle σ of length $p > k/2$ such that p is prime.

Prove that $H = G = S_k$.

Let $\Gamma = (V, E)$ be the simple graph with $V = \{1, \ldots, k\}$ and an edge between any pair $(i, j) \in V \times V$ such that $i \neq j$ and the transposition (ij) is in H. The second assumption ensures that the edge set is not empty.

(i) Show that any connected component in Γ is a complete graph.

(ii) Show that it is enough to prove that Γ is connected to conclude that $H = G$.

(iii) Show that the action of G on $\{1, \ldots, k\}$ induces an action of G on Γ by automorphisms. Then show that G acts transitively on the set of all connected components of Γ. Deduce that all such components are isomorphic.

(iv) Show that the p-cycle $\sigma \in H$ must fix (globally, not necessarily pointwise) each component of Γ, and conclude from this.