Graph Theory

Jan Pantner (jan.pantner@gmail.com)

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1 Independence, matching, covers

1.1 Definitions

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Let $G = (V, E)$ be a graph.

Definition 1.1.1. A set $S \subseteq V$ is an ***independent set*** if $G[S]$ contains no edges. The ***independence number*** $\alpha(G)$ is the maximum possible cardinality of an independent set.

Definition 1.1.2. A set $T \subseteq V$ is a ***vertex cover*** if

$$\forall e \in E.\ T \cap e \neq \emptyset.$$

The ***vertex cover number*** $\beta(G)$ is the minimum possible cardinality of a vertex cover.

Definition 1.1.3. A set $M \subseteq E$ is a ***matching*** if

$$\forall e, f \in M.\ e \neq f \Rightarrow e \cap f = \emptyset.$$

The ***matching number*** $\alpha'(G)$ is the maximum possible cardinality of a matching.

Definition 1.1.4. A set $C \subseteq E$ is an edge cover if every vertex of G is covered by at least one edge from C. If $\delta(G) \geq 1$, we define the ***edge cover number*** $\beta'(G)$ as the minimum possible cardinality of an edge cover.

Lemma 1.1.5. Let G be a graph. The following holds:

- $\alpha(G) + \beta(G) = n(G)$.
- $\alpha'(G) \leq \beta(G)$.
- $\alpha(G) \leq \beta'(G)$.
- If $\delta(G) \geq 1$, then $\alpha'(G) \leq \frac{n}{2} \leq \beta(G)$.

Theorem 1.1.6 (Gallai)**.** If $\delta(G) \geq 1$, then $\alpha'(G) + \beta'(G) = n(G)$.

Definition 1.1.7. Let M be a matching. A path is an ***M-alternating path*** if the edges along the path alternate between M and $\overline{M} = E \setminus M$.

Definition 1.1.8. An M-alternating path is called an ***M-augmenting path*** if both ends of the path are uncovered by M.

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Proposition 1.1.9. Let G be a graph and M a matching. If there exists an M-augmenting path P, then M is not a maximum matching.

Proof. We can construct a bigger matching $M' = M \triangle E(P)$, where $\triangle$ is the disjunctive union. $\square$

Theorem 1.1.10 (König)**.** Let G be a bipartite graph. Then the following holds:

(a) $\alpha'(G) = \beta(G)$.

(a) If M is a matching witn no M-augmenting path, then M is a maximum matching.

Remark 1.1.11. There also exist graphs with $\alpha'(G) = \beta(G)$ that are not bipartite.

Corollary 1.1.12. If G is a bipartite graph, then $\alpha(G) = \beta'(G)$

Proof. We have
$$\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G) = \beta'(G),$$
where the latter two equalities are due to König's and Gallai's theorem respectively. $\square$

Definition 1.1.13. Let G be a bipartite graph with partite classes A and B. ***Hall's condition*** (HC) holds for A, if
$$\forall S \subseteq A.\ |S| \leq |N(S)|\,, \tag{HC}$$
where $N(S)$ is the neighbourhood of S.

Theorem 1.1.14 (Hall)**.** Let $G = (A \cup B, E)$ be a bipartite graph. A matching that covers A exists if and only if Hall's condition holds for A.

Definition 1.1.15. A matching M is a ***perfect matching*** if it covers all vertices.

Corollary 1.1.16. Let G be a bipartite graph. There exists a perfect matching of G if and only if $|A| = |B|$ and A satisfies Hall's condition.

Definition 1.1.17. Let $S \subseteq A$. The ***deficiency*** of S is defined as $\operatorname{def}(S) = |S| - |N(S)|$.

Theorem 1.1.18. Let $G = (A \cup B, E)$ be a bipartite graph and M a matching. Then at most
$$|A| - \max_{S \subseteq A}(\operatorname{def}(S))$$
vertices of A are covered.

Theorem 1.1.19. If G is a regular bipartite graph, then G has a perfect matching.

Theorem 1.1.20. Let M be a matching in G. There exists an M-augmenting path in G if and only of M is not maximum.

The Blossom algorithm is a known algorithm in $O(n\sqrt{n})$ that finds M-augmenting paths (in polynomial time). It also provides a maximum matching, and it allows us to find $\alpha'(G)$ and $\beta'(G)$ in polynomial time.

Let $o(G)$ be the number of odd components ($|V(C)| \equiv 1 \pmod 2$) in G.

Theorem 1.1.21. Tutte A graph G has a perfect matching if and only if
$$\forall S \subseteq V(G).\ |S| \geq o(G \setminus S). \tag{TC}$$

The condition (TC) is called ***Tutte's condition***.

Remark 1.1.22. In bipartite graphs (TC) implies (HC).

Theorem 1.1.23 (Berge-Tutte formula)**.** A maximum matching in a graph G leaves exactly
$$\max_{S \subseteq V(G)} \left(o(G \setminus S) - |S|\right).$$

vertices uncovered. Equivalently,

$$\alpha'(G) = \frac{1}{2}\left(n - \max_{S \subseteq V(G)} \{o(G \setminus S) - |S|\}\right).$$

1.2 Factors

Definition 1.2.1. A ***factor*** is a spanning subgraph (subgraph that contains all vertices). A k***-factor*** is a k-regular spanning subgraph.

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Theorem 1.2.2 (Petersen)**.** If G is a cubic graph witha at most one bridge, then G has a 1-factor.

Theorem 1.2.3 (Petersen)**.** If G is a cubic graph and all cut edges lie on the same path then G has a 1-factor.

Proof. Omitted. □

Example 1.2.4. This is the smallest example of a cubic graph with no 1-factor.

Theorem 1.2.5 (Petersen)**.** Every bridgeless cubic graph decomposes into a 1-factor and 2-factor.

Theorem 1.2.6. If G is a k-regular graph and k is even, then G has a 2-factor.

2 Connectivity

2.1 k-connectivity

Definition 2.1.1. The ***connectivity number*** $\kappa(G)$ is the minimum number of vertices in $S \subseteq V(G)$ such that $G - S$ is disconnected or contains only one vertex.

Remark 2.1.2. The latter conditions handles the case of complete graphs.

Definition 2.1.3. A graph G is k-connected if $\kappa(G) \geq k$.

Alternatively: G is k-connected if the removal of at most $k - 1$ vertices always results in a connected graph with at most two vertices.

Proposition 2.1.4.

1. $\kappa(G) \leq \delta(G)$
2. $\kappa(G) \leq \beta(G)$
3. $\kappa(G) \leq n(G) - 2$

Theorem 2.1.5. The minimum number of edges in a k-connected graph of order $n > k > 2$ is $\lceil nk/2 \rceil$.

Proof. Using ***Harary graphs*** $H_{n,k}$. □

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A Exercises

Exercise sheet 1

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1. For each of the following graphs G, determine $\alpha(G)$, $\alpha'(G)$, $\beta(G)$, and $\beta'(G)$:

   (a) $G = P_n$

   (b) $G = C_n$

2. Prove that $\frac{\beta(G)}{2} \leq \alpha'(G) \leq \beta(G)$ for any graph G.

3. (From lectures) Let G be a bipartite graph with bipartition $\{A, B\}$. The deficiency of the subset $U \subseteq A$ is defined as:

$$\mathrm{def}_G(U) := |U| - |N_G(U)|,$$

   and the deficiency of G is defined as:

$$\mathrm{def}(G) := \max_{U \subseteq A} \mathrm{def}_G(U).$$

   Prove that $\alpha'(G) = |A| - \mathrm{def}(G)$.

4. Let $F = \{F_1, \ldots, F_n\}$ be a family of subsets of a set Y. Prove there are distinct elements $a_1, \ldots, a_n$ such that $a_i \in F_i$ if and only if $|F| \leq |\cup_{S \in F} S|$ for all $F \subseteq S$. (Such a set is called a system of distinct representatives for F.)

5. (Extra) Let Γ be a finite group and $H \leq \Gamma$ with index n. Let $L = \{L_i\}_{i=1}^n$ be the left cosets of H and $R = \{R_i\}_{i=1}^n$ be the right cosets of H. Prove that there is a subset $\{h_1, \ldots, h_n\} \subseteq G$ such that $L = \{h_i H\}_{i=1}^n$ and $R = \{H h_i\}_{i=1}^n$.

Exercise sheet 2

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1. For each of the following graphs G, determine the number of maximum matchings:

   (a) $G = K_n$

   (b) $G = K_{a,b}, a \leq b$

2. Let G be a graph such that $\delta(G) \geq 1$, with maximal matching M and minimal edge cover C. Prove the following equivalences:

   (a) M is a maximum matching if and only if M is contained in a minimum edge cover.

   (b) C is a minimum edge cover if and only if C contains a maximum matching.

3. Use deficiency (see sheet 1) to prove the König-Egerváry theorem. (Hint: find a matching and a vertex cover of the same size using a subset of maximum deficiency).

4. (From lectures) Let G be a bipartite graph with bipartition $\{A, B\}$ such that $|A| = |B|$. Prove that for A, Hall's condition holds if and only if Tutte's condition holds.

5. (From lectures, the Berge-Tutte formula) Prove that a maximum matching in a graph G leaves exactly
$$\max_{S \subseteq V(G)} \left(o(G \setminus S) - |S| \right).$$
vertices uncovered. Equivalently,
$$\alpha'(G) = \frac{1}{2} \left(n - \max_{S \subseteq V(G)} \{ o(G \setminus S) - |S| \} \right).$$

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