

# Operator theory

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## PART I

### Compact and Fredholm operators

#### Preliminaries

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Def.:  $(X, \rho)$  - metric space if  $X$ -set, and  $\rho$  is a metric:

- i)  $\rho(x, y) \geq 0 \quad \forall x, y \in X. \quad \rho(x, y) = 0 \Leftrightarrow x = y$
- ii)  $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$
- iii)  $\rho(x, z) \leq \rho(x, y) + \rho(y, z) \quad \forall x, y, z \in X$

Def.:  $U \subseteq X$  is open if  $\forall x \in U. \exists \delta > 0$ .

s.t.  $B(x, \delta) \subset U \quad (B(x, \delta) = \{y \in X \mid \rho(x, y) < \delta\})$

Def.:  $K \subset X$  is compact if every open cover  $\{U_\alpha\}_{\alpha \in I}$  of  $K$  has a finite subcover.

cover:  $\{U_\alpha\}_\alpha$  is a cover of  $K$  if  $\bigcup_{\alpha \in I} U_\alpha \supset K$

Def.: A precompact set  $A \subset X$  is a set  $A \subset X$

s.t.  $\bar{A}$  is compact.

closure  
of  $A$   
in  $X$

Def:  $\{x_j\}_{j \geq 1}$  is Cauchy sequence in  $X$  if

$\forall \varepsilon > 0. \exists N = N(\varepsilon). \rho(x_j, x_k) < \varepsilon \quad \forall j, k \geq N(\varepsilon).$

Def:  $X$  is complete if  $\forall$  Cauchy  $\{x_j\}_{j \geq 1} \subset X$ .

$\exists x \in X. \rho(x_j, x) \rightarrow 0$  as  $j \rightarrow \infty$

(Every Cauchy sequence converges.)

Ex.  $(\mathbb{R}^n, \rho_{\mathbb{R}^n}(\{x_i\}, \{y_i\}) := \sqrt{\sum |x_i - y_i|^2})$  - complete metric space

Ex.  $(\mathbb{Q}, \rho_{\mathbb{R}})$  - metric space but non-complete

Ex.  $[0, 1]$  is a compact-subset of  $(\mathbb{R}, \rho_{\mathbb{R}})$

Ih:  $K \subseteq \mathbb{R}^n$  is compact  $\Leftrightarrow$  closed and bounded

Def:  $A \subset (X, \rho)$  is bounded if  $\exists x \in X. \exists R > 0. A \subset B(x, R)$

Ex.  $(X, \rho) = \ell^2(\mathbb{Z}) = \left( \left\{ \overset{\uparrow}{\{x_j\}}_{j \in \mathbb{Z}} \mid \sum_{j \in \mathbb{Z}} |x_j|^2 < \infty, \right. \right. \rho(\{x_j\}, \{y_j\}) = \sqrt{\sum_{j \in \mathbb{Z}} |x_j - y_j|^2}$

$B[0, 1] = \{y \in \ell^2(\mathbb{Z}) : \rho(0, y) \leq 1\}$

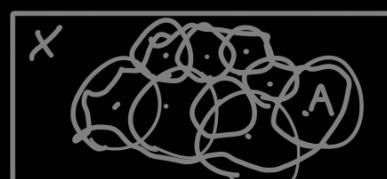
- bounded, closed but not compact HW

Theorem: Let  $(X, \rho)$  be a complete metric space,

$A \subset X$ . The following assertions are equivalent:

i)  $A$  is precompact

ii)  $\forall \varepsilon > 0. \exists N \text{ finite } \varepsilon\text{-net } \{x_j\}_{j=1}^{N_\varepsilon} \text{ in } A$ ,  
that is,  $\bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \supset A$ .



(iii)  $\forall \{x_j\}_{j \geq 1} \subset A$  there is a converging subsequence to some element  $x \in X$ .

Proof: i)  $\Rightarrow$  ii)  $\{U_x\}_{x \in A} = \{B(x, \varepsilon)\}_{x \in A}$  - open cover of  $A$ .

- is an open cover of  $\bar{A}$ :

$$(\forall y \in \bar{A}. \exists x \in A. d(x, y) < \varepsilon_2 \Rightarrow y \in B(x, \varepsilon))$$

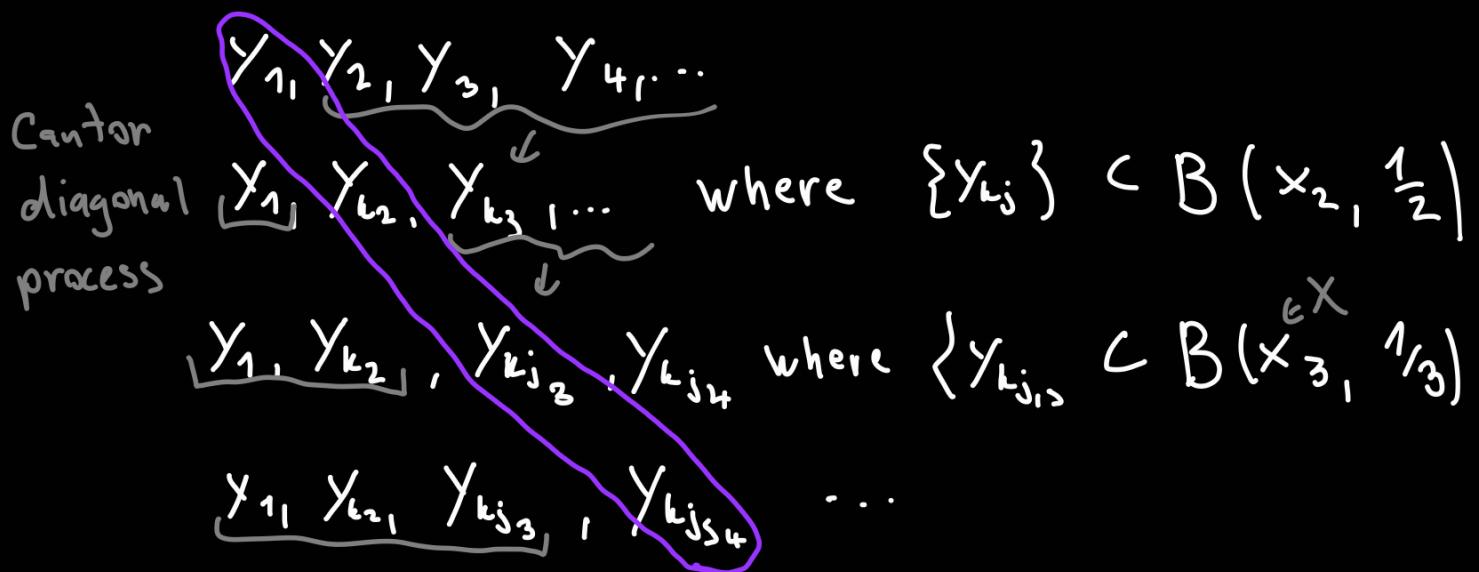
definition of closure

$\Rightarrow \exists \{U_{x_j}\}_{j=1}^N$  - finite subcover of  $\bar{A}$

$\Rightarrow \{U_{x_i}\}_i^N$  is a  $\varepsilon$ -net in  $A$

ii)  $\Rightarrow$  iii) Observe that  $\forall \varepsilon > 0$ . any sequence  $\{y_i\} \subset A$  has an infinite subsequence that is contained in some  $B(x, \varepsilon)$ . (we have a finite  $\varepsilon$ -net)

Assume that  $\{y_k\}$  is arbitrary in  $A$ .



Consider  $z_1 = y_1$

$$z_2 = y_{k_2}$$

$$z_3 = y_{k_{j_3}}$$

$$z_4 = y_{k_{j_4}}$$

⋮

Claim:  $\{z_j\}$  is a Cauchy sequence.

Indeed  $\underset{k < j}{\rho(z_j, z_k)} < \frac{1}{k}$

because  $z_j, z_k \in B(x_k, \frac{1}{k})$   $\rho(z_j, z_k) < \frac{2}{k} \xrightarrow[k \rightarrow \infty]{} 0$

$X$  is complete  $\Rightarrow \{z_j\}$  converges

iii)  $\Rightarrow$  i):

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Plan: a)  $A$  contains a dense countable subset

a) b) c) b) If  $\{U_d\}_{d \in I}$  is an open cover of  $\bar{A}$

under assumption  $\Rightarrow \exists \{U_{d,j}\}_{j \in J}$  - an open countable subcover of  $\bar{A}$

3) c)  $\Rightarrow \{U_{d,j}\}_{j=1}^\infty$  is a cover of  $\bar{A}$

a) Observe that  $\forall \varepsilon > 0$  there exists at most  $N(\varepsilon)$  points

$\Delta \{Y_j(\varepsilon)\}_{1 \leq j \leq N(\varepsilon)}$  s.t.  $\rho(Y_k(\varepsilon), Y_j(\varepsilon)) > \varepsilon \quad \forall k \neq j$ .

(If this is not true, then  $\exists \{Y_j(\varepsilon)\}_{j=1}^\infty$  such that

$\rho(Y_j(\varepsilon), Y_k(\varepsilon)) > \varepsilon$  and it cannot contain a convergent subsequence by Cauchy criterion.)

Now  $E = \left\{ Y_k \left\{ \frac{1}{n} \right\} \mid 1 \leq k \leq N \left( \frac{1}{n} \right), n \geq 1 \right\}$  is a dense countable subset.

( $E$  is dense since  $\forall n. \forall x \in A. \min_{1 \leq k \leq N(\frac{1}{n})} \{\rho(x, Y_k(\frac{1}{n}))\} \leq \frac{1}{n}$  by construction)

b) Assume that  $\{U_d\}_{d \in I}$  is some open cover of  $\bar{A}$ .

For every  $x \in \bar{A}$  define

$$\varepsilon(x) := \sup \left\{ \varepsilon > 0 \mid B(x, \varepsilon) \subset U_d \text{ for some } d \right\} > 0$$

Claim: if  $\{Y_j\}_{j=1}^\infty$  is a countable dense subset in  $A$ , then

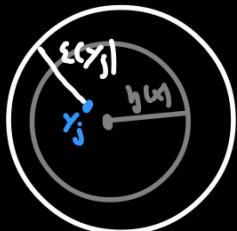
$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$  is an open cover for  $\bar{A}$ .

Take  $x \in \bar{A}$ .  $\exists d_x$ .  $x \in U_{d_x}$ , and let  $h(x) > 0$

such that  $B(x, h(x)) \subset U_{d_x}$  ( $U_{d_x}$  is open  $\Rightarrow h(\varepsilon) \exists$ )

Find  $y_j$  such that  $d(x, y_j) < \frac{h(x)}{10}$ .

Then, since  $h(x) \leq 2\varepsilon(x)$ ,  $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow d(x, y_j) < \varepsilon(y_j)$



$\varepsilon(y_j) \geq \frac{h(x)}{10}$  - Because  $B(y_j, \frac{h(x)}{5}) \subset U_{d_x}$  by triangle inequality and  $\varepsilon(y_j)$  satisfies  $\textcircled{4}$ .

Since  $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$  is an open cover for  $\bar{A}$ , then  $\{U_{d_{y_j}}\}_{j=1}^{\infty}$  is an open cover for  $\bar{A}$ , where  $U_{d_{y_j}}$  is the set  $U_d$  from the definition of  $\varepsilon(y_j)$  (that is,  $U_d \supset B(y_j, \varepsilon(y_j))$ ).

Since we have  $\textcircled{4*}$ ,  $\bigcup_{j=1}^{\infty} U_{d_{y_j}} \supset \bar{A}$ .

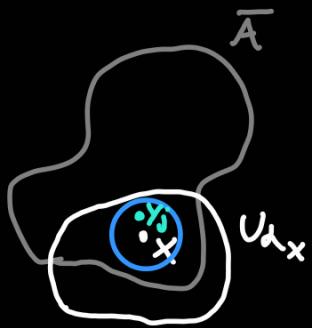
c) Claim:  $\exists N$ .  $\{U_{d_{y_j}}\}_{j=1}^N$  is a cover of  $\bar{A}$ .

Suppose this is not the case  $\Rightarrow \forall j \geq 1$ .  $\exists x_j \in A_j \setminus \bigcup_{k=1}^j U_{d_{y_k}}$ . Consider  $\{x_j\}_{j=1}^{\infty}$ , and assume that the sequence  $\{x_{j_k}\}$  converges to some  $x \in X$ . Note that  $x \in \bar{A}$  ( $x_j \in \bar{A}$ ).  
 $\Rightarrow \exists j_*. x \in U_{d_{y_{j_*}}} \Rightarrow \exists \delta > 0$ . s.t.  $B(x, \delta) \subset U_{d_{y_{j_*}}}$ , but  $x_{j_k} \notin U_{d_{y_{j_*}}}$  for large  $k$  by construction.

(in particular,  $x_{j_k} \notin B(x, \delta)$ , hence  $d(x, x_{j_k}) > \delta$ , but this contradicts the fact that  $x_{j_k} \rightarrow x$ ).

We have shown that  $\{x_j\} \subset \bar{A}$  cannot have a convergent subsequence.

Then if  $\tilde{x}_j \in A$ .  $d(\tilde{x}_j, x_j) < \frac{1}{j}$ , then  $\{\tilde{x}_j\}$  also has no convergent subsequence. So, we assumed there is no



finite subcover  $\{U_{\alpha_j}\}$  and found a sequence  $\{\tilde{x}_j\}_{j \in \mathbb{N}}$  that has no converging subsequence, a contradiction with 3). Therefore  $3) \Rightarrow 1)$ . □

### Examples of compact sets and their properties:

1)  $K \subset (X, \rho)$  is compact  $\Rightarrow K$  is bounded

Indeed, if  $K$  is not bounded, then  $\{B(x, n)\}_{n \geq 1}$  is an open cover without a finite subcover.

2)  $K \subset (X, \rho)$  is compact, then it is closed  
 $(\Leftrightarrow \{x_j\} \subset K \text{ such that } x_j \rightarrow x \text{ in } (X, \rho) \text{ we also have } x \in K)$

Let's check that  $X \setminus K$  is open. Take  $y \in X \setminus K$ , take  $x \in K$ , let  $\delta(x) > 0$ .  $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$   
 $\{B(x, \delta(x))\}_{x \in K}$  is an open cover, let  $\{B(x_j, \delta(x_j))\}_{j=1}^N$  be a finite subcover, then  $\delta := \min_{1 \leq j \leq N} \delta(x_j)$ ,  $B(y, \delta) \cap K = \emptyset$   
 $\Rightarrow X \setminus K$  is open.

Another proof: Suppose  $\{y_j\} \subset K$  s.t.  $y_j \rightarrow y$ ,  $y \notin K$ .

$$U_j = \{x \in X \mid \rho(x, y) > \frac{1}{j}\}$$

$\{U_j\}_{j=1}^\infty$  open cover,  $\{U_{j_k}\}_{k=1}^N$  finite subcover

$$\varepsilon := \min_{1 \leq k \leq N} \left( \frac{1}{j_k} \right), \quad \rho(x, y) > \varepsilon \quad \forall x \in K \quad \text{contradiction} \quad \blacksquare$$

linear space = vector space

3) Let  $X$  be a finite-dimensional complete linear normed space. Then  $E \subset X$  is compact  $\Leftrightarrow E$  is closed and bounded.

$\Leftrightarrow$ : Examples 1+2.

$\Leftarrow$ :  $X = \left\{ \sum_1^N c_k e_k \mid c_k \in \mathbb{C} \right\}$ ,  $N = \dim X$ ,  $\{e_k\}_1^N$ -basis

$$\varphi(\sum c_k e_k) = \max_{1 \leq k \leq N} |c_k| \text{ - norm on } X$$

Since all norms on finite dimensional vcc. spaces are equivalent.

$$\exists A, B > 0. \quad A \|x\| \leq \varphi(x) \leq B \|x\| \quad \forall x \in X.$$

$$\text{in particular the set } \left\{ \left\{ c_k(x) \right\}_1^N, \quad x \in E \right\} \\ x = \sum c_k(x) e_k$$

is bounded (=bdd) in  $\mathbb{C}^n$  for every bounded  $E \subset X$ .

$$\sup_{x \in E} \varphi(x) \leq B \cdot \sup_{x \in E} \|x\| < \infty$$

$\Rightarrow$  for any sequence  $\left\{ \left\{ c_k(x_j) \right\}_{k=1}^N \right\}_{j=1}^\infty$  one can extract a converging subsequence in  $\mathbb{C}^n$ , i.e.

$$c_k(x_{j_n}) \rightarrow c_k \quad n \rightarrow \infty.$$

But then  $\sum c_k(x_{j_n}) e_k \rightarrow \sum c_k e_k$  in  $X$

$\Rightarrow$  bounded subsets are precompact in  $X$

$\Rightarrow$  bdd + closed sets are compact

$$4) \ell^2(\mathbb{Z}) = \left\{ \{c_k\}_{k \in \mathbb{Z}} \mid \sum |c_k|^2 < \infty \right\}$$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^{\infty} |c_k|^2}$$

$$B[0,1] = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \|\{c_k\}\| < 1 \right\}$$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite  $\frac{1}{2}$ -net in  $B[0,1]$ ,  
 bccu  $s(e_i, e_j) > \frac{1}{2}$  for  $e_k = (0, \dots, 0, \overset{\uparrow}{1}, 0, \dots, 0)$ .

Definition:  $X$  is a Banach space if it is a linear normed space such that  $X$  is complete with respect to this norm.

Example: Let  $(K, \rho)$  be a metric compact space.

$$C(K) := \{f : K \rightarrow \mathbb{C} \mid \text{cont. in the metric } \rho\}$$

$$(\Leftrightarrow f(x_j) \rightarrow f(x) \quad \forall x_j \rightarrow x \text{ in } (K, \rho))$$

$$\|f\|_{C(K)} = \|f\| := \max_{x \in K} |f(x)| \quad - \text{norm in } C(K)$$

Theorem: [Arzela-Ascoli]: Assume that  $K$  is a complete compact metric space.  $E \subset C(K)$  is precompact  $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} 1 | E \text{ is bounded in } C \\ 2 | \text{functions in } E \text{ are equicontinuous, that is,} \\ \quad \forall \varepsilon > 0, \exists \delta_\varepsilon > 0. |f(x) - f(y)| < \varepsilon \quad \forall x, y \in K. \rho(x, y) < \delta_\varepsilon \\ \quad \forall f \in E \end{cases}$$

We will need 1+2  $\Rightarrow$  precompactness.

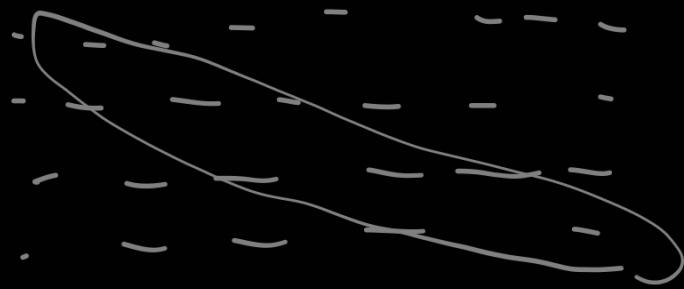
Proof: Find a dense sequence  $\{x_j\}$  in  $K$ .  
 (such sequence exists because  $K$  is compact)

Then take  $\{f_n\}$  arbitrary sequence in  $E$ .

We want to find a converging subsequence of  $\{f_n\}$   
 (then  $E$  - precompact)

For this find a subsequence  $\{f_{n_k}\}$  such that  
 $f_{n_k}(x_j) \rightarrow F(x_j)$  for every  $j$

(Cantor diagonalization process + uniform boundedness)



look at the proof  
from the 1st  
lecture

Claim:  $f_{n_k}$  is Cauchy sequence in  $C(K)$ .

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Aim:  $\|f_{n_s} - f_{n_m}\|_{C(K)} \rightarrow 0$  as  $s, m \rightarrow \infty$

For simplicity let  $g_s = f_{n_s}$   $s \geq 1$ .

Idea:

$$|g_s(x) - g_m(x)| \leq |g_s(x) - g_s(x_j)| + \underbrace{|g_s(x_j) - g_m(x_j)|}_{\substack{\leq \frac{\epsilon}{3} \\ \text{for all } s \text{ if } x_j \\ \text{is close to } x}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\text{take } s, m \text{ large} \\ \text{enough:} \\ \leq \frac{\epsilon}{3}}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\leq \frac{\epsilon}{3} \\ \text{for all } s \text{ if } x_j \\ \text{is close to } x \\ (\text{uniform continuity})}}$$

To make the idea work we need to check that in this construction we can deal only with finite number of points  $x_j$ ,  $j=1, \dots, N(\epsilon)$ .

For this it suffices to find  $N(\delta_\epsilon)$  such that

$\exists (x, x_j) \in \delta_\epsilon$  for every  $x \in K$  and  $x_j$ ,  $j=1 \dots N_\epsilon$ .  
( $\{x_j\}_{j=1}^{N(\delta_\epsilon)}$  is  $\delta_\epsilon$ -net).

So, it remains to show that if  $\{x_j\}_{j=1}^\infty$  is dense then  $\forall \delta_\epsilon > 0. \exists N(\delta_\epsilon). \{x_j\}_{j=1}^{N(\delta_\epsilon)}$  is a  $\delta_\epsilon$ -net.

To this end, let  $\{y_k\}_1^N$  is a  $\delta_{\epsilon/2}$ -net in  $K$  ( $K$  is compact). Let  $\{x_j\}_{j=1}^{N(\delta_\epsilon)}$  be the part of  $\{x_j\}$  such that

$$\text{dist}(\{x_j\}_{j=1}^{N(\delta_\epsilon)}, y_k) \leq \frac{\delta_\epsilon}{2} \quad \forall 1 \leq k \leq N.$$

$\Rightarrow$  then  $\{x_j\}_{j=1}^{N(\delta_\epsilon)}$  is a  $\delta_\epsilon$ -net by triangle inequality.  $\blacksquare$   
( $\|g_s - g_m\|_{C(K)} \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$  for  $s, m$  large enough)

Example:  $K = [0, 1]$ ,  $E_A = \left\{ f \in C[0, 1] \mid f(0) = 0, f \text{ is Lipschitz}, \text{ with constant at most } A \right\}$

$E_A$  is compact

i)  $E_A$  is bounded in  $C[0, 1]$ :

$$|f(x)| \leq |f(x) - f(0)| \leq A|x| \leq A$$

$$E_A \subset B(0, A)$$

ii)  $|f(x) - f(y)| \leq A|x - y| \leq A\delta = \varepsilon$  if  $\varepsilon > 0$ ,  $\delta := \frac{\varepsilon}{A}$ ,  
 $x, y \in [0, 1]: |x - y| \leq \delta$

i + ii + AA theorem  $\Rightarrow E_A$  is precompact

iii)  $E_A$  is closed

If  $f_n \rightarrow f$  in  $C(K)$  then  $f_n(0) \rightarrow f(0) \Rightarrow f(0) = 0$

$$|f_n(x) - f_n(y)| \leq A|x - y|$$

$\downarrow$

$$|f(x) - f(y)| \Rightarrow f \text{ is Lip}(A)$$

Example: Let  $E = \left\{ \sum_{k \in \mathbb{Z}} c_k e^{2\pi i kx}, \text{ where } c_k \in \mathbb{C}: |c_k| \leq \frac{1}{k^2+1} \right\}$

Then  $E$  is compact as well in  $C[0, 1]$ .

i) bbs:  $f \in E$ .  $\|f\| \leq \sum_{k \in \mathbb{Z}} \frac{1}{k^2+1}$

Details: exercise

ii) equicontinuity  $f = \sum_{|k| \leq N} + \sum_{|k| > N}$  small if  $N$  large  
 Lipschitz with some constant  
 $A_N \sim \text{does not depend on } F$

# Compact operators: basic properties

Definition: Let  $X, Y$  be Banach spaces,  $T: X \rightarrow Y$  a linear map.  $T$  is called **bounded** if  $T(B(0,1))$  is a bounded in  $Y$  set in  $Y$ .  $T$  is called **compact** if  $T(B(0,1))$  is a precompact set in  $Y$ . ( $B(0,1) = \{ \|x\|_X < 1\}$ )      bounded linear operator

## Some observations:

- 1) If  $S \subset X$  is bdd then  $T(S)$  is <sup>bdd</sup><sub>precompact</sub> for any <sup>bdd</sup><sub>compact</sub> operator
- 2)  $T$  is compact  $\Rightarrow T$  is bounded  
(precompact sets are bounded)
- 3) with the norm  $\|T\| = \sup_{x \in B(0,1)} \|Tx\|_Y$ , the set of bdd linear operators becomes a linear normed space, to be denoted by  $B(X,Y)$  or  $B(X)$  if  $X=Y$ .
- 4) A linear map between Banach spaces  $X, Y$  is continuous if and only if it is bounded.  
Hint:  $\|Tx - Ty\| \leq \|T\| \cdot \|x - y\|$ , so bounded operators are Lipschitz.

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Definition: **Banach algebra** is an associative algebra which is a linear space with a norm  $\|\cdot\|$  such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and  $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$  for any elements in this algebra.

Proposition: Let  $X$  be a Banach space, then  $B(X,X)$  is a Banach algebra.

Proof:  $\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{B}(X, X)$   $\forall T_1, T_2 \in \mathcal{B}(X, X)$  (proved)  
 $T_1 \cdot T_2 \in \mathcal{B}(X, X)$ , since  $\forall x \in X$ .  $\|T_1 T_2 x\| \leq \|T_1\| \cdot \|T_2 x\|$   
 $\Rightarrow \|T_1\| \cdot \|T_2\| \leq \|T_1\| \cdot \|T_2\|$  since  $T_1 \in \mathcal{B}(X, X)$  we have  
 $\sup_{\substack{y \in X \\ \|y\| \leq 1}} \|T_1 T_2 y\| \leq \|T_1\| \|T_2\| \forall y \in X$

We see that  $T_1 T_2 \in \mathcal{B}(X, X)$  and  $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$ .

Now let us prove that  $\mathcal{B}(X, X)$  is Banach.

Let us show that  $\sum_{n=1}^{\infty} B_k$  converges if  $\sum_{k \in \mathcal{B}(X, X)} \|B_k\| < \infty$ .

$$T_n = \sum_{k=1}^n B_k, \quad x \in X, \quad \|T_N x - T_M x\| = \left\| \sum_{M+1}^N B_k x \right\| \leq \sum_{M+1}^N \|B_k\| \|x\| \xrightarrow[M, N \rightarrow \infty]{} 0,$$

because of (\*).

$\Rightarrow \{T_N x\}_N$  Cauchy in  $X$ , but  $X$ -banach  $\Rightarrow \exists T x = \lim_{N \rightarrow \infty} T_N x$

Moreover,  $\|Tx\| = \lim_{N \rightarrow \infty} \|T_N x\| \leq \lim_{N \rightarrow \infty} \sum_{k=1}^N \|B_k\| \|x\| \leq (\sum_{k=1}^{\infty} \|B_k\|) \|x\|$ .

$\Rightarrow T \in \mathcal{B}(X, X)$ ,  $\|T\| \leq \sum_{k=1}^{\infty} \|B_k\|$

$$\sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx - T_N x\| = \lim_{N \rightarrow \infty} \|T_N x - T_N x\| \leq \sup_{\|x\| \leq 1} \lim_{N \rightarrow \infty} \sum_{k=1}^N \|B_k\| \|x\| = \sum_{k=1}^{\infty} \|B_k\| \xrightarrow[N \rightarrow \infty]{(*)} 0 \quad \square$$

$S_{\infty}(X, X)$  (index  $\infty$  will be explained later)

Proposition: The set  $S_{\infty}(X)$  of all compact operators on  $X$  is a two-sided ideal in  $\mathcal{B}(X) = \mathcal{B}(X, X)$ :  $\forall T_1 \in S_{\infty}(X), \forall T_2 \in \mathcal{B}(X)$ .  $T_1 T_2 \in S_{\infty}(X)$  and  $T_2 T_1 \in S_{\infty}(X)$ .

Proof: Take  $\{x_n\}_{n=1}^{\infty}$  s.t.  $\|x_n\| \leq 1$ , and let us check that there is a subsequence  $\{x_{n_k}\}$ :  $T_1 T_2 x_{n_k}$  converges.

Note that  $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$ .  $T_1$  takes  $B_X(0, \|T_2\|)$  into a precompact subset of  $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$  -convergent subsequence

Now let's consider  $\{T_2 T_1 x_n\}$ . Note that  $\{T_1 x_n\}$  -convergent subsequence ( $T_1 \in S_{\infty}(X)$ ). Then  $\{T_2 T_1 x_n\}$  converges, since  $T_2$  is continuous.

Proposition:  $S_\infty(X, Y)$  is a closed subset in  $\mathcal{B}(X, Y)$ , i.e.

$T_n \in S_\infty(X, Y)$ ,  $T_n \rightarrow T$  in  $\mathcal{B}(X, Y) \Rightarrow T \in S_\infty(X, Y)$ ?

Proof: Let's find a finite  $\varepsilon$ -net in  $T(B_X[0,1])$ .

Take finite  $\varepsilon_3$ -net for  $T_n B_X(0,1)$  for  $n$ :  $\|T - T_n\| \leq \varepsilon_2$ ;  
denote it by  $\{y_k\}_{k=1}^N$ , then

$$\begin{aligned} \|Tx - Tx_k\| &\leq \underbrace{\|Tx - T_nx\|}_{A} + \underbrace{\|T_nx - T_nx_k\|}_{B} + \underbrace{\|T_nx_k - Tx_k\|}_{C} \\ &\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq \varepsilon. \end{aligned}$$

for any  $1 \leq k \leq N$   
so choose  $k$ :  $B \leq \varepsilon_2$   
and note  $A \leq \|T - T_n\| \leq \varepsilon_1$   
for every  $x \in B_X(0,1)$   
 $C \leq \varepsilon_3$

Corollary: If  $T$  is a limit of finite-rank operators in  $\mathcal{B}(X, Y)$ ,  
then  $T \in S_\infty(X, Y)$ .

Proof: Since finite-rank operators are in  $S_\infty(X, Y)$ , we  
have  $T \in S_\infty(X, Y)$  by the previous proposition.  $\square$

Remark: At a general Banach space  $\exists T \in S_\infty(X, Y)$  such  
that  $\nexists \{T_n\}_n$ :  $\text{rank } T_n < \infty$  and  $\|T - T_n\| \rightarrow 0$ .

Definition: Let  $X$  be a Banach space.  $\{e_k\}_{k=1}^\infty$  is a  
**Schauder basis** if  $\forall x \in X \exists \{c_k(x)\}_{k=1}^\infty$  such that  
 $x = \sum_{k=1}^\infty c_k(x)e_k$ , where the series converges in  $X$ .

Theorem: Let  $X$  be a Banach space with Schauder  
basis, then  $T \in S_\infty(X) \Leftrightarrow \exists T_n. \text{rank } T_n \leq n$  and  $\|T - T_n\| \rightarrow 0$ .  
(here  $\text{rank } S = \dim S(X) \quad \forall S \in \mathcal{B}(X)$ )

Proof: ( $\Leftarrow$ ): we already know

$(\Leftrightarrow)$ : Let  $T \in S_\infty(X)$ , and let  $P_n : X \mapsto \sum_{k=1}^{\infty} c_k(x) e_k$ .

P is linear:  $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X. P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$ ?

$$\left. \begin{array}{l} \text{If } x = \sum c_k(x) e_k \\ y = \sum c_k(y) e_k \end{array} \right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha c_k(x) + \beta c_k(y)) e_k$$

$$\alpha x + \beta y = \sum_{k=1}^{\infty} c_k(\alpha x + \beta y) e_k$$

by uniqueness    by def. of Schauder basis

$$c_k(\alpha x + \beta y) = \alpha c_k(x) + \beta c_k(y) \quad \forall k$$

$$\begin{aligned} \text{Then } P_n(\alpha x + \beta y) &= \sum_{n=1}^{\infty} c_k(\alpha x + \beta y) e_k = \sum_{n=1}^{\infty} \alpha c_n(x) e_n + \sum_{n=1}^{\infty} \beta c_n(y) e_n \\ &= \alpha P_n(x) + \beta P_n(y) \quad \Rightarrow P_n \text{ linear} \end{aligned}$$

Note that  $T_n := P_n T$  are such that  $\text{rank}(T_n) \leq n$  because  $\dim P_n T(x) \leq \dim P_n(x) \leq n$ .

It remains to show that  $T_n \rightarrow T$  in  $\mathcal{B}(X)$ . Since  $T$  is compact,  $\forall \varepsilon > 0. \exists \{x_k\}_{k=1}^N$  such that  $\|x_k\| \leq 1 \forall k$  and  $\{Tx_k\}_{k=1}^N$  is a  $\varepsilon$ -net in  $T(B_X(0,1))$ . Now take  $x \in B_X(0,1)$  and write  $\|Tx - T_n x\| \leq \|Tx - Tx_n\| + \|Tx_n - T_n x\| + \|T_n x - T_n x\|$

$$\begin{aligned} &\leq \underbrace{\|Tx - Tx_n\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_n - P_n Tx_n\|}_{\leq \varepsilon \text{ if } n \text{ large enough for any fixed } k} + \underbrace{\|P_n Tx_n - P_n Tx\|}_{\leq \|P_n\| \cdot \|Tx_n - Tx\|} \\ &\leq \varepsilon + \varepsilon + \underbrace{\sup_{n \in \mathbb{N}} \|P_n\|}_{< \infty} \varepsilon \end{aligned}$$

$\sup \|P_n\| < \infty$  by Banach-Schauder theorem on uniform point-wise convergence.

$$\|T - T_n\| \leq \varepsilon (2 + \sup \|P_n\|) \text{ for } n \text{ large enough} \quad \square$$

Theorem [Banach-Schteinhaus]: Assume that  $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$  where  $X$  is a Banach space, such that

$$\sup_n \|T_n x\| \leq C(x) < \infty \quad \begin{matrix} \text{local information} \\ \sim \text{uniform estimate} \end{matrix}$$

For every  $x \in X$ . Then  $\sup_n \|T_n\| < \infty$ . In particular, one can take  $C$  in place of  $C(x)$ .

Remark: In our situation,  $\sup_{1 \leq n < \infty} \|P_n\| \leq C(x) < \infty$  because  $P_n x \rightarrow x$  in  $X$ .

## Banach adjoint operators

October 15, 2025

Definition: Let  $X$  be a Banach space, then  $X^* = \mathcal{B}(X, \mathbb{C})$  is called the dual space to  $X$ . The elements of  $X^*$  are called functionals.

Examples: (can ignore, if one does not know measure theory)

i)  $L^p(\mu) = \left\{ f : S \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is measurable with respect to } \sigma\text{-algebra} \\ \text{of } \mu \end{array}, \int_S |f|^p d\mu < \infty \right\}$

$f=g$   
if  $f(x)=g(x)$   
for  $\mu$ -a.e.xes

$$\|f\|_{L^p(\mu)} = \left( \int_S |f|^p d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \text{ where } \frac{1}{p} + \frac{1}{q} = 1$$

ii)  $\ell^p(\mathbb{Z}) = \left( \left\{ \{x_k\}_{k \in \mathbb{Z}} \mid \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \right\}, \|\{x_k\}\|_{\ell^p(\mathbb{Z})} = \left( \sum_k |x_k|^p \right)^{1/p} \right)$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \text{ where } \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

i)  $g \in L^q(\mu) \leftrightarrow \phi_g : f \mapsto \int_S f g d\mu, \quad \phi_g : L^p(\mu) \rightarrow \mathbb{C}$

ii)  $\{y_k\}_{k=1}^\infty$  in  $\ell^q(\mathbb{Z}) \leftrightarrow \phi_{\{y_k\}} : \{x_k\} \mapsto \sum_{k \in \mathbb{Z}} x_k y_k$

$\phi_{\{y_k\}} : \ell^p(\mathbb{Z}) \rightarrow \mathbb{C}$

Remark: i) is non-trivial measure theory

More examples:

iii)  $C_0(\mathbb{Z}) = \left\{ \{x_k\}_{k \in \mathbb{Z}} \mid x_k \rightarrow 0 \text{ as } |k| \rightarrow \infty \right\}$

$C_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$  same identification

(Hausdorff is actually sufficient  $\Leftrightarrow$  hard)

iv) Let  $K$  be a compact metric space, and  $X = C(K)$ .  
Then  $X^* = M(K)$ .

{ the set of Borel measures on  $K$  (complex valued) }  $\|\mu\|(K) = \sup_{\substack{K = \cup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n \in \mathbb{Z}} |\mu(E_n)| < \infty \}$

set of  
cont. maps  
 $(K, \mathcal{S}) \rightarrow \mathbb{C}$

Riesz - Markov representation theorem

Here  $\mu \in M \xrightarrow{\text{(bi)}} \phi_\mu : f \mapsto \int_K f d\mu$

We can also define  $\ell^p, L^p$  for  $p = \infty$ :

- $\ell^\infty(\mathbb{Z}) := \{ \{x_k\} \subset \mathbb{C} : \sup_{k \in \mathbb{Z}} |x_k| < \infty \}$

- $L^\infty(\mu) := \{ f : \mathbb{Z} \rightarrow \mathbb{C} : \text{ess sup } |f| < \infty \}$

Remark: If  $1 \leq p < \infty$  then  $(L^p)^* = L^q, (L^q)^* = L^p$

Bvt for  $p=1$   $(L^1)^* = L^\infty$ ,  $(L^\infty)^* \neq L^1$   
 $L^1(\mathbb{Z})^* = l^\infty(\mathbb{Z})$ , bvt  $(l^\infty(\mathbb{Z}))^* \neq l^1(\mathbb{Z})$

Definition: Let  $X, Y$  be Banach spaces,  $T \in \mathcal{B}(X, Y)$ . Then  $T^* \in \mathcal{B}(Y^*, X^*)$  is defined by

$$T^* : \Psi_{Y^*} \longmapsto \left( \left( T^* \Psi \right) : x \mapsto \langle Tx, T^* \psi \rangle \right),$$

where  $\langle x, \phi \rangle = \phi(x)$  for  $x \in X, \phi \in X^*$ .  $\Psi(Tx)$

Remark:  $\langle Tx, \psi \rangle = \langle x, T^* \psi \rangle \rightarrow$  this formula is equivalent to the definition of  $T^*$

Remark: Operation that sends  $x, \phi$  into  $\phi(x) = \langle x, \phi \rangle$  for  $x \in X, \phi \in X^*$  is called a pairing of Banach spaces  $X, X^*$ .

Example: For  $f \in C[0,1]$ ,  $\mu$  on  $[0,1]$  then the pairing is  $\langle \phi, \mu \rangle = \int_0^1 f d\mu$ , see (\*).

Theorem: Let  $X, Y$  be Banach spaces,  $T \in \mathcal{B}(X, Y)$ . Then the map  $T^* : Y^* \rightarrow X^*$  defined by  $\langle x, T^* \psi \rangle := \langle Tx, \psi \rangle$ ,  $x \in X$ , is an element of  $\mathcal{B}(Y^*, X^*)$ .  
 $(\Leftrightarrow (T^* \psi)(x) = \psi(Tx))$

Lemma 1 [Hahn-Banach theorem]: Let  $X$  be a Banach space,  $E \subset X$  - subspace in  $X$ ,  $\phi_0 : E \rightarrow \mathbb{C}$  is linear and bdd ( $\Leftrightarrow \phi_0 \in E^*$ ). Then  $\exists \phi \in X^*$  such that  $\phi|_E = \phi_0$  and  $\|\phi\| = \|\phi_0\|$ .

Lemma 2 ["Sufficient amount of functionals"] :

Let  $x \in X$ , then  $\|x\| = \sup_{\|\phi\| \leq 1} |\phi(x)|$ .

Proof:  $|\phi(x)| \leq \|\phi\| \cdot \|x\| \leq \|x\|$ , so  $\|x\| \geq \sup_{\|\phi\| \leq 1} |\phi(x)|$

To prove " $\leq$ ", define  $E = \text{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$ ,  
 $\Phi_0: Y \rightarrow \mathbb{C}$ , if  $y = c_y x \in E$ .

Assume that  $\|x\| = 1$ , then  $\|\Phi_0\|_{E^*} = \sup_{\|y\| \leq 1} |\Phi_0(y)| = (\text{from } (**),$   
 $|c_y| = \|y\| \text{ if } \|x\| = 1) = \sup_{\|y\| \leq 1} \|y\| = 1$ .

Hahn-Banach theorem  $\Rightarrow \exists \tilde{\Phi}_0 \in X^*: \|\tilde{\Phi}_0\| = 1, \Phi_0|_E = \tilde{\Phi}_0$ .

In particular  $\sup_{\|\Phi\| \leq 1} |\Phi(x)| \geq |\tilde{\Phi}| = |\Phi_0(x)| = 1 = \|x\|$ .

We have proved " $\leq$ " in the case where  $\|x\| = 1$ .

The general case follows from consideration of  $\frac{x}{\|x\|}$  in  
place of  $X$ .

October 21, 2025

We are proving that  $T \in \mathcal{B}(X, Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$  and  $\|T\| = \|T^*\|$ .

Let  $T \in \mathcal{B}(X, Y)$ , consider

$$\begin{aligned} \|T^*\| &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \|T^* \psi\|_{X^*} = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |(T^* \psi)(x)| \\ &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\langle x, T^* \psi \rangle| \\ &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\langle T_x, \psi \rangle| \\ &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\psi(Tx)| \\ &= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} |\psi(Tx)| \end{aligned}$$

=  $\|Tx\|$  Lemma "Sufficient amount of functionals"

$$= \sup_{\|x\| \leq 1} \|Tx\| = \|T\| < \infty$$

The claim follows. □

Corollary:  $T \in \mathcal{B}(X, Y)$  is invertible ( $\exists T^{-1} \in \mathcal{B}(Y, X)$ ) iff  $T^* \in \mathcal{B}(Y^*, X^*)$  is invertible ( $\exists (T^*)^{-1} \in \mathcal{B}(X^*, Y^*)$ ).

We prove just  $\Rightarrow$ .

Proof: Assume that  $T$  is invertible  $\Leftrightarrow T^{-1}T = I_X$   
 $TT^{-1} = I_Y$ ,

Let's take adjoint operators and see:

$$\left. \begin{array}{l} (T^{-1}T)^* = (I_X)^* \\ (TT^{-1})^* = (I_Y)^* \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} T^*(T^{-1})^* = I_X^* \\ (T^{-1})^*T^* = I_Y^* \end{array} \right.$$

Exercise:  $(AB)^* = B^*A^*$

It remains to check that  $I_X^* = I_{X^*}$ ,  $I_Y^* = I_{Y^*}$ . Then, by the previous theorem,  $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$ , hence  $T^*$  is invertible and its bounded inverse is  $(T^*)^{-1} = (T^{-1})^*$ .

Let's check that  $I_X^* = I_{X^*}$ . Take  $\bar{x} \in X^*$ ,  $x \in X$ .

$$(I_X^*\bar{x})(x) = \langle x, I_X^*\bar{x} \rangle = \langle I_X x, \bar{x} \rangle = \langle x, \bar{x} \rangle = \bar{x}x$$

$$(I_X^*\bar{x})(x) \stackrel{\text{def}}{=} (\bar{x})x = \bar{x}(x).$$

Similarly,  $I_Y^* = I_{Y^*}$ . □

The „pairing notation” is often not used in literature, but it is very useful to not make mistakes.

Theorem [Schauder]: We have  $T \in S_\infty(X, Y) \Leftrightarrow T^* \in S_\infty(Y^*, X^*)$ .

Proof: We will prove just " $\Rightarrow$ ".

Consider  $K = \overline{TB_X(0,1)}$  - a compact set. Let  $C(K)$  be the Banach space of continuous functions on  $K$  with

$$\|f\|_{C(K)} = \max_{s \in K} |f(s)|, \quad f: K \rightarrow \mathbb{C}$$

Let  $E := \{\psi \in Y^* \mid \|\psi\|_{Y^*} \leq 1, \psi \text{ is considered as a function on } K\}$

$K \subset Y$ ,  $K$  metric space with respect to the metric  $s(x_1, x_2) = \|x_2 - x_1\|_Y$

So,  $E \subset C(K)$  and we claim that  $E$  is precompact.

1) Uniform boundedness:

$\psi \text{ cont.}$

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s \in K} |\psi(s)| = \max_{s \in \overline{TB_X(0,1)}} |\psi(s)| = \sup_{x \in B_X(0,1)} |\psi(Tx)|$$

$$\leq \|\psi\| \sup_{\|x\| \leq 1} \|Tx\| \leq \|\psi\| \cdot \|T\| \leq \underbrace{\|T\|}_{\substack{\text{does not} \\ \text{depend on } \psi}} < \infty$$

2) Equicontinuity: take  $s_1, s_2 \in K$ , let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \leq \|\psi\| \cdot \|s_1 - s_2\| \leq \|s_1 - s_2\|,$$

so maps from  $E$  are Lipschitz with constant 1, hence equicontinuous.

$\Rightarrow$  By Arzela-Ascoli theorem,  $E$  is precompact.

We are now ready to prove  $T^* \in S_\infty(Y^*, X^*)$ . For this, we need to check that if  $\{\psi_n\}$  is a sequence in  $B_{Y^*}(0,1)$ , then  $\exists \{\psi_{n_k}\}$  such that  $T^* \psi_{n_k}$  converges in  $X^*$ . So, take  $\{\psi_n\} \subset B_{Y^*}(0,1)$  and consider it as elements  $E \subset C(K)$ .

Let  $\{\psi_{n_k}\}$  be such that  $\psi_{n_k} \xrightarrow{\text{w*}} \psi$  in  $C(K)$ . ( $E$  is precompact!)

Let's prove that  $\{T^*\psi_{n_k}\}$  is Cauchy in  $X^*$ , then the theorem will follow.

Take  $x \in X$ ,  $\|x\| \leq 1$ , and consider

$$\begin{aligned}
 \| (T^*\psi_{n_k})(x) - (T^*\psi_{n_j})(x) \| &= \| \langle x, T^*\psi_{n_k} \rangle - \langle x, T^*\psi_{n_j} \rangle \| \\
 &= \| \langle Tx, \psi_{n_k} \rangle - \langle Tx, \psi_{n_j} \rangle \| \\
 &= \| \psi_{n_k}(Tx) - \psi_{n_j}(Tx) \| \\
 &\leq \sup_{s \in K} \| \psi_{n_k}(s) - \psi_{n_j}(s) \| \\
 &= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|}_{\varepsilon_{k,j} - \text{does not depend on } x} \Big|_{C(K)} \longrightarrow 0 \quad \text{by } (*).
 \end{aligned}$$

$$\Rightarrow \|T^*\psi_{n_k} - T^*\psi_{n_j}\| \leq \varepsilon_{k,j} \longrightarrow 0.$$

□

## Fredholm alternative

Example: Consider the equation  $f(t) - \int_0^1 e^{t-s} f(s) ds \stackrel{(*)}{=} g(t)$  in  $L^2[0,1]$ .

Question: For which  $g \in L^2[0,1]$  do we have a solution  $f \in L^2[0,1]$ ?

Observation:  $g$  has to satisfy  $\int_0^1 e^{-t} g(t) dt = 0$

Indeed,  $\int_0^1 e^{-t} g(t) dt = \int_0^1 e^{-t} f(t) dt - \int_0^1 e^{-t} \left( \int_0^1 e^{t-s} f(s) ds \right) dt = 0$

It is not clear so far if there are other restrictions.

Theorem [Fredholm alternative]: Let  $X$  be a Banach space,  $T = I - K$  for  $K \in S^\infty(X, X)$ . Then

$$\text{Ran } T = \{x \in X \mid \langle x, \varrho \rangle = 0 \ \forall \varrho \in \ker T^*\}.$$

In other words, either:

- (1)  $\ker T^* = \{0\}$  and the equation  $Tf = g$  has solution  $\forall g \in X$ .
- or (2)  $\ker T^* \neq \{0\}$  and the equation  $Tf = g$  has solutions only for  $g$  s.t.  $\langle g, \varrho \rangle = 0 \ \forall \varrho \in \ker T^*$ .

Let's complete the consideration of the example:  
 we need to check that  $K: f \rightarrow \int_0^1 e^{t-s} f(s) ds$  is compact  
 (exercise) and find  $\text{Ker } T^*$ .

$$\phi \in \text{Ker } T^* \Leftrightarrow T^* \phi = 0$$

Adjoint operator  $T^*$  is defined by

$$\begin{aligned} \langle Tf, g \rangle &= \langle f, T^*g \rangle \quad f, g \in L^2[0,1] \\ \Leftrightarrow \langle f - \int_0^1 e^{t-s} f(s) ds, g \rangle &= \int_0^1 f(t) g(t) dt - \int_0^1 \left( \int_0^1 e^{t-s} f(s) ds \right) g(t) dt \\ &= \int_0^1 f(t) g(t) dt - \int_0^1 f(s) \left( \int_0^1 e^{t-s} g(t) dt \right) ds \\ &= \langle f, g - \int_0^1 e^{t-s} g(t) dt \rangle_{L^2[0,1]} \end{aligned}$$

$$(T^*g): s \mapsto g(s) - \int_0^1 e^{t-s} g(t) dt, \quad s \in [0,1].$$

$$T^*g = 0 \Leftrightarrow g(s) = \int_0^1 e^{t-s} g(t) dt \quad \text{a.e. on } [0,1]$$

$$\Leftrightarrow e^s g(s) = \underbrace{\int_0^1 e^{t-s} g(t) dt}_{\text{constant}} \quad \text{for almost every } s \in [0,1]$$

$$\Rightarrow \text{So, } \text{Ker } T^* = \{c \cdot e^{-s}, c \in \mathbb{C}\}, \quad \dim(\text{Ker } T^*) = 1.$$

By Fredholm theorem, equation  $(**)$  is solvable  $\Leftrightarrow$   
 $\forall c \in \mathbb{C}. \langle g, c \cdot e^{-s} \rangle = 0 \Leftrightarrow \int_0^1 g(s) e^{-s} ds = 0$ , which is  $(***)$ .

## Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let  $X$  be a Banach space,  $E \subseteq X$  - a linear closed subspace,  $\varepsilon > 0$ . Then  $\exists x_0 \in X$  such that  $\|x_0\| = 1$ ,  $\text{dist}(x_0, E) \geq 1 - \varepsilon$ .

Proof: Since  $E \neq X$ , then  $\exists \tilde{x}_0 \in X \setminus E$ . Since  $E$  is closed, we have  $\text{dist}(\tilde{x}_0, E) = \delta > 0$  for some  $\delta > 0$ . Now consider  $\tilde{y}_0 \in E$  such that  $\delta \leq \|\tilde{x}_0 - \tilde{y}_0\| \leq (1+\eta)\delta$  for some  $\eta \in (0, 1)$ .

Now let  $x_\eta := \frac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}$ ,  $\|x_\eta\| = 1$ .

$$\begin{aligned}\text{dist}(x_\eta, E) &= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \text{dist}(\tilde{x}_0 - \tilde{y}_0, E) \\ &= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \text{dist}(\tilde{x}_0, E) \\ &= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta}\end{aligned}$$

Choosing  $\eta$  so that  $\frac{1}{1+\eta} = 1 - \varepsilon$ , we are done.

October 22, 2025

Lemma: Let  $X$  be a Banach space. Then  $I: X \mapsto X$  is compact on  $X \Leftrightarrow \dim X < \infty$ .

Proof:  $\dim X < \infty \Rightarrow I \in S_\infty(X)$  - we already know  $I \in S_\infty(X) \Rightarrow \dim X < \infty$ :

Suppose  $\dim X = +\infty$ , find a sequence  $\{e_n\}: \|e_n\|=1 \quad \forall n \in \mathbb{N}$

$e_1 \in X$  - arbitrary

$e_2: \text{dist}(e_2, \text{span}\{e_1\}) \geq \frac{1}{2}$

$e_3: \text{dist}(e_3, \text{span}\{e_1, e_2\}) \geq \frac{1}{2}$

$e_4: \text{etc}$

existence of  $\{e_n\}$   
follows from previous  
lemma, because  
 $\text{span}\{e_1, \dots, e_k\} \neq X$   
 $\forall k \in \mathbb{N}$

Then  $\{e_n\} \subset B_X[0, 1] = I(B_X[0, 1])$  but there is no convergent subsequence, because  $\|e_k - e_j\| \geq \frac{1}{2} \quad \forall k, j$ .  $\blacksquare$

Lemma: Let  $X$  be a Banach space,  $K \in \mathcal{S}_\infty(X)$ ,  $T = I - K$ .  
 Then: 1)  $\dim(\text{Ker } T) < \infty$ .  
 2)  $\text{Ran } T$  is closed in  $X$ . [closed range Lemma]

Proof: 1): We have  $I|_{\text{Ker } T} = \underbrace{(I-K)|_{\text{Ker } T}}_0 + \underbrace{K|_{\text{Ker } T}}_{\in \mathcal{S}_\infty(\text{Ker } T, S)}$

$$I|_{\text{Ker } T} \in \mathcal{S}_\infty(\text{Ker } T, X) \Rightarrow I \in \mathcal{S}^\infty(\text{Ker } T) \Rightarrow \dim(\text{Ker } T) < \infty$$

2): The statement is equivalent to the fact that if  $\{x_n\} \subset X$  s.t.  $TX_n \rightarrow y$  in  $X$  then  $\exists x \in X. Tx = y$ .

2.a) Let  $\{x_n\} : \|x_n\| \leq c \quad \forall n$ .

Then  $(I-K)(x_n) \rightarrow y$ ,  $(I-K)(x_{n_k}) \xrightarrow{(*)} y$   
 For every subsequence  $x_{n_k}$

Let's choose  $x_{n_k}$ :  $Kx_{n_k}$  converges to  $z \in X$   
 (use  $K \in \mathcal{S}_\infty(X)$ )

Then  $x_{n_k} \rightarrow y+z$  by (\*), take  $x = y+z$ :

$$T(y+z) = \lim_{k \rightarrow \infty} Tx_{n_k} = y, \text{ so } Tx = y. \quad \checkmark$$

2.b)  $\text{dist}(x_n, \text{Ker } T) \leq c \quad \forall n \in \mathbb{Z}$

Take  $\tilde{x}_n := x_n - w_n$ , where  $w_n \in \text{Ker } T : \|\tilde{x}_n\| \leq 2c$ .

We have  $\lim_{n \rightarrow \infty} T\tilde{x}_n = \lim_{n \rightarrow \infty} Tx_n$  by step 2a)  $\exists \tilde{x} : T\tilde{x} = y \quad \checkmark$

2.c)  $\text{dist}(x_n, \text{Ker } T) \rightarrow +\infty$ . Let us show that this situation does not occur. Suppose the converse:

Consider  $\tilde{x}_n = x_n - \underbrace{w_n}_{\text{Ker } T} : \text{dist}(x_n, \text{Ker } T) \leq \|\tilde{x}_n\| \leq 2\text{dist}(x_n, \text{Ker } T)$

For  $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$  we have  $Tz \rightarrow 0$ .

$$Tz_n = T \frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T \frac{x_n}{\|x_n\|} = \frac{Tx_n}{\|x_n\|} \rightarrow \begin{cases} y & \Rightarrow Tx_n \text{ is bdd in } X \\ +\infty & \end{cases}$$

$$\Rightarrow \|Tz_n\| \leq \frac{2\|y\|}{\|\tilde{x}_n\|} \rightarrow 0 \quad \text{for } n \text{ large enough}$$

At the same time,  $Tz_n = z_n - Kz_n$

$\Rightarrow \exists \{z_{n_k}\}$  s.t.  $\{Kz_{n_k}\}$  converges to some  $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{Tz_{n_k}}_{\rightarrow 0} + \underbrace{Kz_{n_k}}_{\rightarrow z} \rightarrow z$$

We have  $Tz = 0$  ( $= \lim Tz_{n_k} = \lim Tz_n = 0$ )

$\Leftrightarrow z \in \text{Ker } T$ ,  $0 = \text{dist}(z, \text{Ker } T) =$

$$= \lim_{k \rightarrow \infty} \text{dist}(z_{n_k}, \text{Ker } T)$$

$$= \lim_{k \rightarrow \infty} \text{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \text{Ker } T\right)$$

$$= \lim \frac{\text{dist}(\tilde{x}_{n_k}, \text{Ker } T)}{\|\tilde{x}_{n_k}\|}$$

$$= \lim \frac{\text{dist}(x_{n_k}, \text{Ker } T)}{\|\tilde{x}_{n_k}\|}$$

$$\stackrel{(*)}{\geq} \frac{1}{2} \quad \leadsto \text{contradiction}$$



Lemma: Let  $X$  be a Banach space,  $T \in \mathcal{B}(X)$ :

$\text{Ker}(T) = \{0\}$  and  $T^{k+1}X = T^kX$  for some  $k \geq 0$ .

Then  $\text{Ran } T = X$ .

Proof: We need to prove that  $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$ .

We know that:  $T^{k+1}a = T^k\tilde{a}$  for every  $a$  and some  $\tilde{a}$  depending on  $a$ .  $\downarrow \text{Ker } T^k \neq \{0\} \Rightarrow \text{Ker } T \neq \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}$$



Theorem [Fredholm]: Let  $X$  be a Banach space.

$K \in S_\infty(X)$ ,  $T = I - K$ . Then TFAE:

1)  $T$  is invertible in  $\mathcal{B}(X)$     2)  $\text{Ker } T = \{0\}$     3)  $\text{Ran } T = X$

1)  $T^*$  is invertible in  $\mathcal{B}(X)$     2)  $\text{Ker } T^* = \{0\}$     3)  $\text{Ran } T^* = X^*$

Proof: We will prove  $2 \Rightarrow 3 \Rightarrow 2' \Rightarrow 3' \Rightarrow 2$ ,  $1 \Rightarrow 1' \Rightarrow 2'$ ,  $2 \wedge 3 \Rightarrow 1$

(2)  $\Rightarrow$  (3): If  $T^k X = T^{k+1} X$  for some  $k$ , we are done by the lemma.

Define  $X_k := T^k X$ ,  $k \geq 0$ , and note that  $X_0 \supset X_1 \supset X_2 \supset X_3 \dots$

Assume that all inclusions are strict, i.e.  $X_k \supsetneq X_{k+1} \forall k$ .

The subspaces  $X_k$  are closed by the closed range lemma (by induction). By the almost orthogonality lemma:  $\exists \{y_k\}_{k=0}^\infty$  s.t.

i)  $y_k \in X_k \forall k$     ii)  $\|y_k\| = 1 \forall k$     iii)  $\text{dist}(y_k, X_{k+1}) \geq \frac{1}{2}$

Since  $K$  is compact  $\{Ky_k\}$  contains a convergent subsequence  $\{Ky_{k_j}\}_{j=1}^\infty$ . On the other hand, if  $j < m$

$$\begin{aligned} Ky_{k_j} - Ky_{k_m} &= (Ky_{k_j} - y_{k_j}) - (Ky_{k_m} - y_{k_m}) + y_{k_j} - y_{k_m} \\ &= y_{k_j} - y_{k_m} - \underbrace{(Ty_{k_j} - Ty_{k_m})}_{\in X_{k_j+1}} \end{aligned}$$

Since  $y_{k_m} \in X_{k_m}$ ,  $j < m$ ,  $k_j < k_m$ ,  $k_j+1 \leq k_m$ ,  $X_{k_m} \subset X_{k_j+1}$   
 $\Rightarrow y_{k_m} \in X_{k_j+1}$ ;  $Ty_{k_m} \subset T X_{k_j+1} \subset X_{k_j+1}$ ;  $Ty_{k_j} \in T(X_{k_j}) = X_{k_j+1}$   
 $\Rightarrow Ky_{k_j} - Ky_{k_m} = y_{k_j} + R$ ,  $R \in X_{k_j+1}$ . Since  $\|y_{k_j} + R\| \geq \frac{1}{2}$  by (iii),  
we get a contradiction ( $\{Ky_{k_j}\}$  is not Cauchy).

(3)  $\Rightarrow$  (2): Take  $\phi \in \text{Ker } T^*$ . We have

$$\begin{aligned} T^* \phi = 0 &\Leftrightarrow \langle x, T^* \phi \rangle = 0 \quad \forall x \in X \Leftrightarrow \langle Tx, \phi \rangle = 0 \quad \forall x \in X \Leftrightarrow \\ &\Leftrightarrow \langle y, \phi \rangle = 0 \quad \forall y \in X \quad (\text{by 3}) \Leftrightarrow \phi = 0 \end{aligned}$$

(2')  $\Rightarrow$  (3'): (the same as  $2 \Rightarrow 3$  using Schauder's theorem)

(3') $\Rightarrow$ (2): Assume that  $\text{Ran } T^* = X^*$  and take  $x \in \text{Ker } T$ . We have  $Tx = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \quad \forall \phi \in X^* \Leftrightarrow \langle x, T^* \phi \rangle \quad \forall \phi \in X^*$   $\Leftrightarrow \langle x, \Psi \rangle = 0 \quad \forall \Psi \in X^* \quad (\text{by 3'}) \Leftrightarrow x = 0$  Lemma on sufficient amount of functionals

Conclusion:  $(2) \Leftrightarrow (3) \Leftrightarrow (2') \Leftrightarrow (3')$

(1) $\Rightarrow$ (1'): We already know for arbitrary  $T \in \mathcal{B}(X)$ .

(1') $\Rightarrow$ (2'): Invertible operators are injective.

$(2') \Leftrightarrow (1), (3)$

(2 & 3) $\Rightarrow$ (1): This holds for every  $T \in \mathcal{B}(X)$  by the following fundamental theorem from general F.A.:

Theorem [linear mapping theorem]: Let  $X, Y$  be Banach spaces,  $T: X \rightarrow Y$  - linear bijection. Then  $T \in \mathcal{B}(X, Y) \Leftrightarrow T^{-1} \in \mathcal{B}(Y, X)$ .

Check injectivity:  $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0$  (by 1)  $\Leftrightarrow x_1 = x_2$

Surjectivity:  $\text{Ran } T = X$  (by 3) □

Remark: If  $T = I - K$ ,  $K \in S_\infty(X)$ , and  $\text{Ker } T = \{0\}$  then the equation  $Tx = y$  has a unique solution for every  $y \in X$ .

Remark:  $\text{Ker } T = \{0\} \Leftrightarrow \text{Ker } T^* = \{0\}$ , so we have proved half of Fredholm Alternative.

Theorem: Let  $X$  be a Banach space,  $T \in \mathcal{B}(X)$ . Then  $\overline{\text{Tx}} = \{x \in X \mid \langle x, \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T\}$ .

Remark: This implies the other half of Fredholm alternative since  $\overline{\text{Tx}} = \text{Tx}$  for  $T = I - K$ ,  $K \in S_\infty(X)$ .

Lemma [seperation lemma]: Let  $Y$  be a Banach space,  $Y_0 \subsetneq Y$  - a closed subspace, then  $\exists \phi \in Y^*: \phi|_{Y_0} = 0$ ,  $\phi(y) \neq 0$  for some  $y \in Y \setminus Y_0$ .

Proof: Take  $y \in Y \setminus Y_0$ , define  $\Phi_0: \text{span}\{y, Y_0\} \rightarrow \mathbb{C}$  by  $\Phi_0(cy + y_0) \mapsto c$ , for  $c \in \mathbb{C}$ ,  $y_0 \in Y_0$ .

$$1) \{cy + y_0 \mid c \in \mathbb{C}, y_0 \in Y_0\} = \text{span}\{y, Y_0\} \quad \text{clear } \checkmark$$

$$2) \underbrace{cy + y_0}_c = \underbrace{\tilde{c}y + \tilde{y}_0}_{\tilde{c}} \Leftrightarrow (c - \tilde{c})y = y_0 - \tilde{y}_0 \in Y_0 \Leftrightarrow c = \tilde{c} \Rightarrow \Phi_0(c) = \Phi_0(\tilde{c})$$

$\Rightarrow$  correctness OK

3)  $\Phi_0$  is linear - clear

$$4) \Phi_0|_{Y_0} = 0 \quad (c=0 \text{ on } Y)$$

$$5) |\Phi_0(cy + y_0)| \stackrel{?}{\leq} A \|cy + y_0\| \quad \forall c, y_0$$

$$|\Phi_0(cy + y_0)| = |c| = (\text{dist}(y, \underbrace{Y_0}_{\text{closed}}))^{-1} \cdot |c| \cdot \text{dist}(y, Y_0)$$

$$= \text{dist}(y, Y_0)^{-1} \cdot \text{dist}(|c|y, Y_0) = \text{dist}(y, Y_0)^{-1} \cdot \text{dist}(|c|y, Y_0)$$

$\nearrow$

$$|c|y = \alpha \cdot e \cdot y_0, |\alpha|=1$$

$$\text{dist}(|c|y, Y_0) = \text{dist}(\underbrace{|c|y_0}_{c}, \underbrace{Y_0}_y)$$

$$\leq \text{dist}(y, Y_0)^{-1} \|cy + y_0\|, \quad \text{so } A = \text{dist}(y, Y_0)^{-1} \text{ works}$$

$$6) \Phi_0(y) = 1 \neq 0$$

$\Rightarrow$  Use the Hahn-Banach theorem and extend  $\Phi_0$  to the whole  $Y$ . □

We actually proved that the lemma holds  $\forall y \in Y \setminus Y_0$ .

We can now prove the theorem from above.

Theorem: Let  $X$  be a Banach space,  $T \in \mathcal{B}(X)$ . Then  
 $\overline{Tx} = \{x \in X \mid \langle x, \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T^*\}$ .

Proof: We have  $Tx \subset E$ ,  $E = \{x \mid \langle x, \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T^*\}$ ,  
because  $\langle Tx, \phi \rangle = \langle x, T^* \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T^*$ .  
Then  $\overline{Tx} \subset \overline{E} = E$  since  $E$  is closed.  
 $(\begin{matrix} x_n \rightarrow x \text{ and } \langle x_n, \phi \rangle = 0 \text{ for some } \phi \in X^* \\ \text{then } \langle x, \phi \rangle = \lim \langle x_n, \phi \rangle = 0 \end{matrix})$

We now need to check  $\overline{Tx} \supset X$ . If not, the inclusion  $\overline{Tx} \subset E$  is proper and by the separation lemma  $\exists \phi : \phi|_{\overline{Tx}} = 0$  <sup>(\*)</sup> but  $\phi(e) \neq 0$  for some  $e \in E$ .

$$\begin{aligned} (*) \Rightarrow \phi|_{Tx} = 0 &\Leftrightarrow \langle Tx, \phi \rangle = 0 \quad \forall x \in X \Leftrightarrow \langle x, T^* \phi \rangle = 0 \quad \forall x \in X \\ &\Leftrightarrow \phi \in \text{Ker } T^* \Rightarrow \phi(e) = 0 \quad \forall e \in E \text{ by definition of } E, \text{ contradiction.} \blacksquare \end{aligned}$$

### Classical form of Fredholm alternative for integral equations

Theorem: Let  $(S, \mu)$  be a space with measure  $\mu$ , and let  $K(x, y) : S \times S \rightarrow \mathbb{C}$ :

$$\iint_{S \times S} |K(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$

Then either equation  $f(y) + \int_S K(x, y) f(x) d\mu(x) = 0$  <sup>(\*\*\*)</sup> has only the trivial solution  $f=0$  and the equation

$$f(y) + \int_S K(x, y) f(x) d\mu(x) = g(y)$$

is solvable for every  $g \in L^2(S, \mu)$  or the equation  $(**)$  has a non-trivial solution in  $L^2(S, \mu)$ .

Remark: Uniqueness implies existence.

Proof: Let's define  $T = I - K$ ,  $(Kf)(y) = \int_S K(x, y) f(x) d\mu(x)$ . Consider  $T$  as an operator on  $L^2(S, \mu)$ .

Then  $(**)$   $\Leftrightarrow \text{Ker } T = \{0\} \Leftrightarrow TL^2(S, \mu) = L^2(S, \mu)$  by Fredholm alternative,  
modulo the fact that  $K \in \text{Soo}(L^2(S, \mu))$ . □

↑ This we postpone until  
Hilbert Spaces theory.

Remarks: Further reading:

- 1)  $\dim \text{Ker } T = \dim \text{Ker } T^*$  if  $T = I - K$ ,  $K \in \text{Soo}$ , it coincides with the dimension of the space of solutions
- 2) There is a version of Fredholm theory for general operators  $T \in \mathcal{B}(X)$ :  $\dim(\text{Ker } T) < \infty$ ,  $\dim(X/\text{Ran } T) < \infty$ .

## Spectrum of compact operators

October 29, 2025

Definition: Let  $T \in \mathcal{B}(X)$ ,  $X$  Banach space.

$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$   
is called the **spectrum** of  $T$ .

Definition: A number  $\lambda \in \mathbb{C}$  is called an **eigenvalue** of  $T$  if  $\exists e \in X \setminus \{0\}$ .  $Te = \lambda e$ .

Definition:  $\sigma_p(T) := \{\lambda \text{ eigenvalue of } T\}$  ... point spectrum of  $T$ .

Remark: We have  $\sigma_p(T) \subset \sigma(T)$  for every  $T \in \mathcal{B}(X)$ .

Proof:  $\lambda \in \sigma_p(T) \Rightarrow \lambda I - T$  is not injective because  $\text{Ker}(\lambda I - T) \neq \{0\}$ .

Remark: In general, we might have  $\sigma_p(T) \neq \sigma(T)$  and even  $\sigma_p(T) = \emptyset$ .

Theorem: Let  $X$  be a Banach space,  $\dim X = +\infty$ , and let  $K \in S_\infty(X)$ . Then  $0 \in \sigma(K)$ ,  $\sigma(K) \setminus \{0\} \subset \sigma_p(T)$ , moreover, each eigenvalue has a finite multiplicity, and  $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$  for every  $r > 0$ .

Proof:  $0 \in \sigma(K)$  since  $0 \notin \sigma(K)$ , then  $\exists K^{-1} \in \mathcal{B}(X)$ :  
 $I = K K^{-1} = K^{-1} K$ , but then  $I \in S_\infty(X) \Rightarrow \dim X < \infty$ , contradiction.

Now let's prove that  $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$ .

Take  $\lambda \in \sigma(K)$  and assume that  $\lambda \notin \sigma_p(K)$ . Then

$$\text{Ker}(\lambda I - K) = \{0\} \Leftrightarrow \text{Ker}(I - \frac{1}{\lambda} K) = \{0\}$$

$\Rightarrow I - \frac{1}{\lambda} K$  is invertible by Fredholm theorem - contradiction.

Now let's prove that  $\dim E_\lambda < \infty$ .

$$E_\lambda := \{e \in X \mid Ke = \lambda e\} \quad \left( \begin{array}{l} \text{definition for :} \\ \lambda \text{ has finite multiplicity} \end{array} \right)$$

If this is not the case, there is a sequence  $\{e_n\}_{n=1}^\infty$  such that  $\text{dist}(e_n, \text{span}\{e_1, \dots, e_{n-1}\}) \geq \frac{1}{2}$ ,  $\|e_n\| = 1$ .

Consider  $\{Ke_n\} = \{\lambda_n e_n\}$ : we cannot choose a convergent subsequence from this sequence - contradiction with  $K \in S_\infty(X)$ .

To prove  $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$  for every  $r > 0$ , assume the converse and let  $e_n \in X$ :  $Ke_n = \lambda_n e_n$ ,  $\|e_n\| = 1$ ,  $\lambda_n \in \sigma_p(K)$ ,  $\lambda_n \neq \lambda_k$  for  $k \neq n$ .

Define  $E_n := \text{span}\{e_1, \dots, e_n\} \quad \forall n$ .

Observation:  $E_{n+1} \supsetneq E_n \quad \forall n$ .

Clearly  $E_{n+1} \supset E_n$ . If  $E_{n+1} = E_n$  for some  $n$ , there exists the first such  $n$ . Then

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^n \lambda_{n+1} \alpha_k e_k$$

$$\lambda_{n+1}e_{n+1} = \sum_{k=1}^n d_k \lambda_k e_k$$

$$\Rightarrow 0 = \sum_{k=1}^n d_k (\lambda_{n+1} - \lambda_k) e_k \Rightarrow d_{n+1} - \lambda_n = 0 \\ \Rightarrow \lambda_{n+1} = \lambda_n \dots \text{contradiction}$$

$\Rightarrow$  The observation is true.

Let's choose  $y_{n+1} \in E_{n+1}$  such that  $\|y_{n+1}\| = 1$ ,  $\text{dist}(y_{n+1}, E_n) \geq \frac{r}{2}$   
It remains to prove that  $\{K_{y_n}\}$  does not have a convergent subsequence.

$$x_{n+1} = d_{n+1}e_{n+1} + R_n, \text{ where } R_n \in E_n$$

$$Ky_{n+1} - Ky_{m+1} = \lambda_{n+1}d_{n+1}e_{n+1} + \tilde{R}_n - \lambda_{m+1}d_{m+1}e_{m+1} - \tilde{R}_m$$

Assume that  $n \geq m+1$ , then

$$\begin{array}{c} \tilde{R}_n - \underbrace{\lambda_{m+1}d_{m+1}e_{m+1}}_{\substack{\in E_n \\ \in E_{m+1} \subset E_n}} + \tilde{R}_m \in E_n \\ \uparrow \quad \uparrow \\ E_n \quad E_{m+1} \subset E_n \end{array}$$

$$\begin{aligned} \Rightarrow \|Ky_{n+1} - Ky_{m+1}\| &\geq \text{dist}(\lambda_{n+1}d_{n+1}e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \text{dist}(d_{n+1}e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \text{dist}(y_{n+1}, E_n) \\ &\geq r \cdot \frac{1}{2} > 0 \end{aligned}$$

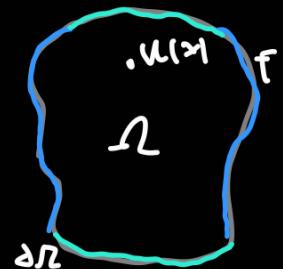
□

Scheme of the solution of Dirichlet problem in  $\mathbb{R}^n, n \geq 3$ , by means of Fredholm theory

Dirichlet problem: find  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  ( $\Omega$  is a domain in  $\mathbb{R}^n$ )  
 $\partial\Omega \in C^2$   
such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} \stackrel{(*)}{=} f \end{cases} \quad \text{where } f \in C(\partial\Omega)$$

Physics interpretation: heat



Scheme for solution: we search for a solution of the form

$$u(x) = \int_{\partial\Omega} K(x,y) \varphi(y) dy, \quad K(x,y) = c_n \frac{(x-y, n_y)}{\|x-y\|_{\mathbb{R}^n}^n}$$

$n_y$  is the outward unit normal,  $c_n \in \mathbb{R}$

$\Delta u = 0$  for every  $\varphi \in C(\partial\Omega)$ , we only need to find good  $\varphi$  (such that  $(*)$  will hold)

$$u_\varphi(x) = -\varphi(x) + \int_{\partial\Omega} K(x,y) \varphi(y) dy \quad \text{if } x \in \partial\Omega$$

$$u_\varphi = (-I + K)\varphi, \quad K\varphi = \int K(x,y) \varphi(y) dy$$

To check that  $\exists \varphi: u_\varphi = f$  on  $\partial\Omega$  we just check that  $K \in S_\infty(C(\partial\Omega))$  and  $\text{Ker}(-I + K) = \{0\}$   
 $\Rightarrow$  we are done by Fredholm alternative.

This is hard to prove (course on PDEs)

# PART II Banach Algebras

November 4, 2025

Definition: A is a **Banach algebra** if A is a Banach space with the operation of multiplication such that:

$$1) (xy)z = x(yz); \quad x, y, z \in A$$

$$2) (x+y)z = xz + yz$$

$$x(y+z) = xy + xz$$

$$3) (\alpha x)y = x(\alpha y); \quad \alpha \in \mathbb{C}$$

$$4) \exists e \in A. xe = ex = x \quad \forall x \in A.$$

$$5) \|xy\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A.$$

Examples: 1) X Banach space  $\Rightarrow \mathcal{B}(X)$  is a Banach algebra with unity  $e=I$  and norm  $\|T\| = \sup_{\|x\|=1} \|Tx\|$

2) Calkin algebra:  $\mathcal{B}(X)/S_\infty(X)$ ,  $e = I + S_\infty(X)$

$$\|T\|_{\mathcal{B}(X)/S_\infty(X)} = \inf_{k \in S_\infty(X)} \|T - k\| = \text{dist}(T, S_\infty(X))$$

3) K-compact Hausdorff space,  $C(K)$  is a Banach algebra,

$$e=1, \|f\|_{C(K)} = \max_{\substack{x \in K \\ f: K \rightarrow \mathbb{C}}} |f(x)|$$

4)  $W(\pi)$  - Wiener algebra on  $\pi := \{z \mid |z|=1\}$ .

$$\left\{ f = \sum_{k \in \mathbb{Z}} c_k z^k \mid c_k \in \mathbb{C}, \sum |c_k| < \infty \right\}$$

$$\|f\|_{W(\pi)} = \sum_{k \in \mathbb{Z}} |c_k|, \quad e=1, \quad \begin{matrix} \text{multiplication is the usual} \\ \text{multiplication of functions} \end{matrix}$$

$$\|f \cdot g\|_{W(\pi)} \leq \|f\|_{W(\pi)} \cdot \|g\|_{W(\pi)}$$

$\uparrow$  exercise

5)  $L^\infty(\mathbb{R})$ ,  $\|f\| = \underset{\mathbb{R}}{\text{esssup}} |f|$

6)  $H^\infty(D)$  - set of bounded analytic functions on  $D := \{|z| < 1\}$

$$\|f\|_{H^\infty(D)} = \sup_{|z| < 1} |f(z)|$$

Remark: 1) & 2) are noncommutative Banach algebras, others are commutative.

Remark: If  $A$  is a Banach space with multiplication and properties 1), 2), 3), 5), then we can always add the identity to convert  $A$  to a Banach algebra as follows:

$$\mathcal{A} = A \times \mathbb{C}, \quad (x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$

$$f(x, \alpha) = (fx, f\alpha)$$

$$(x, \alpha) \cdot (y, \beta) = (xy + \alpha y + \beta x, \alpha \beta)$$

$$((x + \epsilon e)ly + \beta e) = xy + \alpha y + \beta x + \alpha \beta e \quad e = (0, 1)$$

$$\|(x, \alpha)\| = \|x\| + |\alpha| \quad \leftarrow ! \text{ corrected on Nov. 5th}$$

$\Rightarrow \mathcal{A}$  is a Banach algebra with identity and  $(A, 0) \subset \mathcal{A}$ .

Example:  $L^1(\mathbb{R})$ , multiplication  $F * g = \int_{\mathbb{R}} f(y)g(x-y)dy$

$\|F\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f|dx$  - this is a Banach algebra without identity, and the above construction is equivalent to consideration of  $\mathcal{A} = \mathbb{C}\delta_0 + L^1(\mathbb{R})$

$$\left( \begin{array}{l} \text{measure } \delta_0(S) = \begin{cases} 1; & 0 \in S \\ 0; & 0 \notin S \end{cases} \\ \delta_0 * f = f * \delta_0 = \int_{\mathbb{R}} f(y) \delta_0(x-y) dy = f(x) \\ \langle f, \delta_0 \rangle = f(0) = \int_{\mathbb{R}} f(x) \delta_0(dx). \quad f \in C_c(\mathbb{R}) \end{array} \right)$$

measures  $\rightarrow$  no  $dy$

Definition: A Banach algebra,  $x \in A$ . We say  $x \in A$  is invertible if  $\exists x^{-1} \in A$ .  $xx^{-1} = x^{-1}x = e$ .

If  $x$  is invertible, then  $x^{-1}$  is unique.

Definition:  $G(A) := \{x \in A \mid x \text{ is invertible}\}.$

↑ we use  $G$ , because  $G(A)$  is a group

Definition:  $x \in A$ .  $\delta(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\}.$

Example: If  $A$  is the set of  $n \times n$  matrices with complex coefficients, then for  $T \in A$ ,  $\delta(T)$  is the set of eigenvalues (the usual spectrum of the matrix).

Example:  $A = C(K)$ ,  $\sigma(f) = ?$

$\lambda \in \mathbb{C}$ :  $\lambda - f$  is not invertible in  $C(K)$ .

Since  $g \in G(C(K)) \Leftrightarrow \frac{1}{g} \in C(K)$ , we have  $\lambda - f \in G(C(K)) \Leftrightarrow \frac{1}{\lambda - f} \in C(K)$   
 $\Leftrightarrow \lambda \notin f(K)$

$\Rightarrow \sigma(f) = f(K)$  ← compact non-empty subset of  $\mathbb{C}$

## Basic properties

Proposition 1: Let  $A$  be a Banach algebra. Then  $\|e\| \geq 1$ .

Proof:  $\|e\| = \|e \cdot e\| \leq \|e\| \cdot \|e\| \Rightarrow \cancel{\|e\|=0} \text{ or } \|e\| \geq 1$

Definition: A Banach algebra is called **unital** if  $\|e\|=1$ .

Proposition 2: If  $A$  is an arbitrary Banach algebra, then the algebra  $A \times \mathbb{C}$  is unital.

Proof:  $\|(0,1)\| = \|0\| + \|1\| = 1$

↑  
! Note: This was wrong previously and corrected on November 5th.

⚠ From now on, all Banach algebras are unital.

Proposition 3: If  $\frac{x_n}{y_n} \xrightarrow{y}$  in A, then  $x_n \cdot y_n \xrightarrow{} xy$ .

Proof:  $\|x_n \cdot y_n - xy\| \leq \underbrace{\|x_n - x\|}_{\xrightarrow{n} 0} \underbrace{\|y_n\|}_{bdd} + \|x\| \underbrace{\|y_n - y\|}_{\xrightarrow{0}} \xrightarrow{} 0$  □

Proposition 4: Let  $a \in A$ ,  $\|a\| \leq 1$ , then  $e-a \in G(A)$ .

Proof: Define  $(e-a)^{-1} = e+a+a^2+\dots = \sum_{k=0}^{\infty} a^k$  this series converges because  $A$  is a Banach space and  $\sum \|a^k\| \leq \sum \|a\|^k < \infty$

Let us check that  $(e-a)^{-1}(e-a) = e$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \underbrace{\left( \sum_{k=0}^n a^k \right)}_{e-a^{n+1}} (e-a) = e$$

$$e-a^{n+1} \xrightarrow[n]{} e, \text{ since } \|a^{n+1}\| \leq \|a\|^{n+1} \xrightarrow{} 0$$

Similarly,  $(e-a)(e-a)^{-1} = e$ .

Proposition 5:  $G(A)$  is open in A.

Proof: Let  $a \in G(A)$ ,  $b \in A$ , then  $a-b = a(\underbrace{e-a^{-1}b}_{\in G(A)})$  if  $\|a^{-1}b\| < 1$   
 $\|a^{-1}b\| < 1$  holds for all  $b$ :  $\|b\| \leq \frac{1}{\|a^{-1}\|}$   
 $\Rightarrow B(a, \frac{1}{\|a^{-1}\|}) \subset G(A) \Rightarrow G(A)$  is open.

Proposition 6: Let  $x \in G(A)$ ,  $x_n \xrightarrow{} x$  in A. Then  $x_n \in G(A)$  for  $n$  large enough, and  $x_n^{-1} \xrightarrow{} x^{-1}$  as  $n \rightarrow \infty$ .

Proof: Write  $x_n = x + z_n$ , we have  $x_n \in G(A)$  for  $n$  large enough by Proposition 5.

$$(x+z_n)^{-1} - x^{-1} = (x(e+x^{-1}z_n))^{-1} - x^{-1} = \underbrace{(e+x^{-1}z_n)^{-1}}_{y_n} \cdot x^{-1} - x^{-1} \xrightarrow{\substack{\uparrow \\ \text{Prop. 3}}} ex^{-1} - x^{-1} = 0$$

$$x_n \xrightarrow{} e, \text{ see Prop. 4: } \|y_n - e\| \leq \sum_{k=1}^{\infty} \|x^{-1} \cdot z_n\|^k \xrightarrow{} 0 \quad \|z_n\| \xrightarrow{} 0 \quad \boxed{\quad}$$

Theorem: Let  $A$  be a Banach algebra,  $a \in A$ . Then  $\sigma(a)$  is a nonempty compact subset of  $\mathbb{C}$ .

Proof:  $\sigma(a) = \mathbb{C} \setminus S(a)$ ,  $S(a) := \{\lambda \mid \lambda e - a \text{ is invertible}\}$   
resolvent set

$S(a)$  is open by Prop. 5  $\Rightarrow \sigma(a)$  is closed.

Let's check that  $\sigma(a)$  is bounded:  $\sigma(a) \subset \{\lambda \mid |\lambda| \leq \|a\|\}$   
 $\Leftrightarrow S(a) \supset \{\lambda \mid |\lambda| \geq \|a\|\}$ .

Take  $\lambda: |\lambda| \geq \|a\|$ , then  $\lambda e - a = \lambda e \underbrace{(e - \frac{a}{\lambda})}_{\text{invertible by Prop. 4}}$

It remains to check that  $\sigma(a) \neq \emptyset$ .

Assume that  $\sigma(a) = \emptyset$  and consider the function

$$f_\phi(\lambda) = \phi((\lambda e - a)^{-1}) \text{ for some } \phi \in A^*; \lambda \in \mathbb{C}$$

Let's check that  $f_\phi$  is analytic. Take  $\lambda_0 \in \mathbb{C}$ , consider

$$\lim_{\lambda \rightarrow \lambda_0} \frac{f_\phi(\lambda) - f_\phi(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \phi \left( \frac{(\lambda e - a)^{-1} - (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0} \right)$$

$$[x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}] = \lim_{\lambda \rightarrow \lambda_0} \phi \left( \frac{(\lambda e - a)^{-1} ((\lambda e - a) - (\lambda_0 e - a)) (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0} \right)$$

$$= \lim_{\lambda \rightarrow \lambda_0} \phi \left( \underbrace{(\lambda e - a)^{-1}}_{\hookrightarrow (\lambda_0 e - a)^{-1} \text{ Prop. 6}} (\lambda_0 e - a)^{-1} \right)$$

$$\underbrace{\phi \text{ continuous, continuity of multiplication}}_{\Rightarrow} = -\phi((\lambda_0 e - a)^{-2})$$

$\Rightarrow f_\phi$  is analytic ( $f_\phi \in \text{Hol}(\mathbb{C})$ )

Take  $\lambda: |\lambda| \geq 2\|a\|$ ,  $|f_\phi(\lambda)| = \|\phi\| \cdot \|((\lambda e - a)^{-1})\| = \|\phi\| \underbrace{\|(\lambda)^{-1}\|}_{\text{uniformly bounded for}} \underbrace{\|(e - \frac{a}{\lambda})^{-1}\|}_{\lambda: |\lambda| \geq 2\|a\|}$

$f_\phi$  is bdd on  $\mathbb{C}$  by the max. principle

$\Rightarrow F_\phi = c_\phi \in \mathbb{C}$ , but  $c_\phi = 0$  by  $(*)$

$\Rightarrow \phi(\lambda e - a) = 0 \forall \lambda \in \mathbb{C}, \forall \phi \in A^* \Rightarrow \lambda e - a = 0 \forall \lambda \in \mathbb{C} \Rightarrow \text{contradiction}$



Definition: Let  $A_1, A_2$  be Banach algebras.

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We say that  $A_1$  is isomorphic to  $A_2$  ( $A_1 \cong A_2$ ) if there exists a map  $j: A_1 \rightarrow A_2$ :  
 $j(\alpha a + \beta b) = \alpha j(a) + \beta j(b)$  (\*)  
 $j(ab) = j(a) \cdot j(b)$  (\*\*)  
 $\|j(a)\|_{A_2} = \|a\|_{A_1}$   
 $j$  is bijective

Theorem [Banach-Mazur]: Let  $A$  be a Banach algebra s.t.  $G(A) = A \setminus \{0\}$ . Then  $A \cong \mathbb{C}$ .

Proof: For every  $a \in A$  we have  $\sigma(a) \neq \emptyset$ . So there is a  $\lambda(a) \in \mathbb{C}$ :  
 $\lambda(a)e - a$  is not invertible  $\Leftrightarrow \lambda(a)e - a = 0 \Leftrightarrow a = \lambda(a)e$ .  
In particular, such  $\lambda(a)$  is unique.

Define  $j: A \rightarrow \mathbb{C}$ ,  $a \mapsto \lambda(a)$ . We have:

$$\begin{aligned} a+b &= \lambda(a+b)e \\ a+b &= \lambda(a)e + \lambda(b)e \end{aligned} \quad \left. \begin{aligned} \Rightarrow \lambda(a+b) &= \lambda(a) + \lambda(b) \\ \end{aligned} \right\}$$

Similarly,  $\lambda(\alpha a) = \alpha \lambda(a)$ ,  $\lambda(ab) = \lambda(a) \cdot \lambda(b) \Rightarrow (*)$ ,  $(**)$  ✓

$$\|j(a)\|_{\mathbb{C}} = \|a\|_A \Leftrightarrow |\lambda(a)| = \|a\|_A \Leftrightarrow \|\lambda(a)e\|_A = \|a\|_A \quad \checkmark$$

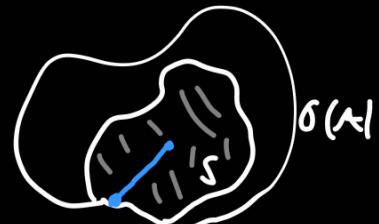
$$\begin{aligned} j(a) = j(b) &\Leftrightarrow \lambda(a)e = \lambda(b)e \Leftrightarrow a = b \\ j(e) = 1 &\Rightarrow j(A) = j(\mathbb{C} \cdot e) = \mathbb{C} \end{aligned} \quad \left. \begin{aligned} &\\ &j \text{ is a bijection} \end{aligned} \right\}$$

Definition: Let  $A$  be a Banach algebra, then  $r(a) := \sup \{|\lambda|, \lambda \in \sigma(a)\}$  is called the spectral radius of  $a$ .

Theorem: Let  $A$  be a Banach algebra. Then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

In particular, the limit above exists.



Proof: Step 1:  $r(a) \leq \|a\|$ , because

$$\lambda e - a = \lambda(e - \underbrace{\frac{a}{\lambda}}_b) \quad \|b\| < 1 \text{ for } \lambda: |\lambda| \geq \|a\|$$

invertible element in  $A$  (Prop. 4)

$$\Rightarrow \lambda e - a \in G(A) \Rightarrow \lambda \in \sigma(a) \Rightarrow \sigma(a) \subset \mathbb{B}[0, \|a\|]. \quad \checkmark$$

Step 2:  $r(a) \leq \|a^n\|^{1/n}$  for every  $n \geq 2$ .

Take  $\lambda \in \mathbb{C}$ , and consider

$$\lambda^n e - a^n = (\lambda e - a) \cdot p_\lambda(a), \quad p_\lambda \text{ is a polynomial}$$

If  $\lambda \in \sigma(a)$ , then  $\lambda^n e - a^n = z_1 \cdot z_2$ , where  $z_1, z_2 \in A$ ,  $z_1 \notin G(A)$ ,  $z_1 z_2 = z_2 z_1$ . If  $z = \lambda^n e - a^n$  is invertible,  $\exists z^{-1} \in A$ .

$$z^{-1} z_1 z_2 = z_1 (z_2 z^{-1}) = e \\ (\underbrace{z_1 z_2}_1 z_1) z^{-1} \Rightarrow z_1 \in G(A) \dots \text{contradiction}$$

$$\Rightarrow \lambda^n \in \sigma(a^n) \stackrel{\text{Step 1}}{\Rightarrow} |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a^n\|^{1/n}. \quad \checkmark$$

Step 3:  $\inf_{n \geq 1} \|a^n\|^{1/n} \leq \liminf \|a^n\|^{1/n} \leq \overline{\lim} \|a^n\|^{1/n} \leq r(a) \leq \inf \|a^n\|^{1/n}$   
 $\hookrightarrow$  This implies the claim.

Notation note:  $\overline{\lim} = \limsup$ ,  $\underline{\lim} = \liminf$ .

All that remains is  $\overline{\lim} \|a^n\|^{1/n} \leq r(a)$ . Take  $\phi \in A^*$ ,  $\|\phi\| \leq 1$ ,  $f_\phi(\lambda) := \phi((\lambda e - a)^{-1})$  for  $\lambda \in \sigma(a)$  (this is an analytic function on  $\sigma(a)$ )

$$\sum_{k \in \mathbb{Z}} : \frac{1}{2\pi i} \oint_{\substack{|\lambda|=r(a)+\varepsilon \\ |\lambda|=2\|a\|}} \lambda^k f_\phi(\lambda) d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^k f_\phi(\lambda) d\lambda =$$

$$= \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^k \phi \left( \lambda \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n} \right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^{k+1} \sum_{n=0}^{\infty} \frac{\phi(a^n)}{\lambda^n} d\lambda$$

$$= \phi(a^{k+2}) \quad (\text{Cauchy formula from complex analysis})$$

$$\Rightarrow \phi(a^{k+2}) \leq \left| \frac{1}{2\pi i} \oint_{\substack{|\lambda|=r(a)+\varepsilon \\ |\lambda|=2\|a\|}} \lambda^k f_\phi(\lambda) d\lambda \right|$$

$$\leq \max_{|\lambda|=r(a)+\varepsilon} (|\lambda^k| \cdot |\Phi(\lambda)|) \frac{1}{2\pi} (2\pi(r(a)+\varepsilon))$$

$$= (r(a) + \varepsilon)^{k+1} \underbrace{\|\Phi\|}_{\leq 1} \cdot \underbrace{\sup_{|\lambda|=r(a)+\varepsilon} \|(\lambda e - a)^{-1}\|}_{\text{constant depending only on } \varepsilon, \text{ because } |\lambda|=r(a)+\varepsilon \text{ is a compact set and } \lambda \mapsto \|(\lambda e - a)^{-1}\| \text{ is continuous}}$$

$$\sup_{\|\Phi\| \leq 1} |\Phi(a^{k+2})| = \|a^{k+2}\|$$

$$C_\varepsilon \cdot \underbrace{(r(a) + \varepsilon)^k}_{\text{constant depending only on } \varepsilon}$$

constant depending only on  $\varepsilon$ , because  $|\lambda|=r(a)+\varepsilon$  is a compact set and  $\lambda \mapsto \|(\lambda e - a)^{-1}\|$  is continuous

$$\Rightarrow \limsup \|a^k\|^{1/k} \leq r(a) + \varepsilon \quad \text{for every } \varepsilon > 0.$$



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Example:  $V: f \mapsto \int_0^x f(s) ds$ ,  $V \in \mathcal{B}(L^2[0,1])$ .

Let's prove that  $r(V) = 0 \Leftrightarrow \sigma(\{V\}) = \{0\}$ .

Proof 1:  $r(V) = \lim_{n \rightarrow \infty} \|V^n\|^{1/n}$ , so we need a formula for  $V^n$ .

By induction:  $(V^n f)(x) \stackrel{(*)}{=} \int_0^x f(s) \frac{(x-s)^{n-1}}{(n-1)!} ds$ .

$n=1$ : ✓

Assume (\*) for some  $n$  and compute

$$\begin{aligned} (V^{n+1} f)(x) &= \int_0^x \left( \int_0^t f(s) \frac{(t-s)^{n-1}}{(n-1)!} ds \right) dt \\ &= \int_0^x \left( f(s) \underbrace{\int_s^t \frac{(t-s)^{n-1}}{(n-1)!} dt}_{\frac{(t-s)^n}{n!} \Big|_s^x = \frac{(x-s)^n}{n!}} \right) ds \\ &= \int_0^x f(s) \frac{(x-s)^n}{n!} ds. \end{aligned}$$

$$\|V^n f\| \leq \max_{x \in (0,1)} \max_{s \in (0,x)} \frac{(x-s)^{n-1}}{(n-1)!} \left( \int_0^1 \left[ \int_0^x |f(s)| ds \right]^2 dx \right)^{1/2}$$

$$\leq \frac{1}{(n-1)!} \left( \int_0^1 \underbrace{\sqrt{x}}_{\leq 1} \underbrace{\int_0^x |f|^2 ds dx}_{\leq \|f\|^2} \right)^{1/2}$$

$$\leq \frac{\|f\|_{L^2[0,1]}}{(n-1)!}$$

$$\Rightarrow \|V^n\| \leq \frac{1}{(n-1)!}, \quad \lim \|V^n\|^{1/n} \leq \lim \left( \frac{1}{(n-1)!} \right)^{1/n} \quad \text{Stirling formula}$$

$$\begin{aligned} &= \lim \frac{1}{\left( \frac{1}{2\pi(n-1)} \left( \frac{n-1}{e} \right)^{n-1} \right)^{1/n} (1+o(1))^{1/n}} \\ &= \lim_{n \rightarrow \infty} \frac{e^{\frac{n-1}{n}}}{(2\pi)^{1/(n-1)}} \cdot \frac{1}{(n-1)^{1/(n-1)}} \xrightarrow[e]{} 0 \\ &= 0 \end{aligned}$$

□

$$\text{Proof 2: } Vf = \int_0^1 K(x,y) f(y) dy, \quad K(x,y) = \chi_{B_r}(x-y)$$

$$\int_0^1 \int_0^1 |K(x,y)|^2 dx dy < \int_0^1 \int_0^1 dx dy = 1 < \infty$$

$$\Rightarrow V \in S_\infty(L^2[0,1]) \Rightarrow \sigma(V) = \left\{ \begin{array}{l} \lambda \text{ is an eigenvalue} \\ \text{of } V \end{array} \right\} \cup \{0\}$$

$$Vf = \lambda f \Leftrightarrow \underbrace{\int_0^x f(s) ds}_{\in C[0,1]} = \lambda f(x) \quad \begin{matrix} \uparrow \\ \text{theorem about the spectrum} \\ \text{of compact operators} \end{matrix}$$

$$\Rightarrow \int_0^x f(s) ds \in C^1[0,1] \Rightarrow \dots \Rightarrow f \in C^\infty[0,1].$$

Differentiating (\*\*), we get  $\lambda f' = f$  on  $[0,1]$ .

Substituting 0 into (\*\*), we get  $\lambda f(0) = 0$ .

$$\text{So, if } \lambda \neq 0, \text{ then } \begin{cases} \lambda f' = f \\ f(0) = 0 \end{cases} \Leftrightarrow \begin{cases} f = C \cdot e^{\lambda x} \\ 0 = C \cdot e^0 \end{cases} \Leftrightarrow f = 0$$

$\Rightarrow$  any  $\lambda \neq 0$  is not an eigenvalue  $\Rightarrow \sigma(V) \cap (\mathbb{C} \setminus \{0\}) = \emptyset$

Since  $\sigma(V) \neq \emptyset$ , we get  $\sigma(V) = \{0\}$ .

# Commutative Banach algebras

Definition: Let  $A$  be a commutative Banach algebra and  $J \subset A$ .  $J$  is called a **proper ideal** in  $A$  if  $J$  is a linear subspace such that  $a \cdot J \subset J \quad \forall a \in A$ , and  $J \neq \{0\}$ ,  $J \neq A$ .

Definition:  $J$  is a **maximal ideal** if  $J$  is a proper ideal and there is no proper ideal  $J'$  such that  $J' \supsetneq J$ .

Proposition: Every proper ideal is contained in some maximal ideal. Take some proper ideal  $J$ ,  $J \neq A$ , and consider all proper ideals  $J'$ :  $J \subsetneq J'$ . This set is partially ordered by inclusion, and for every chain  $\{J'_\alpha\}_{\alpha \in I}$  of ideals ordered by inclusion, the set  $\bigcup_{\alpha \in I} J'_\alpha = J'$  is again a proper ideal.

• **linearity:**  $p \cdot x + q \cdot y \in J$  for every  $p, q \in \mathbb{C}$  and  $x, y \in J'$ , because  $\exists d_{x,y}$ ,  $x \in J_{d_x}$ ,  $y \in J_{d_y} \Rightarrow x, y \in J_{d_x}$  or  $x, y \in J_{d_y}$ , then  $px + qy$  are in the same  $J_{d_x}$  or  $J_{d_y}$  ✓

• **ideal property**  $aJ' = \bigcup_{\alpha \in I} aJ'_\alpha \subset \bigcup_{\alpha \in I} J'_\alpha \quad \forall a \in A$ .

• **properness:**  $J' \neq A$  (If  $J' = A$ , then  $e \in J'$ , then  $e \in J'_\alpha \Rightarrow eA \subseteq J_\alpha \subseteq A$ )

By Zorn's lemma, the set of all proper  $J'$ :  $J \subsetneq J'$  has a maximal element. □

Proposition: If  $M$  is a maximal ideal in  $A$ , then  $M$  is closed.

Proof: Let's prove that  $\overline{M}$  is a proper ideal.

•  $\overline{M}$  is linear ✓

•  $\overline{MA} \subseteq \overline{M}$ : true by continuity of multiplication ✓

•  $\overline{M} \neq A$  (If  $\overline{M} = A$ , then  $e \in \overline{M} \Rightarrow \exists x \in M \text{ . dist}(x, e) < 1 \Rightarrow x \in G(A)$   
 $\Rightarrow c = x \cdot x^{-1} \in xA \subset M \Rightarrow M = A$ , contradiction.) □

Example:  $A = C(K)$ ,  $M_{x_0} = \{f \in A : f(x_0) = 0\}$ .

Then  $M_{x_0}$  is a maximal ideal for every  $x_0 \in K$ .

$\cdot M_{x_0}$  is linear ✓

$\cdot M_{x_0} \cdot A \subseteq M_{x_0}$  ✓

$\cdot M_{x_0} \neq A$ , because  $1 \notin M_{x_0}$  ✓

$\cdot M_{x_0}$  is maximal: If  $\exists J$ -proper:  $J \supsetneq M_{x_0}$  then  $\exists f \in J$ .  $f(x_0) \neq 0$ .

But then  $\forall g \in A$  we have  $g = c \cdot f + h$  for  $c \in \mathbb{C}$  and  $h \in M$ ,  
where  $c$  is such that  $(g - cf)(x_0) = 0$ , i.e.  $c := \frac{g(x_0)}{f(x_0)}$ .

So  $A \subset \mathbb{C} \cdot f + M \subset J$ , contradiction.

Observation:  $M_{x_0} = \text{Ker } \Phi_{x_0}$ ,  $\Phi_{x_0}: f \mapsto f(x_0)$

$\Phi_{x_0}$  is a multiplicative functional:  $\Phi_{x_0}(fg) = \Phi_{x_0}(f) \cdot \Phi_{x_0}(g)$

Definition: Let  $\Phi \in A^*$ . We say  $\Phi$  is a multiplicative functional if  
 $\Phi(fg) = \Phi(f) \cdot \Phi(g)$   $\forall f, g \in A$ , and  $\Phi \neq 0$ .

Theorem: Let  $A$  be a commutative Banach algebra. TFAE

1)  $M$  is a maximal ideal in  $A$ .

2)  $M = \text{Ker } \Phi$  for some multiplicative functional  $\Phi \in A^*$ .

Proof: 2)  $\Rightarrow$  1) obvious: i)  $\text{Ker } \Phi$  is linear

ii)  $x \in \text{Ker } \Phi$ ,  $a \in A$ ,  $\Phi(xa) = \Phi(x)\Phi(a) = 0 \Rightarrow xa \in \text{Ker } \Phi$

iii)  $\text{Ker } \Phi \neq A$  because  $\Phi \neq 0$

iv)  $\text{Ker } \Phi$  is maximal, because  $\text{Ker } \Phi + Ca = A \quad \forall a: \Phi(a) \neq 0$   
(see \*\*\*)

1)  $\Rightarrow$  2) Note that  $A/M$  is a Banach algebra in which every non-zero element is invertible. If  $[a] \in A/M$ , then  $a \cdot A$  is an ideal in  $A$  containing  $M$ , but not equal to  $M \Rightarrow aA + M = A$   
 $aA + M \ni e \Rightarrow ab + M = e$  for some  $b \in A \Rightarrow [a][b] = [e]$ .

By Banach-Mazur theorem, there is an isomorphism  $j$  of Banach algebras  $A/M$  and  $\mathbb{C}$ . Let  $\Phi(x) := j([x])$ .

- i)  $\phi$  is linear, because  $j$  is linear.
- ii)  $\phi$  is multiplicative, because  $j$  is multiplicative.
- iii)  $\text{Ker } \phi = \{x \mid j([x]) = 0\} \Leftrightarrow [x] = 0 \Leftrightarrow x \in M.$

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Notation:  $M(A)$  - set of all maximal ideals in  $A$   
 $A_{\text{mult}}^*$  - set of multiplicative functionals on  $A$

Last time:  $\gamma: \Phi \mapsto \text{Ker } \phi$  maps  $A_{\text{mult}}^*$  onto  $M(A)$ .

Proposition:  $\gamma$  is a bijection.

Proof: We need to check that  $\gamma(\phi_1) = \gamma(\phi_2) \Rightarrow \phi_1 = \phi_2$ .

For this, note that  $\phi(e) = 1 \quad \forall \phi \in A_{\text{mult}}^*$ , because  $\begin{cases} \phi(e) = \phi(e) \cdot \phi(e) \\ \phi(e) \neq 0 \end{cases}$   
for every  $\phi \in A_{\text{mult}}^*$ . So, if  $\gamma(\phi_1) = \gamma(\phi_2)$ , we have  
 $0 = \phi_1(y - \phi_1(y)e) \Rightarrow \phi_2(y - \phi_1(y)e) = 0 \quad \forall y \in A$   
 $\Rightarrow \phi_2(y) = \phi_1(y) \phi_2(e) = \phi_1(y), \text{ so } \phi_1 = \phi_2$ .

Theorem: Let  $A$  be a commutative Banach algebra, and  $a \in A$ .  
TFAE:

- 1)  $a \in A \setminus G(A)$
- 2)  $a \in M$  for some  $M \in M(A)$
- 3)  $\exists \phi \in A_{\text{mult}}^*: \phi(a) = 0$

Proof: 1)  $\Rightarrow$  2):  $J = aA$  - a proper ideal in  $A$  ( $e \notin J$ )  
 $\Rightarrow \exists M \in M(A). M \subset J$

2)  $\Rightarrow$  3): Take  $\phi: \text{Ker } \phi = M \Rightarrow \phi(a) = 0$ .

3)  $\Rightarrow$  1): If  $\exists b \in A. ab = e \Rightarrow \phi(a) \cdot \phi(b) = \phi(e) = 1$ , but  $\phi(a) = 0$ . 

Corollary:  $\sigma(a) = \{\phi(a) \mid \phi \in A_{\text{mult}}^*\}$

Proof:  $\sigma(a) = \{\lambda \mid a - \lambda e \in A \setminus G(A)\} = \{\lambda \mid \exists \phi \in A_{\text{mult}}^*, \phi(a - \lambda e) = 0\}$   
 $= \{\lambda \mid \lambda \in \phi(a) \text{ for some } \phi \in A_{\text{mult}}^*\}.$

□

Remark:  $\forall \phi \in A_{\text{mult}}^*$ ,  $\|\phi\| = 1$ , because  $\phi(e) = 1$  and  $|\phi(a^k)|$  is uniformly bounded for every  $a \in B_A(0,1)$ .

## Applications

Theorem [Wiener]: Let  $f = \sum_{k \in \mathbb{Z}} c_k z^k$ , and  $\sum_{k \in \mathbb{Z}} |c_k| < \infty$ . Assume that  $f(z) \neq 0$  for every  $z \in \Pi = \{|z|=1\}$ . Then  $\frac{1}{f} = \sum_{k \in \mathbb{Z}} b_k z^k$  where  $\sum_{k \in \mathbb{Z}} |b_k| < \infty$ .

Proof: 1.  $W^1(\Pi) = \{\sum c_k z^k \mid \sum |c_k| < \infty\}$  is a Banach algebra:

Indeed,  $W(\Pi)$  is a Banach space with respect to the norm  $\|\sum c_k z^k\| = \sum |c_k|$ , and

$$\begin{aligned} \left\| (\sum c_k z^k)(\sum b_k z^k) \right\| &= \sum_{n \in \mathbb{Z}} \left| \left( \sum_{k \in \mathbb{Z}} c_k b_{n-k} \right) \right| \\ &\leq \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |c_k| |b_{n-k}| \\ &= \sum_{k \in \mathbb{Z}} |c_k| \sum_{j \in \mathbb{Z}} |b_j| \\ &= \left\| \sum c_k z^k \right\| \cdot \left\| \sum b_k z^k \right\| \end{aligned}$$

2. Identification of  $(W^1(\Pi))_{\text{mult}}^*$ :

Let  $\phi \in (W^1(\Pi))_{\text{mult}}^*$ ,  $\lambda = \phi(z)$ , then  $\phi(\frac{1}{z}) \cdot \phi(z) = 1$ ,  $\phi(\frac{1}{z}) = \frac{1}{\lambda}$   
 $|\lambda| \leq \|\phi\| \cdot \|z\| = 1$ ,  $|\frac{1}{\lambda}| \leq \|\phi\| \cdot \|\frac{1}{z}\| = 1$

$$|\lambda| \leq \|\phi\| \cdot \|z\| = 1, |\frac{1}{\lambda}| \leq \|\phi\| \cdot |\frac{1}{z}| = 1 \Rightarrow |\lambda| = 1$$

$\phi\left(\sum_{-N}^N c_n z^n\right) = \sum_{-N}^N c_n \lambda^n$ , and hence  $\phi(f) = f(\lambda)$   $\forall f \in W^1(\Pi)$ , because  $\left\{\sum_{-N}^N c_n z^n\right\}$  is dense in  $W^1(\Pi)$  and  $\phi$  is continuous.

### 3. Application of invertibility criterion:

$f \in W^1(\mathbb{T})$  is invertible  $\Leftrightarrow \exists \phi \in W^1(\mathbb{T})_{\text{mult}}^* . \phi(f) = 0 \Leftrightarrow f(z) = 0 \forall z \in \mathbb{T}$

This is the case in our case.  $\rightarrow$

$\Rightarrow f_g = 1, g \in W^1(\mathbb{T}) \Rightarrow g = \frac{1}{f}, g = \sum b_k z^k, |b_k| < \infty.$  □

Bezout equation: Let  $\{f_k\}_{k=1}^N \subset A(\bar{\mathbb{D}})$ . We are interested if there exists  $\{g_k\}_{k=1}^N \subset A(\bar{\mathbb{D}}) : \sum_1^N f_k g_k = 1.$

Necessary condition:  $\exists z_0 \in \bar{\mathbb{D}} . f_k(z_0) = 0$  for every  $1 \leq k \leq N.$

Theorem: Necessary condition is also sufficient.

Proof: 1.  $A(\mathbb{D})$  is a Banach algebra with respect to the norm

$$\|f\| = \max_{z \in \mathbb{D}} |f(z)| \quad \checkmark$$

2. Identification of  $A(\bar{\mathbb{D}})_{\text{mult}}^*$ :

$\phi \in A(\bar{\mathbb{D}})_{\text{mult}}^* \Leftrightarrow \phi(f) = f(\lambda) \text{ for some } \lambda \in \bar{\mathbb{D}}$  [exercise]

3.  $J = \left\{ \sum_1^N f_k g_k \mid g_k \in A(\bar{\mathbb{D}}) \right\}$  is a proper ideal in  $A(\bar{\mathbb{D}})$

$\Leftrightarrow J \subset M, M \in \mathcal{M}(A(\bar{\mathbb{D}})) \Leftrightarrow \sum_1^N (f_k g_k)(z_0) = 0 \quad \forall g_k \in A(\bar{\mathbb{D}})$   
for some  $z_0 \in \bar{\mathbb{D}}$

$\Leftrightarrow f_k(z_0) = 0 \quad \forall 1 \leq k \leq N$

At the same time:  $J$  is proper  $\Leftrightarrow e \notin J.$

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## Stone-Weierstrass Theorem

Definition:  $C_K(K)$  - the real Banach space of continuous functions on a compact Hausdorff space  $K$ .

Definition:  $A \subset C_{\mathbb{R}}(K)$  is a (real) Stone-Weierstrass algebra, if it is an algebra, and

(1)  $A$  does not vanish at any point  $x \in K$ , that is  $\forall x \in K. \exists f \in A. f(x) \neq 0$ .

(2)  $A$  separates points in  $K$ , that is,  $\forall x, y \in K. \exists f \in A. f(x) \neq f(y)$ .

Theorem [Stone-Weierstrass]: Let  $A \subset C_{\mathbb{R}}(K)$  be an algebra. Then  $A$  is dense in  $C_{\mathbb{R}}(K) \Leftrightarrow A$  is a Stone-Weierstrass algebra.

Example:  $K = [0, 1]$ ,  $A = \mathcal{P}$  ... the set of polynomials. Indeed,  $\mathcal{P}$  is an algebra, and

$\left. \begin{array}{l} \cdot 1 \in \mathcal{P} \Rightarrow (1) \text{ is satisfied} \\ \cdot x \in \mathcal{P} \Rightarrow (2) \text{ is satisfied} \end{array} \right\} \Rightarrow$  polynomials are dense in  $[0, 1]$   
 (Classical Weierstrass theorem)

Example:  $K = [0, 1]$ ,  $A = \text{span}\{\sin(2\pi kx), \cos(2\pi kx) \mid k \in \mathbb{Z}\}$

$A$  is not dense in  $C[0, 1]$ , because 0 and 1 are not separated.

On the other hand,  $A$  is dense in  $C[0, a]$  for every  $0 < a < 1$ .

For the proof we need some lemmas:

Lemma: Let  $a > 0$ . Then  $\exists \{p_k\} \in \mathcal{P}$  such that  $\|p_k - |x|\|_{C[-a, a]} \rightarrow 0$ .

Proof: By scaling, we can assume that  $a = 1$ . Then we need to approximate  $|x| = \sqrt{x^2} = \sqrt{1-y}$ ,  $y = 1-x^2$  by polynomials.

Thus, it suffices to approximate  $y \mapsto \sqrt{1-y}$  by polynomials uniformly on  $[0, 1]$ .

Taylor:  $\sqrt{1-y} = \sum_{k=0}^{\infty} c_k y^k$  for  $c_k = (-1)^k \binom{1/2}{k} = (-1)^k \frac{1/2(1/2-1)\dots(1/2-k+1)}{k!}$ .

Observation:  $c_0 = 1$ ,  $c_k < 0$  for  $k \geq 1$ . (e.g.  $c_1 = -\frac{1}{2}$ ,  $c_2 = \frac{3(-1)}{2} < 0$   
 $c_3 = -1 \cdot \frac{1}{2} \cdot \frac{3(-1)(-3-1)}{3!} < 0$ )

In particular, we have  $\sum_1^{\infty} |c_k| = \sup_{0 \leq y \leq 1} \left( - \sum_{k=1}^{\infty} c_k y^k \right) = \sup_{0 \leq y \leq 1} (c_0 - \sqrt{1-y}) = 1$ .

So  $\sum_{k=0}^{\infty} |c_k| < \infty \Rightarrow p_k = \sum_{j=0}^k c_j y^j$  are such that

$$\|p_k - \sqrt{1-y}\|_{C[0,1]} \leq \sum_{j=k+1}^{\infty} |c_j| \cdot \underbrace{\|y^j\|_{C[0,1]}}_1 \longrightarrow 0.$$
□

Lemma: If A is a SW algebra, then  $\forall x, y \in K. \exists h \in A. h(x)=1, h(y)=0$ .

Proof: We know that  $\exists f, g \in A. F(x) \neq 0, g(x) \neq g(y)$ .

1) If  $f(y)=0$ , then  $h := \frac{f}{f(x)}$ .

2) If  $f(y) \neq 0$ , and  $f(y) \neq f(x)$ , then  $h := \frac{f^2 - f(y)f}{f^2(x) - f(x)f(y)}$

$h = \frac{f - f(y)}{f(x) - f(y)}$  is not okay  $\frac{f}{f(x) - f(y)} \in A, \frac{f(y)}{f(x) - f(y)}$  might not be in A,  
if A does not contain 1

3) If  $f(y) \neq 0$ , and  $f(y) = f(x)$ , then  $\begin{pmatrix} f(x) \\ f(y) \end{pmatrix}, \begin{pmatrix} g(x) \\ g(y) \end{pmatrix}$  are not collinear in  $\mathbb{R}^2 \Rightarrow \exists \alpha, \beta \in \mathbb{R}$ .

$$\alpha \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} + \beta \begin{pmatrix} g(x) \\ g(y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow h := \alpha f + \beta g$$
□

Lemma: Let A be a SW algebra in  $C_R(K)$ . Then  $\forall f, g \in A$  we have  $\min(f, g) \in \bar{A}$ ,  $\max(f, g) \in \bar{A}$ .

Proof:  $\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}, \max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}$

So we only need to show that  $|h| \in \bar{A}$  for every  $h \in A$ .

Let's approximate  $|x|$  by polynomials  $p_k \in \mathcal{P}$  on  $[-a, a]$  for  $a = \|h\|_{C_R(K)}$ . Then  $p_k \circ h \in A, p_k(h) \rightarrow |h|$  on K.  $\Rightarrow |h| \in \bar{A}$ .

Lemma: Let  $x \in K, \varepsilon > 0, f \in C_R(K)$ . Then  $\exists g_x \in \bar{A}$  s.t.  $g_x(x) = f(x)$  and  $g_x(z) \geq f(z) - \varepsilon$  for every  $z \in K$ .

Proof: For every  $y \in K$  define  $g_{x,y}$  to be a function in A :  $\begin{array}{l} g_{x,y}(x) = f(x) \\ g_{x,y}(y) = f(y) \end{array}$   
Such a function is a linear combination of functions

taking values 0 and 1 at points  $x, y$ .

Let  $U = \{z \in K \mid g_{x,y}(z) > f(z) - \epsilon\}, y \in K$ .

For each  $x \in K$  we have  $K = \bigcup_{y \in K} U_y$ , because  $y \in U_y$ .

Each  $U_y$  is open, because  $g_{x,y} - f$  is continuous, and  $U_y$  is the preimage of  $(-\epsilon, +\infty)$  - an open set.

$\Rightarrow$  By compactness,  $\exists \{y_k\}_{k=1}^N. K = \bigcup_{k=1}^N U_{y_k}$ .

Now  $g_x := \max_{1 \leq k \leq N} g_{x,y_k} \in \bar{A}$  works.

Remark: A SW-algebra  $\Rightarrow \bar{A}$  SW-algebra

Proof of SW theorem: Assume that  $A$  is a SW-algebra, and take  $f \in C_R(K), \epsilon > 0$ . For every  $x \in K$  construct  $g_x : g_x(x) = f(x)$ ,

$g_x(z) \geq f(z) - \epsilon \forall z \in K$ . Consider  $V_x = \{z \in K \mid g_x(z) < f(z) + \epsilon\} \ni x$ .

Thus,  $V_x$  is open  $\forall x \in K$ , and  $K = \bigcup_{x \in K} V_x \Rightarrow \exists \{x_k\}_{k=1}^M. K = \bigcup_{k=1}^M V_{x_k}$ .

$$g = \min_{1 \leq k \leq M} g_{x_k} \in \bar{A} \text{ and } f(z) - \epsilon \leq g \leq f(z) + \epsilon$$

$\Rightarrow \|g - f\|_{C_R(K)} < \epsilon \Rightarrow A$  is dense in  $C_R(K)$ . This proves sufficiency.

Necessity: If  $A$  vanishes at some  $x_0 \in K$ , then 1 cannot be approximated uniformly by elements of  $A$ . If  $A$  does not separate points  $x, y$ , then the function  $f : f \in C_R(K), f(x) = 1, f(y) = 0$  cannot be approximated. Such a function exists by Uryson's lemma ( $K_1, K_2 \subset K$ -compact,  $K_1 \cap K_2 = \emptyset \Rightarrow \exists f \in C_R(K). f|_{K_1} = 1, f|_{K_2} = 0$ ).  $\square$

Example: Let  $K = \Pi = \{z \mid |z| = 1\}$ ,  $C(\Pi)$  the complex Banach space of continuous functions. Consider  $A = \mathcal{P} = \text{span} \{z^k \mid z \in \mathbb{Z}_r\}$ ,  $\mathbb{Z}_r = \mathbb{N} \cup \{0\}$ .  $A$  is an algebra,  $A$  does not vanish ( $1 \in A$ ),  $A$  separates points ( $z \in A$  (but we have a complex algebra)).

Observation:  $\int_{\Pi} \bar{z} \overline{p(z)} dm(z) = 0 \quad \forall p \in A$ .

Lobesgue measure on  $\Pi$ , normalised by  $m(\Pi) = 1$

Indeed, this follows from the fact that

$$\int_{-\pi}^{\pi} \bar{z} \cdot \bar{z}^k = 0 \quad \forall k \in \mathbb{Z}_+$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-i(k+1)t} dt = \frac{1}{2\pi} \left. \frac{e^{-i(k+1)t}}{-i(k+1)} \right|_0^{2\pi} = 0$$

Conclusion:  $\bar{z} \perp A$  in  $L^2(m)$ . So,  $A$  cannot be dense, because otherwise  $\bar{z} \perp C(\pi)$  in  $L^2(m) \Rightarrow \bar{z} \perp L^2(m)$ . Contradiction. So,  $A$  is not dense and SW-theorem (real version) does not work in the complex space.

Definition:  $A$  is a complex Stone-Weierstrass algebra if

- 1)  $A$  does not vanish.
- 2)  $A$  separates points
- 3)  $A$  is closed under conjugation  $f \mapsto \bar{f}$ .

In the example above, 3) does not hold.

Theorem [SW, complex]: An algebra  $A$  closed under conjugation is dense in  $C(K) \Leftrightarrow A$  is a SW complex algebra.

Proof:  $A$  is SW complex algebra  $\Rightarrow \text{Re } f, \text{Im } f \in A \quad \forall f \in A \Rightarrow$   
 $\Rightarrow \text{Re } A$  is a real SW-algebra  $\Rightarrow \text{Re } A$  is dense in  $C_R(K) \Rightarrow$   
 $\Rightarrow \text{Re } A + i \text{Re } A$  is dense in  $C(K) = C_R(K) + i C_R(K)$  and it is contained in  $A$ . The other direction is trivial. □