Operator theory

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PART I

Compact and Fredholm operators

Preliminaries

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Def: (X, ρ) - metric space if X-set, and ρ is a metric:

i) $\rho(x,y) \geq 0 \quad \forall x,y \in X$. $\rho(x,y) = 0 \Leftrightarrow x = y$

ii) $\rho(x,y) = \rho(y,x) \quad \forall x,y \in X$

iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \forall x,y,z \in X$

Def: $U \subseteq X$ is open if $\forall x \in U$. $\exists \delta > 0$. s.t. $B(x,\delta) \subset U$ $\quad (B(x,\delta) = \{y \in X \mid \rho(x,y) < \delta\})$

Def: $K \subset X$ is compact if every open cover $\{U_\alpha\}_{\alpha \in I}$ of K has a finite subcover.

cover: $\{U_\alpha\}_\alpha$ is a cover of K if $\bigcup_{\alpha \in I} U_\alpha \supset K$)

Def: A precompact set $A \subset X$ is a set $A \subset X$ s.t. $\overline{A}$ is compact.

closure of A in X

Def: $\{x_j\}_{j\geq 1}$ is Cauchy sequence in X if
$\forall \varepsilon>0.\ \exists N = N(\varepsilon).\ \rho(x_j, x_k) < \varepsilon \ \forall j,k \geq N(\varepsilon)$.

Def: X is complete if $\forall$ Cauchy $\{x_j\}_{j\geq 1} \subset X$.
$\exists x \in X.\ \rho(x_j, x) \to 0$ as $j \to \infty$
(Every Cauchy sequence converges.)

Ex. $\left(\mathbb{R}^n, \rho_{\mathbb{R}^n}(\{x_i\},\{y_j\}) := \sqrt{\sum |x_i - y_i|}\right)$ – complete metric space

Ex. $(\mathbb{Q}, \rho_{\mathbb{R}^n})$ – metric space but non-complete

Ex. $[0,1]$ is a compact-subset of $(\mathbb{R}, \rho_{\mathbb{R}})$

Th: $K \subseteq \mathbb{R}^n$ is compact $\iff$ closed and bounded

Def: $A \subset (X, \rho)$ is bounded if $\exists x \in X.\ \exists R>0.\ A \subset B(x,R)$

Ex.: $(X,\rho) = \ell^2(\mathbb{Z}) = \Big(\Big\{ \{\underset{\mathbb{C}}{x_j}\}_{j\in\mathbb{Z}} \ \Big|\ \sum_{j\in\mathbb{Z}} |x_j|^2 < \infty,$

$$\rho(\{x_j\},\{y_j\}) = \sqrt{\sum_{j\in\mathbb{Z}} |x_j - y_j|^2}\Big)$$

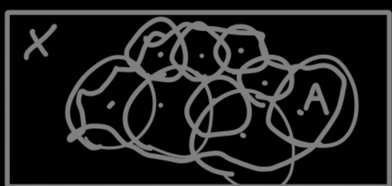
$$B[0,1] = \{ y \in \ell^2(\mathbb{Z}) : \rho(0,y) \leq 1\}$$

– bounded, closed but not compact HW

Theorem: Let (X,ρ) be a complete metric space, $A \subset X$. The following assertions are equivalent:

i) A is precompact

ii) $\forall \varepsilon > 0.\ \exists_N$ a finite ε-net $\{x_j\}_{j=1}^{N_\varepsilon}$ in A, that is, $\bigcup_{j=1}^{N} B(x_j, \varepsilon) \supset A$.



iii) $\forall \{x_j\}_{j\geq 1} \subset A$ there is a converging subsequence to some element $x \in X$.

<u>Proof</u>: i) $\Rightarrow$ ii) $\{U_x\}_{x\in A} = \{B(x,\varepsilon)\}_{x\in A}$ - open cover of A.

- is an open cover of $\bar{A}$:

$(\forall y \in \bar{A}.\ \exists x \in A.\ \rho(x,y) < \varepsilon/2 \Rightarrow y \in B(x,\varepsilon))$

definition of closure

$\Rightarrow \exists\ \{U_{x_j}\}_{j=1}^{N}$ - finite subcover of $\bar{A}$

$\Rightarrow\ \{U_{x_j}\}_j^N$ is a ε-net in A

ii) $\Rightarrow$ iii) Observe that $\forall \varepsilon > 0.$ any sequence $\{y_i\} \subset A$ has an infinite subsequence that is contained in some $B(x,\varepsilon)$. (we have a finite ε-net)

Assume that $\{y_k\}$ is arbitrary in A.

Cantor diagonal process

$y_1, \underbrace{y_2, y_3, y_4, \dots}$

$\underbrace{y_1}, y_{k_2}, \underbrace{y_{k_3}, \dots}$ where $\{y_{k_j}\} \subset B(x_2, \frac{1}{2})$

$\underbrace{y_1, y_{k_2}}, y_{kj_3}, y_{kj_4}$ where $\{y_{kj_s} \subset B(x_3, 1/3)$ $\in X$

$\underbrace{y_1, y_{k_2}, y_{kj_3}}, y_{kjs_4}$ ...

Consider $z_1 = y_1$

$z_2 = y_{k_2}$

$z_3 = y_{kj_3}$

$z_4 = y_{kjs_4}$

$\vdots$

<u>Claim</u>: $\{z_j\}$ is a Cauchy sequence.

Indeed $\underset{k<j}{\rho(z_j, z_k)} < \frac{1}{k}$

because $z_j, z_k \in B(x_k, \frac{1}{k})$ $\rho(z_j, z_k) < \frac{2}{K} \xrightarrow{} \infty$ $\Rightarrow$ ok

X is complete $\Rightarrow$ $\{z_j\}$ converges

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<u>iii) ⇒ i)</u>:

Plan: a) A contains a dense countable subset

a)+b)+c) under assumption 3)

b) If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $\bar{A}$ $\Rightarrow \exists \{U_{\alpha_j}\}_{j \in J}$ - an open coutable subcover of $\bar{A}$

c) $\Rightarrow \{U_{\alpha_j}\}_{j=1}^{r}$ is a cover of $\bar{A}$

a) Observe that $\forall \varepsilon > 0$ there exists at most $N(\varepsilon)$ points $A \supset \{y_j(\varepsilon)\}_{1 \le j \le N(\varepsilon)}$ s.t. $\rho(y_k(\varepsilon), y_j(\varepsilon)) > \varepsilon$ $\forall k \ne j$.

(If this is not true, then $\exists \{y_j(\varepsilon)\}_{j=1}^{\infty}$ such that $\rho(y_j(\varepsilon), y_k(\varepsilon)) > \varepsilon$ and it cannot contain a convergent subsequence by Cauchy criterion.)

Now $E = \left\{ y_k\{\frac{1}{n}\} \;\middle|\; 1 \le k \le N(\frac{1}{n}),\ n \ge 1 \right\}$ is a dense countable subset.

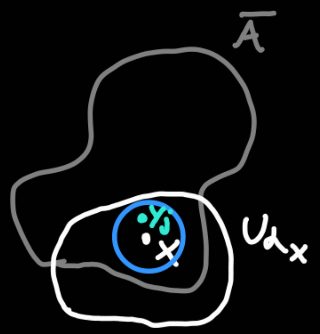
(E is dense since $\forall n. \forall x \in A.\ \min\limits_{1 \le k \le N(\frac{1}{n})} (\rho(x, y_k(\frac{1}{n})) \le \frac{1}{n}$ by construction)

b) Assume that $\{U_\alpha\}_{\alpha \in I}$ is some open cover of $\bar{A}$. For every $x \in \bar{A}$ define

$$\varepsilon(x) \overset{(*)}{:=} \frac{\sup\{\varepsilon > 0 \mid B(x, \varepsilon) \subset U_\alpha \text{ for some } \alpha\}}{2} > 0$$

<u>Claim</u>: if $\{y_j\}_{j=1}^{\infty}$ is coutable dense subset in A, then

$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$.



Take $x \in \bar{A}$ $\exists d_x.\ x \in U_{d_x}$, and let $h(x) > 0$ such that $B(x, h(x)) \subset U_{d_x}$ (U_{d_x} is open $\Rightarrow h(\varepsilon)\ \exists$)

Find y_j such that $\rho(x, y_j) < \frac{h(x)}{10}$.

Then, since $h(x) \leq 2\varepsilon(x)$, $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow \rho(x, y_j) < \varepsilon(y_j)$

$\varepsilon(y_j) \geq \frac{h(x)}{10}$ - Because $B(y_j, \frac{h(x)}{5}) \subset U_{d_x}$ by triangle inequality and $\varepsilon(y_j)$ satisfies $(*)$.

Since $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, then $\{U_{d_{y_j}}\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, where $U_{d_{y_j}}$ is the set U_d from the definition of $\varepsilon(y_j)$

(that is, $U_{d(x)} \supseteq B(y_j, \varepsilon(y_j))$).

Since we have $(**)$, $\bigcup_{j=1}^{\infty} U_{y_{d_i}} \supset \bar{A}$.

c) <u>Claim</u>: $\exists N.\ \{U_{d_{y_j}}\}_{j=1}^{N}$ is a cover of $\bar{A}$.

Suppose this is not the case $\Rightarrow \forall j \geq 1.\ \exists x_j \in A_j \setminus \bigcup_{k=1}^{j} U_{y_j}$.
Consider $\{x_j\}_{j=1}^{\infty}$, and assume that the sequence $\{x_{j_k}\}$ - convergent to some $x \in X$. Note that $x \in \bar{A}$ ($x_j \in \bar{A}$).
$\Rightarrow \exists j_*.\ x \in U_{d_{j_*}} \Rightarrow \exists \delta > 0.$ s.t. $B(x, \delta) \subset U_{d_{j_*}}$, but $x_{j_k} \notin U_{d_{j*}}$ for large k by construction.
(in particular, $x_{j_k} \notin B(x, \delta)$, hence $\rho(x, x_{j_k}) > \delta$, but this contradicts the fact that $x_{j_k} \longrightarrow x$).
We have shown that $\{x_j\} \subset \bar{A}$ cannot have a convergent subsequence.
Then if $\tilde{x}_j \in A.\ \rho(\tilde{x}_j, x_j) < \frac{1}{j}$, then $\{\tilde{x}_j$ also has no convergent subsequence. So, we assumed there is no

finite subcover $\{U_{\alpha_{y_j}}\}$ and found a sequence $\{\tilde{x}_j\} \subset A$ that has no converging subsequence, a contradiction with 3). Therefore 3) $\Rightarrow$ 1). $\blacksquare$

Examples of compact sets and their properties:

1) $K \subset (X, \rho)$ is compact $\Rightarrow$ K is bounded

Indeed, if K is not bounded, then $\{B(x, n)\}_{n \geq 1}$ is an open cover without a finite subcover.

2) $K \subset (X, \rho)$ is compact, then it is closed
($\Leftrightarrow$ $\{x_j\} \subset K$ such that $x_j \to x$ in (X, ρ) we also have $x \in K$)

Let's check that $X \setminus K$ is open. Take $y \in X \setminus K$, take $x \in K$, let $\delta(x) > 0$. $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$
$\{B(x, \delta(x))\}_{x \in K}$ is an open cover, let $\{B(x_j, \delta(x_j))\}_{j=1}^{N}$ be a finite subcover, then $\delta := \min\limits_{1 \leq j \leq K} \delta(x_j)$, $B(y, \delta) \cap K = \emptyset$
$\Rightarrow$ $X \setminus K$ is open.

<u>Another proof</u>: Suppose $\{y_j\} \subset K$ s.t. $y_j \to y$, $y \notin K$.

$$U_j = \{x \in X \mid \rho(x, y) > \tfrac{1}{j}\}$$

$\{U_j\}_{j=1}^{\infty}$ open cover, $\{U_{j_k}\}_{k=1}^{N}$ finite subcover

$$\varepsilon := \min_{1 \leq k \leq N} \left(\frac{1}{j_k}\right), \quad \rho(x, y) > \varepsilon \quad \forall x \in K \quad \text{contradiction} \quad \blacksquare$$

linear space = vector space

3) Let X be a finite-dimensional complete linear normed space. Then $E \subset X$ is compact $\Leftrightarrow$ E is closed and bounded.

($\Rightarrow$): Examples 1+2.

($\Leftarrow$): $X = \{ \sum_1^N a_k e_k \mid c_k \in \mathbb{C} \}$, $N = \dim X$, $\{e_k\}_1^N$ - basis

$$\varphi\left(\sum a_k e_k\right) = \max_{1 \le k \le N} |c_k| \quad \text{- norm on } X$$

Since all norms on finite dimensional vec. spaces are equivalent.

$\exists\, A, B > 0.\quad A\|x\| \le \varphi(x) \le B\|x\| \quad \forall x \in X.$

in particular the set $\left\{ \{c_k(x)\}_1^N,\ \begin{array}{l} x \in E \\ x = \sum c_k(x) e_k \end{array} \right\}$

is bounded (= bdd) in $\mathbb{C}^n$ For every bounded $E \subset X$.

$$\sup_{x \in E} \varphi(x) \le B \cdot \sup_{x \in E} \|x\| < \infty$$

$\Rightarrow$ for any sequence $\{ \{c_k(x_j)\}_{k=1}^N \}_{j=1}^\infty$ one can extract a converging subsequence in $\mathbb{C}^n$, i.e.

$$c_k(x_{j_n}) \longrightarrow c_k \qquad n \to \infty.$$

But then $\sum_1^N c_k(x_{j_n}) e_k \longrightarrow \sum c_k e_k$ in X

$\Rightarrow$ bounded subsets are precompact in X

$\Rightarrow$ bdd + closed sets are compact

4) $\ell^2(\mathbb{Z}) = \{ \{c_k\}_{k \in \mathbb{Z}} \mid \sum |c_k|^2 < \infty \}$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^\infty |c_k|^2}$$

$$B[0,1] = \{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \|\{c_k\}\| < 1 \}$$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite $\frac{1}{2}$-net in $B[0,1]$, beca $\rho(e_k, e_j) > 1/2$ for $e_k = (0, \dots, 0, \underset{k\text{th place}}{\hat{1}}, 0, \dots, 0)$.

Definition: X is a Banach space if it is a linear normed space such that X is complete with respect to this norm.

Example: Let (K, ρ) be a metric compact space.

$$C(K) := \{ f : K \to \mathbb{C} \mid \text{cont. in the metric } \rho \}$$
$$(\Leftrightarrow f(x_j) \to f(x) \quad \forall x_j \to x \quad \text{in } (K, \rho))$$

$$\|f\|_{C(K)} = \|f\| := \max_{x \in K} |f(x)| \quad \text{- norm in } C(K)$$

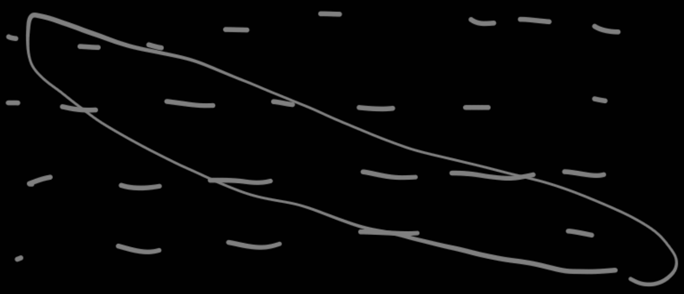
Theorem: [Arzela-Askoli]: Assume that K is a complete compact metric space. $E \subset C(K)$ is precompact $\Leftrightarrow$

$\Leftrightarrow \begin{cases} 1) & E \text{ is bounded in } C \\ 2) & \text{Functions in } E \text{ are equicontinuous, that is,} \end{cases}$

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0. \ |f(x) - f(y)| < \varepsilon \quad \forall x, y \in K. \ \rho(x,y) < \delta_\varepsilon$
$\forall f \in E$

We will need $1) + 2) \Rightarrow$ precompactness.

Proof: Find a dense sequence $\{x_j\}$ in K.
(such sequence exists because K is compact)
Then take $\{f_n\}$ arbitrary sequence in E.
We want to find a converging subsequence of $\{f_n\}$
(then E - precompact)
For this find a subsequence $\{f_{n_k}\}$ such that
$$f_{n_k}(x_j) \longrightarrow f(x_j) \quad \text{for every } j$$
(cantor diagonalization process + uniform boundedness)



look at the proof from the 1st lecture

<u>Claim</u>: f_{n_k} is Cauchy sequence in $C(K)$.

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<u>Aim</u>: $\|f_{n_s} - f_{n_m}\|_{C(K)} \longrightarrow 0$ as $s, m \to \infty$

For simplicity let $g_s = f_{n_s}$ $s \geq 1$.

Idea:

$$|g_s(x) - g_m(x)| \leq \underset{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x}}{|g_s(x) - g_s(x_j)|} + \underbrace{|g_s(x_j) - g_m(x_j)|}_{\substack{\text{take } s,m \text{ large} \\ \text{enough:} \\ \leq \varepsilon/3}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x \\ \text{(equicontinuity)}}}$$

To make the idea work we need to check that in this construction we can deal only with finite number of points x_j, $j = 1, \dots, N(\varepsilon)$.

For this it suffices to find $N(\delta_\varepsilon)$ such that $\rho(x, x_j) < \delta_\varepsilon$ for every $x \in K$ and x_j, $j = 1 \dots N_\varepsilon$.
($\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is δ_ε-net).

So, it remains to show that if $\{x_j\}_{j=1}^{\infty}$ is dense then $\forall \delta_\varepsilon > 0 . \exists N(\delta_\varepsilon) . \{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net.
To this end, let $\{y_k\}_1^N$ is a $\delta_\varepsilon/2$-net in K (K is compact)
Let $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ be the part of $\{x_j\}$ such that

$$\operatorname{dist}\left(\{x_j\}_{j=1}^{N(\delta_\varepsilon)}, y_k\right) \leq \delta_\varepsilon/2 \quad \forall 1 \leq k \leq N.$$

$\Rightarrow$ then $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net by triangle inequality. ∎

($\|g_s - g_m\|_{C(K)} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ for s, m large enough)

<u>Example</u>: $K=[0,1]$, $E_A = \left\{ f \in C[0,1],\ f(0)=0,\ f \text{ is Lipshitz with constant at most } A \right\}$, $A>0$

<u>E_A is compact</u>

i) E_A is bounded in $C[0,1]$:

$|f(x)| \leq |f(x)-f(0)| \leq A|x| \leq A$

$E_A \subset B(0,A)$

ii) $|f(x)-f(y)| \leq A|x-y| \leq A\delta = \varepsilon$ if $\varepsilon>0$, $\delta := \frac{\varepsilon}{A}$, $x,y \in [0,1]$: $|x-y| \leq \delta$

$i + ii + AA$ theorem $\Rightarrow E_A$ is precompact

iii) E_A is <u>closed</u>

if $f_n \rightrightarrows f$ in $C(K)$ then $f_n(0) \to f(0) \Rightarrow f(0)=0$

$$|f_n(x)-f_n(y)| \leq A|x-y|$$

$$\downarrow$$

$$|f(x)-f(y)| \Rightarrow f \text{ is } Lip(A)$$

<u>Example</u>: Let $E = \left\{ \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}, \text{ where } c_k \in \mathbb{C} : |c_k| \leq \frac{1}{k^2+1} \right\}$

Then E is compact as well in $C[0,1]$.

i) bbs: $f \in E$. $\|f\| \leq \sum_{k\in\mathbb{Z}} \frac{1}{k^2+1}$

Details: exercise

ii) equicontinuity $f = \sum_{|k|\leq N} + \sum_{|k|>N}$ small if N large

Lipshitz with some constant A_N ~ does not depend on f

Compact operators: basic properties

<u>Definition</u>: Let X, Y be Banach spaces, $T: X \to Y$ a linear map. T is called bounded if $T(B(0,1))$ is a bounded in Y set in Y. T is called compact if $T(B(0,1))$ is a precompact set in Y. ($B(0,1) = \{\|x\|_X < 1\}$) bounded linear operator

<u>Some observations</u>:

1) If $S \subset X$ is bdd then $T(S)$ is bdd for any bounded operator
precompact compact

2) T is compact $\Rightarrow$ T is bounded
(precompact sets are bounded)

3) with the norm $\|T\| = \sup\limits_{x \in B(0,1)} \|Tx\|_Y$, the set of bdd linear operators becomes a linear normed space, to be denoted by $B(X,Y)$ or $B(X)$ if $X=Y$.

4) A linear map between Banach spaces X, Y is continuous if and only if it is bounded.
Hint: $\|Tx - Ty\| \leq \|T\| \cdot \|x-y\|$, so bounded operators are Lipshitz.

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<u>Definition</u>: Banach algebra is an associative algebra which is a linear space with a norm $\|\cdot\|$ such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$ for any elements in this algebra.

<u>Proposition</u>: Let X be a Banach space, then $B(X, X)$ is a Banach algebra.

Proof: $\alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{B}(X, X)$ $\forall T_1 T_2 \in \mathcal{B}(X, X)$ (proved)

$T_1 \cdot T_2 \in \mathcal{B}(X,X)$, since $\forall x \in X$. $\|T_1 T_2 x\| \leq \|T_1\| \cdot \|T_2 x\|$

$\Rightarrow \|T_1 T_2\| \leq \|T_1\| \cdot \|T_2\|$

$$\|T_1 T_2\| = \sup_{\substack{y \in X \\ \|y\| \leq 1}} \|T_1 T_2 x\|$$

since $T_1 \in \mathcal{B}(X,X)$ we have $\|T_1 y\| \leq \|T_1\| \|y\|$ $\forall y \in X$

We see that $T_1 T_2 \in \mathcal{B}(X,X)$ and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

Now let us prove that $\mathcal{B}(X,X)$ is Banach.

Let us show that $\sum_{n=1}^{\infty} B_k$ converges in $\mathcal{B}(X,X)$ if $\sum \|B_k\| < \infty$. (*)

$T_n = \sum_{k=1}^{N} B_k$, $x \in X$, $\|T_N x - T_M x\| = \|\sum_{M+1}^{N} B_k x\| \leq \sum_{M+1}^{N} \|B_k\| \cdot \|x\| \xrightarrow[M,N \to \infty]{} 0$, $N > M$

because of (*).

$\Rightarrow \{T_N x\}_N$ Cauchy in X, but X-banach $\Rightarrow \exists Tx = \lim_{N \to \infty} T_N x$

Moreover, $\|Tx\| = \lim \|T_N x\| \leq \lim_{N \to \infty} \sum_{1}^{N} \|B_k\| \|x\| \leq \left(\sum_{1}^{\infty} \|B_k\|\right) \cdot \|x\|$.

$\Rightarrow T \in \mathcal{B}(X,X)$, $\|T\| \leq \sum_{1}^{\infty} \|B_k\|$

$$\sup_{\substack{x \in X \\ \|x\| < 1}} \|Tx - T_N x\| = \lim_{M \to \infty} \|T_N x - T_M x\| \leq \sup_{\|x\| < 1} \lim_{N \to \infty} \sum_{M+1}^{N} \|B_k\| \cdot \|x\| = \sum_{M+1}^{\infty} \|B_k\| \xrightarrow[(*)]{M \to \infty} 0$$

$\blacksquare$

Proposition: The set $S_\infty(X)$ $= S_\infty(X,X)$ (index ∞ will be explained later) of all compact operators on X is a two-sided ideal in $\mathcal{B}(X) = \mathcal{B}(X,X)$: $\forall T_1 \in S_\infty(X)$. $\forall T_2 \in \mathcal{B}(X)$. $T_1 T_2 \in S_\infty(X)$ and $T_2 T_1 \in S_\infty(X)$.

Proof: Take $\{x_n\}_1^\infty$ s.t. $\|x_n\| \leq 1$, and let us check that there is a subsequence $\{x_{n_k}\}$: $T_1 T_2 x_{n_k}$ converges.

Note that $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$. T_1 takes $B_X(0, \|T_2\|)$ into a precompact subset of $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$ - convergent subsequence

Now let's consuder $\{T_2 T_1 x_n\}$. Note that $\{T_1 x_{n_k}\}$ - convergent subsequence ($T_1 \in S_\infty(X)$). Then $\{T_2 T_1 x_{n_k}\}$ converges, since T_2 is continuous.

$\blacksquare$

Proposition: $S_\infty(X,Y)$ is a closed subset in $\mathcal{B}(X,Y)$, i.e. $T_n(X,Y)$, $T_n \longrightarrow T$ in $\mathcal{B}(X,Y) \Rightarrow T \in S_\infty(X,Y)$?

Proof: Let's find a finite ε-net in $T(B_X[0,1])$.
Take finite $\varepsilon/3$-net for $T_n B_X(0,1)$ for $n : \|T-T_n\| \leq \varepsilon/2$; denote it by $\{y_k\}_{k=1}^N$, then

$$\|Tx - Tx_k\| \leq \underbrace{\|Tx - T_n x\|}_{A} + \underbrace{\|T_n x - T_n x_k\|}_{B} + \underbrace{\|T_n x_k - T x_k\|}_{C} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \leq \varepsilon.$$

for any $1 \leq k \leq N$
so choose $k : B \leq \varepsilon/2$
and note $A \leq \|T-T_n\| \leq \varepsilon/3$
for every $x \in B_X(0,1)$
$C \leq \varepsilon/3$

Corollary: If T is a limit of finite-rank operators in $\mathcal{B}(X,Y)$, then $T \in S_\infty(X,Y)$.

Proof: Since finite-rank operators are in $S_\infty(X,Y)$, we have $T \in S_\infty(X,Y)$ by the previous proposition. ☒

Remark: At a general Banach space $\exists T \in S_\infty(X,Y)$ such that $\nexists (T_n)_n : \operatorname{rank} T_n < \infty$ and $\|T-T_n\| \longrightarrow 0$.

Definition: Let X be a Banach space. $\{e_k\}_{k=1}^\infty$ is a *Schauder basis* if $\forall x \in X.\ \exists! \{c_k(x)\}_{k=1}^\infty$ such that $x = \sum_{k=1}^\infty c_k(x) e_k$, where the series converges in X.

Theorem: Let X be a Banach space with Schauder basis, then $T \in S_\infty(X) \Leftrightarrow \exists T_n .\ \operatorname{rank} T_n \leq n$ and $\|T-T_n\| \to 0$.
(here $\operatorname{rank} S = \dim S(X)$ $\forall S \in \mathcal{B}(X)$)

Proof: ($\Leftarrow$): we already know

($\Leftarrow$): Let $T \in S_\infty(X)$, and let $P_n : x \longmapsto \sum_{k=1}^{\infty} c_k(x) e_k$.

<u>P is linear</u>: $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X$. $P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$?

$$\left.\begin{array}{l} \text{If } x = \sum c_k(x) e_k \\ \quad y = \sum c_k(y) e_k \end{array}\right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha c_k(x) + \beta c_k(y)) e_k$$

$$\alpha x + \beta y = \sum_{k=1}^{\infty} c_k(\alpha x + \beta y) e_k$$

by uniqueness by def. of Schauder basis

$$c_k(\alpha x + \beta y) = \alpha c_k(x) + \beta c_k(y) \quad \forall k$$

Then $P_n(\alpha x + \beta y) = \sum_{n=1}^{n} c_k(\alpha x + \beta y) e_k = \sum_{k=1}^{n} \alpha c_k(x) e_k + \sum_{n=1}^{k} \beta c_k(y) e_k$

$= \alpha P_n(x) + \beta P_n(y) \Rightarrow P_n$ linear

Note that $T_n := P_n T$ are such that $\operatorname{rank}(T_n) \leq n$ because $\dim P_n T(X) \leq \dim P_n(X) \leq n$.

It remains to show that $T_n \longrightarrow T$ in $\mathcal{B}(X)$. Since T is compact, $\forall \varepsilon > 0$. $\exists \{x_k\}_{k=1}^{N}$ such that $\|x_k\| \leq 1$ $\forall k$ and $\{T x_k\}_{k=1}^{N}$ is a ε-net in $T(B_X(0,1))$. Now take $x \in B_X(0,1)$ and write $\|Tx - T_n x\| \leq \|Tx - Tx_k\| + \|Tx_k - T_n x\| + \|T_n x_k - T_n x\|$

$$\leq \underbrace{\|Tx - Tx_k\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_k - P_n T x_k\|}_{\substack{\leq \varepsilon \text{ if } n \\ \text{large enough} \\ \text{for any fixed } k}} + \underbrace{\|P_n T x_k - P_n T x\|}_{\leq \|P_n\| \cdot \|Tx_k - Tx\|}$$

$$\leq \varepsilon + \varepsilon + \underbrace{\sup \|P_n\|}_{<\infty} \varepsilon$$

$\sup \|P_n\| < \infty$ by Banach-Schteinhaus theorem theorem on uniform point-wise convergence.

$\|T - T_n\| \leq \varepsilon(2 + \sup \|P_n\|)$ for n large enough $\blacksquare$

Theorem [Banach-Schteinhaus]: Assume that $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ where X is a Banach space, such that

$$\sup_n \|T_n x\| \leq C(x) < \infty$$

local information
$\rightsquigarrow$ uniform estimate

For every $x \in X$. Then $\sup_n \|T_n\| < \infty$. In particular, one can take C in place of $C(x)$.

Remark: In our situation $\sup_{1 \leq n < \infty} \|P_n\| < C(x) < \infty$ because $P_n x \longrightarrow x$ in X.

Banach adjoint operators

October 15, 2025

Definition: Let X be a Banach space, then $X^* = \mathcal{B}(X, \mathbb{C})$ is called the dual space to X. The elements of X^* are called functionals.

$$= \left\{\phi : X \to \mathbb{C} \;\middle|\; \begin{matrix}\phi \text{ linear} \\ \|\phi\| < \infty\end{matrix}\right\}$$

Examples: (can ignore, if one does not know measure theory)

i) $\underset{1 \leq p < \infty}{L^p(\mu)} = \left\{ f : S \longrightarrow \mathbb{C} \;\middle|\; \begin{matrix}f \text{ is measurable with} \\ \text{respect to } \sigma\text{-algebra} \\ \text{of } \mu\end{matrix},\ \int_S |f|^p \, d\mu < \infty \right\} \big/ \underset{\substack{\text{if } f(x) = g(x) \\ \text{for } \mu\text{-a.e. } x \in S}}{f = g}$

$$\|f\|_{L^p(\mu)} = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

ii) $\underset{1 \leq p < \infty}{\ell^p(\mathbb{Z})} = \left(\left\{ \{x_k\}_{k \in \mathbb{Z}} \;\middle|\; \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \right\},\ \|x\|_{\ell^p(\mathbb{Z})} = \left(\sum |x|^p \right)^{1/p} \right)$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

i) $g \in L^q(\mu) \leftrightsquigarrow \Phi_g : f \longmapsto \int_S f g \, d\mu$, $\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$

ii) $\{y_k\}_{k=1}^{\infty}$ in $\ell^q(\mathbb{Z}) \leftrightsquigarrow \Phi_{\{y_k\}} : \{x_k\} \longmapsto \sum_{k \in \mathbb{Z}} x_k y_k$

$\Phi_{\{y_k\}} : \ell^p(\mathbb{Z}) \to \mathbb{C}$

Remark: i) is non-trivial measure theory

More examples:

iii) $c_0(\mathbb{Z}) = \{ \{x_k\}_{k \in \mathbb{Z}} \mid x_k \to 0 \text{ as } |k| \to \infty \}$

$c_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$ same identification

(Hausdorff is actually sufficient ⇝ hard)

iv) Let K be a compact metric space, and $X = C(K)$.
Then $X^* = \mathcal{M}(K)$.

set of cont. maps $(K, \delta) \to \mathbb{C}$

$= \left\{ \begin{array}{l} \text{the set of Borel} \\ \text{measures on } K \\ \text{(complex valued)} \end{array} \,\middle|\, |\mu|(K) = \sup_{\substack{K = \cup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n \in \mathbb{Z}} |\mu(E_n)| < \infty \right\}$

$\|\mu\| = |\mu|(K)$

Riezs-Markov representation theorem

Here $\mu \in \mathcal{M} \overset{(*)}{\leftrightsquigarrow} \Phi_\mu : f \longrightarrow \int_K f \, d\mu$

We can also define ℓ^p, L^p for $p = \infty$:

- $\ell^\infty(\mathbb{Z}) := \{ \{x_k\} \subset \mathbb{C} : \sup_{k \in \mathbb{Z}} |x_k| < \infty$
- $L^\infty(\mu) := \{ f : \text{—}, \text{———} : \operatorname{ess\,sup} |f| < \infty \}$

Remark: If $1 < p < \infty$ then $(L^p)^* = L^q$, $(L^q)^* = L^p$

But for $p=1$ $(L^1)^* = L^\infty$, $(L^\infty)^* \neq L^1$

$\ell^1(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})$, but $(\ell^\infty(\mathbb{Z}))^* \neq \ell^1(\mathbb{Z})$

<u>Definition</u>: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ is defined by

$$T^* : \underset{\in Y^*}{\Psi} \longmapsto \left(\underset{\in X^*}{(T^*\Psi)} : x \mapsto \langle Tx, T^*\psi \rangle \right),$$

where $\langle x, \phi \rangle = \phi(x)$ for $x \in X$, $\phi \in X^*$. $\overset{\shortparallel}{\psi(Tx)}$

<u>Remark</u>: $\langle Tx, \psi \rangle = \langle x, T^*\psi \rangle$ → this formula is equivalent to the definition of T^*

<u>Remark</u>: Operation that sends x, ϕ into $\phi(x) = \langle x, \phi \rangle$ for $x \in X$, $\phi \in X^*$ is called a pairing of Banach spaces X, X^*.

<u>Example</u>: For $\varphi \in C[0,1]$, μ on $[0,1]$ then the pairing is $\langle \varphi, \mu \rangle = \int_0^1 \varphi \, d\mu$, see (✱).

<u>Theorem</u>: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then the map $T^* : Y^* \longrightarrow X^*$ defined by $\langle x, T^*\psi \rangle := \langle Tx, \psi \rangle$, $x \in X$, is an element of $\mathcal{B}(Y^*, X^*)$.
($\Leftrightarrow$ $(T^*\psi)(x) = \psi(Tx)$)

<u>Lemma 1 [Hahn-Banach theorem]</u>: Let X be a Banach space, $E \subset X$ - subspace in X, $\phi_0 : E \to \mathbb{C}$ is linear and bdd ($\Leftrightarrow \phi_0 \in E^*$). Then $\exists \, \phi \in X^*$ such that $\phi|_E = \phi_0$ and $\|\phi\| = \|\phi_0\|$.

Lemma 2 ["Sufficient amount of functionals"]:

Let $x \in X$, then $\|x\| = \sup_{\|\phi\| \le 1} |\phi(x)|$.

Proof: $|\phi(x)| \le \|\phi\| \cdot \|x\| \le \|x\|$, so $\|x\| \ge \sup_{\|\phi\| \le 1} |\phi(x)|$

To prove "$\le$", define $E = \operatorname{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$,
$\phi_0 : y \longrightarrow c_y$ if $y = c_y x \in E$.
Assume that $\|x\| = 1$, then $\|\phi_0\|_{E^*} = \sup_{\|y\| \le 1} |c_y| =$ (from (**),
$|c_y| = \|y\|$ if $\|x\| = 1$) $= \sup_{\|y\| \le 1} \|y\| = 1$.

Hahn-Banach theorem $\Rightarrow \exists\, \tilde{\phi}_0 \in X^* : \|\tilde{\phi}_0\| = 1$, $\tilde{\phi}|_E = \phi_0$.
In particular $\sup_{\|\phi\| \le 1} |\phi(x)| \ge |\tilde{\phi}| = |\phi_0(x)| = 1 = \|x\|$.

We have proved "$\le$" in the case where $\|x\| = 1$.
The general case follows from consideration of $\frac{x}{\|x\|}$ in place of X.

October 21, 2025

We are proving that $T \in \mathcal{B}(X, Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.

Let $T \in \mathcal{B}(X, Y)$, consider

$$\|T^*\| = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} \|T^* \psi\|_{X^*} = \sup_{\|\psi\| \le 1} \sup_{\substack{x \in X \\ \|x\| \le 1}} |(T^*\psi)(x)|$$

$$= \sup \sup |\langle x, T^*\psi \rangle|$$

$$= \sup \sup |\langle Tx, \psi \rangle|$$

$$= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} \sup_{\substack{x \in X \\ \|x\| \le 1}} |\psi(Tx)|$$

$$= \sup_{\substack{x \in X \\ \|x\| \le 1}} \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} |\psi(Tx)|$$

Lemma "sufficient amount of functionals" $= \|Tx\|$

$$= \sup_{\|x\|\le 1} \|Tx\| = \|T\| < \infty$$

The claim follows. ▨

Corollary: $T \in \mathcal{B}(X,Y)$ is invertible $(\exists T^{-1} \in \mathcal{B}(Y,X))$ iff $T^* \in \mathcal{B}(Y^*, X^*)$ is invertible $(\exists (T^*)^{-1} \in \mathcal{B}(X^*, Y^*))$.

We prove just $\Rightarrow$.

Proof: Assume that T is invertible $\Leftrightarrow$ $T^{-1}T = I_X$
$TT^{-1} = I_Y$

Let's take adjoint operators and see:

$$\left.\begin{aligned} (T^{-1}T)^* &= (I_X)^* \\ (TT^{-1})^* &= (I_Y)^* \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} T^*(T^{-1})^* &= I_X^* \\ (T^{-1})^* T^* &= I_Y^* \end{aligned}\right.$$

Exercise: $(AB)^* = B^*A^*$

It remains to check that $I_X^* = I_{X^*}$, $I_Y^* = I_{Y^*}$. Then, by the previous theorem, $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$, hence T^* is invertible and its bounded inverse is $(T^*)^{-1} = (T^{-1})^*$.

Let's check that $I_X^* = I_{X^*}$. Take $\Phi \in X^*$, $x \in X$.

$$(I_X^* \Phi)(x) = \langle x, I_X^* \Phi \rangle = \langle I_X x, \Phi \rangle = \langle x, \Phi \rangle = \Phi x$$

$$(I_{X^*}\Phi(x)) \underset{\text{def}}{=} (\Phi)(x) = \Phi(x).$$

Similarly, $I_Y^* = I_{Y^*}$. ▨

The „pairing notation" is often not used in literature, but it is very usefull to not make mistakes.

Theorem [Schauder]: We have $T \in S_\infty(X,Y) \Leftrightarrow T^*_\infty(Y^*, X^*)$.

Proof: We will prove just "$\Rightarrow$".

Consider $K = \overline{TB_X(0,1)}$ – a compact set. Let $C(K)$ be the Banach space of continuous functions on K with

$$\|f\|_{C(K)} = \max_{s\in K}|f(s)|, \quad f: K \to \mathbb{C}$$

Let $E := \{\psi \in Y^* \mid \|\psi\|_{Y^*} \leq 1, \ \psi \text{ is considered as a function on } K\}$

$K \subset Y$, K metric space with respect to the metric $\rho(y_1,y_2) = \|y_2 - y_1\|_Y$

So, $E \subset C(K)$ and we claim that E is precompact.

1) uniform boundedness:

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s\in K}|\psi(s)| = \max_{s\in \overline{TB_X(0,1)}}|\psi(s)| \overset{\psi \text{ cont.}}{=} \sup_{x\in B_X(0,1)}|\psi(Tx)|$$

$$\leq \|\psi\| \sup_{\|x\|\leq 1}\|Tx\| \leq \|\psi\|\cdot\|T\| \leq \underbrace{\|T\| < \infty}_{\text{does not depent on } \psi}$$

2) Equicontinuity: take $s_1, s_2 \in K$, let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \leq \|\psi\|\cdot\|s_1 - s_2\| \leq \|s_1 - s_2\|,$$

so maps from E are Lipschitz with constant 1, hence equicontinuous.

$\Rightarrow$ By Arzela-Askoli theorem, E is precompact.

We are now ready to prove $T^* \in S_\infty(Y^*, X^*)$. For this, we need to check that if $\{\psi_n\}$ is a sequence in $B_{Y^*}(0,1)$, then $\exists\{\psi_{n_k}\}$ such that $T^*\psi_{n_k}$ converges in X^*. So, take $\{\psi_n\} \subset B_{Y^*}(0,1)$ and consider it as elements $E \subset C(K)$.

Let $\{\psi_{n_k}\}$ be such that $\psi_{n_k} \xrightarrow{(*)} \psi$ in $C(K)$. (E is precompact!)

Let's prove that $\{T^*\psi_{n_k}\}$ is Cauchy in X^*, then the theorem will follow.

Take $x \in X$, $\|x\| \le 1$, and consider

$$\|(T^*\psi_{n_k})(x) - (T^*\psi_{n_j})(x)\| = \|\langle x, T^*\psi_{n_k}\rangle - \langle x, T^*\psi_{n_j}\rangle\|$$

$$= \|\langle Tx, \psi_{n_k}\rangle - \langle Tx, \psi_{n_j}\rangle\|$$

$$= \|\psi_{n_k}(Tx) - \psi_{n_j}(Tx)\|$$

$$\le \sup_{s \in K} |\psi_{n_k}(s) - \psi_{n_j}(s)|$$

$$= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|_{C(K)}}_{= \varepsilon_{kj} \text{ - does not depend on } x} \longrightarrow 0 \text{ by } (*).$$

$$\Rightarrow \|T^*\psi_{n_k} - T^*\psi_{n_j}\| \le \varepsilon_{k,j} \longrightarrow 0.$$ ▨

Fredholm alternative

<u>Example</u>: Consider the equation $f(t) - \int_0^1 e^{t-s} f(s)\,ds \overset{(**)}{=} g(t)$ in $L^2[0,1]$.

Question: For which $g \in L^2[0,1]$ do we have a solution $f \in L^2[0,1]$?

Observation: g has to satisfy $\int_0^1 e^{-t} g(t)\,dt \overset{(***)}{=} 0$ $\qquad \int_0^1 c\,dt = c$

Indeed, $\int_0^1 e^{-t} g(t)\,dt = \int_0^1 e^{-t} f(t)\,dt - \int_0^1 \cancel{e^{-t}} \left(\int_0^1 \cancel{e^{t}}{}^{-s} f(s)\,ds\right) dt = 0$

It is not clear so far if there are other restrictions.

<u>Theorem [Fredholm alternative]</u>: Let X be a Banach space, $T = I - K$ for $K \in \mathcal{S}^\infty(X,X)$. Then

$$\operatorname{Ran} T = \{x \in X \mid \langle x, \Phi\rangle = 0 \ \forall \Phi \in \ker T^*\}.$$

In other words, either:

(1) $\ker T^* = \{0\}$ and the equation $Tf = g$ has solution $\forall g \in X$.

or

(2) $\ker T^* \neq \{0\}$ and the equation $Tf = g$ has solutions only for $g : \langle g, \Phi\rangle = 0 \ \forall \phi \in \ker T^*$.

Let's complete the consideration of the example: we need to check that $K: f \longrightarrow \int_0^1 e^{t-s} f(s)\,ds$ is compact (exercise) and find $\operatorname{Ker} T^*$.

$$\phi \in \operatorname{Ker} T^* \iff T^* \varphi = 0$$

Adjoint operator T^* is defined by

$$\langle Tf, g\rangle = \langle f, T^* g\rangle \qquad f, g \in L^2[0,1]$$

$(L^2[0,1]^* = L^2[0,1])$

$$\iff \langle f - \int_0^1 e^{t-s} f, g\rangle = \int_0^1 f(t)g(t)\,dt - \int_0^1\int_0^1 e^{t-s} f(s)\,ds\, g(t)\,dt$$

$$= \int_0^1 f(t)g(t)\,dt - \int_0^1 f(s)\left(\int_0^1 e^{t-s} g(t)\,dt\right)ds$$

$$= \langle f, g - \int_0^1 e^{t-s} g(t)\,dt\rangle_{L^2[0,1]}$$

$$(T^* g): s \longmapsto g(s) - \int_0^1 e^{t-s} g(t)\,dt, \quad s \in [0,1].$$

$$T^* g = 0 \iff g(s) = \int_0^1 e^{t-s} g(t)\,dt \quad \text{a.e. on } [0,1]$$

$$\iff e^s g(s) = \underbrace{\int_0^1 e^t g(t)\,dt}_{\text{constant}} \quad \text{for almost every } s \in [0,1]$$

$\Rightarrow$ So, $\ker T^* = \{c \cdot e^{-s}, c \in \mathbb{C}\}$, $\dim(\operatorname{Ker} T^*) = 1$.
By Fredholm theorem, equation (**) is solvable $\iff$ $\forall c \in \mathbb{C}.\ \langle g, c \cdot e^{-s}\rangle = 0 \iff \int_0^1 g(s) e^{-s}\,ds = 0$, which is (***).

Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let X be a Banach space, $E \lneq X$ - a linear closed subspace, $\varepsilon > 0$. Then $\exists x_0 \in X$ such that $\|x_0\| = 1$, $\operatorname{dist}(x_0, E) \geq 1 - \varepsilon$.

Proof: Since $E \neq X$, then $\exists\, \tilde{x}_0 \in X \setminus E$. Since E is closed, we have $\mathrm{dist}(\tilde{x}_0, E) = \delta > 0$ for some $\delta > 0$. Now consider $\tilde{y}_0 \in E$ such that $\delta \leq \|\tilde{x}_0 - \tilde{y}_0\| \leq (1+\eta)\delta$ for some $\eta \in (0,1)$.

Now let $x_\eta := \dfrac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}$, $\|x_\eta\| = 1$.

$$\mathrm{dist}(x_\eta, E) = \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|}\,\mathrm{dist}(\tilde{x}_0 - \tilde{y}_0, E)$$

$$= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|}\,\mathrm{dist}(\tilde{x}_0, E)$$

$$= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta}$$

Choosing η so that $\frac{1}{1+\eta} = 1 - \varepsilon$, we are done.

October 22, 2025

Lemma: Let X be a Banach space. Then $I: x \mapsto x$ is compact on $X \iff \dim X < \infty$.

Proof: $\dim X < \infty \Rightarrow I \in S_\infty(X)$ – we already know

$I \in S_\infty(X) \Rightarrow \dim X < \infty$:

Suppose $\dim X = +\infty$, find a sequence $\{e_n\}$: $\|e_n\| = 1$ $\forall n \in \mathbb{N}$

$e_1 \in X$ – arbitrary

e_2: $\mathrm{dist}(e_2, \mathrm{span}\{e_1\}) \geq 1/2$

e_3: $\mathrm{dist}(e_3, \mathrm{span}\{e_1, e_2\}) \geq 1/2$

e_4: etc

existence of $\{e_n\}$ follows from previous lemma, because $\mathrm{span}\{e_1, \dots, e_k\} \neq X$ $\forall k \in \mathbb{N}$

Then $\{e_n\} \subset B_X[0,1] = I(B_X[0,1])$ but there is no convergent subsequence, because $\|e_k - e_j\| \geq 1/2$ $\forall k, j$. $\square$

Lemma: Let X be a Banach space, $K \in S_\infty(X)$, $T = I - K$.
Then: 1) $\dim(\operatorname{Ker} T) < \infty$.
2) $\operatorname{Ran} T$ is closed in X.

Proof: 1): We have $I|_{\operatorname{Ker}T} = \underbrace{(I-K)|_{\operatorname{Ker}T}}_{0} + \underbrace{K|_{\operatorname{Ker}T}}_{\in S_\infty(\operatorname{Ker}T, S)}$

$$I|_{\operatorname{Ker}T} \in S_\infty(\operatorname{Ker}T, X) \Rightarrow I \in S^\infty(\operatorname{Ker}T) \Rightarrow \dim(\operatorname{Ker}T) < \infty$$

2): The statement is equivalent to the fact that if $\{x_n\} \subset X$ s.t. $Tx_n \longrightarrow y$ in X then $\exists x \in X.\ Tx = y$.

2.a) Let $\{x_n\}: \|x_n\| \leq C \quad \forall n$.

Then $(I-K)(x_n) \longrightarrow y$, $(I-K)(x_{n_k}) \xrightarrow{(*)} y$
For every subsequence x_{n_k}

Let's choose x_{n_k}: Kx_{n_k} converges to $z \in X$
(use $K \in S_\infty(X)$)

Then $x_{n_k} \longrightarrow y + z$ by $(*)$, take $x = y + z$:

$T(y+z) = \lim_{k\to\infty} Tx_{n_k} = y$, so $Tx = y$. ✓

2.b) $\operatorname{dist}(x_n, \operatorname{Ker}T) \leq C \quad \forall n \in \mathbb{Z}$
Take $\tilde{x}_n := x_n - w_n$, where $w_n \in \operatorname{Ker}T$: $\|\tilde{x}_n\| \leq 2C$.
We have $\lim_{n\to\infty} T\tilde{x}_n = \lim_{n\to\infty} Tx_n$ by step 2a) $\exists \tilde{x}$: $T\tilde{x} = y$ ✓

2.c) $\operatorname{dist}(x_n, \operatorname{Ker}T) \longrightarrow +\infty$. Let us show that this situation does not occur. Suppose the converse:
Consider $\tilde{x}_n = x_n - \underset{\operatorname{Ker}T}{w_n}$: $\underbrace{\operatorname{dist}(x_n, \operatorname{Ker}T) \leq \|\tilde{x}_n\| \leq 2\operatorname{dist}(x_n, \operatorname{Ker}T)}_{(**)}$

For $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ we have $Tz \longrightarrow 0$.

$$Tz_n = T\frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T\frac{x_n}{\|\tilde{x}_n\|} = \frac{Tx_n}{\|\tilde{x}_n\|} \quad \begin{array}{l} Tx_n \to y \Rightarrow Tx_n \text{ is bdd in } X \\ \|\tilde{x}_n\| \to +\infty \end{array}$$

$$\Rightarrow \|Tz_n\| \leq \frac{2\|y\|}{\|\tilde{x}_n\|} \longrightarrow 0 \quad \text{For } n \text{ large enough}$$

At the same time, $Tz_n = z_n - Kz_n$

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $\{Kz_{n_k}\}$ converges to some $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{Tz_{n_k}}_{\to 0} + \underbrace{Kz_{n_k}}_{\to z} \longrightarrow z$$

We have $Tz = 0$ $\left(= \lim Tz_{n_k} = \lim Tz_n = 0\right)$

$$\Leftrightarrow z \in \operatorname{Ker}T, \quad 0 = \operatorname{dist}(z, \operatorname{Ker}T) =$$

$$= \lim_{k\to\infty} \operatorname{dist}(z_{n_k}, \operatorname{Ker}T)$$

$$= \lim_{k\to\infty} \operatorname{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \operatorname{Ker}T\right)$$

$$= \lim \frac{\operatorname{dist}(\tilde{x}_{n_k}, \operatorname{Ker}T)}{\|\tilde{x}_{n_k}\|}$$

$$= \lim \frac{\operatorname{dist}(x_{n_k}, \operatorname{Ker}T)}{\|\tilde{x}_{n_k}\|}$$

$$\overset{(**)}{\geq} \frac{1}{2} \quad \rightsquigarrow \text{contradiction}$$

▨

Lemma: Let X be a Banach space, $T \in \mathcal{B}(X)$: $\operatorname{Ker}(T) = \{0\}$ and $T^{k+1}X = T^k X$ for some $k \geq 0$. Then $\operatorname{Ran}T = X$.

Proof: We need to prove that $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$.

We know that: $T^{k+1}a = T^k\tilde{a}$ for every a and some $\tilde{a}$ depending on a.

$\operatorname{Ker}T^k \neq \{0\} \Rightarrow \operatorname{Ker}T \neq \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}.$$

▨