PART III Applications

Classical moment problems

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Hamburger's moment problem:
Given a sequence of numbers $\{h_k\}_{k=0}^{\infty}$, decide if there exists a measure μ on $\mathbb{R}$ s.t. $h_k = \int_{\mathbb{R}} x^k d\mu$, $k \geq 0$.
($\Leftrightarrow$ which sequences of reals are moment sequences of measures?)

Trigonometric moment problem:
Given a sequence $\{t_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$, is there a $\mu \geq 0$ s.t. $t^k = \int_{\mathbb{T}} z^k d\mu$?

Obvious restrictions:

- Hamburger case: $\sum_{k,j=0}^{N} h_{k+j} a_k \bar{a}_j \overset{(*)}{\geq} 0 \quad \forall \{a_k\}_{k=0}^{N} \subset \mathbb{C},\ N \geq 0$

$$0 \leq \int_{\mathbb{R}} \Big|\sum_{k=0}^{N} a_k x^k\Big|^2 d\mu = \sum_{1}^{N} a_k \bar{a}_j \int_{\mathbb{R}} x^{k+j} d\mu^N = \sum_{k,j=0} a_k \bar{a}_j h_{k+j}$$

- Trigonometric case: $\sum_{k,j=-N}^{N} t_{k-j}\, a_k \bar{a}_j \overset{(**)}{\geq} 0 \quad \forall \{a_k\}_{-N}^{N} \subset \mathbb{C},\ N \geq 0$

$$0 \leq \int_{\mathbb{T}} \Big|\sum_{-N}^{N} a_k z^k\Big|^2 d\mu = \sum_{-N}^{N} a_k \bar{a}_j \int_{\mathbb{R}} z^k \bar{z}^j d\mu^N = \sum_{-N}^{N} a_k \bar{a}_j t_{k-j}$$

Theorem [Hamburger]: The assumption (*) is sufficient for the solvability of the Hamburger case.

Theorem: The assumption (**) is sufficient for the solvability of the Trigonometric moment problem. Moreover, we have
$\sum_{-N}^{N} t_{k-j} a_k \bar{a}_j > 0 \ \forall \{a_k\}_{k=0}^{N} \Leftrightarrow \{t_k\}$ is the moment sequence of a measure μ such that $\# \operatorname{supp} \mu = +\infty$. (***)

Our goal is to prove (***).

Proof: $H_0 = \left(\operatorname{span}\{z^k\}_{k\in\mathbb{Z}}, \left\langle \sum_{-N}^{N} a_k z^k, \sum_{-N}^{N} \tilde{a}_k z^k \right\rangle := \sum t_{k-j} a_k \overline{\tilde{a}_j} \right)$

↳ pre Hilbert space, because it is linear, and $\langle\cdot,\cdot\rangle$ is the inner product on H_0, but H_0 is not complete w.r.t. $\|\sum a_z z^k\| = \sqrt{\left\langle \sum_{-N}^{N} a_z z^k, \sum_{-N}^{N} a_z z^k \right\rangle}$

General Functional analysis implies that $\exists H$ - Hilbert space such that $H_0 \subset H$ as a dense linear subset.

$T: \sum_{k=0}^{N} a_k z^k \longmapsto \sum_{k=0}^{N} a_k z^{k+1}$ - densely defined operator on H:

$$\left\|T\left(\sum_{-N}^{N} a_k z^k\right)\right\|^2 = \left\|\sum_{-N}^{N} a_k z^{k+1}\right\|^2 = \sum_{-N+1}^{N+1} t_{k-j} a_{k-1}\overline{a_{j-1}} = \sum_{-N+1}^{N+1} t_{(k-1)-(j-1)} a_{k-1}\overline{a_{j-1}} =$$

$$\left\|\sum_{-N+1}^{N+1} a_{k-1} z^k\right\|^2 \qquad = \sum_{-N}^{N} t_{k-j} a_k \overline{a_j} = \left\|\sum_{-N}^{N} a_k z^k\right\|^2$$

$\Rightarrow T$ is an isometry initially defined on H_0.

Let's extend it to the whole space H. $\Rightarrow T$ is isometry on H, $T(H) \supset H_0$ - dense in H, since $T(H)$ is closed, we have $T(H) = H$.

$\Rightarrow T$ is unitary. Moreover, there is $h=1$ s.t. $\underset{= (H_0)}{\operatorname{span}\{T^k T^{*j} h\}}$ is dense in H.

By the spectral theorem, there is a measure μ s.t. $\operatorname{supp}\mu = \mathbb{T} \subset \mathbb{T}$: $T \cong M_z$ on $L^2(\mu)$.

$$\langle T^k h, h\rangle_H = \langle M_z^k 1, 1\rangle_{L^2(\mu)} \quad \forall k \geq 0 \text{ for } h=1 \text{ in } H.$$

$$\langle T^k 1, 1\rangle_H = \langle z^k, 1\rangle_H = \sum_{0}^{k} t_{i-j}\,\delta_k(i)\,\delta_0(j) = t_k$$

$$\langle M_z^k 1, 1\rangle_{L^2(\mu)} = \langle z^k, 1\rangle_{L^2(\mu)} = \int_{\mathbb{T}} z^k d\mu$$

$\Rightarrow t_k = \int_{\mathbb{T}} z^k d\mu, \quad k \geq 0$

$\underline{t_{-k} = \overline{t_k}} = \overline{\int_{\mathbb{T}} z^k d\mu} = \int_{\mathbb{T}} z^{-k} d\mu, \; k \geq 0$

$\Rightarrow t_k$ is the moment sequence

$$\begin{cases} t_k = \langle z^k, 1\rangle \\ t_{-k} = \langle z^{-k}, 1\rangle = \langle T^k z^{-k}, T^k 1\rangle = \langle 1, z^k\rangle = \overline{\langle z^k, 1\rangle} = \overline{t_k} \Rightarrow \overline{t_{-k}} = t_k \end{cases}$$

It remains to show that the measure μ is such that $\#\operatorname{supp}\mu = \infty$.

$\Leftrightarrow \int \left|\sum_{-N}^{N} a_k z^k\right|^2 d\mu > 0$ (true by assumption). ▨

Characters on compact Abelian groups

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<u>Definition</u>: G is a topological group if G is a group with topology whose operation is continuous in the product topology $G \times G$, and the operation of taking the inverses is also continuous.

<u>Definition</u>: G is a compact group if G is a topological group such that G with its topology is a compact Hausdorff space.

<u>Definition</u>: A map $\gamma : G \longrightarrow \mathbb{T}$ is a character if γ is a group homomorphism and $\gamma \in \mathcal{C}(G, \mathbb{T})$.

<u>Remark</u>: We will deal with the abelian (commutative) case, and we will denote the group operation by "+", the inverse element to $x \in G$ by $-x$, and the identity of the group by 0.

<u>Definition</u>: $\hat{G} = \{\gamma \text{ character of } G\}$ is called the dual group to G.

<u>Remark</u>: In our notation, for every $\gamma \in \hat{G}$ we have

$$\left.\begin{array}{l} \gamma(x+y) = \gamma(x) \cdot \gamma(y) \quad \forall x, y \in G \\ |\gamma(x)| = 1 \quad \forall x \in G \\ \gamma \in \mathcal{C}(G, \mathbb{T}) \end{array}\right] \text{ equivalent to } \gamma \in \hat{G}$$

<u>Remark</u>: $\gamma_0 : x \longmapsto 1$ is always in $\hat{G}$

<u>Definition</u>: Let G be a locally compact topological group. Then μ is the Haar measure on G if $\mu(U + x) = \mu(x + U) = \mu(U)$ for every Borel set U, $\mu \neq 0$, μ is regular ($\Rightarrow$ finite on compact subsets).

Theorem [Weyl]: Every locally compact topological group has a Haar measure μ, which is unique up to multiplication by a constant.

<u>Agreement</u>: If G is compact, we normalize μ: $\mu(G)=1$. With this normalization the Haar measure is unique.

<u>Theorem [Peter and Weyl]</u>: If G is a commutative compact group then characters form an orthogonal basis in $L^2(G,\mu)$, where μ is the Haar measure of G.

<u>Examples of characters</u>:

- $G=\mathbb{R}$ (locally compact), $\hat{G}=\{e^{i\lambda x} \mid \lambda\in\mathbb{R}\}$, $\mathbb{R}\cong\hat{\mathbb{R}}$.
- $G=\mathbb{T}$ (compact), $\hat{\mathbb{T}}=\{z^n \mid n\in\mathbb{Z}\}$, $\hat{\mathbb{T}}\cong\mathbb{Z}$.
- $G=\mathbb{Z}/n\mathbb{Z}$ (compact), $\hat{G}=G_n$.
- $G_n=\{\xi\in\mathbb{T} \mid \xi^m=1\}$, $\hat{G}=\mathbb{Z}/n\mathbb{Z}$.

The decomposition of $f=\sum_{k\in\mathbb{Z}} c_k z^k$ for every $f\in L^2(\mathbb{T})$ is just the Fourier decomposition, the map $f\longmapsto\{c_k\}$ is the discrete Fourier transform. In the continuous case $(G=\mathbb{R})$ $f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)e^{ix\lambda}dx$ is the Fourier decomposition, and the map $f\longmapsto g$ is the Fourier transform $\left(g(t)=\int_{\mathbb{R}} f(x)e^{-ixt}dx\right)$.

Characters form an orthonormal system:

$$\int_G \gamma_1(x)\overline{\gamma_2(x)}\,d\mu \overset{\mu \text{ is Haar}}{=} \int_G \gamma_1(x-y)\overline{\gamma_2(x-y)}\,d\mu(x)$$
$$= \gamma_1(-y)\overline{\gamma_2(-y)}\int_G \gamma_1(x)\overline{\gamma_2(x)}\,d\mu$$

$$\Rightarrow \langle \gamma_1(x),\gamma_2(x)\rangle_{L^2(\mu)} = \gamma_1(-y)\overline{\gamma_2(-y)}\langle\gamma_1,\gamma_2\rangle_{L^2(\mu)}$$

$$\Rightarrow \langle\gamma_1(x),\gamma_2(x)\rangle\neq 0 \Leftrightarrow \gamma_1(-y)\overline{\gamma_2(-y)}=1 \quad \forall y\in G$$
$$\Leftrightarrow \gamma_1(-y)=\overline{\overline{\gamma_2(-y)}} \quad \forall y\in G$$
$$\Leftrightarrow \gamma_1=\gamma_2$$

If $\gamma_1=\gamma_2=\gamma$, then $\|\gamma\|^2=\int_G \underbrace{|\gamma|^2}_{=1}d\mu=\mu(G)=1$

So, the problem is completeness of $\{\gamma\}_{\gamma\in\hat{G}}$.

<u>Definition</u>: Let $s\in G$. The shift operator T_s is $T_s: f\longmapsto f(\cdot - s)$.

<u>Lemma 1</u>: T_s is unitary on $L^2(G) = L^2(G,\mu)$ (μ is Haar).

<u>Proof</u>: $T_s L^2(G) = L^2(G)$ and

$$\|T_s f\|^2_{L^2(G)} = \int_G |f(x-s)|^2 d\mu = \int_G |f(x)|^2 d\mu = \|f\|^2_{L^2(G)}$$

$\Rightarrow T_s$ is an isometry. $\blacksquare$

<u>Lemma 2</u>: For every $f\in L^2(G)$, we have $T_s f \longrightarrow f$ as $s\longrightarrow 0$ in G
($\Leftrightarrow \forall \varepsilon>0.\ \exists U_\varepsilon$ - open neighbourhood of 0. $\forall s\in U_\varepsilon.\ \|T_s f - f\| < \varepsilon$).

To prove this lemma, we will use a version of Cantors theorem for compact groups:

<u>Theorem</u>: If $f\in \mathcal{C}(G)$, G is a compact group, then $\forall\varepsilon>0.\ \exists U_\varepsilon$ a neighborhood of 0 such that $|f(x)-f(y)|<\varepsilon$ if $x-y\in U_\varepsilon$. (Without proof.)

<u>Proof of lemma 2</u>: Take $f\in L^2(G)$ and find $g\in\mathcal{C}(G): \|f-g\|_{L^2(G)} \le \frac{\varepsilon}{3}$
(this is possible, because μ is a regular measure).

$$\|T_s f - f\|_{L^2(G)} \le \underbrace{\|T_s(f-g)\|}_{\le \varepsilon/3} + \underbrace{\|f-g\|}_{\le\varepsilon/3} + \underbrace{\|T_s g - g\|}_{\le ?}$$

$$\|T_s g - g\|_{L^2(G)} \le \|T_s g - g\|_{L^\infty(G)} = \max_{x\in G}|g(x-s)-g(x)| \le \varepsilon/3$$

by Cantors theorem if $s\in U_{\varepsilon/3}$ ($U_{\varepsilon/3}$ from Cantors theorem) $\blacksquare$

<u>Definition</u>: Let $f,g\in L^1(G)$, then $(f*g)(y) = \int_G f(x)g(y-x)\,d\mu(x)$.

<u>Remark</u>: $L^p(G)\subset L^1(G)$, because G is compact, so

$$\left(\int |f|\, d\mu\right) \le \left(\int 1^q\right)^{1/q}\left(\int |f|^p\right)^{1/p} = \|f\|_{L^p(G)}$$

In particular, we can also define $f*g$ for every $f\in L^{p_1}(G)$, $g\in L^{p_2}(G)$

Lemma 3 [Young inequality]: $1\le p\le\infty$

$$\|f*g\|_{L^p}\le \|f\|_{L^p(G)}\|g\|_{L^1(G)}\quad \forall f\in L^p(G),\ g\in L^1(G).$$

Proof: We may assume that $\|g\|_{L^1(G)}=1$. Then

$$\|g*f\|_{L^p}^p = \int_G\Big|\int_G f(y-x)\underbrace{g(x)\,d\mu(x)}_{\text{probability measure}}\Big|^p d\mu(y) \overset{\text{Jensen}}{\le} \int_G\int_G |f(y-x)|^p|g(x)|\,d\mu(x)\,d\mu(y)$$

$$\overset{\text{Fubini}}{=} \int_G |g(x)|\underbrace{\int_G |f(y-x)|^p\,d\mu(y)}_{\int_G |f(y)|^p d\mu}\,d\mu(x) = \underbrace{\|g\|_{L^1}}_{=1}\cdot\|f\|_{L^p}^p$$

$\Rightarrow \|g*f\|_{L^p}\le\|f\|_{L^p} = \|g\|_{L^1}\cdot\|f\|_{L^p}$.

It remains to note that $g*f=f*g$:

$$f*g=\int f(x)\,g(y-x)\,d\mu(x) \underset{x=y-\tilde{x}}{=} \int f(y-\tilde{x})\,g(\tilde{x})\,d\mu(\tilde{x})$$ $\square$

Lemma 4 [Approximation lemma]: $\forall\varphi\in L^2(G,\mu)$ we have

$$\inf_{\substack{U\ni 0\\ U=-U\\ U\text{ open}}}\Big\|\varphi-\varphi*\frac{\chi_U}{\mu(U)}\Big\| = 0.$$

Proof: Take $\varphi\in L^2(G,\mu)$ and $\psi\in\mathcal{C}(G)$: $\|\varphi-\psi\|_{L^2(G)}<\varepsilon$.

$$\Big\|\varphi-\varphi*\frac{\chi_U}{\mu(U)}\Big\|_{L^2}\le\Big\|\psi-\psi*\frac{\chi_U}{\mu(U)}\Big\|_{L^2}+\underbrace{\|\varphi-\psi\|_2}_{\le\varepsilon}+\underbrace{\Big\|(\varphi-\psi)*\frac{\chi_U}{\mu(U)}\Big\|_{L^2}}_{\le\|\varphi-\psi\|_{L^2}\underbrace{\|\frac{\chi_U}{\mu(U)}\|_{L^1}}_{=1}\le\varepsilon}$$

$$\Big\|\psi-\psi*\frac{\chi_U}{\mu(U)}\Big\|_{L^2(G)}\le\Big\|\psi(y)-\int_G\psi(x)\frac{\chi_U(y-x)}{\mu(U)}\,d\mu\Big\|_{L^2(G)}\le$$

$$\le\sup_{y\in G}\int_G\underbrace{|\psi(y)-\psi(x)|}_{\le\varepsilon}\frac{\chi_U(y-x)}{\mu(x)}\,d\mu\le\varepsilon$$

if $U=U_\varepsilon$ for the function $f=\psi$ in Cantor's theorem $\square$

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