

PART III. Applications

Classical moment problems

December 10, 2025

Hamburger's moment problem:

Given a sequence of numbers $\{h_k\}_{k=0}^{\infty}$, decide if there exists a measure μ on \mathbb{R} s.t. $h_k = \int_{\mathbb{R}} x^k d\mu$, $k \geq 0$.

(\Leftrightarrow which sequences of reals are moment sequences of measures?)

Trigonometric moment problem:

Given a sequence $\{t_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$, is there a $\mu \geq 0$ s.t. $t^k = \int_{\mathbb{T}} z^k d\mu$?

Obvious restrictions:

• Hamburger case: $\sum_{k,j=0}^N h_{k+j} a_k \bar{a}_j \stackrel{(*)}{\geq} 0 \quad \forall \{a_k\}_{k=0}^N \subset \mathbb{C}, N \geq 0$

$$0 \leq \int_{\mathbb{R}} \left| \sum_{k=0}^N a_k x^k \right|^2 d\mu = \sum_{k=0}^N a_k \bar{a}_k \int_{\mathbb{R}} x^{k+k} d\mu = \sum_{k,j=0}^N a_k \bar{a}_j h_{k+j}$$

• Trigonometric case: $\sum_{k,j=-N}^N t_{k-j} a_k \bar{a}_j \stackrel{(**)}{\geq} 0 \quad \forall \{a_k\}_{k=-N}^N \subset \mathbb{C}, N \geq 0$

$$0 \leq \int_{\mathbb{T}} \left| \sum_{k=-N}^N a_k z^k \right|^2 d\mu = \sum_{k=-N}^N a_k \bar{a}_k \int_{\mathbb{T}} z^{k+k} d\mu = \sum_{k,j=-N}^N a_k \bar{a}_j t_{k-j}$$

Theorem [Hamburger]: The assumption $(*)$ is sufficient for the solvability of the Hamburger case.

Theorem: The assumption $(**)$ is sufficient for the solvability of the Trigonometric moment problem. Moreover, we have
 $\sum_{k,j=-N}^N t_{k-j} a_k \bar{a}_j \geq 0 \quad \forall \{a_k\}_{k=-N}^N \Leftrightarrow \{t_k\}$ is the moment sequence of a measure μ such that $\#\text{supp } \mu = +\infty$. [Herglotz]

Our goal is to prove $(***)$.

Proof: $H_0 = \left(\text{span} \{z^k\}_{k \in \mathbb{Z}}, \left\langle \sum_{-N}^N a_k z^k, \sum_{-N}^N \bar{a}_k z^k \right\rangle := \sum_{k \in \mathbb{Z}} t_{k-j} a_k \bar{a}_j \right)$

↪ pre Hilbert space, because it is linear, and $\langle \cdot, \cdot \rangle$ is the inner product on H_0 , but H_0 is not complete w.r.t. $\|\sum a_k z^k\| = \sqrt{\langle \sum a_k z^k, \sum a_k z^k \rangle}$

General functional analysis implies that $\exists H$ -Hilbert space such that $H_0 \subset H$ as a dense linear subset.

$T: \sum_{k=0}^N a_k z^k \longmapsto \sum_{k=0}^{N+1} a_k z^{k+1}$ - densely defined operator on H :

$$\left\| T \left(\sum_{-N}^N a_k z^k \right) \right\|^2 = \left\| \sum_{-N}^N a_k z^{k+1} \right\|^2 = \sum_{-N+1}^{N+1} t_{k-j} a_{k-1} \bar{a}_{j-1} = \sum_{-N+1}^{N+1} t_{(k-1)-(j-1)} a_{k-1} \bar{a}_{j-1} = \\ \sum_{-N+1}^{N+1} a_{k-1} z^k = \sum_{-N}^N t_{k-j} a_k \bar{a}_j = \left\| \sum_{-N}^N a_k z^k \right\|^2$$

$\Rightarrow T$ is an isometry initially defined on H_0 .

Let's extend it to the whole space H . $\Rightarrow T$ is isometry on H ,

$T(H) = H_0$ - dense in H , since $T(H)$ is closed, we have $T(H) = H$.

$\Rightarrow T$ is unitary. Moreover, there is $h=1$ s.t. $\text{span} \{T^k T^{*j} h\}$ is dense in H .

By the spectral theorem, there is a measure μ s.t. $\text{Supp } \mu = T \subset T$:

$T \cong M_z$ on $L^2(\mu)$.

$$\langle T^k h, h \rangle_H = \langle M_z^k 1, 1 \rangle_{L^2(\mu)} \quad \forall k \geq 0 \text{ for } h=1 \text{ in } H.$$

||

$$\langle T^k 1, 1 \rangle_H = \langle z^k, 1 \rangle_H = \sum_0^k t_{i-j} \delta_k(i) \delta_0(j) = t_k$$

$$\langle M_z^k 1, 1 \rangle_{L^2(\mu)} = \langle z^k, 1 \rangle_{L^2(\mu)} = \int z^k d\mu$$

$$\Rightarrow t_k = \int z^k d\mu, \quad k \geq 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow t_k \text{ is the moment sequence}$$

$$t_{-k} = \overline{t_k} = \overline{\int z^k d\mu} = \int z^{-k} d\mu, \quad k \geq 0$$

$$\left. \begin{array}{l} t_k = \langle z^k, 1 \rangle \\ t_{-k} = \langle z^{-k}, 1 \rangle \end{array} \right\}$$

$$\langle t_{-k}, \langle z^{-k}, 1 \rangle \rangle = \langle T^k z^{-k}, T^k 1 \rangle = \langle 1, z^k \rangle = \langle z^k, 1 \rangle = \overline{\langle z^k, 1 \rangle} = \overline{t_k} = t_{-k}$$

It remains to show that the measure μ is such that $\text{Supp } \mu = \mathbb{C}$.

$\Leftrightarrow \int \left| \sum_{-N}^N a_k z^k \right|^2 d\mu > 0$ (true by assumption). □

Characters on compact Abelian groups

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Definition: G is a **topological group** if G is a group with topology whose operation is continuous in the product topology $G \times G$, and the operation of taking the inverses is also continuous.

Definition: G is a **compact group** if G is a topological group such that G with its topology is a compact Hausdorff space.

Definition: A map $\gamma: G \rightarrow \mathbb{T}$ is a **character** if γ is a group homomorphism and $\gamma \in \mathcal{C}(G, \mathbb{T})$.

Remark: We will deal with the abelian (commutative) case, and we will denote the group operation by "+", the inverse element to $x \in G$ by $-x$, and the identity of the group by 0.

Definition: $\hat{G} = \{\text{character of } G\}$ is called the **dual group** to G .

Remark: In our notation, for every $\gamma \in \hat{G}$ we have

$$\begin{aligned} \gamma(x+y) &= \gamma(x) \cdot \gamma(y) \quad \forall x, y \in G \\ |\gamma(x)| &= 1 \quad \forall x \in G \\ \gamma &\in \mathcal{C}(G, \mathbb{T}) \end{aligned} \quad \left. \begin{array}{l} \text{equivalent to } \gamma \in \hat{G} \\ \gamma \in \mathcal{C}(G, \mathbb{T}) \end{array} \right]$$

Remark: $\delta_0: x \mapsto 1$ is always in \hat{G}

Definition: Let G be a locally compact topological group. Then μ is the **Haar measure** on G if $\mu(U+x) = \mu(x+U) = \mu(U)$ for every Borel set U , $\mu \neq 0$, μ is regular (\Rightarrow finite on compact subsets).

Theorem [Weyl]: Every locally compact topological group has a Haar measure μ , which is unique up to multiplication by a constant.

Agreement: If G is compact, we normalize μ : $\mu(G) = 1$. With this normalization the Haar measure is unique.

Theorem [Peter-Weyl]: If G is a commutative compact group, then characters form an orthonormal basis in $L^2(G, \mu)$, where μ is the Haar measure of G .

Examples of characters:

- $G = \mathbb{R}$ (locally compact), $\hat{G} = \{e^{i\lambda x} \mid \lambda \in \mathbb{R}\}$, $\mathbb{R} \cong \hat{\mathbb{R}}$.
- $G = \mathbb{T}$ (compact), $\hat{\mathbb{T}} = \{z^n \mid n \in \mathbb{Z}\}$, $\hat{\mathbb{T}} \cong \mathbb{Z}$.
- $G = \mathbb{Z}/n\mathbb{Z}$ (compact), $\hat{G} = G_n$.
- $G_n = \{\xi \in \mathbb{T} \mid \xi^m = 1\}$, $\hat{G} = \mathbb{Z}/n\mathbb{Z}$.

The decomposition of $f = \sum_{k \in \mathbb{Z}} c_k z^k$ for every $f \in L^2(\mathbb{T})$ is just the Fourier decomposition, the map $f \mapsto \{c_k\}$ is the discrete Fourier transform. In the continuous case ($G = \mathbb{R}$) $F(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)e^{ix\lambda} dx$ is the Fourier decomposition, and the map $f \mapsto g$ is the Fourier transform ($g(t) = \int_{\mathbb{R}} f(x)e^{-ixt} dx$).

Characters form an orthonormal system:

$$\begin{aligned} \int_G g_1(x) \overline{g_2(x)} d\mu &\stackrel{\mu \text{ is Haar}}{=} \int_G \gamma_1(x-y) \overline{\gamma_2(x-y)} d\mu(x) \\ &= \gamma_1(-y) \overline{\gamma_2(-y)} \int_G \gamma_1(x) \overline{\gamma_2(x)} d\mu \end{aligned}$$

$$\Rightarrow \langle \gamma_1(x), \gamma_2(x) \rangle_{L^2(\mu)} = \gamma_1(-y) \overline{\gamma_2(-y)} \langle \gamma_1, \gamma_2 \rangle_{L^2(\mu)}$$

$$\begin{aligned} \Rightarrow \langle \gamma_1(x), \gamma_2(x) \rangle \neq 0 &\Leftrightarrow \gamma_1(-y) \overline{\gamma_2(-y)} = 1 \quad \forall y \in G \\ &\Leftrightarrow \gamma_1(-y) = \overline{\gamma_2(-y)} \quad \forall y \in G \\ &\Leftrightarrow \gamma_1 = \gamma_2 \end{aligned}$$

$$\text{If } \gamma_1 = \gamma_2 = \gamma, \text{ then } \|\gamma\|^2 = \int_G |\gamma|^2 d\mu = \mu(G) = 1$$

So, the problem is completeness of $\{g\}_{g \in G}$.

Definition: Let $s \in G$. The shift operator T_s is $T_s: f \mapsto f(\cdot - s)$.

Lemma 1: T_s is unitary on $L^2(G) = L^2(G, \mu)$ (μ is Haar).

Proof: $T_s L^2(G) = L^2(G)$ and

$$\|T_s f\|_{L^2(G)}^2 = \int_G |f(x-s)|^2 d\mu = \int_G |f(x)|^2 d\mu = \|f\|_{L^2(G)}^2$$

$\Rightarrow T_s$ is an isometry. □

Lemma 2: For every $f \in L^2(G)$, we have $T_s f \rightarrow f$ as $s \rightarrow 0$ in G ($\Leftrightarrow \forall \varepsilon > 0. \exists U_\varepsilon$ -open neighbourhood of 0. $\forall s \in U_\varepsilon. \|T_s f - f\| < \varepsilon$).

To prove this lemma, we will use a version of Cantor's theorem for compact groups:

Theorem: If $f \in C(G)$, G is a compact group, then $\forall \varepsilon > 0. \exists U_\varepsilon$ a neighbourhood of 0 such that $|f(x) - f(y)| < \varepsilon$ if $x - y \in U_\varepsilon$. (Without proof.)

Proof of lemma 2: Take $f \in L^2(G)$ and find $g \in C(G)$: $\|f - g\|_{L^2(G)} \leq \frac{\varepsilon}{3}$ (this is possible, because μ is a regular measure).

$$\|T_s f - f\|_{L^2(G)} \leq \underbrace{\|T_s(f-g)\|}_{\leq \varepsilon_3} + \underbrace{\|f-g\|}_{\leq \varepsilon_3} + \underbrace{\|T_s g - g\|}_{\leq ?}$$

$$\|T_s g - g\|_{L^2(G)} \leq \|T_s g - g\|_{L^\infty(G)} = \max_{x \in G} |g(x-s) - g(x)| \leq \varepsilon_3$$

by Cantor's theorem if $s \in U_{\varepsilon_3}$ (U_{ε_3} from Cantor's theorem) □

Definition: Let $F, g \in L^1(G)$, then $(F * g)(y) = \int_G f(x) g(y-x) d\mu(x)$.

Remark: $L^p(G) \subset L^1(G)$, because G is compact, so

$$(\int_{\mathbb{G}} |f| d\mu)^p \leq (\int_{\mathbb{G}} 1^p)^{1/p} (\int_{\mathbb{G}} |f|^p)^{1/p} = \|f\|_{L^p(\mu)}^p$$

In particular, we can also define $f*g$ for every $f \in L^p(\mu)$, $g \in L^1(\mu)$

Lemma 3 [Young inequality]: $1 < p \leq \infty$

$$\|f*g\|_{L^p} \leq \|f\|_{L^p(\mu)} \|g\|_{L^1(\mu)} \quad \forall f \in L^p(\mu), g \in L^1(\mu).$$

Proof: We may assume that $\|g\|_{L^1(\mu)} = 1$. Then

$$\begin{aligned} \|g*f\|_{L^p}^p &= \int_{\mathbb{G}} \left| \int_{\mathbb{G}} f(y-x) g(x) d\mu(x) \right|^p d\mu(y) \stackrel{\text{Jensen}}{\leq} \int_{\mathbb{G}} \int_{\mathbb{G}} |f(y-x)|^p |g(x)| d\mu(x) d\mu(y) \\ &\stackrel{\text{Fubini}}{=} \int_{\mathbb{G}} |g(x)| \underbrace{\int_{\mathbb{G}} |f(y-x)|^p d\mu(y)}_{\int_{\mathbb{G}} |f(y)|^p d\mu} d\mu(x) = \underbrace{\|g\|_{L^1}}_1 \cdot \underbrace{\|f\|_{L^p}}_1^p \end{aligned}$$

$$\Rightarrow \|g*f\|_{L^p} \leq \|f\|_{L^p} = \|g\|_{L^1} \cdot \|f\|_{L^p}.$$

It remains to note that $g*f = f*g$:

$$f * g = \int f(x) g(y-x) d\mu_x = \int_{x=y-\tilde{x}} f(y-\tilde{x}) g(\tilde{x}) d\mu(\tilde{x})$$

□

Lemma 4 [Approximation lemma]: $\forall \varphi \in L^2(\mathbb{G}, \mu)$ we have

$$\inf_{\substack{u \geq 0 \\ u = -u \\ u \text{ open}}} \left\| \varphi - \varphi * \frac{\chi_u}{\mu(u)} \right\| = 0.$$

Proof: Take $\varphi \in L^2(\mathbb{G}, \mu)$ and $\tau \in \mathcal{C}(\mathbb{G})$: $\|\varphi - \tau\|_{L^2(\mu)} < \varepsilon$.

$$\begin{aligned} \left\| \varphi - \varphi * \frac{\chi_u}{\mu(u)} \right\|_{L^2} &\leq \left\| \varphi - \varphi * \frac{\chi_u}{\mu(u)} \right\|_{L^2} + \underbrace{\|\varphi - \tau\|_{L^2}}_{\leq \varepsilon} + \left\| (\varphi - \tau) * \frac{\chi_u}{\mu(u)} \right\|_{L^2} \\ &\leq \|\varphi - \tau\|_{L^2} \underbrace{\left\| \frac{\chi_u}{\mu(u)} \right\|_{L^1}}_{1} \leq \varepsilon \end{aligned}$$

$$\left\| \varphi - \varphi * \frac{\chi_u}{\mu(u)} \right\|_{L^2(\mathbb{G})} \leq \left\| \tau(y) - \int_{\mathbb{G}} \tau(x) \frac{\chi_u(y-x)}{\mu(u)} d\mu(x) \right\|_{L^2(\mathbb{G})} \leq$$

$$\leq \sup_{y \in \mathbb{G}} \int_{\mathbb{G}} |\tau(y) - \tau(x)| \frac{\chi_u(y-x)}{\mu(u)} d\mu(x) \leq \varepsilon$$

if $u = u_\varepsilon$ for the function $\varphi = \tau$ in Cantor's theorem

□

Goal: G -compact abelian group with Haar measure μ , then \hat{G} is an QNB in $L^2(G, \mu)$.

Lemma: Let G be as above, $f \in L^2(G, \mu)$. Then F.A.E:

- i) $f = c \cdot g$, $g \in \hat{G}$
- ii) $T_s f = \lambda_s f$ in $L^2(G, \mu)$ $\forall s \in G$ $T_s f = f(\cdot - s)$, $s \in G$

Proof: (1) \Rightarrow (2): $T_s(cg) = cg(x-s) = c g(-s)g(x)$, so $\lambda_s := g(-s)$.

$$(2) \Rightarrow (1): T_{s+s'} = T_s T_{s'} \quad s, s' \in G$$

$$\Rightarrow \lambda_{s+s'} f(x) = \lambda_s \lambda_{s'} f(x) \text{ for } \mu\text{-a.e. } x \in G$$

$$\Rightarrow \lambda_{s+s'} = \lambda_s \cdot \lambda_{s'} \text{ because } \exists x. f(x) \neq 0 \quad \begin{matrix} \text{otherwise one can take } c=0 \\ f=1 \end{matrix}$$

$$\|T_s f - T_{s'} f\| = |\lambda_s - \lambda_{s'}| \|f\| \Rightarrow \text{the map } s \mapsto \lambda_s \text{ is continuous from } G \text{ to } \mathbb{T}$$

$$\Leftrightarrow |\lambda_s - \lambda_{s'}| \rightarrow 0 \text{ if } s \rightarrow s' \text{ in } G \Leftrightarrow \|T_s f - T_{s'} f\|_{L^2(\mu, G)} \rightarrow 0 \text{ if } s \rightarrow s'$$

which is true for $f \in \mathcal{C}(G)$ by Cantor's theorem and

$$\|T_s f - T_{s'} f\| \leq \|T_s(f - \bar{f})\| + \|T_s \cdot (f - \bar{f})\| + \|T_s \bar{f} - T_{s'} \bar{f}\|$$

$$\leq \underbrace{2\|\bar{f} - \hat{f}\|}_{\leq \frac{\epsilon}{4}} + \underbrace{\|T_s \hat{f} - T_{s'} \hat{f}\|}_{\leq \frac{\epsilon}{2} \text{ for } s' \text{ close to } s}$$

$\Rightarrow \lambda_s$ is a continuous function from G to \mathbb{C}

$$\|f\| = \|T_s f\| = |\lambda_s| \cdot \|f\| \quad \left. \begin{array}{l} \text{isometry} \\ \text{by } T_s f = \lambda_s f \end{array} \right\} \Rightarrow |\lambda_s| = 1 \text{ for every } s \in G$$

$$\Rightarrow \lambda_s \in \mathcal{C}(G, \mathbb{T}), \lambda_{s+s'} = \lambda_s \cdot \lambda_{s'} \Rightarrow \lambda_s \in \hat{G}.$$

Let us prove that $f = c \overline{\lambda_x}$ for some $c \in \mathbb{C}$ ($g := \overline{\lambda_x}$).

$h(x) := \lambda_x f(x)$. We have $T_s h = \lambda_{x-s} f(x-s) = \lambda_x \lambda_{-s} f(x) = \lambda_x f(x) = h$ on a set $E_s \subset G : \mu(E_s) = \mu(G) = 1$. Unfortunately E_s might depend on s and we cannot say $h(x) = h(0 - (-x)) = (T_{-x} h)(0) = h(0)$. So we need to argue differently.
↑ problem

Take $g = \frac{\chi_U}{\mu(U)}$ for some U -open set in G , $U = -U$:

$$(h * g)(y) = \int_G h(x) g(y-x) d\mu(x) = \int_G h(x+y) g(y-(x+y)) d\mu(x)$$

$\uparrow \quad G$
 $\mu \text{ is Haar}$

$$= \int_G h(x)g(-x)d\mu(x) = \int_G \underset{u \in -U}{\overset{u}{\int}} h(x)g(x)d\mu = C_U$$

$\Rightarrow (h \ast g)(y) = C_U \quad \forall y \in G$. In fact, $C_U = C$, C does not depend on U :

$$\begin{aligned} C_U &= \int_G C_U d\mu = \int_G \int_G h(x)g(y-x)d\mu d\mu \stackrel{\text{Fubini}}{=} \int_G h(x) \underbrace{\left(\int_G g(y-x)d\mu(y) \right)}_{\int_G g(y)d\mu(y) = 1} d\mu(x) \\ &= \int_G h(x)d\mu = C \end{aligned}$$

$\Rightarrow h \ast \frac{X_U}{\mu(U)} = C$, C does not depend on U .

From the approximation lemma, $\inf_h \|h - h \ast \frac{X_U}{\mu(U)}\|_{L^1(\mu)} = 0$ we have

$$\|h - C\|_{L^1(\mu)} = 0 \Rightarrow h = C \text{ a.e. on } G.$$

$$\Rightarrow \lambda_x \varphi_x = C \text{ a.e. on } G$$

$$\Rightarrow \varphi = C \cdot \varphi_x \text{ for } \varphi = \bar{\lambda}_x.$$

□

Lemma: $A: F \longrightarrow F \ast \frac{X_U}{\mu(U)}$, $\mu(U) > 0$ - a compact self-adjoint operator on $L^2(G, \mu)$.

Proof: $Af = \int_G k(x, y)f(x)d\mu(x)$ for $k(x, y) = \frac{X_U(y-x)}{\mu(U)}$

Note that $k(x, y) = k(y, x) = \overline{k(y, x)}$ $\Rightarrow A = A^*$ ($A^* f = \int_G \overline{k(y, x)}f(x)d\mu(x)$).

$$\int_G \int_G |k(x, y)|^2 d\mu(x) d\mu(y) < \infty \quad (\text{in our case, } \int_G \int_G |k(x, y)|^2 d\mu d\mu = \frac{1}{\mu(U)} < \infty)$$

\Rightarrow From Hilbert-Schmidt test (to be proved later) $A \in S_\infty(L^2(G, \mu))$. □

Lemma: If H is a separable Hilbert space, $A \in S_\infty(H)$, $A = A^* \Rightarrow A = \sum_{\lambda_k \in \sigma(A)} \lambda_k P_{E_k}$, where $E = \{h \in H \mid Ah = \lambda_k h\}$, and the series converges in operator norm.

Proof: This is an exercise from the homework.

Lemma: Let H be a finite-dimensional Hilbert space, $\dim H = N < \infty$. Let $\{U_\alpha\}_{\alpha \in I}$ be a family of unitary operators on H , $U_\alpha U_\beta = U_\beta U_\alpha \quad \forall \alpha, \beta \in I$. $\Rightarrow \exists \{e_n\}_{n=1}^N$ - an ONB in H : $U_\alpha e_n = \lambda_{\alpha, n} e_n \quad \forall \alpha \in I$.

Proof: Induction on N .

$$\cdot N=1 \quad U_\lambda = c_\lambda I \quad \forall \lambda \in \mathbb{C}$$

$$\cdot N-1 \rightarrow N, N \geq 2$$

either $U_\lambda = c_\lambda I \quad \forall \lambda \in \mathbb{C}$

or $\exists d. E_d = \{h \in H \mid U_\lambda h = \lambda_d h\}$ satisfies $E_d \neq \{0\}, E_d \neq H$

$\forall \beta \in I, \forall h \in E_d$, we have $U_\beta(U_\beta h) = U_\beta(U_\lambda h) = U_\beta(\lambda_d h) = \lambda_d U_\beta h$

$\Rightarrow U_\beta h \in E_d$ by definition of E_d

$\Rightarrow U_\beta E_d \subset E_d \Rightarrow U_\beta E_d = E_d \quad (\dim U_\beta E_d = \dim E_d)$.

Moreover, $U_\beta^* E_d = U_\beta^{-1}(E_d) = E_d$. So, E_d is a reducing subspace for U_β , in particular, $U_\beta = (U_\beta|_{E_d}) \oplus (U_\beta|_{E_d^\perp})$ $\forall \beta \in I$. \square

\Rightarrow by induction assumption, ok.



Proof of Peter-Weyl: We need to prove that \hat{G} is an ONB in $L^2(G, \mu)$.

We know that $\forall f \neq g \quad (f, g)_{L^2(G, \mu)} = 0$, so we need to check that $\forall \varphi \in L^2(G, \mu), \varphi \in \text{clos}_{L^2(\mu)}(\text{span } \hat{G})$.

Take an open neighbourhood U of 0 s.t. $U = -U$. Consider the compact self-adjoint operator $A_U : f \mapsto f * \frac{\chi_U}{\mu(U)}$. We have $A_U = \sum_{\lambda \in \sigma(A_U)} \lambda_k P_{E_{\lambda_k}}$. Let us show that $A_U \varphi \in \text{clos}_{L^2(\mu)}(\text{span } \hat{G})$. It is enough to check that $P_{E_{\lambda_k}} \varphi \in \text{span } \hat{G}$. Observe that $T_S A_U = A_U T_S$:

$$A_U T_S f = \int f(x-s) \frac{\chi_U(y-x)}{\mu(U)} dx = \int f(x-s) \frac{\chi_U(y-s-\tilde{x})}{\mu(U)} dx = \int f(\tilde{x}) \frac{\chi_U(y-s-\tilde{x})}{\mu(U)} d\mu = T_S A_U f$$

\Rightarrow If $h : A_U h = \lambda_k h \Rightarrow A_U T_S h = T_S A_U h = T_S(\lambda_k h) = \lambda_k T_S h \Rightarrow T_S h \in E_k$

$\Rightarrow T_S E_{\lambda_k} \subset E_{\lambda_k} \rightarrow [\dim E_{\lambda_k} < \infty \text{ because } A \text{ is compact}] \Rightarrow T_S E_{\lambda_k} = E_{\lambda_k}$.

Now we can use lemma for $\{U_\lambda\}_{\lambda \in I} = \{T_S|_{E_{\lambda_k}}\}_{S \in G}$. There is a ONB $\{e_{\lambda_k, n}\}$: $T_S e_{\lambda_k, n} = h_n e_{\lambda_k, n} \quad \forall n \leq \dim E_{\lambda_k} \quad \forall S \in G$.

By lemma, $e_{\lambda_k, n} = c_{\lambda_k, n} f_{\lambda_k, n}$ for some $c_{\lambda_k, n} \in \mathbb{C}, f_{\lambda_k, n} \in \hat{G}$.

In particular E_{λ_k} is spanned by $\{f_{\lambda_k, n}\}_{n \leq \dim E_{\lambda_k}} \Rightarrow P_{E_{\lambda_k}} \varphi \in E \subset \text{span } G$.

So, $A_U \varphi \in \text{clos}_{L^2(G, \mu)}(\text{span } \hat{G})$. By the approximation lemma,

$\inf_{\lambda} \|A_U \varphi - \varphi\|_{L^2(\mu)} = 0$ and $A_U \varphi \in \text{clos } \text{span } \hat{G} \Rightarrow \varphi \in \text{clos } \text{span } \hat{G}$. \square

Minmax principle

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Theorem [minmax principle]: Let H be a separable Hilbert space, $A \in S_\infty(H)$, $A = A^*$, $\sigma(A) = \{-\lambda_n^-\}_{n=1}^{N_-} \cup \{\lambda_n^+\}_{n=1}^{N_+}$ where $N_\pm \subseteq \mathbb{N} \cup \{\infty\} \cup \{0\}$, $\lambda^\pm > n \forall n$, and each point $\lambda \in \sigma(A)$ appears in $\{-\lambda_n^-\} \cup \{\lambda_n^+\}$ exactly $k(\lambda)$ times, where $k(\lambda)$ is the multiplicity of $\lambda = \dim \{\varphi \mid A\varphi = \lambda\varphi\}$. Assume, moreover, that $\{\lambda_n^+\}, \{\lambda_n^-\}$ are non-increasing. Then

$$\pm \lambda_n^\pm = \min_{\substack{L \subset H \\ \text{codim } L \leq n-1}} \max_{x \in L \setminus \{0\}} \frac{\pm \langle Ax, x \rangle}{\langle x, x \rangle}. \quad (\text{codim } L = \dim(H \ominus L))$$

Proof: Let's consider the decomposition $A = \bigoplus_{\lambda \in \sigma(A)} \lambda P_{E_\lambda}$ ($E_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$, see exercises).

Let's choose orthonormal sequence e_1^\pm, e_2^\pm, \dots such that

$$A = - \sum_{n=1}^{N_-} \lambda_n^- \langle \cdot, e_n^- \rangle e_n^- + \sum_{n=1}^{N_+} \lambda_n^+ \langle \cdot, e_n^+ \rangle e_n^+. \quad (*)$$

Here we use the fact that in $E \subset H$ - subspace, $\{\psi_1, \dots, \psi_m\}$ -ONB in E , then $P_E = \sum_1^m \langle \cdot, \psi_k \rangle \psi_k$ - orthogonal projector in H to E (Proof: add orthonormal sequence e_1^\pm, e_2^\pm, \dots so that $\{\psi_k\} \cup \{e_n^\pm\}$ is ONB in H and consider the action P_E on $h = \sum_{k=1}^m c_k \psi_k$).

Let's prove that $\lambda_n^+ = \min_{\substack{L \subset H \\ \text{codim } L \leq n-1}} \max_{x \in L \setminus \{0\}} \frac{\langle Ax, x \rangle}{\langle x, x \rangle}$.

(the case $-\lambda_n^- = \dots$ follows from this, because $-\lambda_n^-(A) = \lambda_n^+(-A)$, see (x))
Set $F_n = \text{span} \{e_k^+\}_{k=1}^n$. For every $L \subset H : \text{codim } L \leq n-1$ we have $F_n \cap L \neq \{0\}$. (**)

$$H = L \oplus (H \ominus L)^{\text{dim } L \leq n-1}$$

$$H = (H \ominus F_n) \oplus F_n^{\text{dim } F_n}$$

Indeed $P_{H \ominus L} : F_n \rightarrow H \ominus L$ has a nonzero kernel $\Rightarrow \exists h \in F_n \setminus \{0\}, P_{H \ominus L} h = 0 \Leftrightarrow h \in L \Rightarrow h \in F_n \cap L$, (***) ok.

Take $h \in (F_n \cap L) \setminus \{0\}$ and consider $h = \sum_{k=1}^n d_k e_k^+$, $d_k \in \mathbb{C}$ ($h \in F_n$).

$$\langle Ah, h \rangle \stackrel{(xxx)}{=} \left\langle \sum_{k=1}^n \lambda_k^+ \langle h, \varphi_k^+ \rangle, h \right\rangle$$

$$= \left\langle \sum_{k=1}^n \lambda_k^+ d_k \varphi_k^+, \sum_{k=1}^n d_k \varphi_k^+ \right\rangle$$

$$= \sum_{k=1}^n \lambda_k^+ |d_k|^2$$

$$\geq \lambda_n^+ \left(\sum_{k=1}^n |d_k|^2 \right)$$

↳ norm

$$= \lambda_n^+ \langle h, h \rangle$$

$$\Rightarrow \min_{L \subset H} \max_{\substack{h \neq 0 \\ \text{codim } L \leq n-1}} \frac{\langle Ah, h \rangle}{\langle h, h \rangle} \geq \lambda_n^+$$

(sequence is non-increasing)

To prove the converse inequality we take $L := H \ominus F_{n-1}$. For every $h \in H \ominus F_{n-1}$ we have

$$P_{(\ker A)^\perp} h = \sum_{k=n}^{N_+} d_k \varphi_k^+ + \sum_{k=1}^{N_-} \beta_k \varphi_k^-.$$

here we use the fact that
 $\{\varphi_k^+\}_{k=1}^{N_+} \cup \{\varphi_k^-\}_{k=1}^{N_-}$ is an ONB in
 $H \ominus \ker A$, exercise

$$\langle Ah, h \rangle = \langle A\tilde{h}, \tilde{h} \rangle = \dots \text{argument similar to (xxx)} \dots$$

$$= \sum_{k=n}^{N_+} \lambda_k^+ |d_k|^2 - \underbrace{\sum_{k=1}^{N_-} \lambda_k^- |\beta_k|^2}_{\leq 0} \leq 0$$

$$\leq \lambda_n^+ \cdot \sum_{k=n}^{N_+} |d_k|^2$$

$$= \lambda_n^+ \langle h, h \rangle, \text{ equality holds for } h = \varphi_n^+$$

For this $L = H \ominus F_{n-1}$ we proved $\max_{h \neq 0} \frac{\langle Ah, h \rangle}{\langle h, h \rangle} \leq \lambda_n^+$, ok. □

Square root of a nonnegative operator

Definition: $A \in \mathcal{B}(H)$ is **nonnegative** if $\langle Ax, x \rangle \geq 0 \quad \forall x \in H$
positive if $\langle Ax, x \rangle > 0 \quad \forall x \in H$

Theorem: For every $A \in \mathcal{B}(H)$ s.t. $A \geq 0 \exists! \sqrt{A}$ s.t.

1) $\sqrt{A} \in \mathcal{B}(H)$.

2) $\sqrt{A} \geq 0$.

3) $\sqrt{A} \sqrt{A} = A$.

Lemma 1 [polarization identity]: For every $A \in \mathcal{B}(H)$, $\forall x, y \in H$ we have

$$\langle Ax, y \rangle = \left\langle \frac{A(x+y)}{2}, \frac{x+y}{2} \right\rangle - \left\langle A \frac{x-y}{2}, \frac{x-y}{2} \right\rangle + i \left(\left\langle A \frac{x+iy}{2}, \frac{x+iy}{2} \right\rangle - \left\langle A \frac{x-iy}{2}, \frac{x-iy}{2} \right\rangle \right).$$

Proof: $\forall z \in \mathbb{C}$ we have $z = \operatorname{Re} z + i \operatorname{Im} z = \operatorname{Re} z + i \operatorname{Re}(-iz)$

$$\begin{aligned}\langle Ax, y \rangle &= \operatorname{Re} \langle Ax, y \rangle + i \operatorname{Re}(-i \langle Ax, y \rangle) \\ &= \operatorname{Re} \langle Ax, y \rangle + i \operatorname{Re}(\langle Ax, iy \rangle)\end{aligned}$$

$$\operatorname{Re} \langle Ax, y \rangle = \frac{1}{4} (\langle A(x+y), x+y \rangle - \langle A(x-y), x-y \rangle) \quad \square$$

Corollary 1: Let $A_1, A_2 \in \mathcal{B}(H)$: $\langle A_1 x, x \rangle = \langle A_2 x, x \rangle \forall x \in H \Rightarrow A_1 = A_2$.

Proof: By Lemma 1, $\langle A_1 x, y \rangle = \langle A_2 x, y \rangle \forall x, y \in H$,

$$\|A_1 - A_2\| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle (A_1 - A_2)x, y \rangle| = 0 \Rightarrow A_1 = A_2 \quad \square$$

Corollary 2: $A \geq 0 \Rightarrow A = A^*$.

Proof: $\langle Ax, x \rangle = \underbrace{\langle x, Ax \rangle}_{\langle Ax, x \rangle = \overline{\langle Ax, x \rangle}} = \langle A^* x, x \rangle \forall x \in H \Rightarrow A = A^*$ by Corollary 1. \square

Proof of theorem: Take $A \geq 0$, by Corollary 2 we have $A = A^*$.

By spectral theorem, $A \cong \bigoplus_k M_{X_k}$, where M_{X_k} is the multiplication operator $f \mapsto xf$ on $L^2(\mu_k)$ for some $\mu_k \geq 0$, $\operatorname{supp} \mu_k \subset \mathbb{R}$.

$$A \geq 0 \Rightarrow M_{X_k} \geq 0$$

$$\Leftrightarrow \langle xF, F \rangle_{L^2(\mu_k)} \geq 0 \quad \forall f \in L^2(\mu_k)$$

$$\Leftrightarrow \int_{\mathbb{R}} x |f(x)|^2 d\mu_k \geq 0 \quad \forall f \in L^2(\mu_k)$$

$$\Leftrightarrow \operatorname{supp} \mu_k \subset [0, +\infty)$$

$$\Rightarrow \sigma(A) = \overline{\bigcup \operatorname{supp} \mu_k} \subset [0, \infty)$$

Now consider $t = \sqrt{A} \in C(\sigma(A))$. $\sqrt{A} := t(A)$

Then $\sqrt{A} \in \mathcal{B}(H)$ because $\|\sqrt{A}\| = \|\varphi\|_{C(\sigma(A))}$

$$\sqrt{A} \cdot \sqrt{A} = \varphi(A) \cdot \varphi(A) = \varphi^2(A) = \varphi(A) = A$$

$\sqrt{A} \geq 0$ because $\varphi(A) = \bigoplus_k M_{\sqrt{\lambda_k}}$ and $\langle \sqrt{x}f, f \rangle_{L^2(\mu)} \geq 0 \quad \forall f \in L^2(\mu)$

Uniqueness: Suppose there is $\tilde{\sqrt{A}}$ with the same properties 1) \Rightarrow
 $K := \sigma(\sqrt{A}) \cup \sigma(\tilde{\sqrt{A}})$ - compact in $\mathbb{R}_+ = [0, \infty)$.

Find p_n - polynomials: $p_n(x) \rightarrow \sqrt{x}$ on $[0, L] \subseteq K$, $L \geq 1$.

Define $g_n(x) := p_n(x^2)$, then $g_n(x) \rightarrow x$ on $[0, L^2] \supset K$. We have

$$\begin{aligned} \|g_n(\sqrt{A}) - \sqrt{A}\| &\rightarrow 0 && \text{by functional} \\ \|g_n(\tilde{\sqrt{A}}) - \tilde{\sqrt{A}}\| &\rightarrow 0 && \text{calculus} \end{aligned}$$

But $g_n(\sqrt{A}) = p_n((\sqrt{A})^2) = p_n(A) = p_n((\tilde{\sqrt{A}})^2) = g_n(\tilde{\sqrt{A}}) \quad \forall n$
 $\Rightarrow \sqrt{A} = \tilde{\sqrt{A}}$. □

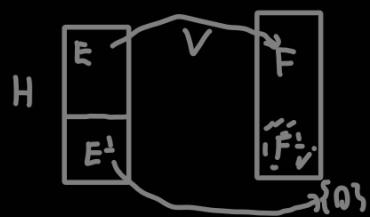
Polar decomposition of bounded operators

January 7, 2026

Motivation: $z = |z| \cdot e^{i \operatorname{Arg} z}$, $z \in \mathbb{C}$

$$? \quad T = V|T|, \quad T \in \mathcal{B}(H)$$

Definition: Let H be a Hilbert space and $E, F \subset H$ subspaces in H .
 $V \in \mathcal{B}(H)$ is a partial isometry with domain of isometricity E and range F , if $V|_E$ is a unitary operator from E onto F and $V|_{E^\perp} = 0$.



Theorem [polar decomposition]: $\forall T \in \mathcal{B}(H) \exists$ partial isometry V with domain of isometricity $\overline{\operatorname{Ran} T^*}$ and the range $\overline{\operatorname{Ran} T}$ such that $T = V|T|$, $|T| = \sqrt{T^*T}$. Moreover $\overline{\operatorname{Ran} T^*} = \overline{\operatorname{Ran} |T|}$.

Remark: $T^*T \geq 0$, so $|T| = \sqrt{T^*T}$ is defined correctly
 $\langle T^*Th, h \rangle = \langle Th, Th \rangle = \|Th\|^2 \geq 0$

Remark: We might have $|T^*| \neq |T|$.

Proof of theorem: For every $T \in \mathcal{B}(H)$ we have

$$H = \overline{\text{Ran } T^*} \otimes \text{Ker } T \quad \left(h \perp \overline{\text{Ran } T^*} \Leftrightarrow h \perp \text{Ran } T^* \Leftrightarrow \langle h, T^*h \rangle = 0 \right) \\ \Leftrightarrow \langle Th, h \rangle = 0 \Leftrightarrow Th = 0 \Leftrightarrow h \in \text{Ker } T$$

From this formula, $\overline{\text{Ran } |T|} = \overline{\text{Ran } T^*}$. Indeed this is equivalent to $\text{Ker}(|T|) \stackrel{(*)}{=} \text{Ker } T$, but

$$\langle |T|x, |T|x \rangle = \langle |T|^2 x, x \rangle = \underbrace{\langle T^*Tx, x \rangle}_{\substack{\text{def. of } |T|}} = \langle Tx, Tx \rangle,$$

so $(*)$ is ok.

Now define V on $\text{Ran } |T|$ as follows:

$$V: |T|x \longrightarrow Tx, \quad x \in H.$$

1) The definition is correct: if $|T|x_1 = |T|x_2 \Rightarrow Tx_1 = Tx_2$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ |T|(x_1 - x_2) = 0 & & T(x_1 - x_2) = 0 \\ \uparrow & & \uparrow \\ x_1 - x_2 \in \text{Ker } |T| & \xrightarrow{\text{by } (*)} & x_1 - x_2 \in \text{Ker } T \end{array}$$

2) V is linear - ok.

3) V is an isometry on $\text{Ran } |T|$:

$$\|V(|T|x)\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle |T|^2 x, x \rangle = \langle |T|x, |T|x \rangle = \| |T|x \|^2$$

Next, extend V to $\overline{\text{Ran } |T|} = \overline{\text{Ran } T^*}$ by continuity and set $V|_{\overline{\text{Ran } T^*}^\perp} = 0$. Then V is a partial isometry with the domain of isometricity $\overline{\text{Ran } T^*}$ and range $V = \overline{\text{Ran } T}$. We also have $V|T|x = Tx$ by the definition of V . □

Remark: If $\dim \text{Ker } T = \dim \text{Ker } T^*$ then $\exists U$ -unitary operator on H s.t. $T = U|T|$. (exercise)

Singular numbers of compact operators. Finite rank approximation.

Problem I: Let $T \in \mathcal{B}(H)$. What is $\text{dist}(T, \mathcal{F}_n)$ where $\mathcal{F}_n = \{A \in \mathcal{B}(H) \mid \text{rank } A \leq n\}$.

Problem II: How to choose best approximant to T from \mathcal{F}_n ?

Definition: Let $T \in \mathcal{S}_{\infty}(H)$, $|T| = \sqrt{T^*T}$, and let $\{\lambda_k(|T|)\}_{k=1}^{N_+}$, $N_+ \in \mathbb{Z}_+ \cup \{\infty\}$ be the sequence of eigenvalues, enumerated so that $\lambda_n \geq \lambda_{n+1} \forall n$, and so that each eigenvalue $\lambda \in \sigma(|T|)$ appears in $\{\lambda_k(|T|)\}_{k=1}^{N_+}$ exactly $k(\lambda)$ times, where $k(\lambda)$ is the multiplicity of $\lambda = \dim \{\varphi \mid |T|\varphi = \lambda \varphi\}$.

Then $s_k(T) = \lambda_k(|T|)$ is called the k -th singular value of T .

Remark: $s_k(T) = \sqrt{\lambda_k(T^*T)}$ because $f(\sigma(A)) = \sigma(f(A)) \quad \forall A \in \mathcal{S}_{\infty}(H)$
(take $F = \sqrt{x}$, $A = T^*T = |T|^2 \geq 0$) $f \in C(\sigma(A))$
(spectral mapping theorem)

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