

Operator theory

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PART I

Compact and Fredholm operators

Preliminaries

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Def.: (X, ρ) - metric space if X -set, and ρ is a metric:

- i) $\rho(x, y) \geq 0 \quad \forall x, y \in X. \quad \rho(x, y) = 0 \Leftrightarrow x = y$
- ii) $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$
- iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z) \quad \forall x, y, z \in X$

Def.: $U \subseteq X$ is open if $\forall x \in U. \exists \delta > 0$.

s.t. $B(x, \delta) \subset U \quad (B(x, \delta) = \{y \in X \mid \rho(x, y) < \delta\})$

Def.: $K \subset X$ is compact if every open cover $\{U_\alpha\}_{\alpha \in I}$ of K has a finite subcover.

cover: $\{U_\alpha\}_\alpha$ is a cover of K if $\bigcup_{\alpha \in I} U_\alpha \supset K$

Def.: A precompact set $A \subset X$ is a set $A \subset X$

s.t. \bar{A} is compact.

closure
of A
in X

Def: $\{x_j\}_{j \geq 1}$ is Cauchy sequence in X if

$\forall \varepsilon > 0. \exists N = N(\varepsilon). \rho(x_j, x_k) < \varepsilon \quad \forall j, k \geq N(\varepsilon).$

Def: X is complete if \forall Cauchy $\{x_j\}_{j \geq 1} \subset X$.

$\exists x \in X. \rho(x_j, x) \rightarrow 0$ as $j \rightarrow \infty$

(Every Cauchy sequence converges.)

Ex. $(\mathbb{R}^n, \rho_{\mathbb{R}^n}(\{x_i\}, \{y_i\}) := \sqrt{\sum |x_i - y_i|^2})$ - complete metric space

Ex. $(\mathbb{Q}, \rho_{\mathbb{R}})$ - metric space but non-complete

Ex. $[0, 1]$ is a compact-subset of $(\mathbb{R}, \rho_{\mathbb{R}})$

Ih: $K \subseteq \mathbb{R}^n$ is compact \Leftrightarrow closed and bounded

Def: $A \subset (X, \rho)$ is bounded if $\exists x \in X. \exists R > 0. A \subset B(x, R)$

Ex. $(X, \rho) = \ell^2(\mathbb{Z}) = \left(\left\{ \overset{\uparrow}{\{x_j\}}_{j \in \mathbb{Z}} \mid \sum_{j \in \mathbb{Z}} |x_j|^2 < \infty, \right. \right. \rho(\{x_j\}, \{y_j\}) = \sqrt{\sum_{j \in \mathbb{Z}} |x_j - y_j|^2}$

$B[0, 1] = \{y \in \ell^2(\mathbb{Z}) : \rho(0, y) \leq 1\}$

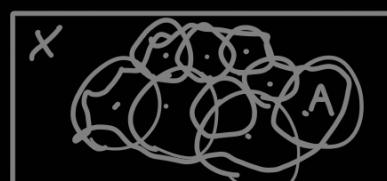
- bounded, closed but not compact HW

Theorem: Let (X, ρ) be a complete metric space,

$A \subset X$. The following assertions are equivalent:

i) A is precompact

ii) $\forall \varepsilon > 0. \exists N \text{ finite } \varepsilon\text{-net } \{x_j\}_{j=1}^{N_\varepsilon} \text{ in } A$,
that is, $\bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \supset A$.



(iii) $\forall \{x_j\}_{j \geq 1} \subset A$ there is a converging subsequence to some element $x \in X$.

Proof: i) \Rightarrow ii) $\{U_x\}_{x \in A} = \{B(x, \varepsilon)\}_{x \in A}$ - open cover of A .

- is an open cover of \bar{A} :

$(\forall y \in \bar{A}. \exists x \in A. d(x, y) < \varepsilon_2 \Rightarrow y \in B(x, \varepsilon))$
definition of closure

$\Rightarrow \exists \{U_{x_j}\}_{j=1}^N$ - finite subcover of \bar{A}

$\Rightarrow \{U_{x_i}\}_i^N$ is a ε -net in A

ii) \Rightarrow iii) Observe that $\forall \varepsilon > 0$. any sequence $\{y_i\} \subset A$ has an infinite subsequence that is contained in some $B(x, \varepsilon)$. (we have a finite ε -net)

Assume that $\{y_k\}$ is arbitrary in A .

Cantor diagonal process

where $\{y_{k_j}\} \subset B(x_2, \frac{1}{2})$

where $\{y_{k_{j_s}}\} \subset B(x_3, \frac{1}{3})$

Consider $z_1 = y_1$

$$z_2 = y_{k_2}$$

$$z_3 = y_{k_{j_3}}$$

$$z_4 = y_{k_{j_{s_4}}} \\ \vdots$$

Claim: $\{z_j\}$ is a Cauchy sequence.

Indeed $\underset{k < j}{\rho(z_j, z_k)} < \frac{1}{k}$

because $z_j, z_k \in B(x_k, \frac{1}{k})$ $\rho(z_j, z_k) < \frac{2}{k} \xrightarrow[k \rightarrow \infty]{} 0$

X is complete $\Rightarrow \{z_j\}$ converges

iii) \Rightarrow i):

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Plan: a) A contains a dense countable subset

a) b) c) b) If $\{U_d\}_{d \in I}$ is an open cover of \bar{A}

under assumption $\Rightarrow \exists \{U_{d,j}\}_{j \in J}$ - an open countable subcover of \bar{A}

3) c) $\Rightarrow \{U_{d,j}\}_{j=1}^\infty$ is a cover of \bar{A}

a) Observe that $\forall \varepsilon > 0$ there exists at most $N(\varepsilon)$ points

$\Delta \{Y_j(\varepsilon)\}_{1 \leq j \leq N(\varepsilon)}$ s.t. $\rho(Y_k(\varepsilon), Y_j(\varepsilon)) > \varepsilon \quad \forall k \neq j$.

(If this is not true, then $\exists \{Y_j(\varepsilon)\}_{j=1}^\infty$ such that

$\rho(Y_j(\varepsilon), Y_k(\varepsilon)) > \varepsilon$ and it cannot contain a convergent subsequence by Cauchy criterion.)

Now $E = \left\{ Y_k \left\{ \frac{1}{n} \right\} \mid 1 \leq k \leq N \left(\frac{1}{n} \right), n \geq 1 \right\}$ is a dense countable subset.

(E is dense since $\forall n. \forall x \in A. \min_{1 \leq k \leq N(\frac{1}{n})} \{\rho(x, Y_k(\frac{1}{n}))\} \leq \frac{1}{n}$ by construction)

b) Assume that $\{U_d\}_{d \in I}$ is some open cover of \bar{A} .

For every $x \in \bar{A}$ define

$$\varepsilon(x) := \sup \left\{ \varepsilon > 0 \mid B(x, \varepsilon) \subset U_d \text{ for some } d \right\} > 0$$

Claim: if $\{Y_j\}_{j=1}^\infty$ is a countable dense subset in A , then

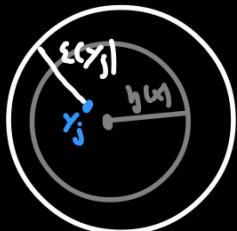
$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for \bar{A} .

Take $x \in \bar{A}$. $\exists d_x$. $x \in U_{d_x}$, and let $h(x) > 0$

such that $B(x, h(x)) \subset U_{d_x}$ (U_{d_x} is open $\Rightarrow h(\varepsilon) \exists$)

Find y_j such that $d(x, y_j) < \frac{h(x)}{10}$.

Then, since $h(x) \leq 2\varepsilon(x)$, $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow d(x, y_j) < \varepsilon(y_j)$



$\varepsilon(y_j) \geq \frac{h(x)}{10}$ - Because $B(y_j, \frac{h(x)}{5}) \subset U_{d_x}$ by triangle inequality and $\varepsilon(y_j)$ satisfies $\textcircled{4}$.

Since $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for \bar{A} , then $\{U_{d_{y_j}}\}_{j=1}^{\infty}$ is an open cover for \bar{A} , where $U_{d_{y_j}}$ is the set U_d from the definition of $\varepsilon(y_j)$ (that is, $U_d \supset B(y_j, \varepsilon(y_j))$).

Since we have $\textcircled{4*}$, $\bigcup_{j=1}^{\infty} U_{d_{y_j}} \supset \bar{A}$.

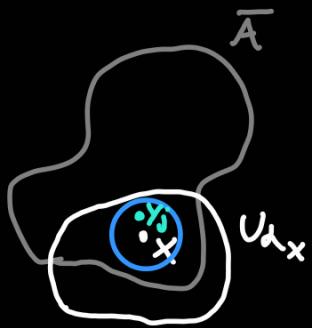
c) Claim: $\exists N$. $\{U_{d_{y_j}}\}_{j=1}^N$ is a cover of \bar{A} .

Suppose this is not the case $\Rightarrow \forall j \geq 1$. $\exists x_j \in A_j \setminus \bigcup_{k=1}^j U_{d_{y_k}}$. Consider $\{x_j\}_{j=1}^{\infty}$, and assume that the sequence $\{x_{j_k}\}$ converges to some $x \in X$. Note that $x \in \bar{A}$ ($x_j \in \bar{A}$).
 $\Rightarrow \exists j_*$. $x \in U_{d_{y_{j_*}}} \Rightarrow \exists \delta > 0$. s.t. $B(x, \delta) \subset U_{d_{y_{j_*}}}$, but $x_{j_k} \notin U_{d_{y_{j_*}}}$ for large k by construction.

(in particular, $x_{j_k} \notin B(x, \delta)$, hence $d(x, x_{j_k}) > \delta$, but this contradicts the fact that $x_{j_k} \rightarrow x$).

We have shown that $\{x_j\} \subset \bar{A}$ cannot have a convergent subsequence.

Then if $\tilde{x}_j \in A$. $d(\tilde{x}_j, x_j) < \frac{1}{j}$, then $\{\tilde{x}_j\}$ also has no convergent subsequence. So, we assumed there is no



finite subcover $\{U_{\alpha_j}\}$ and found a sequence $\{\tilde{x}_j\}_{j \in \mathbb{N}}$ that has no converging subsequence, a contradiction with 3). Therefore $3) \Rightarrow 1)$. □

Examples of compact sets and their properties:

1) $K \subset (X, \rho)$ is compact $\Rightarrow K$ is bounded

Indeed, if K is not bounded, then $\{B(x, n)\}_{n \geq 1}$ is an open cover without a finite subcover.

2) $K \subset (X, \rho)$ is compact, then it is closed
 $(\Leftrightarrow \{x_j\} \subset K \text{ such that } x_j \rightarrow x \text{ in } (X, \rho) \text{ we also have } x \in K)$

Let's check that $X \setminus K$ is open. Take $y \in X \setminus K$, take $x \in K$, let $\delta(x) > 0$. $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$
 $\{B(x, \delta(x))\}_{x \in K}$ is an open cover, let $\{B(x_j, \delta(x_j))\}_{j=1}^N$ be a finite subcover, then $\delta := \min_{1 \leq j \leq N} \delta(x_j)$, $B(y, \delta) \cap K = \emptyset$
 $\Rightarrow X \setminus K$ is open.

Another proof: Suppose $\{y_j\} \subset K$ s.t. $y_j \rightarrow y$, $y \notin K$.

$$U_j = \{x \in X \mid \rho(x, y) > \frac{1}{j}\}$$

$\{U_j\}_{j=1}^\infty$ open cover, $\{U_{j_k}\}_{k=1}^N$ finite subcover

$$\varepsilon := \min_{1 \leq k \leq N} \left(\frac{1}{j_k} \right), \quad \rho(x, y) > \varepsilon \quad \forall x \in K \quad \text{contradiction} \quad \blacksquare$$

linear space = vector space

3) Let X be a finite-dimensional complete linear normed space. Then $E \subset X$ is compact $\Leftrightarrow E$ is closed and bounded.

\Leftrightarrow : Examples 1+2.

\Leftarrow : $X = \left\{ \sum_1^N c_k e_k \mid c_k \in \mathbb{C} \right\}$, $N = \dim X$, $\{e_k\}_1^N$ -basis

$$\varphi(\sum c_k e_k) = \max_{1 \leq k \leq N} |c_k| \text{ - norm on } X$$

Since all norms on finite dimensional vcc. spaces are equivalent.

$$\exists A, B > 0. \quad A \|x\| \leq \varphi(x) \leq B \|x\| \quad \forall x \in X.$$

$$\text{in particular the set } \left\{ \left\{ c_k(x) \right\}_1^N, \quad x \in E \right\} \\ x = \sum c_k(x) e_k$$

is bounded (=bdd) in \mathbb{C}^n for every bounded $E \subset X$.

$$\sup_{x \in E} \varphi(x) \leq B \cdot \sup_{x \in E} \|x\| < \infty$$

\Rightarrow for any sequence $\left\{ \left\{ c_k(x_j) \right\}_{k=1}^N \right\}_{j=1}^\infty$ one can extract a converging subsequence in \mathbb{C}^n , i.e.

$$c_k(x_{j_n}) \rightarrow c_k \quad n \rightarrow \infty.$$

But then $\sum c_k(x_{j_n}) e_k \rightarrow \sum c_k e_k$ in X

\Rightarrow bounded subsets are precompact in X

\Rightarrow bdd + closed sets are compact

$$4) \ell^2(\mathbb{Z}) = \left\{ \{c_k\}_{k \in \mathbb{Z}} \mid \sum |c_k|^2 < \infty \right\}$$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^{\infty} |c_k|^2}$$

$$B[0,1] = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \|\{c_k\}\| < 1 \right\}$$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite $\frac{1}{2}$ -net in $B[0,1]$,
 bccu $s(e_i, e_j) > \frac{1}{2}$ for $e_k = (0, \dots, 0, \overset{\uparrow}{1}, 0, \dots, 0)$.

Definition: X is a Banach space if it is a linear normed space such that X is complete with respect to this norm.

Example: Let (K, ρ) be a metric compact space.

$$C(K) := \{f : K \rightarrow \mathbb{C} \mid \text{cont. in the metric } \rho\}$$

$$(\Leftrightarrow f(x_j) \rightarrow f(x) \quad \forall x_j \rightarrow x \text{ in } (K, \rho))$$

$$\|f\|_{C(K)} = \|f\| := \max_{x \in K} |f(x)| \quad - \text{norm in } C(K)$$

Theorem: [Arzela-Ascoli]: Assume that K is a complete compact metric space. $E \subset C(K)$ is precompact \Leftrightarrow

$$\Leftrightarrow \begin{cases} 1 | E \text{ is bounded in } C \\ 2 | \text{functions in } E \text{ are equicontinuous, that is,} \\ \quad \forall \varepsilon > 0, \exists \delta_\varepsilon > 0. |f(x) - f(y)| < \varepsilon \quad \forall x, y \in K. \rho(x, y) < \delta_\varepsilon \\ \quad \forall f \in E \end{cases}$$

We will need 1+2 \Rightarrow precompactness.

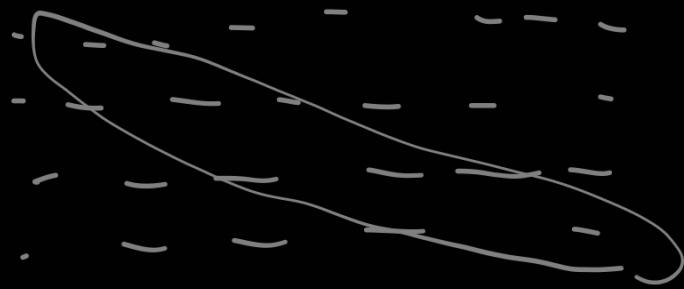
Proof: Find a dense sequence $\{x_j\}$ in K .
 (such sequence exists because K is compact)

Then take $\{f_n\}$ arbitrary sequence in E .

We want to find a converging subsequence of $\{f_n\}$
 (then E - precompact)

For this find a subsequence $\{f_{n_k}\}$ such that
 $f_{n_k}(x_j) \rightarrow F(x_j)$ for every j

(Cantor diagonalization process + uniform boundedness)



look at the proof
from the 1st
lecture

Claim: f_{n_k} is Cauchy sequence in $C(K)$.

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Aim: $\|f_{n_s} - f_{n_m}\|_{C(K)} \rightarrow 0$ as $s, m \rightarrow \infty$

For simplicity let $g_s = f_{n_s}$ $s \geq 1$.

Idea:

$$|g_s(x) - g_m(x)| \leq |g_s(x) - g_s(x_j)| + \underbrace{|g_s(x_j) - g_m(x_j)|}_{\substack{\leq \frac{\epsilon}{3} \\ \text{for all } s \text{ if } x_j \\ \text{is close to } x}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\text{take } s, m \text{ large} \\ \text{enough:} \\ \leq \frac{\epsilon}{3}}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\leq \frac{\epsilon}{3} \\ \text{for all } s \text{ if } x_j \\ \text{is close to } x \\ (\text{uniform continuity})}}$$

To make the idea work we need to check that in this construction we can deal only with finite number of points x_j , $j=1, \dots, N(\epsilon)$.

For this it suffices to find $N(\delta_\epsilon)$ such that

$\exists (x, x_j) \in \delta_\epsilon$ for every $x \in K$ and x_j , $j=1 \dots N_\epsilon$.
($\{x_j\}_{j=1}^{N(\delta_\epsilon)}$ is δ_ϵ -net).

So, it remains to show that if $\{x_j\}_{j=1}^\infty$ is dense then $\forall \delta_\epsilon > 0. \exists N(\delta_\epsilon). \{x_j\}_{j=1}^{N(\delta_\epsilon)}$ is a δ_ϵ -net.

To this end, let $\{y_k\}_1^N$ is a $\delta_{\epsilon/2}$ -net in K (K is compact). Let $\{x_j\}_{j=1}^{N(\delta_\epsilon)}$ be the part of $\{x_j\}$ such that

$$\text{dist}(\{x_j\}_{j=1}^{N(\delta_\epsilon)}, y_k) \leq \frac{\delta_\epsilon}{2} \quad \forall 1 \leq k \leq N.$$

\Rightarrow then $\{x_j\}_{j=1}^{N(\delta_\epsilon)}$ is a δ_ϵ -net by triangle inequality. \blacksquare
($\|g_s - g_m\|_{C(K)} \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$ for s, m large enough)

Example: $K = [0, 1]$, $E_A = \left\{ f \in C[0, 1] \mid f(0) = 0, f \text{ is Lipschitz}, \text{ with constant at most } A \right\}$

E_A is compact

i) E_A is bounded in $C[0, 1]$:

$$|f(x)| \leq |f(x) - f(0)| \leq A|x| \leq A$$

$$E_A \subset B(0, A)$$

ii) $|f(x) - f(y)| \leq A|x - y| \leq A\delta = \varepsilon$ if $\varepsilon > 0$, $\delta := \frac{\varepsilon}{A}$,
 $x, y \in [0, 1]: |x - y| \leq \delta$

i + ii + AA theorem $\Rightarrow E_A$ is precompact

iii) E_A is closed

If $f_n \rightarrow f$ in $C(K)$ then $f_n(0) \rightarrow f(0) \Rightarrow f(0) = 0$

$$|f_n(x) - f_n(y)| \leq A|x - y|$$

\downarrow

$$|f(x) - f(y)| \Rightarrow f \text{ is Lip}(A)$$

Example: Let $E = \left\{ \sum_{k \in \mathbb{Z}} c_k e^{2\pi i kx}, \text{ where } c_k \in \mathbb{C}: |c_k| \leq \frac{1}{k^2+1} \right\}$

Then E is compact as well in $C[0, 1]$.

i) bbs: $f \in E$. $\|f\| \leq \sum_{k \in \mathbb{Z}} \frac{1}{k^2+1}$

Details: exercise

ii) equicontinuity $f = \sum_{|k| \leq N} + \sum_{|k| > N}$ small if N large
 Lipschitz with some constant
 $A_N \sim \text{does not depend on } F$

Compact operators: basic properties

Definition: Let X, Y be Banach spaces, $T: X \rightarrow Y$ a linear map. T is called **bounded** if $T(B(0,1))$ is a bounded in Y set in Y . T is called **compact** if $T(B(0,1))$ is a precompact set in Y . ($B(0,1) = \{ \|x\|_X < 1\}$) bounded linear operator

Some observations:

- 1) If $S \subset X$ is bdd then $T(S)$ is ^{bdd}_{precompact} for any ^{bdd}_{compact} operator
- 2) T is compact $\Rightarrow T$ is bounded
(precompact sets are bounded)
- 3) with the norm $\|T\| = \sup_{x \in B(0,1)} \|Tx\|_Y$, the set of bdd linear operators becomes a linear normed space, to be denoted by $B(X,Y)$ or $B(X)$ if $X=Y$.
- 4) A linear map between Banach spaces X, Y is continuous if and only if it is bounded.
Hint: $\|Tx - Ty\| \leq \|T\| \cdot \|x - y\|$, so bounded operators are Lipschitz.

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Definition: **Banach algebra** is an associative algebra which is a linear space with a norm $\|\cdot\|$ such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$ for any elements in this algebra.

Proposition: Let X be a Banach space, then $B(X,X)$ is a Banach algebra.

Proof: $\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{B}(X, X)$ $\forall T_1, T_2 \in \mathcal{B}(X, X)$ (proved)
 $T_1 \cdot T_2 \in \mathcal{B}(X, X)$, since $\forall x \in X$. $\|T_1 T_2 x\| \leq \|T_1\| \cdot \|T_2 x\|$
 $\Rightarrow \|T_1\| \cdot \|T_2\| \leq \|T_1\| \cdot \|T_2\|$ since $T_1 \in \mathcal{B}(X, X)$ we have
 $\sup_{\substack{y \in X \\ \|y\| \leq 1}} \|T_1 T_2 y\| \leq \|T_1\| \|T_2\| \forall y \in X$

We see that $T_1 T_2 \in \mathcal{B}(X, X)$ and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

Now let us prove that $\mathcal{B}(X, X)$ is Banach.

Let us show that $\sum_{n=1}^{\infty} B_k$ converges if $\sum_{k \in \mathcal{B}(X, X)} \|B_k\| < \infty$.

$$T_n = \sum_{k=1}^n B_k, \quad x \in X, \quad \|T_N x - T_M x\| = \left\| \sum_{M+1}^N B_k x \right\| \leq \sum_{M+1}^N \|B_k\| \|x\| \xrightarrow[M, N \rightarrow \infty]{} 0,$$

because of (*).

$\Rightarrow \{T_N x\}_N$ Cauchy in X , but X -banach $\Rightarrow \exists T x = \lim_{N \rightarrow \infty} T_N x$

Moreover, $\|Tx\| = \lim_{N \rightarrow \infty} \|T_N x\| \leq \lim_{N \rightarrow \infty} \sum_{k=1}^N \|B_k\| \|x\| \leq (\sum_{k=1}^{\infty} \|B_k\|) \|x\|$.

$\Rightarrow T \in \mathcal{B}(X, X)$, $\|T\| \leq \sum_{k=1}^{\infty} \|B_k\|$

$$\sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx - T_N x\| = \lim_{M \rightarrow \infty} \|T_N x - T_M x\| \leq \sup_{\|x\| \leq 1} \lim_{M \rightarrow \infty} \sum_{N+1}^M \|B_k\| \|x\| = \sum_{k=1}^{\infty} \|B_k\| \xrightarrow[k \rightarrow \infty]{(*)} 0 \quad \square$$

$S_{\infty}(X, X)$ (index ∞ will be explained later)

Proposition: The set $S_{\infty}(X)$ of all compact operators on X is a two-sided ideal in $\mathcal{B}(X) = \mathcal{B}(X, X)$: $\forall T_1 \in S_{\infty}(X), \forall T_2 \in \mathcal{B}(X)$. $T_1 T_2 \in S_{\infty}(X)$ and $T_2 T_1 \in S_{\infty}(X)$.

Proof: Take $\{x_n\}_{n=1}^{\infty}$ s.t. $\|x_n\| \leq 1$, and let us check that there is a subsequence $\{x_{n_k}\}$: $T_1 T_2 x_{n_k}$ converges.

Note that $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$. T_1 takes $B_X(0, \|T_2\|)$ into a precompact subset of $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$ -convergent subsequence

Now let's consider $\{T_2 T_1 x_n\}$. Note that $\{T_1 x_n\}$ -convergent subsequence ($T_1 \in S_{\infty}(X)$). Then $\{T_2 T_1 x_n\}$ converges, since T_2 is continuous.

Proposition: $S_\infty(X, Y)$ is a closed subset in $\mathcal{B}(X, Y)$, i.e.

$T_n \in S_\infty(X, Y)$, $T_n \rightarrow T$ in $\mathcal{B}(X, Y) \Rightarrow T \in S_\infty(X, Y)$?

Proof: Let's find a finite ε -net in $T(B_X[0, 1])$.

Take finite $\varepsilon/3$ -net for $T_n B_X(0, 1)$ for n : $\|T - T_n\| \leq \varepsilon/2$; denote it by $\{y_k\}_{k=1}^N$, then

$$\begin{aligned} \|Tx - Tx_k\| &\leq \underbrace{\|Tx - T_nx\|}_{A} + \underbrace{\|T_nx - T_nx_k\|}_{B} + \underbrace{\|T_nx_k - Tx_k\|}_{C} \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \leq \varepsilon. \end{aligned}$$

for any $1 \leq k \leq N$
so choose k : $B \leq \varepsilon/2$
and note $A \leq \|T - T_n\| \leq \varepsilon/2$
for every $x \in B_X(0, 1)$
 $C \leq \varepsilon/3$

Corollary: If T is a limit of finite-rank operators in $\mathcal{B}(X, Y)$, then $T \in S_\infty(X, Y)$.

Proof: Since finite-rank operators are in $S_\infty(X, Y)$, we have $T \in S_\infty(X, Y)$ by the previous proposition. \square

Remark: At a general Banach space $\exists T \in S_\infty(X, Y)$ such that $\nexists \{T_n\}_n$: $\text{rank } T_n < \infty$ and $\|T - T_n\| \rightarrow 0$.

Definition: Let X be a Banach space. $\{e_k\}_{k=1}^\infty$ is a **Schauder basis** if $\forall x \in X \exists \{c_k(x)\}_{k=1}^\infty$ such that $x = \sum_{k=1}^\infty c_k(x)e_k$, where the series converges in X .

Theorem: Let X be a Banach space with Schauder basis, then $T \in S_\infty(X) \Leftrightarrow \exists T_n. \text{rank } T_n \leq n$ and $\|T - T_n\| \rightarrow 0$. (here $\text{rank } S = \dim S(X) \quad \forall S \in \mathcal{B}(X)$)

Proof: (\Leftarrow): we already know

(\Leftrightarrow) : Let $T \in S_\infty(X)$, and let $P_n : X \mapsto \sum_{k=1}^{\infty} c_k(x) e_k$.

P is linear: $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X. P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$?

$$\left. \begin{array}{l} \text{If } x = \sum c_k(x) e_k \\ y = \sum c_k(y) e_k \end{array} \right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha c_k(x) + \beta c_k(y)) e_k$$

$$\alpha x + \beta y = \sum_{k=1}^{\infty} c_k(\alpha x + \beta y) e_k$$

by uniqueness by def. of Schauder basis

$$c_k(\alpha x + \beta y) = \alpha c_k(x) + \beta c_k(y) \quad \forall k$$

$$\begin{aligned} \text{Then } P_n(\alpha x + \beta y) &= \sum_{n=1}^{\infty} c_k(\alpha x + \beta y) e_k = \sum_{n=1}^{\infty} \alpha c_n(x) e_n + \sum_{n=1}^{\infty} \beta c_n(y) e_n \\ &= \alpha P_n(x) + \beta P_n(y) \quad \Rightarrow P_n \text{ linear} \end{aligned}$$

Note that $T_n := P_n T$ are such that $\text{rank}(T_n) \leq n$ because $\dim P_n T(x) \leq \dim P_n(x) \leq n$.

It remains to show that $T_n \rightarrow T$ in $\mathcal{B}(X)$. Since T is compact, $\forall \varepsilon > 0. \exists \{x_k\}_{k=1}^N$ such that $\|x_k\| \leq 1 \forall k$ and $\{Tx_k\}_{k=1}^N$ is a ε -net in $T(B_X(0,1))$. Now take $x \in B_X(0,1)$ and write $\|Tx - T_n x\| \leq \|Tx - Tx_n\| + \|Tx_n - T_n x\| + \|T_n x - T_n x\|$

$$\begin{aligned} &\leq \underbrace{\|Tx - Tx_n\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_n - P_n Tx_n\|}_{\leq \varepsilon \text{ if } n \text{ large enough for any fixed } k} + \underbrace{\|P_n Tx_n - P_n Tx\|}_{\leq \|P_n\| \cdot \|Tx_n - Tx\|} \\ &\leq \varepsilon + \varepsilon + \sup_{n \in \mathbb{N}} \|P_n\| \varepsilon \end{aligned}$$

$\sup \|P_n\| < \infty$ by Banach-Schauder theorem on uniform point-wise convergence.

$$\|T - T_n\| \leq \varepsilon (2 + \sup \|P_n\|) \text{ for } n \text{ large enough} \quad \square$$

Theorem [Banach-Schteinhaus]: Assume that $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ where X is a Banach space, such that

$$\sup_n \|T_n x\| \leq C(x) < \infty \quad \begin{matrix} \text{local information} \\ \sim \text{uniform estimate} \end{matrix}$$

For every $x \in X$. Then $\sup_n \|T_n\| < \infty$. In particular, one can take C in place of $C(x)$.

Remark: In our situation, $\sup_{1 \leq n < \infty} \|P_n\| \leq C(x) < \infty$ because $P_n x \rightarrow x$ in X .

Banach adjoint operators

October 15, 2025

Definition: Let X be a Banach space, then $X^* = \mathcal{B}(X, \mathbb{C})$ is called the dual space to X . The elements of X^* are called functionals.

Examples: (can ignore, if one does not know measure theory)

i) $L^p(\mu) = \left\{ f : S \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is measurable with respect to } \sigma\text{-algebra} \\ \text{of } \mu \end{array}, \int_S |f|^p d\mu < \infty \right\}$

$f=g$
if $f(x)=g(x)$
for μ -a.e.xes

$$\|f\|_{L^p(\mu)} = \left(\int_S |f|^p d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \text{ where } \frac{1}{p} + \frac{1}{q} = 1$$

ii) $\ell^p(\mathbb{Z}) = \left(\left\{ \{x_k\}_{k \in \mathbb{Z}} \mid \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \right\}, \|\{x_k\}\|_{\ell^p(\mathbb{Z})} = \left(\sum_k |x_k|^p \right)^{1/p} \right)$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \text{ where } \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

i) $g \in L^q(\mu) \leftrightarrow \phi_g : f \mapsto \int_S f g d\mu, \quad \phi_g : L^p(\mu) \rightarrow \mathbb{C}$

ii) $\{y_k\}_{k=1}^\infty$ in $\ell^q(\mathbb{Z}) \leftrightarrow \phi_{\{y_k\}} : \{x_k\} \mapsto \sum_{k \in \mathbb{Z}} x_k y_k$

$\phi_{\{y_k\}} : \ell^p(\mathbb{Z}) \rightarrow \mathbb{C}$

Remark: i) is non-trivial measure theory

More examples:

iii) $C_0(\mathbb{Z}) = \left\{ \{x_k\}_{k \in \mathbb{Z}} \mid x_k \rightarrow 0 \text{ as } |k| \rightarrow \infty \right\}$

$C_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$ same identification

(Hausdorff is actually sufficient \Leftrightarrow hard)

iv) Let K be a compact metric space, and $X = C(K)$.
Then $X^* = M(K)$.

{ the set of Borel measures on K (complex valued) } $\|\mu\|(K) = \sup_{\substack{K = \cup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n \in \mathbb{Z}} |\mu(E_n)| < \infty \}$

set of
cont. maps
 $(K, \mathcal{S}) \rightarrow \mathbb{C}$

Riesz - Markov representation theorem

Here $\mu \in M \xrightarrow{\text{(bi)}} \phi_\mu : f \mapsto \int_K f d\mu$

We can also define ℓ^p, L^p for $p = \infty$:

- $\ell^\infty(\mathbb{Z}) := \{ \{x_k\} \subset \mathbb{C} : \sup_{k \in \mathbb{Z}} |x_k| < \infty \}$

- $L^\infty(\mu) := \{ f : \mathbb{Z} \rightarrow \mathbb{C} : \text{ess sup } |f| < \infty \}$

Remark: If $1 \leq p < \infty$ then $(L^p)^* = L^q, (L^q)^* = L^p$

Bvt for $p=1$ $(L^1)^* = L^\infty$, $(L^\infty)^* \neq L^1$
 $L^1(\mathbb{Z})^* = l^\infty(\mathbb{Z})$, bvt $(l^\infty(\mathbb{Z}))^* \neq l^1(\mathbb{Z})$

Definition: Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ is defined by

$$T^* : \Psi_{Y^*} \longmapsto \left(\left(T^* \Psi \right) : x \mapsto \langle Tx, T^* \psi \rangle \right),$$

where $\langle x, \phi \rangle = \phi(x)$ for $x \in X, \phi \in X^*$. $\Psi(Tx)$

Remark: $\langle Tx, \psi \rangle = \langle x, T^* \psi \rangle \rightarrow$ this formula is equivalent to the definition of T^*

Remark: Operation that sends x, ϕ into $\phi(x) = \langle x, \phi \rangle$ for $x \in X, \phi \in X^*$ is called a pairing of Banach spaces X, X^* .

Example: For $f \in C[0,1]$, μ on $[0,1]$ then the pairing is $\langle \phi, \mu \rangle = \int_0^1 f d\mu$, see (*).

Theorem: Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then the map $T^* : Y^* \rightarrow X^*$ defined by $\langle x, T^* \psi \rangle := \langle Tx, \psi \rangle$, $x \in X$, is an element of $\mathcal{B}(Y^*, X^*)$.
 $(\Leftrightarrow (T^* \psi)(x) = \psi(Tx))$

Lemma 1 [Hahn-Banach theorem]: Let X be a Banach space, $E \subset X$ - subspace in X , $\phi_0 : E \rightarrow \mathbb{C}$ is linear and bdd ($\Leftrightarrow \phi_0 \in E^*$). Then $\exists \phi \in X^*$ such that $\phi|_E = \phi_0$ and $\|\phi\| = \|\phi_0\|$.

Lemma 2 ["Sufficient amount of functionals"] :

Let $x \in X$, then $\|x\| = \sup_{\|\phi\| \leq 1} |\phi(x)|$.

Proof: $|\phi(x)| \leq \|\phi\| \cdot \|x\| \leq \|x\|$, so $\|x\| \geq \sup_{\|\phi\| \leq 1} |\phi(x)|$

To prove " \leq ", define $E = \text{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$,
 $\Phi_0: Y \rightarrow \mathbb{C}$, if $y = c_y x \in E$.

Assume that $\|x\| = 1$, then $\|\Phi_0\|_{E^*} = \sup_{\|y\| \leq 1} |\Phi_0(y)| = (\text{from } (**),$
 $|c_y| = \|y\| \text{ if } \|x\| = 1) = \sup_{\|y\| \leq 1} \|y\| = 1$.

Hahn-Banach theorem $\Rightarrow \exists \tilde{\Phi}_0 \in X^*: \|\tilde{\Phi}_0\| = 1, \Phi_0|_E = \tilde{\Phi}_0$.

In particular $\sup_{\|\Phi\| \leq 1} |\Phi(x)| \geq |\tilde{\Phi}| = |\Phi_0(x)| = 1 = \|x\|$.

We have proved " \leq " in the case where $\|x\| = 1$.

The general case follows from consideration of $\frac{x}{\|x\|}$ in
place of X .

October 21, 2025

We are proving that $T \in \mathcal{B}(X, Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.

Let $T \in \mathcal{B}(X, Y)$, consider

$$\begin{aligned} \|T^*\| &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \|T^* \psi\|_{X^*} = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |(T^* \psi)(x)| \\ &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\langle x, T^* \psi \rangle| \\ &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\langle T_x, \psi \rangle| \\ &= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\psi(Tx)| \\ &= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} |\psi(Tx)| \end{aligned}$$

= $\|Tx\|$ Lemma "Sufficient amount of functionals"

$$= \sup_{\|x\| \leq 1} \|Tx\| = \|T\| < \infty$$

The claim follows. □

Corollary: $T \in \mathcal{B}(X, Y)$ is invertible ($\exists T^{-1} \in \mathcal{B}(Y, X)$) iff $T^* \in \mathcal{B}(Y^*, X^*)$ is invertible ($\exists (T^*)^{-1} \in \mathcal{B}(X^*, Y^*)$).

We prove just \Rightarrow .

Proof: Assume that T is invertible $\Leftrightarrow T^{-1}T = I_X$
 $TT^{-1} = I_Y$,

Let's take adjoint operators and see:

$$\left. \begin{array}{l} (T^{-1}T)^* = (I_X)^* \\ (TT^{-1})^* = (I_Y)^* \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} T^*(T^{-1})^* = I_X^* \\ (T^{-1})^*T^* = I_Y^* \end{array} \right.$$

Exercise: $(AB)^* = B^*A^*$

It remains to check that $I_X^* = I_{X^*}$, $I_Y^* = I_{Y^*}$. Then, by the previous theorem, $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$, hence T^* is invertible and its bounded inverse is $(T^*)^{-1} = (T^{-1})^*$.

Let's check that $I_X^* = I_{X^*}$. Take $\bar{x} \in X^*$, $x \in X$.

$$(I_X^*\bar{x})(x) = \langle x, I_X^*\bar{x} \rangle = \langle I_X x, \bar{x} \rangle = \langle x, \bar{x} \rangle = \bar{x}x$$

$$(I_X^*\bar{x})(x) \stackrel{\text{def}}{=} (\bar{x})x = \bar{x}(x).$$

Similarly, $I_Y^* = I_{Y^*}$. □

The „pairing notation” is often not used in literature, but it is very useful to not make mistakes.

Theorem [Schauder]: We have $T \in S_\infty(X, Y) \Leftrightarrow T^* \in S_\infty(Y^*, X^*)$.

Proof: We will prove just " \Rightarrow ".

Consider $K = \overline{TB_X(0,1)}$ - a compact set. Let $C(K)$ be the Banach space of continuous functions on K with

$$\|f\|_{C(K)} = \max_{s \in K} |f(s)|, \quad f: K \rightarrow \mathbb{C}$$

Let $E := \{\psi \in Y^* \mid \|\psi\|_{Y^*} \leq 1, \psi \text{ is considered as a function on } K\}$

$K \subset Y$, K metric space with respect to the metric $s(x_1, x_2) = \|x_2 - x_1\|_Y$

So, $E \subset C(K)$ and we claim that E is precompact.

1) Uniform boundedness:

$\psi \text{ cont.}$

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s \in K} |\psi(s)| = \max_{s \in \overline{TB_X(0,1)}} |\psi(s)| = \sup_{x \in B_X(0,1)} |\psi(Tx)|$$

$$\leq \|\psi\| \sup_{\|x\| \leq 1} \|Tx\| \leq \|\psi\| \cdot \|T\| \leq \underbrace{\|T\|}_{\substack{\text{does not} \\ \text{depend on } \psi}} < \infty$$

2) Equicontinuity: take $s_1, s_2 \in K$, let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \leq \|\psi\| \cdot \|s_1 - s_2\| \leq \|s_1 - s_2\|,$$

so maps from E are Lipschitz with constant 1, hence equicontinuous.

\Rightarrow By Arzela-Ascoli theorem, E is precompact.

We are now ready to prove $T^* \in S_\infty(Y^*, X^*)$. For this, we need to check that if $\{\psi_n\}$ is a sequence in $B_{Y^*}(0,1)$, then $\exists \{\psi_{n_k}\}$ such that $T^* \psi_{n_k}$ converges in X^* . So, take $\{\psi_n\} \subset B_{Y^*}(0,1)$ and consider it as elements $E \subset C(K)$.

Let $\{\psi_{n_k}\}$ be such that $\psi_{n_k} \xrightarrow{\text{w*}} \psi$ in $C(K)$. (E is precompact!)

Let's prove that $\{T^*\psi_{n_k}\}$ is Cauchy in X^* , then the theorem will follow.

Take $x \in X$, $\|x\| \leq 1$, and consider

$$\begin{aligned}
 \| (T^*\psi_{n_k})(x) - (T^*\psi_{n_j})(x) \| &= \| \langle x, T^*\psi_{n_k} \rangle - \langle x, T^*\psi_{n_j} \rangle \| \\
 &= \| \langle Tx, \psi_{n_k} \rangle - \langle Tx, \psi_{n_j} \rangle \| \\
 &= \| \psi_{n_k}(Tx) - \psi_{n_j}(Tx) \| \\
 &\leq \sup_{s \in K} \| \psi_{n_k}(s) - \psi_{n_j}(s) \| \\
 &= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|}_{\varepsilon_{k,j} - \text{does not depend on } x} \Big|_{C(K)} \longrightarrow 0 \quad \text{by } (*).
 \end{aligned}$$

$$\Rightarrow \|T^*\psi_{n_k} - T^*\psi_{n_j}\| \leq \varepsilon_{k,j} \longrightarrow 0.$$

□

Fredholm alternative

Example: Consider the equation $f(t) - \int_0^1 e^{t-s} f(s) ds \stackrel{(*)}{=} g(t)$ in $L^2[0,1]$.

Question: For which $g \in L^2[0,1]$ do we have a solution $f \in L^2[0,1]$?

Observation: g has to satisfy $\int_0^1 e^{-t} g(t) dt = 0$

Indeed, $\int_0^1 e^{-t} g(t) dt = \int_0^1 e^{-t} f(t) dt - \int_0^1 e^{-t} \left(\int_0^1 e^{t-s} f(s) ds \right) dt = 0$

It is not clear so far if there are other restrictions.

Theorem [Fredholm alternative]: Let X be a Banach space, $T = I - K$ for $K \in S^\infty(X, X)$. Then

$$\text{Ran } T = \{x \in X \mid \langle x, \varrho \rangle = 0 \ \forall \varrho \in \ker T^*\}.$$

In other words, either:

- (1) $\ker T^* = \{0\}$ and the equation $Tf = g$ has solution $\forall g \in X$.
- or (2) $\ker T^* \neq \{0\}$ and the equation $Tf = g$ has solutions only for g s.t. $\langle g, \varrho \rangle = 0 \ \forall \varrho \in \ker T^*$.

Let's complete the consideration of the example:
 we need to check that $K: f \rightarrow \int_0^1 e^{t-s} f(s) ds$ is compact
 (exercise) and find $\text{Ker } T^*$.

$$\phi \in \text{Ker } T^* \Leftrightarrow T^* \phi = 0$$

Adjoint operator T^* is defined by

$$\begin{aligned} \langle Tf, g \rangle &= \langle f, T^*g \rangle \quad f, g \in L^2[0,1] \\ \Leftrightarrow \langle f - \int_0^1 e^{t-s} f(s) ds, g \rangle &= \int_0^1 f(t) g(t) dt - \int_0^1 \left(\int_0^1 e^{t-s} f(s) ds \right) g(t) dt \\ &= \int_0^1 f(t) g(t) dt - \int_0^1 f(s) \left(\int_0^1 e^{t-s} g(t) dt \right) ds \\ &= \langle f, g - \int_0^1 e^{t-s} g(t) dt \rangle_{L^2[0,1]} \end{aligned}$$

$$(T^*g): s \mapsto g(s) - \int_0^1 e^{t-s} g(t) dt, \quad s \in [0,1].$$

$$T^*g = 0 \Leftrightarrow g(s) = \int_0^1 e^{t-s} g(t) dt \quad \text{a.e. on } [0,1]$$

$$\Leftrightarrow e^s g(s) = \underbrace{\int_0^1 e^{t-s} g(t) dt}_{\text{constant}} \quad \text{for almost every } s \in [0,1]$$

$$\Rightarrow \text{So, } \text{Ker } T^* = \{c \cdot e^{-s}, c \in \mathbb{C}\}, \quad \dim(\text{Ker } T^*) = 1.$$

By Fredholm theorem, equation $(**)$ is solvable \Leftrightarrow
 $\forall c \in \mathbb{C}. \langle g, c \cdot e^{-s} \rangle = 0 \Leftrightarrow \int_0^1 g(s) e^{-s} ds = 0$, which is $(***)$.

Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let X be a Banach space, $E \subseteq X$ - a linear closed subspace, $\varepsilon > 0$. Then $\exists x_0 \in X$ such that $\|x_0\| = 1$, $\text{dist}(x_0, E) \geq 1 - \varepsilon$.

Proof: Since $E \neq X$, then $\exists \tilde{x}_0 \in X \setminus E$. Since E is closed, we have $\text{dist}(\tilde{x}_0, E) = \delta > 0$ for some $\delta > 0$. Now consider $\tilde{y}_0 \in E$ such that $\delta \leq \|\tilde{x}_0 - \tilde{y}_0\| \leq (1+\eta)\delta$ for some $\eta \in (0, 1)$.

Now let $x_\eta := \frac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}$, $\|x_\eta\| = 1$.

$$\begin{aligned}\text{dist}(x_\eta, E) &= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \text{dist}(\tilde{x}_0 - \tilde{y}_0, E) \\ &= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \text{dist}(\tilde{x}_0, E) \\ &= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta}\end{aligned}$$

Choosing η so that $\frac{1}{1+\eta} = 1 - \varepsilon$, we are done.

October 22, 2025

Lemma: Let X be a Banach space. Then $I: X \mapsto X$ is compact on $X \Leftrightarrow \dim X < \infty$.

Proof: $\dim X < \infty \Rightarrow I \in S_\infty(X)$ - we already know $I \in S_\infty(X) \Rightarrow \dim X < \infty$:

Suppose $\dim X = +\infty$, find a sequence $\{e_n\}: \|e_n\| = 1 \quad \forall n \in \mathbb{N}$

$e_1 \in X$ - arbitrary

$e_2: \text{dist}(e_2, \text{span}\{e_1\}) \geq \frac{1}{2}$

$e_3: \text{dist}(e_3, \text{span}\{e_1, e_2\}) \geq \frac{1}{2}$

$e_4: \text{etc}$

existence of $\{e_n\}$
follows from previous
lemma, because
 $\text{span}\{e_1, \dots, e_k\} \neq X$
 $\forall k \in \mathbb{N}$

Then $\{e_n\} \subset B_X[0, 1] = I(B_X[0, 1])$ but there is no convergent subsequence, because $\|e_k - e_j\| \geq \frac{1}{2} \quad \forall k, j$. \blacksquare

Lemma: Let X be a Banach space, $K \in \mathcal{S}_\infty(X)$, $T = I - K$.
 Then: 1) $\dim(\text{Ker } T) < \infty$.
 2) $\text{Ran } T$ is closed in X . [closed range Lemma]

Proof: 1): We have $I|_{\text{Ker } T} = \underbrace{(I-K)|_{\text{Ker } T}}_0 + \underbrace{K|_{\text{Ker } T}}_{\in \mathcal{S}_\infty(\text{Ker } T, S)}$

$$I|_{\text{Ker } T} \in \mathcal{S}_\infty(\text{Ker } T, X) \Rightarrow I \in \mathcal{S}^\infty(\text{Ker } T) \Rightarrow \dim(\text{Ker } T) < \infty$$

2): The statement is equivalent to the fact that if $\{x_n\} \subset X$ s.t. $TX_n \rightarrow y$ in X then $\exists x \in X. Tx = y$.

2.a) Let $\{x_n\} : \|x_n\| \leq c \quad \forall n$.

Then $(I-K)(x_n) \rightarrow y$, $(I-K)(x_{n_k}) \xrightarrow{(*)} y$
 For every subsequence x_{n_k}

Let's choose x_{n_k} : Kx_{n_k} converges to $z \in X$
 (use $K \in \mathcal{S}_\infty(X)$)

Then $x_{n_k} \rightarrow y+z$ by (*), take $x = y+z$:

$$T(y+z) = \lim_{k \rightarrow \infty} Tx_{n_k} = y, \text{ so } Tx = y. \quad \checkmark$$

2.b) $\text{dist}(x_n, \text{Ker } T) \leq c \quad \forall n \in \mathbb{Z}$

Take $\tilde{x}_n := x_n - w_n$, where $w_n \in \text{Ker } T : \|\tilde{x}_n\| \leq 2c$.

We have $\lim_{n \rightarrow \infty} T\tilde{x}_n = \lim_{n \rightarrow \infty} Tx_n$ by step 2a) $\exists \tilde{x} : T\tilde{x} = y \quad \checkmark$

2.c) $\text{dist}(x_n, \text{Ker } T) \rightarrow +\infty$. Let us show that this situation does not occur. Suppose the converse:

Consider $\tilde{x}_n = x_n - \underbrace{w_n}_{\text{Ker } T} : \text{dist}(x_n, \text{Ker } T) \leq \|\tilde{x}_n\| \leq 2\text{dist}(x_n, \text{Ker } T)$

For $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ we have $Tz \rightarrow 0$.

$$Tz_n = T \frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T \frac{x_n}{\|x_n\|} = \frac{Tx_n}{\|x_n\|} \rightarrow \begin{cases} y & \Rightarrow Tx_n \text{ is bdd in } X \\ +\infty & \end{cases}$$

$$\Rightarrow \|Tz_n\| \leq \frac{2\|y\|}{\|\tilde{x}_n\|} \rightarrow 0 \quad \text{for } n \text{ large enough}$$

At the same time, $Tz_n = z_n - Kz_n$

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $\{Kz_{n_k}\}$ converges to some $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{Tz_{n_k}}_{\rightarrow 0} + \underbrace{Kz_{n_k}}_{\rightarrow z} \rightarrow z$$

We have $Tz = 0$ ($= \lim Tz_{n_k} = \lim Tz_n = 0$)

$\Leftrightarrow z \in \text{Ker } T$, $0 = \text{dist}(z, \text{Ker } T) =$

$$= \lim_{k \rightarrow \infty} \text{dist}(z_{n_k}, \text{Ker } T)$$

$$= \lim_{k \rightarrow \infty} \text{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \text{Ker } T\right)$$

$$= \lim \frac{\text{dist}(\tilde{x}_{n_k}, \text{Ker } T)}{\|\tilde{x}_{n_k}\|}$$

$$= \lim \frac{\text{dist}(x_{n_k}, \text{Ker } T)}{\|\tilde{x}_{n_k}\|}$$

$$\stackrel{(*)}{\geq} \frac{1}{2} \quad \leadsto \text{contradiction}$$



Lemma: Let X be a Banach space, $T \in \mathcal{B}(X)$:

$\text{Ker}(T) = \{0\}$ and $T^{k+1}X = T^kX$ for some $k \geq 0$.

Then $\text{Ran } T = X$.

Proof: We need to prove that $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$.

We know that: $T^{k+1}a = T^k\tilde{a}$ for every a and some \tilde{a} depending on a . $\downarrow \text{Ker } T^k \neq \{0\} \Rightarrow \text{Ker } T \neq \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}$$



Theorem [Fredholm]: Let X be a Banach space.

$K \in S_\infty(X)$, $T = I - K$. Then TFAE:

1) T is invertible in $\mathcal{B}(X)$ 2) $\text{Ker } T = \{0\}$ 3) $\text{Ran } T = X$

1) T^* is invertible in $\mathcal{B}(X)$ 2) $\text{Ker } T^* = \{0\}$ 3) $\text{Ran } T^* = X^*$

Proof: We will prove $2 \Rightarrow 3 \Rightarrow 2' \Rightarrow 3' \Rightarrow 2$, $1 \Rightarrow 1' \Rightarrow 2'$, $2 \wedge 3 \Rightarrow 1$

(2) \Rightarrow (3): If $T^k X = T^{k+1} X$ for some k , we are done by the lemma.

Define $X_k := T^k X$, $k \geq 0$, and note that $X_0 \supset X_1 \supset X_2 \supset X_3 \dots$

Assume that all inclusions are strict, i.e. $X_k \supsetneq X_{k+1} \forall k$.

The subspaces X_k are closed by the closed range lemma (by induction). By the almost orthogonality lemma: $\exists \{y_k\}_{k=0}^\infty$ s.t.

i) $y_k \in X_k \forall k$ ii) $\|y_k\| = 1 \forall k$ iii) $\text{dist}(y_k, X_{k+1}) \geq \frac{1}{2}$

Since K is compact $\{Ky_k\}$ contains a convergent subsequence $\{Ky_{k_j}\}_{j=1}^\infty$. On the other hand, if $j < m$

$$\begin{aligned} Ky_{k_j} - Ky_{k_m} &= (Ky_{k_j} - y_{k_j}) - (Ky_{k_m} - y_{k_m}) + y_{k_j} - y_{k_m} \\ &= y_{k_j} - y_{k_m} - \underbrace{(Ty_{k_j} - Ty_{k_m})}_{\in X_{k_j+1}} \end{aligned}$$

Since $y_{k_m} \in X_{k_m}$, $j < m$, $k_j < k_m$, $k_j+1 \leq k_m$, $X_{k_m} \subset X_{k_j+1}$
 $\Rightarrow y_{k_m} \in X_{k_j+1}$; $Ty_{k_m} \subset T X_{k_j+1} \subset X_{k_j+1}$; $Ty_{k_j} \in T(X_{k_j}) = X_{k_j+1}$
 $\Rightarrow Ky_{k_j} - Ky_{k_m} = y_{k_j} + R$, $R \in X_{k_j+1}$. Since $\|y_{k_j} + R\| \geq \frac{1}{2}$ by (iii),
we get a contradiction ($\{Ky_{k_j}\}$ is not Cauchy).

(3) \Rightarrow (2): Take $\phi \in \text{Ker } T^*$. We have

$$\begin{aligned} T^* \phi = 0 &\Leftrightarrow \langle x, T^* \phi \rangle = 0 \quad \forall x \in X \Leftrightarrow \langle Tx, \phi \rangle = 0 \quad \forall x \in X \Leftrightarrow \\ &\Leftrightarrow \langle y, \phi \rangle = 0 \quad \forall y \in X \quad (\text{by 3}) \Leftrightarrow \phi = 0 \end{aligned}$$

(2') \Rightarrow (3'): (the same as $2 \Rightarrow 3$ using Schauder's theorem)

(3') \Rightarrow (2): Assume that $\text{Ran } T^* = X^*$ and take $x \in \text{Ker } T$. We have $Tx = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \quad \forall \phi \in X^* \Leftrightarrow \langle x, T^*\phi \rangle \quad \forall \phi \in X^*$ $\Leftrightarrow \langle x, \Psi \rangle = 0 \quad \forall \Psi \in X^* \quad (\text{by 3'}) \Leftrightarrow x = 0$ Lemma on sufficient amount of functionals

Conclusion: $(2) \Leftrightarrow (3) \Leftrightarrow (2') \Leftrightarrow (3')$

(1) \Rightarrow (1'): We already know for arbitrary $T \in \mathcal{B}(X)$.

(1') \Rightarrow (2'): Invertible operators are injective.

$(2') \Leftrightarrow (1), (3)$

(2 & 3) \Rightarrow (1): This holds for every $T \in \mathcal{B}(X)$ by the following fundamental theorem from general F.A.:

Theorem [linear mapping theorem]: Let X, Y be Banach spaces, $T: X \rightarrow Y$ - linear bijection. Then $T \in \mathcal{B}(X, Y) \Leftrightarrow T^{-1} \in \mathcal{B}(Y, X)$.

Check injectivity: $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0 \quad (\text{by 1}) \Leftrightarrow x_1 = x_2$

Surjectivity: $\text{Ran } T = X \quad (\text{by 3})$



Remark: If $T = I - K$, $K \in S_\infty(X)$, and $\text{Ker } T = \{0\}$ then the equation $Tx = y$ has a unique solution for every $y \in X$.

Remark: $\text{Ker } T = \{0\} \Leftrightarrow \text{Ker } T^* = \{0\}$, so we have proved half of Fredholm Alternative.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{\text{Tx}} = \{x \in X \mid \langle x, \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T\}$.

Remark: This implies the other half of Fredholm alternative since $\overline{\text{Tx}} = \text{Tx}$ for $T = I - K$, $K \in S_\infty(X)$.

Lemma [seperation lemma]: Let Y be a Banach space, $Y_0 \subsetneq Y$ - a closed subspace, then $\exists \phi \in Y^*: \phi|_{Y_0} = 0$, $\phi(y) \neq 0$ for some $y \in Y \setminus Y_0$.

Proof: Take $y \in Y \setminus Y_0$, define $\Phi_0: \text{span}\{y, Y_0\} \rightarrow \mathbb{C}$ by $\Phi_0(cy + y_0) \mapsto c$, for $c \in \mathbb{C}$, $y_0 \in Y_0$.

$$1) \{cy + y_0 \mid c \in \mathbb{C}, y_0 \in Y_0\} = \text{span}\{y, Y_0\} \quad \text{clear } \checkmark$$

$$2) \underbrace{cy + y_0}_c = \underbrace{\tilde{c}y + \tilde{y}_0}_{\tilde{c}} \Leftrightarrow (c - \tilde{c})y = y_0 - \tilde{y}_0 \in Y_0 \Leftrightarrow c = \tilde{c} \Rightarrow \Phi_0(c) = \Phi_0(\tilde{c})$$

\Rightarrow correctness OK

3) Φ_0 is linear - clear

$$4) \Phi_0|_{Y_0} = 0 \quad (c=0 \text{ on } Y)$$

$$5) |\Phi_0(cy + y_0)| \stackrel{?}{\leq} A \|cy + y_0\| \quad \forall c, y_0$$

$$|\Phi_0(cy + y_0)| = |c| = (\text{dist}(y, \underbrace{Y_0}_{\text{closed}}))^{-1} \cdot |c| \cdot \text{dist}(y, Y_0)$$

$$= \text{dist}(y, Y_0)^{-1} \cdot \text{dist}(|c|y, Y_0) = \text{dist}(y, Y_0)^{-1} \cdot \text{dist}(|c|y, Y_0)$$

\nearrow

$$|c|y = \alpha \cdot e \cdot y_0, |\alpha|=1$$

$$\text{dist}(|c|y, Y_0) = \text{dist}(\underbrace{|c|y_0}_{c}, \underbrace{Y_0}_y)$$

$$\leq \text{dist}(y, Y_0)^{-1} \|cy + y_0\|, \quad \text{so } A = \text{dist}(y, Y_0)^{-1} \text{ works}$$

$$6) \Phi_0(y) = 1 \neq 0$$

\Rightarrow Use the Hahn-Banach theorem and extend Φ_0 to the whole Y . □

We actually proved that the lemma holds $\forall y \in Y \setminus Y_0$.

We can now prove the theorem from above.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then
 $\overline{Tx} = \{x \in X \mid \langle x, \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T^*\}$.

Proof: We have $Tx \subset E$, $E = \{x \mid \langle x, \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T^*\}$,
because $\langle Tx, \phi \rangle = \langle x, T^* \phi \rangle = 0 \quad \forall \phi \in \text{Ker } T^*$.
Then $\overline{Tx} \subset \overline{E} = E$ since E is closed.
 $(\begin{matrix} x_n \rightarrow x \text{ and } \langle x_n, \phi \rangle = 0 \text{ for some } \phi \in X^* \\ \text{then } \langle x, \phi \rangle = \lim \langle x_n, \phi \rangle = 0 \end{matrix})$

We now need to check $\overline{Tx} \supset X$. If not, the inclusion $\overline{Tx} \subset E$ is proper and by the separation lemma $\exists \phi : \phi|_{\overline{Tx}} = 0$ ^(*) but $\phi(e) \neq 0$ for some $e \in E$.

$$\begin{aligned} (*) \Rightarrow \phi|_{Tx} = 0 &\Leftrightarrow \langle Tx, \phi \rangle = 0 \quad \forall x \in X \Leftrightarrow \langle x, T^* \phi \rangle = 0 \quad \forall x \in X \\ \Leftrightarrow \phi \in \text{Ker } T^* &\Rightarrow \phi(e) = 0 \quad \forall e \in E \text{ by definition of } E, \text{ contradiction.} \blacksquare \end{aligned}$$

Classical form of Fredholm alternative for integral equations

Theorem: Let (S, μ) be a space with measure μ , and let $K(x, y) : S \times S \rightarrow \mathbb{C}$:

$$\iint_{S \times S} |K(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$

Then either equation $f(y) + \int_S K(x, y) f(x) d\mu(x) = 0$ ^(***) has only the trivial solution $f=0$ and the equation

$$f(y) + \int_S K(x, y) f(x) d\mu(x) = g(y)$$

is solvable for every $g \in L^2(S, \mu)$ or the equation $(**)$ has a non-trivial solution in $L^2(S, \mu)$.

Remark: Uniqueness implies existence.

Proof: Let's define $T = I - K$, $(Kf)(y) = \int_S K(x, y) f(x) d\mu(x)$. Consider T as an operator on $L^2(S, \mu)$.

Then $(**)$ $\Leftrightarrow \text{Ker } T = \{0\} \Leftrightarrow TL^2(S, \mu) = L^2(S, \mu)$ by Fredholm alternative,
modulo the fact that $K \in \text{Soo}(L^2(S, \mu))$. □

↑ This we postpone until
Hilbert Spaces theory.

Remarks: Further reading:

- 1) $\dim \text{Ker } T = \dim \text{Ker } T^*$ if $T = I - K$, $K \in \text{Soo}$, it coincides with the dimension of the space of solutions
- 2) There is a version of Fredholm theory for general operators $T \in \mathcal{B}(X)$: $\dim(\text{Ker } T) < \infty$, $\dim(X/\text{Ran } T) < \infty$.

Spectrum of compact operators

October 29, 2025

Definition: Let $T \in \mathcal{B}(X)$, X Banach space.

$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$
is called the **spectrum** of T .

Definition: A number $\lambda \in \mathbb{C}$ is called an **eigenvalue** of T if $\exists e \in X \setminus \{0\}$. $Te = \lambda e$.

Definition: $\sigma_p(T) := \{\lambda \text{ eigenvalue of } T\}$... point spectrum of T .

Remark: We have $\sigma_p(T) \subset \sigma(T)$ for every $T \in \mathcal{B}(X)$.

Proof: $\lambda \in \sigma_p(T) \Rightarrow \lambda I - T$ is not injective because $\text{Ker}(\lambda I - T) \neq \{0\}$.

Remark: In general, we might have $\sigma_p(T) \neq \sigma(T)$ and even $\sigma_p(T) = \emptyset$.

Theorem: Let X be a Banach space, $\dim X = +\infty$, and let $K \in S_\infty(X)$. Then $0 \in \sigma(K)$, $\sigma(K) \setminus \{0\} \subset \sigma_p(T)$, moreover, each eigenvalue has a finite multiplicity, and $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$.

Proof: $0 \in \sigma(K)$ since $0 \notin \sigma(K)$, then $\exists K^{-1} \in \mathcal{B}(X)$:
 $I = K K^{-1} = K^{-1} K$, but then $I \in S_\infty(X) \Rightarrow \dim X < \infty$, contradiction.

Now let's prove that $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$.

Take $\lambda \in \sigma(K)$ and assume that $\lambda \notin \sigma_p(K)$. Then

$$\text{Ker}(\lambda I - K) = \{0\} \Leftrightarrow \text{Ker}(I - \frac{1}{\lambda} K) = \{0\}$$

$\Rightarrow I - \frac{1}{\lambda} K$ is invertible by Fredholm theorem - contradiction.

Now let's prove that $\dim E_\lambda < \infty$.

$$E_\lambda := \{e \in X \mid Ke = \lambda e\} \quad \left(\begin{array}{l} \text{definition for :} \\ \lambda \text{ has finite multiplicity} \end{array} \right)$$

If this is not the case, there is a sequence $\{e_n\}_{n=1}^\infty$ such that $\text{dist}(e_n, \text{span}\{e_1, \dots, e_{n-1}\}) \geq \frac{1}{2}$, $\|e_n\| = 1$.

Consider $\{Ke_n\} = \{\lambda_n e_n\}$: we cannot choose a convergent subsequence from this sequence - contradiction with $K \in S_\infty(X)$.

To prove $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$, assume the converse and let $e_n \in X$: $Ke_n = \lambda_n e_n$, $\|e_n\| = 1$, $\lambda_n \in \sigma_p(K)$, $\lambda_n \neq \lambda_k$ for $k \neq n$.

Define $E_n := \text{span}\{e_1, \dots, e_n\} \quad \forall n$.

Observation: $E_{n+1} \supsetneq E_n \quad \forall n$.

Clearly $E_{n+1} \supset E_n$. If $E_{n+1} = E_n$ for some n , there exists the first such n . Then

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^n \lambda_{n+1} \alpha_k e_k$$

$$\lambda_{n+1}e_{n+1} = \sum_{k=1}^n d_k \lambda_k e_k$$

$$\Rightarrow 0 = \sum_{k=1}^n d_k (\lambda_{n+1} - \lambda_k) e_k \Rightarrow d_{n+1} - \lambda_n = 0 \\ \Rightarrow \lambda_{n+1} = \lambda_n \dots \text{contradiction}$$

\Rightarrow The observation is true.

Let's choose $y_{n+1} \in E_{n+1}$ such that $\|y_{n+1}\| = 1$, $\text{dist}(y_{n+1}, E_n) \geq \frac{r}{2}$
It remains to prove that $\{K_{y_n}\}$ does not have a convergent subsequence.

$$x_{n+1} = d_{n+1}e_{n+1} + R_n, \text{ where } R_n \in E_n$$

$$Ky_{n+1} - Ky_{m+1} = \lambda_{n+1}d_{n+1}e_{n+1} + \tilde{R}_n - \lambda_{m+1}d_{m+1}e_{m+1} - \tilde{R}_m$$

Assume that $n \geq m+1$, then

$$\begin{array}{c} \tilde{R}_n - \underbrace{\lambda_{m+1}d_{m+1}e_{m+1}}_{E_{m+1} \subset E_n} + \tilde{R}_m \in E_n \\ \uparrow \quad \uparrow \quad \uparrow \\ E_n \quad E_{m+1} \subset E_n \quad E_m \subset E_n \end{array}$$

$$\begin{aligned} \Rightarrow \|Ky_{n+1} - Ky_{m+1}\| &\geq \text{dist}(\lambda_{n+1}d_{n+1}e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \text{dist}(d_{n+1}e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \text{dist}(y_{n+1}, E_n) \\ &\geq r \cdot \frac{1}{2} > 0 \end{aligned}$$

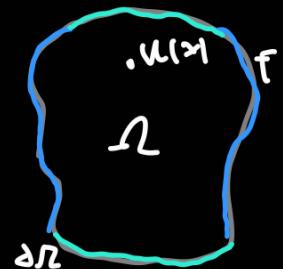
□

Scheme of the solution of Dirichlet problem in $\mathbb{R}^n, n \geq 3$, by means of Fredholm theory

Dirichlet problem: find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ (Ω is a domain in \mathbb{R}^n)
 $\partial\Omega \in C^2$
such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} \stackrel{(*)}{=} f \end{cases} \quad \text{where } f \in C(\partial\Omega)$$

Physics interpretation: heat



Scheme for solution: we search for a solution of the form

$$u(x) = \int_{\partial\Omega} K(x,y) \varphi(y) dy, \quad K(x,y) = c_n \frac{(x-y, n_y)}{\|x-y\|_{\mathbb{R}^n}^n}$$

n_y is the outward unit normal, $c_n \in \mathbb{R}$

$\Delta u = 0$ for every $\varphi \in C(\partial\Omega)$, we only need to find good φ (such that $(*)$ will hold)

$$u_\varphi(x) = -\varphi(x) + \int_{\partial\Omega} K(x,y) \varphi(y) dy \quad \text{if } x \in \partial\Omega$$

$$u_\varphi = (-I + K)\varphi, \quad K\varphi = \int K(x,y) \varphi(y) dy$$

To check that $\exists \varphi: u_\varphi = f$ on $\partial\Omega$ we just check that $K \in S_\infty(C(\partial\Omega))$ and $\text{Ker}(-I + K) = \{0\}$
 \Rightarrow we are done by Fredholm alternative.

This is hard to prove (course on PDEs)

PART II Banach Algebras

November 4, 2025

Definition: A is a **Banach algebra** if A is a Banach space with the operation of multiplication such that:

$$1) (xy)z = x(yz); \quad x, y, z \in A$$

$$2) (x+y)z = xz + yz$$

$$x(y+z) = xy + xz$$

$$3) (\alpha x)y = x(\alpha y); \quad \alpha \in \mathbb{C}$$

$$4) \exists e \in A. xe = ex = x \quad \forall x \in A.$$

$$5) \|xy\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A.$$

Examples: 1) X Banach space $\Rightarrow \mathcal{B}(X)$ is a Banach algebra with unity $e=I$ and norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$

2) Calkin algebra: $\mathcal{B}(X)/S_\infty(X)$, $e = I + S_\infty(X)$

$$\|T\|_{\mathcal{B}(X)/S_\infty(X)} = \inf_{k \in S_\infty(X)} \|T - k\| = \text{dist}(T, S_\infty(X))$$

3) K-compact Hausdorff space, $C(K)$ is a Banach algebra,

$$e=1, \|f\|_{C(K)} = \max_{\substack{x \in K \\ f: K \rightarrow \mathbb{C}}} |f(x)|$$

4) $W(\mathbb{T})$ - Wiener algebra on $\mathbb{T} := \{z \mid |z|=1\}$.

$$\left\{ f = \sum_{k \in \mathbb{Z}} c_k z^k \mid c_k \in \mathbb{C}, \sum |c_k| < \infty \right\}$$

$$\|f\|_{W(\mathbb{T})} = \sum_{k \in \mathbb{Z}} |c_k|, \quad e=1, \quad \begin{matrix} \text{multiplication is the usual} \\ \text{multiplication of functions} \end{matrix}$$

$$\|f \cdot g\|_{W(\mathbb{T})} \leq \|f\|_{W(\mathbb{T})} \cdot \|g\|_{W(\mathbb{T})}$$

\uparrow exercise

5) $L^\infty(\mathbb{R})$, $\|f\| = \underset{\mathbb{R}}{\text{esssup}} |f|$

6) $H^\infty(D)$ - set of bounded analytic functions on $D := \{|z| < 1\}$

$$\|f\|_{H^\infty(D)} = \sup_{|z| < 1} |f(z)|$$

Remark: 1) & 2) are noncommutative Banach algebras, others are commutative.

Remark: If A is a Banach space with multiplication and properties 1), 2), 3), 5), then we can always add the identity to convert A to a Banach algebra as follows:

$$\mathcal{A} = A \times \mathbb{C}, \quad (x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$

$$f(x, \alpha) = (fx, f\alpha)$$

$$(x, \alpha) \cdot (y, \beta) = (xy + \alpha y + \beta x, \alpha \beta)$$

$$((x + \epsilon e)ly + \beta e) = xy + \alpha y + \beta x + \alpha \beta e \quad e = (0, 1)$$

$$\|(x, \alpha)\| = \|x\| + |\alpha| \quad \leftarrow ! \text{ corrected on Nov. 5th}$$

$\Rightarrow \mathcal{A}$ is a Banach algebra with identity and $(A, 0) \subset \mathcal{A}$.

Example: $L^1(\mathbb{R})$, multiplication $F * g = \int_{\mathbb{R}} f(y)g(x-y)dy$

$\|F\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f|dx$ - this is a Banach algebra without identity, and the above construction is equivalent to consideration of $\mathcal{A} = \mathbb{C}\delta_0 + L^1(\mathbb{R})$

$$\left(\begin{array}{l} \text{measure } \delta_0(S) = \begin{cases} 1; & 0 \in S \\ 0; & 0 \notin S \end{cases} \\ \delta_0 * f = f * \delta_0 = \int_{\mathbb{R}} f(y) \delta_0(x-y) dy = f(x) \\ \langle f, \delta_0 \rangle = f(0) = \int_{\mathbb{R}} f(x) \delta_0(dx). \quad f \in C_c(\mathbb{R}) \end{array} \right)$$

measures \rightarrow no dy

Definition: A Banach algebra, $x \in A$. We say $x \in A$ is invertible if $\exists x^{-1} \in A$. $xx^{-1} = x^{-1}x = e$.

If x is invertible, then x^{-1} is unique.

Definition: $G(A) := \{x \in A \mid x \text{ is invertible}\}.$

↑ we use G , because $G(A)$ is a group

Definition: $x \in A$. $\delta(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\}.$

Example: If A is the set of $n \times n$ matrices with complex coefficients, then for $T \in A$, $\delta(T)$ is the set of eigenvalues (the usual spectrum of the matrix).

Example: $A = C(K)$, $\sigma(f) = ?$

$\lambda \in \mathbb{C}$: $\lambda - f$ is not invertible in $C(K)$.

Since $g \in G(C(K)) \Leftrightarrow \frac{1}{g} \in C(K)$, we have $\lambda - f \in G(C(K)) \Leftrightarrow \frac{1}{\lambda - f} \in C(K)$
 $\Leftrightarrow \lambda \notin f(K)$

$\Rightarrow \sigma(f) = f(K)$ ← compact non-empty subset of \mathbb{C}

Basic properties

Proposition 1: Let A be a Banach algebra. Then $\|e\| \geq 1$.

Proof: $\|e\| = \|e \cdot e\| \leq \|e\| \cdot \|e\| \Rightarrow \cancel{\|e\|=0} \text{ or } \|e\| \geq 1$

Definition: A Banach algebra is called **unital** if $\|e\|=1$.

Proposition 2: If A is an arbitrary Banach algebra, then the algebra $A \times \mathbb{C}$ is unital.

Proof: $\|(0,1)\| = \|0\| + \|1\| = 1$

↑
! Note: This was wrong previously and corrected on November 5th.

⚠ From now on, all Banach algebras are unital.

Proposition 3: If $\frac{x_n}{y_n} \xrightarrow{y}$ in A , then $x_n \cdot y_n \xrightarrow{} xy$.

Proof: $\|x_n \cdot y_n - xy\| \leq \underbrace{\|x_n - x\|}_{\xrightarrow{n} 0} \underbrace{\|y_n\|}_{bdd} + \|x\| \underbrace{\|y_n - y\|}_{\xrightarrow{0}} \xrightarrow{} 0$ □

Proposition 4: Let $a \in A$, $\|a\| \leq 1$, then $e-a \in G(A)$.

Proof: Define $(e-a)^{-1} = e+a+a^2+\dots = \sum_{k=0}^{\infty} a^k$ this series converges because A is a Banach space and $\sum \|a^k\| \leq \sum \|a\|^k < \infty$

Let us check that $(e-a)^{-1}(e-a) = e$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \underbrace{\left(\sum_{k=0}^n a^k \right)}_{e-a^{n+1}} (e-a) = e$$

$$e-a^{n+1} \xrightarrow{n \rightarrow \infty} e, \text{ since } \|a^{n+1}\| \leq \|a\|^{n+1} \xrightarrow{n \rightarrow \infty} 0$$

Similarly, $(e-a)(e-a)^{-1} = e$.

Proposition 5: $G(A)$ is open in A .

Proof: Let $a \in G(A)$, $b \in A$, then $a-b = a(\underbrace{e-a^{-1}b}_{\in G(A)})$ if $\|a^{-1}b\| < 1$
 $\|a^{-1}b\| < 1$ holds for all b : $\|b\| \leq \frac{1}{\|a^{-1}\|}$
 $\Rightarrow B(a, \frac{1}{\|a^{-1}\|}) \subset G(A) \Rightarrow G(A)$ is open.

Proposition 6: Let $x \in G(A)$, $x_n \xrightarrow{} x$ in A . Then $x_n \in G(A)$ for n large enough, and $x_n^{-1} \xrightarrow{} x^{-1}$ as $n \rightarrow \infty$.

Proof: Write $x_n = x + z_n$, we have $x_n \in G(A)$ for n large enough by Proposition 5.

$$(x+z_n)^{-1} - x^{-1} = (x(e+x^{-1}z_n))^{-1} - x^{-1} = \underbrace{(e+x^{-1}z_n)^{-1}}_{y_n} \cdot x^{-1} - x^{-1} \xrightarrow{\substack{\uparrow \\ \text{Prop. 3}}} ex^{-1} - x^{-1} = 0$$

$$x_n \xrightarrow{} e, \text{ see Prop. 4: } \|y_n - e\| \leq \sum_{k=1}^{\infty} \|x^{-1} \cdot z_n\|^k \xrightarrow{} 0 \quad \|z_n\| \xrightarrow{} 0 \quad \boxed{\square}$$

Theorem: Let A be a Banach algebra, $a \in A$. Then $\sigma(a)$ is a nonempty compact subset of \mathbb{C} .

Proof: $\sigma(a) = \mathbb{C} \setminus S(a)$, $S(a) := \{\lambda \mid \lambda e - a \text{ is invertible}\}$
resolvent set

$S(a)$ is open by Prop. 5 $\Rightarrow \sigma(a)$ is closed.

Let's check that $\sigma(a)$ is bounded: $\sigma(a) \subset \{\lambda \mid |\lambda| \leq \|a\|\}$
 $\Leftrightarrow S(a) \supset \{\lambda \mid |\lambda| \geq \|a\|\}$.

Take $\lambda: |\lambda| \geq \|a\|$, then $\lambda e - a = \lambda e \underbrace{(e - \frac{a}{\lambda})}_{\text{invertible by Prop. 4}}$

It remains to check that $\sigma(a) \neq \emptyset$.

Assume that $\sigma(a) = \emptyset$ and consider the function

$$f_\phi(\lambda) = \phi((\lambda e - a)^{-1}) \text{ for some } \phi \in A^*; \lambda \in \mathbb{C}$$

Let's check that f_ϕ is analytic. Take $\lambda_0 \in \mathbb{C}$, consider

$$\lim_{\lambda \rightarrow \lambda_0} \frac{f_\phi(\lambda) - f_\phi(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \phi \left(\frac{(\lambda e - a)^{-1} - (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0} \right)$$

$$[x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}] = \lim_{\lambda \rightarrow \lambda_0} \phi \left(\frac{(\lambda e - a)^{-1} ((\lambda e - a) - (\lambda_0 e - a)) (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0} \right)$$

$$= \lim_{\lambda \rightarrow \lambda_0} \phi \left(\underbrace{(\lambda e - a)^{-1}}_{\hookrightarrow (\lambda_0 e - a)^{-1} \text{ Prop. 6}} (\lambda_0 e - a)^{-1} \right)$$

$$\underbrace{\phi \text{ continuous, continuity of multiplication}}_{\Rightarrow} = -\phi((\lambda_0 e - a)^{-2})$$

$\Rightarrow f_\phi$ is analytic ($f_\phi \in \text{Hol}(\mathbb{C})$)

Take $\lambda: |\lambda| \geq 2\|a\|$, $|f_\phi(\lambda)| = \|\phi\| \cdot \|((\lambda e - a)^{-1})\| = \|\phi\| \underbrace{\|(\lambda)^{-1}\|}_{\text{uniformly bounded for}} \underbrace{\|(e - \frac{a}{\lambda})^{-1}\|}_{\lambda: |\lambda| \geq 2\|a\|}$

f_ϕ is bdd on \mathbb{C} by the max. principle

$\Rightarrow F_\phi = c_\phi \in \mathbb{C}$, but $c_\phi = 0$ by $(*)$

$\Rightarrow \phi(\lambda e - a) = 0 \forall \lambda \in \mathbb{C}, \forall \phi \in A^* \Rightarrow \lambda e - a = 0 \forall \lambda \in \mathbb{C} \Rightarrow \text{contradiction}$



Definition: Let A_1, A_2 be Banach algebras.

November 5, 2025

We say that A_1 is isomorphic to A_2 ($A_1 \cong A_2$) if there exists a map $j: A_1 \rightarrow A_2$:
 $j(\alpha a + \beta b) = \alpha j(a) + \beta j(b)$ (*)
 $j(ab) = j(a) \cdot j(b)$ (**)
 $\|j(a)\|_{A_2} = \|a\|_{A_1}$
 j is bijective

Theorem [Banach-Mazur]: Let A be a Banach algebra s.t. $G(A) = A \setminus \{0\}$. Then $A \cong \mathbb{C}$.

Proof: For every $a \in A$ we have $\sigma(a) \neq \emptyset$. So there is a $\lambda(a) \in \mathbb{C}$:
 $\lambda(a)e - a$ is not invertible $\Leftrightarrow \lambda(a)e - a = 0 \Leftrightarrow a = \lambda(a)e$.
In particular, such $\lambda(a)$ is unique.

Define $j: A \rightarrow \mathbb{C}$, $a \mapsto \lambda(a)$. We have:

$$\begin{aligned} a+b &= \lambda(a+b)e \\ a+b &= \lambda(a)e + \lambda(b)e \end{aligned} \quad \left. \begin{aligned} \Rightarrow \lambda(a+b) &= \lambda(a) + \lambda(b) \end{aligned} \right\}$$

Similarly, $\lambda(\alpha a) = \alpha \lambda(a)$, $\lambda(ab) = \lambda(a) \cdot \lambda(b) \Rightarrow (*)$, $(**)$ ✓

$$\|j(a)\|_{\mathbb{C}} = \|a\|_A \Leftrightarrow |\lambda(a)| = \|a\|_A \Leftrightarrow \|\lambda(a)e\|_A = \|a\|_A \quad \checkmark$$

$$j(a) = j(b) \Leftrightarrow \lambda(a)e = \lambda(b)e \Leftrightarrow a = b \quad \left. \begin{aligned} \end{aligned} \right\} j \text{ is a bijection}$$

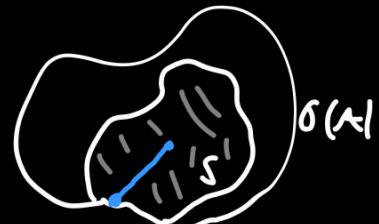
$$j(e) = 1 \Rightarrow j(A) \supset j(\mathbb{C} \cdot e) = \mathbb{C}$$

Definition: Let A be a Banach algebra, then $r(a) := \sup \{|\lambda|, \lambda \in \sigma(a)\}$ is called the spectral radius of a .

Theorem: Let A be a Banach algebra. Then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

In particular, the limit above exists.



Proof: Step 1: $r(a) \leq \|a\|$, because

$$\lambda e - a = \lambda(e - \underbrace{\frac{a}{\lambda}}_b) \quad \|b\| < 1 \text{ for } \lambda: |\lambda| \geq \|a\|$$

invertible element in A (Prop. 4)

$$\Rightarrow \lambda e - a \in G(A) \Rightarrow \lambda \in \sigma(a) \Rightarrow \sigma(a) \subset \mathbb{B}[0, \|a\|]. \quad \checkmark$$

Step 2: $r(a) \leq \|a^n\|^{1/n}$ for every $n \geq 2$.

Take $\lambda \in \mathbb{C}$, and consider

$$\lambda^n e - a^n = (\lambda e - a) \cdot p_\lambda(a), \quad p_\lambda \text{ is a polynomial}$$

If $\lambda \in \sigma(a)$, then $\lambda^n e - a^n = z_1 \cdot z_2$, where $z_1, z_2 \in A$, $z_1 \notin G(A)$, $z_1 z_2 = z_2 z_1$. If $z = \lambda^n e - a^n$ is invertible, $\exists z^{-1} \in A$.

$$z^{-1} z_1 z_2 = z_1 (z_2 z^{-1}) = e \\ (\underbrace{z_1 z_2}_1 z_1) z^{-1} \Rightarrow z_1 \in G(A) \dots \text{contradiction}$$

$$\Rightarrow \lambda^n \in \sigma(a^n) \stackrel{\text{Step 1}}{\Rightarrow} |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a^n\|^{1/n}. \quad \checkmark$$

Step 3: $\inf_{n \geq 1} \|a^n\|^{1/n} \leq \liminf_{n \geq 1} \|a^n\|^{1/n} \leq \limsup_{n \geq 1} \|a^n\|^{1/n} \leq r(a) \leq \inf_{n \geq 1} \|a^n\|^{1/n}$
 \hookrightarrow This implies the claim.

Notation note: $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$.

All that remains is $\limsup_{n \geq 1} \|a^n\|^{1/n} \leq r(a)$. Take $\phi \in A^*$, $\|\phi\| \leq 1$, $f_\phi(\lambda) := \phi((\lambda e - a)^{-1})$ for $\lambda \in \sigma(a)$ (this is an analytic function on $\sigma(a)$)

$$\sum_{k \in \mathbb{Z}} : \frac{1}{2\pi i} \oint_{\substack{|\lambda|=r(a)+\varepsilon \\ |\lambda|=2\|a\|}} \lambda^k f_\phi(\lambda) d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^k f_\phi(\lambda) d\lambda =$$

$$= \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^k \phi \left(\lambda \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n} \right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^{k+1} \sum_{n=0}^{\infty} \frac{\phi(a^n)}{\lambda^n} d\lambda$$

$$= \phi(a^{k+2}) \quad (\text{Cauchy formula from complex analysis})$$

$$\Rightarrow \phi(a^{k+2}) \leq \left| \frac{1}{2\pi i} \oint_{\substack{|\lambda|=r(a)+\varepsilon \\ |\lambda|=2\|a\|}} \lambda^k f_\phi(\lambda) d\lambda \right|$$

$$\leq \max_{|\lambda|=r(a)+\varepsilon} (|\lambda^k| \cdot |\Phi(\lambda)|) \frac{1}{2\pi} (2\pi(r(a)+\varepsilon))$$

$$= (r(a) + \varepsilon)^{k+1} \underbrace{\|\Phi\|}_{\leq 1} \cdot \underbrace{\sup_{|\lambda|=r(a)+\varepsilon} \|(\lambda e - a)^{-1}\|}_{\text{constant depending only on } \varepsilon, \text{ because } |\lambda|=r(a)+\varepsilon \text{ is a compact set and } \lambda \mapsto \|(\lambda e - a)^{-1}\| \text{ is continuous}}$$

$$\sup_{\|\Phi\| \leq 1} |\Phi(a^{k+2})| = \|a^{k+2}\|$$

$$C_\varepsilon \cdot \underbrace{(r(a) + \varepsilon)^k}_{\text{constant depending only on } \varepsilon}$$

constant depending only on ε , because $|\lambda|=r(a)+\varepsilon$ is a compact set and $\lambda \mapsto \|(\lambda e - a)^{-1}\|$ is continuous

$$\Rightarrow \limsup \|a^k\|^{1/k} \leq r(a) + \varepsilon \quad \text{for every } \varepsilon > 0.$$



November 11, 2025

Example: $V: f \mapsto \int_0^x f(s) ds$, $V \in \mathcal{B}(L^2[0,1])$.

Let's prove that $r(V) = 0 \Leftrightarrow \sigma(\{V\}) = \{0\}$.

Proof 1: $r(V) = \lim_{n \rightarrow \infty} \|V^n\|^{1/n}$, so we need a formula for V^n .

By induction: $(V^n f)(x) \stackrel{(*)}{=} \int_0^x f(s) \frac{(x-s)^{n-1}}{(n-1)!} ds$.

$n=1$: ✓

Assume (*) for some n and compute

$$\begin{aligned} (V^{n+1} f)(x) &= \int_0^x \left(\int_0^t f(s) \frac{(t-s)^{n-1}}{(n-1)!} ds \right) dt \\ &= \int_0^x \left(f(s) \underbrace{\int_s^t \frac{(t-s)^{n-1}}{(n-1)!} dt}_{\frac{(t-s)^n}{n!} \Big|_s^x = \frac{(x-s)^n}{n!}} \right) ds \\ &= \int_0^x f(s) \frac{(x-s)^n}{n!} ds. \end{aligned}$$

$$\|V^n f\| \leq \max_{x \in (0,1)} \max_{s \in (0,x)} \frac{(x-s)^{n-1}}{(n-1)!} \left(\int_0^1 \left[\int_0^x |f(s)| ds \right]^2 dx \right)^{1/2}$$

$$\leq \frac{1}{(n-1)!} \left(\int_0^1 \underbrace{\sqrt{x}}_{\leq 1} \underbrace{\int_0^x |f|^2 ds dx}_{\leq \|f\|^2} \right)^{1/2}$$

$$\leq \frac{\|f\|_{L^2[0,1]}}{(n-1)!}$$

$$\Rightarrow \|V^n\| \leq \frac{1}{(n-1)!}, \quad \lim \|V^n\|^{1/n} \leq \lim \left(\frac{1}{(n-1)!} \right)^{1/n} \quad \text{Stirling formula}$$

$$\begin{aligned} &= \lim \frac{1}{\left(\frac{1}{2\pi(n-1)} \left(\frac{n-1}{e} \right)^{n-1} \right)^{1/n} (1+o(1))^{1/n}} \\ &= \lim_{n \rightarrow \infty} \frac{e^{\frac{n-1}{n}}}{(2\pi)^{1/(n-1)}} \cdot \frac{1}{(n-1)^{1/(n-1)}} \xrightarrow[e]{} 0 \\ &= 0 \end{aligned}$$

□

$$\text{Proof 2: } Vf = \int_0^1 K(x,y) f(y) dy, \quad K(x,y) = \chi_{B_r}(x-y)$$

$$\int_0^1 \int_0^1 |K(x,y)|^2 dx dy < \int_0^1 \int_0^1 dx dy = 1 < \infty$$

$$\Rightarrow V \in S_\infty(L^2[0,1]) \Rightarrow \sigma(V) = \left\{ \begin{array}{l} \lambda \text{ is an eigenvalue} \\ \text{of } V \end{array} \right\} \cup \{0\}$$

$$Vf = \lambda f \Leftrightarrow \underbrace{\int_0^x f(s) ds}_{\in C[0,1]} = \lambda f(x) \quad \begin{matrix} \uparrow \\ \text{theorem about the spectrum} \\ \text{of compact operators} \end{matrix}$$

$$\Rightarrow \int_0^x f(s) ds \in C^1[0,1] \Rightarrow \dots \Rightarrow f \in C^\infty[0,1].$$

Differentiating (**), we get $\lambda f' = f$ on $[0,1]$.

Substituting 0 into (**), we get $\lambda f(0) = 0$.

$$\text{So, if } \lambda \neq 0, \text{ then } \begin{cases} \lambda f' = f \\ f(0) = 0 \end{cases} \Leftrightarrow \begin{cases} f = C \cdot e^{\lambda x} \\ 0 = C \cdot e^0 \end{cases} \Leftrightarrow f = 0$$

\Rightarrow any $\lambda \neq 0$ is not an eigenvalue $\Rightarrow \sigma(V) \cap (\mathbb{C} \setminus \{0\}) = \emptyset$

Since $\sigma(V) \neq \emptyset$, we get $\sigma(V) = \{0\}$.

Commutative Banach algebras

Definition: Let A be a commutative Banach algebra and $J \subset A$. J is called a **proper ideal** in A if J is a linear subspace such that $a \cdot J \subset J \quad \forall a \in A$, and $J \neq \{0\}$, $J \neq A$.

Definition: J is a **maximal ideal** if J is a proper ideal and there is no proper ideal J' such that $J' \supsetneq J$.

Proposition: Every proper ideal is contained in some maximal ideal. Take some proper ideal J , $J \neq A$, and consider all proper ideals J' : $J \subsetneq J'$. This set is partially ordered by inclusion, and for every chain $\{J'_\alpha\}_{\alpha \in I}$ of ideals ordered by inclusion, the set $\bigcup_{\alpha \in I} J'_\alpha = J'$ is again a proper ideal.

• **linearity:** $p \cdot x + q \cdot y \in J$ for every $p, q \in \mathbb{C}$ and $x, y \in J'$, because $\exists d_{x,y}$, $x \in J_{d_x}$, $y \in J_{d_y} \Rightarrow x, y \in J_{d_x}$ or $x, y \in J_{d_y}$, then $px + qy$ are in the same J_{d_x} or J_{d_y} ✓

• **ideal property** $aJ' = \bigcup_{\alpha \in I} aJ'_\alpha \subset \bigcup_{\alpha \in I} J_\alpha \quad \forall a \in A$.

• **properness:** $J' \neq A$ (If $J' = A$, then $e \in J'$, then $e \in J'_\alpha \Rightarrow eA \subseteq J_\alpha \subseteq A$)

By Zorn's lemma, the set of all proper J' : $J \subsetneq J'$ has a maximal element. □

Proposition: If M is a maximal ideal in A , then M is closed.

Proof: Let's prove that \overline{M} is a proper ideal.

• \overline{M} is linear ✓

• $\overline{MA} \subseteq \overline{M}$: true by continuity of multiplication ✓

• $\overline{M} \neq A$ (If $\overline{M} = A$, then $e \in \overline{M} \Rightarrow \exists x \in M \text{ . dist}(x, e) < 1 \Rightarrow x \in G(A)$
 $\Rightarrow c = x \cdot x^{-1} \in xA \subset M \Rightarrow M = A$, contradiction.) □

Example: $A = C(K)$, $M_{x_0} = \{f \in A : f(x_0) = 0\}$.

Then M_{x_0} is a maximal ideal for every $x_0 \in K$.

$\cdot M_{x_0}$ is linear ✓

$\cdot M_{x_0} \cdot A \subseteq M_{x_0}$ ✓

$\cdot M_{x_0} \neq A$, because $1 \notin M_{x_0}$ ✓

$\cdot M_{x_0}$ is maximal: If $\exists J$ -proper: $J \supsetneq M_{x_0}$ then $\exists f \in J$. $f(x_0) \neq 0$.

But then $\forall g \in A$ we have $g = c \cdot f + h$ for $c \in \mathbb{C}$ and $h \in M$,
where c is such that $(g - cf)(x_0) = 0$, i.e. $c := \frac{g(x_0)}{f(x_0)}$.

So $A \subset \mathbb{C} \cdot f + M \subset J$, contradiction.

Observation: $M_{x_0} = \text{Ker } \Phi_{x_0}$, $\Phi_{x_0}: f \mapsto f(x_0)$

Φ_{x_0} is a multiplicative functional: $\Phi_{x_0}(fg) = \Phi_{x_0}(f) \cdot \Phi_{x_0}(g)$

Definition: Let $\Phi \in A^*$. We say Φ is a multiplicative functional if $\Phi(fg) = \Phi(f) \cdot \Phi(g)$ $\forall f, g \in A$, and $\Phi \neq 0$.

Theorem: Let A be a commutative Banach algebra. TFAE

1) M is a maximal ideal in A .

2) $M = \text{Ker } \Phi$ for some multiplicative functional $\Phi \in A^*$.

Proof: 2) \Rightarrow 1) obvious: i) $\text{Ker } \Phi$ is linear

ii) $x \in \text{Ker } \Phi, a \in A$, $\Phi(xa) = \Phi(x)\Phi(a) = 0 \Rightarrow xa \in \text{Ker } \Phi$

iii) $\text{Ker } \Phi \neq A$ because $\Phi \neq 0$

iv) $\text{Ker } \Phi$ is maximal, because $\text{Ker } \Phi + Ca = A \quad \forall a: \Phi(a) \neq 0$
(see ***)

1) \Rightarrow 2) Note that A/M is a Banach algebra in which every non-zero element is invertible. If $[a] \in A/M$, then $a \cdot A$ is an ideal in A containing M , but not equal to $M \Rightarrow aA + M = A$
 $aA + M \ni e \Rightarrow ab + M = e$ for some $b \in A \Rightarrow [a][b] = [e]$.

By Banach-Mazur theorem, there is an isomorphism j of Banach algebras A/M and \mathbb{C} . Let $\Phi(x) := j([x])$.

- i) ϕ is linear, because j is linear.
- ii) ϕ is multiplicative, because j is multiplicative.
- iii) $\text{Ker } \phi = \{x \mid j([x]) = 0\} \Leftrightarrow [x] = 0 \Leftrightarrow x \in M.$

November 12, 2025 

Notation: $M(A)$ - set of all maximal ideals in A
 A_{mult}^* - set of multiplicative functionals on A

Last time: $\gamma: \Phi \mapsto \text{Ker } \phi$ maps A_{mult}^* onto $M(A)$.

Proposition: γ is a bijection.

Proof: We need to check that $\gamma(\phi_1) = \gamma(\phi_2) \Rightarrow \phi_1 = \phi_2$.

For this, note that $\phi(e) = 1 \quad \forall \phi \in A_{\text{mult}}^*$, because $\begin{cases} \phi(e) = \phi(e) \cdot \phi(e) \\ \phi(e) \neq 0 \end{cases}$
for every $\phi \in A_{\text{mult}}^*$. So, if $\gamma(\phi_1) = \gamma(\phi_2)$, we have
 $0 = \phi_1(y - \phi_1(y)e) \Rightarrow \phi_2(y - \phi_1(y)e) = 0 \quad \forall y \in A$
 $\Rightarrow \phi_2(y) = \phi_1(y) \phi_2(e) = \phi_1(y), \text{ so } \phi_1 = \phi_2$.

Theorem: Let A be a commutative Banach algebra, and $a \in A$.
TFAE:

- 1) $a \in A \setminus G(A)$
- 2) $a \in M$ for some $M \in M(A)$
- 3) $\exists \phi \in A_{\text{mult}}^*: \phi(a) = 0$

Proof: 1) \Rightarrow 2): $J = aA$ - a proper ideal in A ($e \notin J$)
 $\Rightarrow \exists M \in M(A). M \subset J$

2) \Rightarrow 3): Take $\phi: \text{Ker } \phi = M \Rightarrow \phi(a) = 0$.

3) \Rightarrow 1): If $\exists b \in A. ab = e \Rightarrow \phi(a) \cdot \phi(b) = \phi(e) = 1$, but $\phi(a) = 0$. 

Corollary: $\sigma(a) = \{\phi(a) \mid \phi \in A_{\text{mult}}^*\}$

Proof: $\sigma(a) = \{\lambda \mid a - \lambda e \in A \setminus G(A)\} = \{\lambda \mid \exists \phi \in A_{\text{mult}}^*, \phi(a - \lambda e) = 0\}$
 $= \{\lambda \mid \lambda \in \phi(a) \text{ for some } \phi \in A_{\text{mult}}^*\}.$

□

Remark: $\forall \phi \in A_{\text{mult}}^*$, $\|\phi\| = 1$, because $\phi(e) = 1$ and $|\phi(a^k)|$ is uniformly bounded for every $a \in B_A(0,1)$.

Applications

Theorem [Wiener]: Let $f = \sum_{k \in \mathbb{Z}} c_k z^k$, and $\sum_{k \in \mathbb{Z}} |c_k| < \infty$. Assume that $f(z) \neq 0$ for every $z \in \Pi = \{|z|=1\}$. Then $\frac{1}{f} = \sum_{k \in \mathbb{Z}} b_k z^k$ where $\sum_{k \in \mathbb{Z}} |b_k| < \infty$.

Proof: 1. $W^1(\Pi) = \{\sum c_k z^k \mid \sum |c_k| < \infty\}$ is a Banach algebra:

Indeed, $W(\Pi)$ is a Banach space with respect to the norm $\|\sum c_k z^k\| = \sum |c_k|$, and

$$\begin{aligned} \left\| (\sum c_k z^k)(\sum b_k z^k) \right\| &= \sum_{n \in \mathbb{Z}} \left| \left(\sum_{k \in \mathbb{Z}} c_k b_{n-k} \right) \right| \\ &\leq \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |c_k| |b_{n-k}| \\ &= \sum_{k \in \mathbb{Z}} |c_k| \sum_{j \in \mathbb{Z}} |b_j| \\ &= \|\sum c_k z^k\| \cdot \|\sum b_k z^k\| \end{aligned}$$

2. Identification of $(W^1(\Pi))_{\text{mult}}^*$:

Let $\phi \in (W^1(\Pi))^*_{\text{mult}}$, $\lambda = \phi(z)$, then $\phi(\frac{1}{z}) \cdot \phi(z) = 1$, $\phi(\frac{1}{z}) = \frac{1}{\lambda}$
 $|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot \|\frac{1}{z}\| = 1$

$$|\lambda| \leq \|\phi\| \cdot \|z\| = 1, |\frac{1}{\lambda}| \leq \|\phi\| \cdot |\frac{1}{z}| = 1 \Rightarrow |\lambda| = 1$$

$\phi\left(\sum_{-N}^N c_n z^n\right) = \sum_{-N}^N c_n \lambda^n$, and hence $\phi(f) = f(\lambda)$ $\forall f \in W^1(\Pi)$, because $\left\{\sum_{-N}^N c_n z^n\right\}$ is dense in $W^1(\Pi)$ and ϕ is continuous.

3. Application of invertibility criterion:

$f \in W^1(\mathbb{T})$ is invertible $\Leftrightarrow \exists \phi \in W^1(\mathbb{T})_{\text{mult}}^* . \phi(f) = 0 \Leftrightarrow f(z) = 0 \forall z \in \mathbb{T}$

This is the case in our case. \rightarrow

$\Rightarrow f_g = 1, g \in W^1(\mathbb{T}) \Rightarrow g = \frac{1}{f}, g = \sum b_k z^k, |b_k| < \infty.$ □

Bezout equation: Let $\{f_k\}_{k=1}^N \subset A(\bar{\mathbb{D}})$. We are interested if there exists $\{g_k\}_{k=1}^N \subset A(\bar{\mathbb{D}}) : \sum_1^N f_k g_k = 1$.

Necessary condition: $\exists z_0 \in \bar{\mathbb{D}} . f_k(z_0) = 0$ for every $1 \leq k \leq N$.

Theorem: Necessary condition is also sufficient.

Proof: 1. $A(\mathbb{D})$ is a Banach algebra with respect to the norm

$$\|f\| = \max_{z \in \mathbb{D}} |f(z)| \quad \checkmark$$

2. Identification of $A(\bar{\mathbb{D}})_{\text{mult}}^*$:

$\phi \in A(\bar{\mathbb{D}})_{\text{mult}}^* \Leftrightarrow \phi(f) = f(\lambda) \text{ for some } \lambda \in \bar{\mathbb{D}}$ [exercise]

3. $J = \left\{ \sum_1^N f_k g_k \mid g_k \in A(\bar{\mathbb{D}}) \right\}$ is a proper ideal in $A(\bar{\mathbb{D}})$

$\Leftrightarrow J \subset M, M \in \mathcal{M}(A(\bar{\mathbb{D}})) \Leftrightarrow \sum_1^N (f_k g_k)(z_0) = 0 \quad \forall g_k \in A(\bar{\mathbb{D}})$
for some $z_0 \in \bar{\mathbb{D}}$

$\Leftrightarrow f_k(z_0) = 0 \quad \forall 1 \leq k \leq N$

At the same time: J is proper $\Leftrightarrow e \notin J$.

November 18, 2025 □

Stone-Weierstrass Theorem

Definition: $C_K(K)$ - the real Banach space of continuous functions on a compact Hausdorff space K .

Definition: $A \subset C_{\mathbb{R}}(K)$ is a (real) Stone-Weierstrass algebra, if it is an algebra, and

(1) A does not vanish at any point $x \in K$, that is $\forall x \in K. \exists f \in A. f(x) \neq 0$.

(2) A separates points in K , that is, $\forall x, y \in K. \exists f \in A. f(x) \neq f(y)$.

Theorem [Stone-Weierstrass]: Let $A \subset C_{\mathbb{R}}(K)$ be an algebra. Then A is dense in $C_{\mathbb{R}}(K) \Leftrightarrow A$ is a Stone-Weierstrass algebra.

Example: $K = [0, 1]$, $A = \mathcal{P}$... the set of polynomials. Indeed, \mathcal{P} is an algebra, and

$\left. \begin{array}{l} \cdot 1 \in \mathcal{P} \Rightarrow (1) \text{ is satisfied} \\ \cdot x \in \mathcal{P} \Rightarrow (2) \text{ is satisfied} \end{array} \right\} \Rightarrow$ polynomials are dense in $[0, 1]$
 (Classical Weierstrass theorem)

Example: $K = [0, 1]$, $A = \text{span}\{\sin(2\pi kx), \cos(2\pi kx) \mid k \in \mathbb{Z}\}$

A is not dense in $C[0, 1]$, because 0 and 1 are not separated.

On the other hand, A is dense in $C[0, a]$ for every $0 < a < 1$.

For the proof we need some lemmas:

Lemma: Let $a > 0$. Then $\exists \{p_k\} \in \mathcal{P}$ such that $\|p_k - |x|\|_{C[-a, a]} \rightarrow 0$.

Proof: By scaling, we can assume that $a = 1$. Then we need to approximate $|x| = \sqrt{x^2} = \sqrt{1-y}$, $y = 1-x^2$ by polynomials.

Thus, it suffices to approximate $y \mapsto \sqrt{1-y}$ by polynomials uniformly on $[0, 1]$.

Taylor: $\sqrt{1-y} = \sum_{k=0}^{\infty} c_k y^k$ for $c_k = (-1)^k \binom{1/2}{k} = (-1)^k \frac{1/2(1/2-1)\dots(1/2-k+1)}{k!}$.

Observation: $c_0 = 1$, $c_k < 0$ for $k \geq 1$. (e.g. $c_1 = -\frac{1}{2}$, $c_2 = \frac{3(-1)}{2} < 0$
 $c_3 = -1 \cdot \frac{1}{2} \cdot \frac{3(-1)(-1-1)(-1-2)}{3!} < 0$)

In particular, we have $\sum_1^{\infty} |c_k| = \sup_{0 \leq y \leq 1} \left(- \sum_{k=1}^{\infty} c_k y^k \right) = \sup_{0 \leq y \leq 1} (c_0 - \sqrt{1-y}) = 1$.

So $\sum_{k=0}^{\infty} |c_k| < \infty \Rightarrow p_k = \sum_{j=0}^k c_j y^j$ are such that

$$\|p_k - \sqrt{1-y}\|_{C[0,1]} \leq \sum_{j=k+1}^{\infty} |c_j| \cdot \underbrace{\|y^j\|_{C[0,1]}}_1 \longrightarrow 0.$$
□

Lemma: If A is a SW algebra, then $\forall x, y \in K. \exists h \in A. h(x)=1, h(y)=0$.

Proof: We know that $\exists f, g \in A. F(x) \neq 0, g(x) \neq g(y)$.

1) If $f(y)=0$, then $h := \frac{f}{f(x)}$.

2) If $f(y) \neq 0$, and $f(y) \neq f(x)$, then $h := \frac{f^2 - f(y)f}{f^2(x) - f(x)f(y)}$

$h = \frac{f - f(y)}{f(x) - f(y)}$ is not okay $\frac{f}{f(x) - f(y)} \in A, \frac{f(y)}{f(x) - f(y)}$ might not be in A,
if A does not contain 1

3) If $f(y) \neq 0$, and $f(y) = f(x)$, then $\begin{pmatrix} f(x) \\ f(y) \end{pmatrix}, \begin{pmatrix} g(x) \\ g(y) \end{pmatrix}$ are not collinear in $\mathbb{R}^2 \Rightarrow \exists \alpha, \beta \in \mathbb{R}$.

$$\alpha \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} + \beta \begin{pmatrix} g(x) \\ g(y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow h := \alpha f + \beta g$$
□

Lemma: Let A be a SW algebra in $C_R(K)$. Then $\forall f, g \in A$ we have $\min(f, g) \in \bar{A}$, $\max(f, g) \in \bar{A}$.

Proof: $\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}, \max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}$

So we only need to show that $|h| \in \bar{A}$ for every $h \in A$.

Let's approximate $|x|$ by polynomials $p_k \in \mathcal{P}$ on $[-a, a]$ for $a = \|h\|_{C_R(K)}$. Then $p_k \circ h \in A, p_k(h) \rightarrow |h|$ on K. $\Rightarrow |h| \in \bar{A}$.

Lemma: Let $x \in K, \varepsilon > 0, f \in C_R(K)$. Then $\exists g_x \in \bar{A}$ s.t. $g_x(x) = f(x)$ and $g_x(z) \geq f(z) - \varepsilon$ for every $z \in K$.

Proof: For every $y \in K$ define $g_{x,y}$ to be a function in A : $\begin{array}{l} g_{x,y}(x) = f(x) \\ g_{x,y}(y) = f(y) \end{array}$
Such a function is a linear combination of functions

taking values 0 and 1 at points x, y .

Let $U = \{z \in K \mid g_{x,y}(z) > f(z) - \epsilon\}, y \in K$.

For each $x \in K$ we have $K = \bigcup_{y \in K} U_y$, because $y \in U_y$.

Each U_y is open, because $g_{x,y} - f$ is continuous, and U_y is the preimage of $(-\epsilon, +\infty)$ - an open set.

\Rightarrow By compactness, $\exists \{y_k\}_{k=1}^N. K = \bigcup_{k=1}^N U_{y_k}$.

Now $g_x := \max_{1 \leq k \leq N} g_{x,y_k} \in \bar{A}$ works.

Remark: A SW-algebra $\Rightarrow \bar{A}$ SW-algebra

Proof of SW theorem: Assume that A is a SW-algebra, and take $f \in C_R(K), \epsilon > 0$. For every $x \in K$ construct $g_x : g_x(x) = f(x)$,

$g_x(z) \geq f(z) - \epsilon \forall z \in K$. Consider $V_x = \{z \in K \mid g_x(z) < f(z) + \epsilon\} \ni x$.

Thus, V_x is open $\forall x \in K$, and $K = \bigcup_{x \in K} V_x \Rightarrow \exists \{x_k\}_{k=1}^M. K = \bigcup_{k=1}^M V_{x_k}$.

$$g = \min_{1 \leq k \leq M} g_{x_k} \in \bar{A} \text{ and } f(z) - \epsilon \leq g \leq f(z) + \epsilon$$

$\Rightarrow \|g - f\|_{C_R(K)} < \epsilon \Rightarrow A$ is dense in $C_R(K)$. This proves sufficiency.

Necessity: If A vanishes at some $x_0 \in K$, then 1 cannot be approximated uniformly by elements of A . If A does not separate points x, y , then the function $f : f \in C_R(K), f(x) = 1, f(y) = 0$ cannot be approximated. Such a function exists by Uryson's lemma ($K_1, K_2 \subset K$ -compact, $K_1 \cap K_2 = \emptyset \Rightarrow \exists f \in C_R(K). f|_{K_1} = 1, f|_{K_2} = 0$). \square

Example: Let $K = \Pi = \{z \mid |z| = 1\}$, $C(\Pi)$ the complex Banach space of continuous functions. Consider $A = \mathcal{P} = \text{span} \{z^k \mid z \in \mathbb{Z}_r\}$, $\mathbb{Z}_r = \mathbb{N} \cup \{0\}$. A is an algebra, A does not vanish ($1 \in A$), A separates points ($z \in A$ (but we have a complex algebra)).

Observation: $\int_{\Pi} \bar{z} \overline{p(z)} dm(z) = 0 \quad \forall p \in A$.

Lobesgue measure on Π , normalised by $m(\Pi) = 1$

Indeed, this follows from the fact that

$$\int_{-\pi}^{\pi} \bar{z} \cdot \bar{z}^k = 0 \quad \forall k \in \mathbb{Z}_+$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-i(k+1)t} dt = \frac{1}{2\pi} \left. \frac{e^{-i(k+1)t}}{-i(k+1)} \right|_0^{2\pi} = 0$$

Conclusion: $\bar{z} \perp A$ in $L^2(m)$. So, A cannot be dense, because otherwise $\bar{z} \perp C(\pi)$ in $L^2(m) \Rightarrow \bar{z} \perp L^2(m)$. Contradiction. So, A is not dense and SW-theorem (real version) does not work in the complex space.

Definition: A is a complex Stone-Weierstrass algebra if

- 1) A does not vanish.
- 2) A separates points
- 3) A is closed under conjugation $f \mapsto \bar{f}$.

In the example above, 3) does not hold.

Theorem [SW, complex]: An algebra A closed under conjugation is dense in $C(K) \Leftrightarrow A$ is a SW complex algebra.

Proof: A is SW complex algebra $\Rightarrow \text{Re } f, \text{Im } f \in A \quad \forall f \in A \Rightarrow$
 $\Rightarrow \text{Re } A$ is a real SW-algebra $\Rightarrow \text{Re } A$ is dense in $C_R(K) \Rightarrow$
 $\Rightarrow \text{Re } A + i \text{Re } A$ is dense in $C(K) = C_R(K) + i C_R(K)$ and it is contained in A . The other direction is trivial. □

C^* -algebras: Gelfand-Naimark theorem

November 19, 2025

Definition: A Banach algebra A is called an algebra with involution, if there is an operation $*$ s.t.

$$1) (\alpha x + \beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^* \quad \forall \alpha, \beta \in \mathbb{C} \quad \forall x, y \in A$$

$$2) (xy)^* = y^*x^* \quad \forall x, y \in A$$

$$3) (x)^{**} = x \quad \forall x \in A$$

If moreover,

$$4) \|x^*\| = \|x\| \quad \forall x \in A$$

$$5) \|x^*x\| = \|x^*\| \cdot \|x\| \quad \forall x \in A$$

then A is called a C^* -algebra.

Example: Let K be a Hausdorff compact. Then $C(K)$ is a C^* -algebra, $f^* := \overline{f}$.

Proposition: Let $T \in \mathcal{B}(H)$, where H is a Hilbert space. Then

$$\|T\| \stackrel{(*)}{=} \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle Tx, y \rangle|.$$

Proof: We will use the fact that $\forall h \in H. \|h\| = \sup_{\|x\| \leq 1} |\langle h, x \rangle|$.

(indeed, $|\langle h, x \rangle| \leq \|h\| \cdot \|x\| \leq \|h\|$ by CS inequality, and

taking $x = \frac{h}{\|h\|}$, we get $\langle h, x \rangle = \|h\|$.

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} |\langle Tx, y \rangle| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle Tx, y \rangle| \quad \square$$

Definition: Let $T \in \mathcal{B}(H)$, then $T^* \in \mathcal{B}(H)$ is the operator such that $\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x, y \in H$.

Proposition: $T^* \in \mathcal{B}(H)$ exists for every $T \in \mathcal{B}(H)$, moreover, $\|T^*\| = \|T\|$.

Proof: We will use Riesz theorem, which says that $\forall \varphi \in H^*$

$\exists! h \in H$ such that $\phi(x) = \langle x, h \rangle \quad \forall x \in H$. Having this theorem, define: $\phi_y(x) := \langle Tx, y \rangle$, $x \in H$, where $y \in H$ is fixed.

$$\phi_y \in H^*: \sup_{\|x\| \leq 1} |\phi_y(x)| = \sup_{\|x\| \leq 1} |\langle Tx, y \rangle| \leq \|T\| \cdot \|y\|.$$

$\Rightarrow \exists! h \in H$. $\phi_y(x) = \langle x, h \rangle$. Let's define $T^*y := h$.

$$\text{Since } \phi_{\alpha y_1 + \beta y_2} = \bar{\alpha} \phi_{y_1} + \bar{\beta} \phi_{y_2}$$

$$\begin{aligned} \langle x, T^*(\alpha y_1 + \beta y_2) \rangle &= \bar{\alpha} \underbrace{\langle x, T^* y_1 \rangle}_{\langle x, \alpha T^* y_1 \rangle} + \bar{\beta} \underbrace{\langle x, T^* y_2 \rangle}_{\langle x, \beta T^* y_2 \rangle} = \langle x, \alpha T^* y_1 + \beta T^* y_2 \rangle \quad \forall x \in H \\ &\quad \langle x, \alpha T^* y_1 \rangle \quad \langle x, \beta T^* y_2 \rangle \end{aligned}$$

$$\Rightarrow T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2.$$

We have used the fact that $\langle h, z_1 \rangle = \langle h, z_2 \rangle \quad \forall h \Rightarrow z_1 = z_2$.

(Proof: $h: z_1 - z_2 \quad \langle z_1 - z_2, z_1 - z_2 \rangle = 0$)

To compute the norm of T^* , we note that

$$\|T^*\| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle T^* y, x \rangle| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle y, Tx \rangle| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\overline{\langle Tx, y \rangle}| = \|T\|$$

$$\Rightarrow \|T^*\| = \|T\|, \text{ in particular, } T^* \in \mathcal{B}(H).$$



Proposition: $\|T^*T\| = \|T^*\| \cdot \|T\| \quad \forall T \in \mathcal{B}(H)$

$$\begin{aligned} \text{Proof: } \|T^*T\| &= \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle T^*Tx, y \rangle| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} \|\langle Tx, Ty \rangle\| \\ &\geq \sup_{\|x\| \leq 1} \|\langle Tx, Tx \rangle\| \\ &= \|T\|^2 \end{aligned}$$

$$\Rightarrow \|T\|^2 \leq \|T^*T\| \leq \|T^*\| \cdot \|T\| = \|T\|^2 \Rightarrow \|T\|^2 = \|T^*\| \cdot \|T\|$$

$$\text{In particular, } \|T^*T\| = \|T^*\| \cdot \|T\|.$$



Example: $\mathcal{B}(H)$ is a C^* -algebra. We only need to check that $(\alpha T_1 + \beta T_2)^* = \bar{\alpha} T_1^* + \bar{\beta} T_2^*$, $(T_1 T_2)^* = T_2^* T_1^*$, $(T^*)^* = T$.

This follows directly from the definition

e.g. $\langle Tx, y \rangle = \langle x, T^*y \rangle = \overline{\langle T^*y, x \rangle} = \overline{\langle y, (T^*)^*x \rangle} = \langle (T^*)^*x, y \rangle$
 $\Rightarrow T^{**}$ and T have the same bilinear forms \Rightarrow
 $\Rightarrow \|T^{**} - T\| = 0$ by (*) for $T^{**} - T$ in place of T .

Definition: $T \in \mathcal{B}(H)$ is called **normal** if $T^*T = TT^*$.

Example: $A = \overline{\text{span}\{T^k(T^*)^j \mid k, j \geq 0\}}$ is a commutative C^* -algebra for every normal operator $T \in \mathcal{B}(H)$.

The only nontrivial thing here is that $S \in A \Rightarrow S^* \in A$. This follows from the fact that $S_n \rightarrow S$ in $\mathcal{B}(H) \Rightarrow S_n^* \rightarrow S^*$ in $\mathcal{B}(H)$ ($\|S_n^* - S^*\| = \|S_n - S\|$).

November 20, 2025

Goal: A is a commutative C^* -algebra $\Rightarrow A \cong C(K)$ for some Hausdorff compact K .

In fact, we will see that $K = A_{\text{mult}}^*$ with some topology, that we will now define.

Definition: Let X be a Banach space, X^* the dual space to X . Then $\sigma(X^*, X)$, the **weak*-topology**, is defined on X^* as follows: For $\phi \in X^*$, $\varepsilon > 0$, $F \subset X$ a finite subset, we define.

$$V_{F, \varepsilon}(\phi) := \left\{ \tau \in X^* \mid |\phi(x) - \tau(x)| < \varepsilon \quad \forall x \in F \right\} \quad \begin{matrix} (\text{neighbourhood of } \phi) \\ (\text{corresponding to } F, \varepsilon) \end{matrix}$$

Open subsets in $\sigma(X^*, X)$ are precisely those subsets S that have the following property: $\forall \phi \in S \exists F, \varepsilon. V_{F, \varepsilon}(\phi) \subset S$.

Remarks: (1) If X is separable, then this topology is metrizable on each bounded subset of X , and closed subsets could be defined as follows: S is closed $\Leftrightarrow \forall \phi_n \in S. \phi_n(x) \rightarrow \phi(x) \quad \forall x \in X$ we have $\phi \in S$.

(2) In the general situation $\sigma(X^*, X)$ is Hausdorff.

Indeed, take $\phi_1, \phi_2 \in X^*$, $\phi_1 \neq \phi_2 \Rightarrow \exists x \in X. \phi_1(x) \neq \phi_2(x) \Leftrightarrow$
 $\Rightarrow V_{F_1, \varepsilon}(\phi_1) \cap V_{F_2, \varepsilon}(\phi_2) = \emptyset$ for $F_1 = \{x\} = F_2$, $\varepsilon = \frac{|\phi_1(x) - \phi_2(x)|}{2}$

(3) The importance of this topology is explained by the following theorem.

→ Proof: Functional Analysis

in norm (in the initial topology)

Theorem [Banach-Alaoglu]: Any bounded closed subset of X^* is compact in the $\sigma(X^*, X)$ -topology for every Banach space X .

Lemma: Let A be a commutative Banach algebra.

Then A^*_{mult} with the induced topology from $\sigma(X^*, X)$ is Hausdorff compact.

Proof: Since $A^*_{\text{mult}} \subset \mathcal{B}_{X^*}[0, 1]$, by B-A theorem, we only need to check that A^*_{mult} is closed. Assume that $\phi_n \in A^*_{\text{mult}}$, $\phi_n \rightarrow \phi$ in $X^* \Rightarrow \phi \in X^*$ and $\phi(xy) = \lim_{n \rightarrow \infty} \phi_n(xy) = \phi(x) \cdot \phi(y)$ $\forall x, y \in X$. □

Definition: Let A be a commutative Banach algebra, and for every $x \in A$ define $\hat{x}: \phi \rightarrow \phi(x)$, $\phi \in A^*_{\text{mult}}$. The mapping $x \mapsto \hat{x}$ is called the Gelfand transform.

Lemma: For every $x \in A$ we have $\hat{x} \in C(A^*_{\text{mult}})$.

Proof: Take $x \in A$, $U \subset \mathbb{C}$ -open subset. We need to show that $\hat{x}^{-1}(U)$ is open in $\sigma(X^*, X)$. It is enough to prove that $\hat{x}^{-1}(B(z_0, \varepsilon))$ is open $\forall z_0 \in U \cap \hat{x}(A^*_{\text{mult}}) \forall \varepsilon > 0$.

Let $z_0 \in \hat{x}(A^*_{\text{mult}})$, i.e., $\exists \tau. z_0 = \hat{x}(\tau) = \psi(x)$.

Then $\hat{x}^{-1}(B(z_0, \varepsilon)) = \{\phi \in A^*_{\text{mult}} \mid |\hat{x}(\phi) - z_0| < \varepsilon\} = V_{F_1, \varepsilon}(\tau) \cap A^*_{\text{mult}}$,
where $F = \{x\}$. □

open in induced topology

Lemma: Let A be a C^* -algebra and $x \in A$ normal. Then
 $\|x\| = r(x)$. $(x^*x = xx^*)$

Proof: At first assume $x^* = x$, then

$$\begin{aligned} \|x^{2^n}\| &= \|x^{2^n-1} \cdot x^{2^{n-1}}\| = \|(x^{2^{n-1}})^* x^{2^{n-1}}\| = \|x^{2^{n-1}}\|^2 = [\text{iterate}] = \|x\|^{2^n} \\ \Rightarrow r(x) &= \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \|x\| = \|x\|. \end{aligned}$$

Now the general case: $x^*x = xx^*$. Consider $y = x^*x$, and note that $\|y\| = r(y)$, because $y^* = y$. We have $\|x\|^2 = \|y\|$, and

$$\begin{aligned} r(y) &= \lim_{n \rightarrow \infty} \|(x^*)^n x^n\|^{\frac{1}{n}} = r(x)^2 \\ \Rightarrow r(x) &= \|x\|. \end{aligned}$$

\square

Lemma: Let A be a Banach algebra, $x, y \in A$, $xy = yx$. Then $e^x \cdot e^y = e^{x+y}$, where $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in A$.

$$\begin{aligned} \text{Proof: } e^x \cdot e^y &= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{j=0}^{\infty} \frac{y^j}{j!} \right) = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{k+j=n} x^k \cdot y^j \frac{1}{k!} \cdot \frac{1}{j!} \cdot (k+j)! \right)}_{xy=yx \rightarrow (x+y)^n} \frac{1}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e^{x+y} \end{aligned}$$

\square

Theorem [Gelfand-Naimark]: Let A be a commutative C^* -algebra. Then the Gelfand transform is the isomorphism of C^* -algebras A and $C(A^*_{\text{mult}})$, i.e.

$$1) \widehat{\alpha x + \beta y} = \alpha \widehat{x} + \beta \widehat{y}$$

$$2) \widehat{xy} = \widehat{x} \widehat{y}$$

$$3) \widehat{x^*} = \widehat{x}$$

$$4) \|\widehat{x}\|_{C(A^*_{\text{mult}})} = \|x\|$$

5) $x \mapsto \widehat{x}$ is a bijection between A and A^*_{mult} .

Proof: 1) $\widehat{\alpha x + \beta y}(\phi) = \phi(\alpha x + \beta y) - \alpha \phi(x) + \beta \phi(y) = \alpha \widehat{x} + \beta \widehat{y}$

$$2) \widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \widehat{x}\widehat{y}$$

3) Take $x \in A$, and observe that $x = \operatorname{Re}x + i\operatorname{Im}x$.

$$\operatorname{Re}x = \frac{x+x^*}{2}, \quad \operatorname{Re}x = (\operatorname{Re}x)^*$$

$$\operatorname{Im}x = \frac{x-x^*}{2}, \quad \operatorname{Im}x = (\operatorname{Im}x)^*$$

$$\text{Moreover, } (\operatorname{Re}x + i\operatorname{Im}x)^* = \widehat{\operatorname{Re}x} - i\widehat{\operatorname{Im}x}$$

\Rightarrow it suffices to prove that $\widehat{x} = \overline{\widehat{x}}$ for every $x = x^*$.

$$\widehat{x} = \overline{\widehat{x}} \Leftrightarrow \phi(x) = \overline{\phi(x)} \Leftrightarrow |e^{i\phi(x)}| = 1 \Leftrightarrow |\phi(e^{ix})| = 1$$

$$1 = |e_a| = |\phi(e_a)| = |\phi(e^{ix}) \cdot \phi(e^{-ix})| \leq \|e^{-ix}\| \cdot \|e^{ix}\| = \|(e^{ix})^* e^{ix}\| = \|e^{ix}\| = 1$$

identity lemma, $ix - ix = 0$ $e^0 = e_a$ $C^*-property \& (e^{ix})^* = e^{-ix}$

\Rightarrow all inequalities are in fact equalities, $|\phi(e^{ix})| \cdot |\phi(e^{-ix})| = 1$.

$$4) \|x\| = r(A) = \sup \{|\lambda|, \lambda \in \sigma(x)\}$$

$$\begin{aligned} &\text{description of } \sup \{|\lambda|, \lambda \in \{\phi(x), \phi \in A_{\text{mult}}^*\}\} \\ &\text{spectrum in commutative Banach algebra} \end{aligned}$$

$$= \sup \{|\phi(x)|, \phi \in A_{\text{mult}}^*\}$$

$$= \sup_{\phi \in A_{\text{mult}}^*} |\widehat{x}(\phi)| = \|\widehat{x}\|_{C(A_{\text{mult}}^*)}$$

5) \widehat{A} is a closed subalgebra in $C(A_{\text{mult}}^*)$ because it is an isometric image of a closed algebra ($= A$).

It remains to check that \widehat{A} does not vanish and separates points (S-W-theorem then implies $A = \text{clos } A = C(A_{\text{mult}}^*)$).

i) \widehat{A} does not vanish $\Leftrightarrow \forall \phi \in A_{\text{mult}}^* \exists x \in A. \widehat{x}(\phi) \neq 0 \Leftrightarrow \phi \neq 0$. This holds, because $\phi(e_\lambda) = 1$.

ii) A separates points $\Leftrightarrow \forall \phi \neq \psi. \exists x \in A. \widehat{x}(\phi) \neq \widehat{x}(\psi) \Leftrightarrow \phi \neq \psi$. □

Application: functional calculus for C^* -algebras

Definition: Let A be a commutative C^* -algebra, $x \in A$, $\sigma(x)$ the spectrum of x , $f \in C(\sigma(x))$. Then we define:

$$f(x) := (f(\hat{x}))^\vee,$$

where " \vee " is the inverse Gelfand transform.

Remark: $f(x)$ is defined correctly, in other words, $\text{Ran } \hat{x} \subset \text{Dom } f$.
Indeed, $\text{Ran } \hat{x} = \{\hat{x}(\phi), \phi \in A^*_{\text{mult}}\} = \{\phi(x), \phi \in A^*_{\text{mult}}\} = \sigma(x)$.

Remark: If f is a polynomial, $f = c_0 + c_1 z + \dots + c_n z^n$, then
 $f(x) = c_0 + c_1 x + \dots + c_n x^n$ for every $x \in A$.

Indeed, it suffices to check this for $f_n = z^n$, $z \in \mathbb{C}$:

$$f_n(x) = (z^n(\hat{x}))^\vee = (\hat{x}^n)^\vee = x^n.$$

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