Introduction to algebraic geometry

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I. AFFINE VARIETIES (Afine raznoterosti)

1. Recap of basic notions

A polynomial over a ring R is a formal expression $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$, where $a_i \in R$, n = degree of p (if $a_n \neq 0$). The set of all polynomials in x over R is denoted $R[x]$. All rings will be commutative and with 1. $R[x]$ is a ring for usual addition and multiplication of polynomials.

Suppose R is a subring of a ring S and $a \in S$. Then we can compute $a_n a^n + a_{n-1}a^{n-1} + \dots + a_0 \in S$. We denote this element $p(a)$ and call it the value of p at a. If $p(a) = 0$, then a is a root/zero of p. We have a polynomial function $S \longrightarrow S$, $a \longmapsto p(a)$.

Polynomial in two variables x and y with coefficients in a ring R is an expression $p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}x^i y^j$, where $a_{ij} \in R$. The expression $a_{ij}x^i y^j$ is called a monomial. The degree of a monomial $x^i y^j$ is $i+j$. The degree of the polynomial $p(x,y) = \max\{i+j \mid a_{ij} \neq 0\}$. The set of all polynomials in two variables is a ring. We denote it by $R[x,y]$.

$R[x,y] \cong R[x][y] \cong R[y][x]$.

Similarly we define polynomials in more variables.

$R[x_1, \dots, x_n] \cong R[x_1, \dots, x_{n-1}][x_n]$.

If R is a subring of S and $p \in R[x_1, \dots, x_n]$ and $a = (a_1, \dots, a_n) \in S^n$, we can compute $p(a_1, \dots, a_n) \in S$. We get a function $S^n \longrightarrow S$, $a \longmapsto p(a)$.

Let R be a ring. An ideal of R is a subset $I \subseteq R$ s.t.

i) If $x, y \in I$, then $x+y \in I$.
ii) If $a \in R$ and $x \in I$, then $ax \in I$.
$I \trianglelefteq R$.
If an ideal I contains an invertible element, then $I = R$.
If $M \subseteq R$ is some set, then
$(M) := \{ \sum_{i=0}^{n} a_i x_i \mid n \in \mathbb{N},\ a_i \in R,\ x_i \in M \}$ is an ideal.
We call it the ideal generated by M. If $M = \{m_1, \dots, m_n\}$, we write $(x_1, \dots, x_n)$ instead of $(\{x_1, \dots, x_n\})$.
$I \trianglelefteq R$ is finitely generated if $I = (M)$ where M is a finite set. I is a principal ideal, if $I = (a)$ for some $a \in R$.
A domain where every ideal is principal is called a principal ideal domain (PID).

$F[x]$ is a PID if F is a field. A polynomial ring in more variables is not a PID.

Let R be a domain. An element $0 \neq a \in R$ is irreducible if it is not invertible and it cannot be written as a product of non-invertible elements. R is a unique factorization domain (UFD) if:
i) Each $0 \neq a \in R$ can be written in a form $a = u p_1 \cdots p_n$, where u is invertible and $p_1, \dots, p_n$ are irreducible.
ii) If $a = v q_1 \cdots q_m$ is another such expression, then $m = n$ and there exists a permutation π of elements $w_1, \dots, w_n$ s.t. $q_i = w_i p_{\pi(i)}$ for each i. We say q_i and $p_{\pi(i)}$ are associated.
Polynomial rings in any number of variables over a field are UFD.

$$\text{PID} \Rightarrow \text{UFD}$$
$$\not\Leftarrow$$

Proposition: For a ring R the following are eqivalent:
i) Each ideal is finitely generated.
ii) Each increasing sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ terminates, i.e. $I_m = I_{m+1} = I_{m+2} = \cdots$ for some m.
iii) Each family of ideals in R has a maximal element (for inclusion).

Proof: Commutative algebra.

Definition: A ring satisfying the above properties is called a noetherian ring (noetherski kolobar).

1) Each PID is noetherian.
$F[x]$ is noetherian if F is a field
2) Each quotient R/I of a noetherian ring R is noetherian.

Fact from commutative algebra:
Theorem [Hilbert basis theorem]: If R is noetherian, then $R[x]$ is noetherian.

Remark: If R is noetherian, then the power series ring $R[[x]]$ is noetherian.

Corollary: If F is a field, then $F[x_1, \dots, x_n]$ is noetherian.

Corollary: Every finitely generated algebra over a field is noetherian.

2. Affine varieties and Zariski topology

afine raznoterosti

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We will always work over an algebraically closed field $\mathbb{k}$.

<u>Definition</u>: The n-dimensional affine space over $\mathbb{k}$ is $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{k}} = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{k} \text{ for each } i\}$.
As a set, $\mathbb{A}^n = \mathbb{k}^n$, but $\mathbb{k}^n$ has an additional structure of a vector space, so we use a different notation.

affine space – translated vec. space
– not important where the origin is

$n=1$: $\mathbb{A}^1$: affine line
$n=2$: $\mathbb{A}^2$: affine plane

<u>Definition</u>: Let $S \subseteq \mathbb{k}[x_1, \dots, x_n]$ be a set of polynomials. The (affine) zero locus of S is the set
$V(S) := \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid \forall f \in S.\ f(a_1, \dots, a_n) = 0\}$.

This set contains the set off all common zeroes of polynomials in S. If $S = \{f_1, \dots, f_m\}$, then we write $V(f_1, \dots, f_n)$ instead of $V(\{f_1, \dots, f_n\})$.

<u>Definition</u>: A set $X \subseteq \mathbb{A}^n$ is an affine variety if $X = V(S)$ for some set $S \subseteq \mathbb{k}[x_1, \dots, x]$.
some authors also additionally assume irreducibility

<u>Examples</u>: i) $\emptyset = V(1)$
ii) $\mathbb{A}^n = V(0)$
iii) Each point is an affine variety:
$\{(a_1, \dots, a_n)\} = V(x_1 - a_1, \dots, x_n - a_n)$

iv) All affine spaces (and in particular all vector spaces) are affine spaces, as they are solutions of systems of linear equations.

v) Plane affine algebraic curves are affine varieties. They are defined as $\underbrace{\{(a,b)\in \mathbb{A}^2 \mid p(a,b)=0\}}_{=V(p)}$ where $p\in \Bbbk[x,y]$.

vi) If $p\in \Bbbk[x_1,\dots,x_n]$, then $V(p)=\{(a_1,\dots,a_n)\in \mathbb{A}^n \mid p(a_1,\dots,a_n)=0\}$ is called a hypersurface.

Lemma: If $S_1\subseteq S_2\subseteq k[x_1,\dots,x_n]$, then $V(S_2)\subseteq V(S_1)$.
Proof: Obvious.

Proposition: Let $S\subseteq k[x_1,\dots,x_n]$ and I the ideal generated by S. Then $V(I)=V(S)$.

Proof: $S\subseteq I$, so $V(I)\subseteq V(S)$ follows from the Lemma.

$(\supseteq)$: Assume we have $a\in V(S)\setminus V(I)$. Then there is $f\in I$ such that $f(a)\neq 0$. I is generated by S, so there exist $g_1,\dots,g_m\in \Bbbk[x_1,\dots,x_n]$ and $h_1,\dots,h_m\in S$ s.t. $f=g_1h_1+\dots+g_mh_m$.

$$0\neq f(a)=g_1(a)\underbrace{h_1(a)}_{=0,\text{ because } a\in V(S)}+\dots+g_m(a)h_m(a)=0 \;\lightning$$

We get $V(S)=V(I)$. ▨

Corollary: Varieties in $\mathbb{A}^n$ are exactly sets of the form $V(I)$, where $I\trianglelefteq \Bbbk[x_1,\dots,x_n]$.

It may happen that $I_1 \neq I_2$, but $V(I_1) = V(I_2)$.

<u>Example</u>: in $\mathbb{A}^1$: $V(x^2) = \{0\} = V(x)$

$\mathbb{k}[x_1,\dots,x_n]$ is noetherian, so all ideals are finitely generated. If $I = (f_1,\dots,f_m)$, then $V(f_1,\dots,f_n) = V(I)$.

<u>Corollary</u>: Affine varieties in $\mathbb{A}^n$ are exactly the sets $V(S)$ where S is a finite set.

<u>Lemma</u>: (i) For any family $\{S_j\}_{j\in J}$ of subsets of $\mathbb{k}[x_1,\dots,x_n]$ we have $V(\bigcup_{j\in J} S_j) = \bigcap_{j\in J} V(S_j)$.

ii) For any polynomials $f_1,\dots,f_s, g_1,\dots,g_t \in \mathbb{k}[x_1,\dots,x_n]$ we have $V(f_1,\dots,f_s) \cup V(g_1,\dots,g_s) = V(f_i g_j \mid 1 \leq i \leq s,\ 1 \leq j \leq t)$.

<u>Proof</u>: $a \in V(\bigcup_{j\in J} S_j) \iff \forall f \in \bigcup_{j\in J} S_j .\ f(a) = 0$

$$\iff \forall j \in J.\ \forall f \in S_j .\ f(a) = 0$$

$$\iff \forall j \in J.\ a \in V(S_j)$$

$$\iff a \in \bigcap_{j\in J} V(S_j)$$

(ii): Assume $a \in V(f_1,\dots,f_s) \cup V(g_1,\dots,g_t)$. Then

$$a \in V(f_1,\dots,f_s) \quad \text{or} \quad a \in V(g_1,\dots,g_t).$$

$$\Rightarrow \quad \forall i.\ f_i(a) = 0 \quad \text{or} \quad \forall j.\ g_j(a) = 0$$

In both cases $f_i(a) g_j(a) = 0 \ \forall i. \forall j$

$\Rightarrow \quad a \in V(f_i g_j \mid 1 \le i \le s,\ 1 \le j \le t)$.

Conversely, assume $a \notin V(f_1, \dots, f_s) \cup V(g_1, \dots, g_t)$.

$\exists i$. st. $f_i(a) \neq 0$ and $\exists j$ s.t. $g_j(a) \neq 0 \Rightarrow f_i(a)\, g_j(a) \neq 0$

$\Rightarrow a \notin V(f_i g_j \mid 1 \le i \le s,\ 1 \le j \le t)$. ▨

Corollary: (1) $\emptyset$, $\mathbb{A}^n$ are affine varieties

(2) If $\{X_j\}_{j \in J}$ is any family of verieties, then $\bigcap_{j \in J} X_j$ is also an affine variety.

(3) If $X, Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \cup Y$ is an affine variety.

(1)-(3) are axioms of closed sets of some topology, so affine varieties are exactly the closed sets of some topology on $\mathbb{A}^n$. This topology is called the Zariski topology on $\mathbb{A}^n$.

On subsets of $\mathbb{A}^n$ we define the Zariski topology as a relative topology: Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. A subset $Z \subseteq X$ is Zariski closed in X if there exists an affine variety $Y \subseteq \mathbb{A}^n$ s.t. $Z = X \cap Y$. In particular, if X is an affine variety, then a set $Z \subseteq X$ is closed $\Leftrightarrow$ it is an affine variety.

If the topology is not mentioned explicitly, we will always mean the Zariski topology.

<u>Examples</u>: Zariski topology on $\mathbb{A}^1$:
Zariski closed sets are common zeroes of finitely many polynomials. Each nonzero in 1 variable has finitely many zeroes. $\Rightarrow$ All closed sets are finite.
Converse is clear: given a finite set in $\mathbb{A}^1$, it is easy to find a polynomial whose zeroes are precisely the elements of the given set.
$\Rightarrow$ On $\mathbb{A}^1$ the Zariski topology is equal to the topology of finite complements.

The example shows that the Zariski topology is NOT Hausdorff: every two open sets of $\mathbb{A}^1$ intersect (and the same holds for open subsets in $\mathbb{A}^n$).

<u>Example</u>: Zariski closed sets in $\mathbb{A}^2$ are $\mathbb{A}^2$, $\emptyset$, finite unions of points and affine algebraic curves.

Open sets in the Zariski topology are complements of varieties.

<u>Definition</u>: Let $p \in \mathbb{k}[x_1, \dots, x_n]$. The set $D(p) := \mathbb{A}^n \setminus V(p) = \{a \in \mathbb{A}^n \mid p(a) \neq 0\}$ is called a distinguished open set /subset (odlikovana odprta množica /podmnožica) of p in $\mathbb{A}^n$.

<u>Example</u>: Distinguished open sets in $\mathbb{A}^2$ are complements of algebraic curves.

$f, g \in \mathbb{k}[x_1, \dots, x_n]$
$D(f) \cap D(g) = \{a \in \mathbb{A}^n \mid f(a) \neq 0 \wedge g(a) \neq 0\}$

$= \{a \in \mathbb{A}^n \mid f(a)g(a) \neq 0\}$
$= D(f \cdot g)$

$\Rightarrow$ The intersection of distinguished open subsets is a distinguished open subset.

Distinguished open subsets form a basis of the Zariski topology: every open subset is a finite union of distinguished open subsets.

Let $U \subseteq \mathbb{A}^n$ be an open subset. Then $Z = \mathbb{A}^n \setminus U$ is closed, so an affine variety. Therefore there exist polynomials $f_1, \dots, f_m$ s.t. $Z = V(f_1, \dots, f_m) = \{a \in \mathbb{A}^n \mid \forall i.\ f_i(a) = 0\} =$
$= \bigcap_{i=1}^{m} V(f_i) \Rightarrow U = \bigcup_{i=1}^{m} D(f_i)$.

3. V–I correspondence and Nullstellensatz

„Če tega ne prevajajo v angleščino, tudi v slovenščino ne bomo."

<u>Definition</u>: For each subset $X \subseteq \mathbb{A}^n$ we define $I(X) := \{f \in \mathbb{k}[x_1, \dots, x_n] \mid \forall a \in X.\ f(a) = 0\}$. This is an ideal in $\mathbb{k}[x_1, \dots, x_n]$ called the ideal of X. (easy exercise)

$$\{\text{varieties in } X\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{ideals in } \mathbb{k}[x_1, \dots, x_n]\}$$

These two maps are not inverse to each other. For example, we know that $V(x^2) = V(x) \Rightarrow I(V(x^2)) = I(V(x))$.

<u>Definition</u>: Let $I_1, I_2 \trianglelefteq R$. The product of ideals I_1, I_2 is $I_1 I_2 = \{\sum_{i=1}^{m} a_i b_i \mid m \in \mathbb{N}, a_i \in I_1, b_i \in I_2\}$.

Lemma: Product of ideals is an ideal.
Proof: exercise

Definition: Let $I \trianglelefteq R$. The radical of I is $\sqrt{I} = \mathrm{rad}(I) =$
$= \{a \in R \mid a^m \in I \text{ for some } m \in \mathbb{N}\}$. The ideal $I \trianglelefteq R$ is radical if $I = \sqrt{I}$.

Lemma: Radical of an ideal is an ideal.
Exercise: show that if $a^n \in I$ and $b^m \in I$, then $(a+b)^{m+n-1} \in I$.

Example: If $I = ((x-a_1)^{k_1}(x-a_2)^{k_2} \cdots (x-a_r)^{k_r})$, then $\sqrt{I} = ((x-a_1)(x-a_2)\cdots(x-a_r))$. *Proof:* exercise.

Proposition: (1) $I(\emptyset) = \mathbb{k}[x_1, \dots, x_n]$.
(2) $I(\mathbb{A}^n) = (0)$
(3) If $I_1 \subseteq I_2$, then $V(I_2) \subseteq V(I_1)$.
(4) If $X_1 \subseteq X_2$, then $I(X_2) \subseteq I(X_1)$.
(5) $X \subseteq V(I(X))$ for each $X \subseteq \mathbb{A}^n$.
(6) $S \subseteq I(V(S))$ for each $S \subseteq \mathbb{k}[x_1, \dots, x_n]$.
(7) $V(S) = V(I(V(S)))$ $\forall S \subseteq \mathbb{k}[x_1, \dots, x_n]$.
(8) If $X \subseteq \mathbb{A}^n$ is a variety, then $X = V(I(X))$.
(9) If $X \subseteq \mathbb{A}^n$ is any set, then $V(I(X)) = \overline{X}$ (Zariski closure of X), $I(X) = I(\overline{X})$.
(10) $I(X)$ is always a radical ideal.
(11) $I(X) = I(V(I(X)))$ $\forall X \subseteq \mathbb{A}^n$.
(12) $V(I) = V(\sqrt{I})$ for each ideal.
(13) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$ for all ideals I_1, I_2.
(14) $V(I_1) \cap V(I_2) = V(I_1 + I_2)$ for all ideals I_1, I_2.
(15) $I(X \cup Y) = I(X) \cap I(Y)$ $\forall$ varieties X, Y
(16) $I(X) + I(Y) \subseteq I(X \cap Y)$ $\forall$ varieties X, Y

Proof: (2) It is clear that 0 vanishes everywhere. We have to prove that it is the only such polynomial! We prove it with induction on n.

$n=1$: Let f be a polynomial in 1 variable that vanishes everywhere on $\mathbb{A}^1$. Since $\mathbb{k}$ is algebraically closed, it is infinite and the only polynomial that vanishes everywhere it is the zero polynomial.

$n \mapsto n+1$: Let $f \in \mathbb{k}[x_1, \dots, x_{n+1}]$ vanish everywhere on $\mathbb{A}^{n+1}$.

Write $f(x_1, \dots, x_n) = \sum_{i=0}^{d} g_i(x_i, \dots, x_n) x_{n+1}^i$.

Take any $(a_1, \dots, a_n) \in \mathbb{A}^n$. Then $f(a_1, \dots, a_n, x_{n+1})$ is a polynomial in 1 variable that vanishes everywhere by the assumption. By case $n=1$, all coefficients of $f(a_1, \dots, a_n, x_{n+1})$ are zero.

$\Rightarrow \forall i \quad g_i(a_1, \dots, a_n) = 0 \quad \forall (a_1, \dots, a_n) \in \mathbb{A}^n$

$\underset{\text{ind. assumption}}{\Rightarrow} \quad g_i$ is the zero polynomial $\forall i$

$\Rightarrow$ f is the zero polynomial. ☒

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(11): exercise class

(7): similar proof

(8): the same as 7

(9): $\bar{X} = V(I(X))$

Clear: $X \subseteq V(I(X))$

Suppose that Z is a variety such that $X \subset Z \quad | I$

$I(Z) \subseteq I(X) \quad | V$

$V(I(Z)) \subseteq V(I(Z)) \underset{\substack{\uparrow \\ Z \text{ is a variety}}}{=} Z$

$\Rightarrow V(I(X))$ is the smallest closed set that contains X

(13) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$

Suppose $I_1 = (f_1, \dots, f_s)$ and $I_2 = (g_1, \dots, g_t)$.

Then: $I_1 I_2 = (f_i g_j \,;\, 1 \leq i \leq s,\ 1 \leq j \leq t)$.

We know that $V(\underbrace{f_1, \dots, f_s}_{I_1}) \cup V(\underbrace{g_1, \dots, g_t}_{I_2}) =$

$$= V(\underbrace{f_i g_j \mid 1 \leq i \leq s,\ 1 \leq j \leq t}_{I_1 I_2})$$

If $a \in V(I_1) \cup V(I_2)$, then $f(a) = 0\ \forall f \in I$ or $g(a) = 0\ \forall g \in I_2$.

In both cases $h(a) = 0\ \ \forall h \in I_1 \cap I_2 \Rightarrow a \in V(I_1 \cap I_2) \Rightarrow$

$\Rightarrow V(I_1) \cup V(I_2) \subseteq V(I_1 \cap I_2)$.

$I_1 I_2 \subseteq I_1 \cap I_2 \quad \Rightarrow V(I_1 \cap I_2) \subseteq V(I_1 I_2)$

Other parts: exercise. ▨

$V(I(X)) = X$ for a variety X

$I(V(J)) \neq J$ for ideal J

in general

We will use the following result from commutative algebra.

Proposition: Let F be a field and let E be a finitely generated F-algebra which is also a field. Then E is a finite algebraic extension of F.

Corollary: Let A be a finitely generated commutative algebra over $\Bbbk$ ($\Bbbk$ alg. closed) and let M be some maximal ideal in A. Then $A/M \cong \Bbbk$.

Proof: A/M is finitely generated and a field, as M is a maximal ideal. By proposition A/M is a finite algebraic extension of $\Bbbk$. $\Bbbk$ is algebraically closed $\Rightarrow$ $A/M \cong \Bbbk$. ▨

Theorem [Weak Nullstelensatz]:

(1) Maximal ideals in the polynomial ring are exactly the ideals of the form $(x_1-a_1, \dots, x_n-a_n)$ for some $a_1, \dots, a_n \in \mathbb{k}$.

(2) If J is a proper ideal of $\mathbb{k}[x_1, \dots, x_n]$ then $V(J) \neq \emptyset$.

It is crucial to have an algebraically closed field:

- over $\mathbb{R}$: $V(x^2+1) = \emptyset$

  (x^2+1) is a maximal ideal in $\mathbb{R}[x]$

<u>Proof:</u> (1) Let $a_1, \dots, a_n \in \mathbb{k}$. We want to prove that $M = (x_1-a_1, \dots, x_n-a_n)$ is a maximal ideal.

Define a ring homomorphism

$$\begin{aligned} \varphi : \mathbb{k}[x_1, \dots, x_n] &\longrightarrow \mathbb{k} \\ x_i &\longmapsto a_i \\ f(x_1, \dots, x_n) &\longmapsto f(a_1, \dots, a_n) \end{aligned}$$

φ is surjective: for each $a \in \mathbb{k}$ the constant polynomial a maps to a

$\Rightarrow \underset{\text{field}}{\mathbb{k}} \cong \mathbb{k}[x_1, \dots, x_n]/\ker\varphi$

$\Rightarrow \ker\varphi$ is a maximal ideal

It is enough to show that $\ker\varphi = (x_1-a_1, \dots, x_n-a_n)$.

Obviously $(x_1-a_1, \dots, x_n-a_n) \subseteq \ker\varphi$.

For the other inclusion take $f \in \ker\varphi$.

We divide f by x_1-a_1 and the remainder belongs to $\mathbb{k}[x_2, \dots, x_n]$. We divide the remainder by x_2-a_2 and get the remainder in $\mathbb{k}[x_3, \dots, x_n], \dots$

$\Rightarrow f = \sum_{i=1}^{n} g_i(x_i-a_i) + b$ for some $g_i \in \mathbb{k}[x_i, x_{i+1}, \dots, x_n]$ and $b \in \mathbb{k}$

$f \in \ker\varphi \Rightarrow$

$0 = \varphi(f) = \sum_{i=1}^{n} \varphi(g_i)\varphi(x_i-a_i) + \varphi(b) = b \Rightarrow b = 0$

$\Rightarrow f \in (x_1 - a_1, \dots, x_n - a_n)$

(We didn't use algebraic closure, $(x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal in $F[x_1, \dots, x_n]$ for any field F.)

We have to prove there are no other maximal ideals.
Let $M \trianglelefteq \mathbb{k}[x_1, \dots, x_n]$ be an arbitrary maximal ideal.
$\mathbb{k}$ is alg. closed, $\mathbb{k}$ is fin. generated $\mathbb{k}$-algebra so $\mathbb{k}[x_1, \dots, x_n]/M \cong \mathbb{k}$ by previous corollary.

Define the maps $\pi : \mathbb{k}[x_1, \dots, x_n] \xrightarrow{\text{canonical projection}} \mathbb{k}[x_1, \dots, x_n]/M = \mathbb{k}$
Denote $a_i = \pi(x_i) \in \mathbb{k}$ for each i.
π is a ring homomorphism $\Rightarrow \pi(f) = f(\pi(x_1), \dots, \pi(x_n))$
$= f(a_1, \dots, a_n)$
We already proved $\ker \pi = (x_1 - a_1, \dots, x_n - a_n)$. By the construction $\ker \pi = M$.

(2): If J is a proper ideal in $\mathbb{k}[x_1, \dots, x_n]$, it is contained in some maximal ideal M.
By (1), M is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in \mathbb{k}$.

$$V(M) = V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \neq \emptyset$$

$\Rightarrow \emptyset \neq V(M) \subseteq V(J)$. ∎

Corollary: We have mutually inverse bijections

$$\{\text{points in } \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{ \begin{matrix} \text{maximal ideals} \\ \text{in } \mathbb{k}[x_1, \dots, x_n] \end{matrix} \right\}$$

$$(a_1, \dots, a_n) \longleftrightarrow (x_1 - a_1, \dots, x_n - a_n)$$

Theorem [Hilbert's Nullstelensatz]:
$I(V(J)) = \sqrt{J}$ for each $J \trianglelefteq \mathbb{k}[x_1, \dots, x_n]$.

<u>Proof</u>: One inclusion is easy.

If $f \in \sqrt{J}$, then $f^m \in J$ for some m.

If $a \in V(J)$, then $f^m(a) = 0 \Rightarrow f(a) = 0 \Rightarrow f \in I(V(J))$.

($\subseteq$): Let $f \in I(V(J))$.

We consider the ring $\mathbb{k}[x_1, \dots, x_n, y]$ (with a variable added) and the ideal

$$\tilde{J} = (J) + (fy - 1) \trianglelefteq \mathbb{k}[x_1, \dots, x_n, y]$$

one generator is added to J

First we show that $V(\tilde{J})$ is empty.

Suppose $(a_1, \dots, a_n, a_{n+1}) \in V(\tilde{J})$.

$$\Rightarrow \forall g \in \tilde{J}.\ g(a_1, \dots, a_n, a_{n+1}) = 0$$

If $g \in J \trianglelefteq \mathbb{k}[x_1, \dots, x_n]$, we get $g(a_1, \dots, a_n) = 0$.

$\Rightarrow (a_1, \dots, a_n) \in V(J)$

$f \in (I(V(J))) \Rightarrow f(a_1, \dots, a_n) = 0$

$(a_1, \dots, a_n, a_{n+1}) \in V(fy - 1) \Rightarrow f(a_1, \dots, a_n) \cdot a_{n+1} = 1$

$\Rightarrow f(a_1, \dots, a_n) \neq 0$ ... contradiction $\Rightarrow V(\tilde{J}) = \emptyset$

By weak Nullstelensatz we get that $\tilde{J}$ is not proper, so $1 \in \tilde{J}$.

$\mathbb{k}[x_1, \dots, x_n]$ is noetherian $\Rightarrow$ there exist $g_1, \dots, g_m \in \mathbb{k}[x_1, \dots, x_n]$ such that $J = (g_1, \dots, g_m)$.

$1 \in (g_1, \dots, g_m, fy - 1) \Rightarrow \exists\, p_1, \dots, p_m, q \in \mathbb{k}[x_1, \dots, x_n, y]$.

$$1 = p_1 g_1 + \dots + p_m g_m + q(fy - 1) \qquad (*)$$

Let N be the largest number such that y^N appears in each $p_1, \dots, p_m$. We multiply $(*)$ with f^N and rearrange the terms in such a way that each y appears together with f as fy. We get

$$f^N = P_1(x_1, \dots, x_n, fy) g_1 + \dots + P_m(x_1, \dots, x_n, fy) g_m + Q(x_1, \dots, x_n, fy) \cdot (fy - 1)$$

We look at this equation modulo $(fy-1)$:

$$f^N \equiv P_1(x_1,\dots,x_n,fy)g_1+\dots+P_m(x_1,\dots,x_n,fy)g_m$$

$$\equiv P_1(x_1,\dots,x_n,1)g_1+\dots+P_m(x_1,\dots,x_n,1)g_m \pmod{(fy-1)}$$

$$\Rightarrow \underbrace{f^N-\sum_{i=1}^m P_i(x_1,\dots,x_n,1)g_i}_{\text{we don't have } y \text{ here}} \in (fy-1)\cap \mathbb{k}[x_1,\dots,x_n]=(0)$$

$$\Rightarrow f^N=\sum_{i=1}^m \underbrace{P_i(x_1,\dots,x_n,1)}_{\in \mathbb{k}[x_1,\dots,x_n]}\cdot g_i \in J \Rightarrow f\in\sqrt{J} \qquad \boxed{/\!/}$$

Corollary: V and J are mutually reverse bijections

$$\{\text{radical ideals}\} \underset{J}{\overset{V}{\rightleftarrows}} \{\text{affine varieties}\}.$$

Corollary: $I(X\cap Y)=\sqrt{I(X)+I(Y)}$ if X,Y are affine affine varieties.

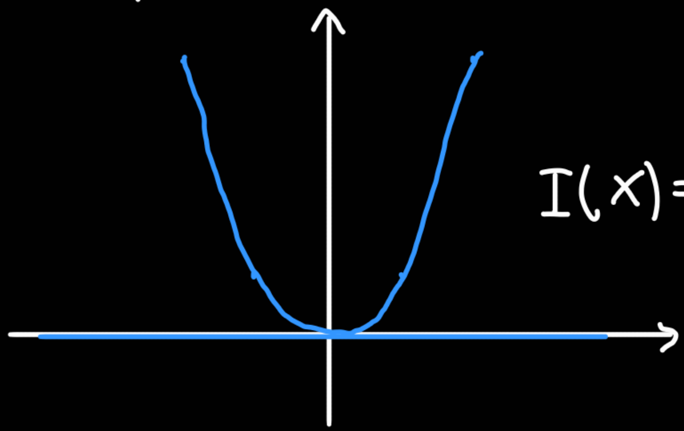
Proof: $I(X\cap Y) = I(V(I(X))\cap V(I(Y)))$

$$= I(V(I(X)+I(Y)))$$

Nullstelensatz $\rightarrow$

$$=\sqrt{I(X)+I(Y)} \qquad \boxed{/\!/}$$

Example: $X=V(y^2-x)$, $Y=V(y)$



$X\cap Y=\{(0,0)\}$

$I(X)=I(V(y-x^2))=\sqrt{(y-x^2)}=(y-x^2)$

principal ideal generated by polynomials without multiple factors

$I(Y)=(y)$

$I(X) + I(Y) = (y - x^2, y) = (y, x^2)$

This ideal is not radical, as it contains x^2, but not x
$\Rightarrow I(X) + I(Y) \neq I(X \cap Y)$.

$$I(X \cap Y) = I(\{(0,0)\}) = (x, y) = \sqrt{(x^2, y)}$$

$I(X) + I(Y)$ is not radical, because X and Y have a common tangent in $(0,0)$.

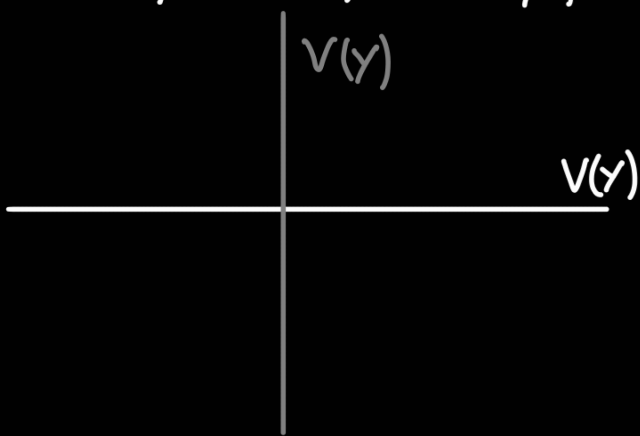
4. Irreducibility of varieties

A topological space X is disconnected if it is a union of two disjoint closed subsets. It is connected otherwise.

<u>Definition</u>: A topological space X is reducible if there exist proper closed subsets $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$. X is irreducible otherwise.

<u>Definition</u>: An affine variety X is reducible if there exist varieties $X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$. X is irreducible otherwise.

<u>Example</u>: $V(xy) = V(x) \cup V(y)$ is reducible



Lemma: If X is an irreducible variety, then the following holds:
1) Each non-empty open subset of X is dense in X. (in the Zariski topology)
2) Every two non-empty open subsets of X intersect.

Proof: (1) If U is open and $\overline{U} = X$, then we have a decomposition $X = \overline{U} \cup (X \setminus U)$

(2): If U_1, U_2 are open, non-empty, and $U_1 \cup U_2 = \emptyset$, then $X = (X \setminus U_1) \cup (X \setminus U_2)$. $\blacksquare$

Definition: Let $I \triangleleft R$. slovene: „praideal"
(1) I is a *prime ideal* if $I \neq R$ and the following holds: If $ab \in I$ for some $a, b \in R$, then $a \in I$ or $b \in I$.

2) I is a *primary ideal* if $I \neq R$ and the following holds: If $ab \in I$ for some $a, b \in R$, then $a \in R$ or $b^m \in I$ for some $m \in \mathbb{N}$. slovene: „primarni ideal"

Lemma: A radical of a primary ideal is a prime ideal.

Proof: Let I be primary and $ab \in \sqrt{I}$. Then $(ab)^r = a^r b^r \in I$ for some $r \in \mathbb{N}$.

Since I is primary, we get $\underset{\substack{\Downarrow \\ a \in \sqrt{I}}}{a^r \in I}$ or $\underset{\substack{\Downarrow \\ b \in \sqrt{I}}}{(b^r)^m = b^{rm} \in I}$ for some m. $\blacksquare$

Corollary: A primary ideal which is radical is a prime ideal.

Theorem: $X \subseteq \mathbb{A}^n$ is an irreducible variety $\iff I(X)$ is a prime ideal.

Proof: ($\Rightarrow$): Assume X is irreducible and let $fg \in I(X)$. Define $X_1 = X \cap V(f)$ and $X_2 = X \cap V(g)$. Then $X_1, X_2 \subseteq X$
$\Rightarrow X_1 \cup X_2 \subseteq X$.
Let $a \in X$. $fg \in I(X)$ so $f(a)g(a) = 0$ so $f(a) = 0$ or
$g(a) = 0 \Rightarrow a \in V(f)$ or $a \in V(g) \Rightarrow X = X_1 \cup X_2$.
X_1, X_2 are closed, so by irreducibility of X one of them is equal to X.
WLOG: $X_1 = X$ $(X_1 = X \cap V(f))$
$\Rightarrow X \subseteq V(f) \Rightarrow \forall a \in X.\ f(a) = 0 \Rightarrow f \in I(X)$

($\Leftarrow$): Assume $I(X)$ is a prime ideal. Let $X = X_1 \cup X_2$ for some $X_1, X_2 \subseteq X$ varieties and suppose that $X_1 \neq X$.
We will show that $I(X) = I(X_2)$.
$X_2 \subseteq X \Rightarrow I(X) \subseteq I(X_2)$
For the other inclusion take $g \in I(X_2)$.

$$X_1 \subsetneq X \Rightarrow I(X) \subsetneq I(X_1) \Rightarrow \exists f \in I(X_1) \setminus I(X)$$

Let $a \in X$ be arbitrary. Then $a \in X_1$ or $a \in X_2$, so $f(a) = 0$ or $g(a) = 0$. So we have $f(a)g(a) = 0$ for each $a \in X$. $\Rightarrow fg \in I(X)$.
$I(X)$ is prime and $f \notin I(X)$, so $g \in I(X)$

$$\Rightarrow I(X_2) \subseteq I(X) \Rightarrow I(X_2) = I(X)$$

$$\Rightarrow \underbrace{V(I(X_2))}_{X_2} = \underbrace{V(I(X))}_{X} \Rightarrow X \text{ is irreducible.}$$

$\blacksquare$

Corollary: A hypersurface $V(f)$ where f is a square-free polynomial is irreducible $\iff$ f is irreducible.

$f = g_1^{k_1} \cdots g_r^{k_r}$, g_i irreducible $\Rightarrow k_i = 1 \ \forall i$

Corollary: $\mathbb{A}^n$ is irreducible

Proof: $I(\mathbb{A}^n) = (0)$ is a prime ideal. $\blacksquare$

Corollary: V and I are mutually inverse bijections

$$\left\{\begin{matrix}\text{irreducible}\\ \text{varieties}\end{matrix}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{prime ideals}\}.$$

Remark:

algebra	geometry
$\Bbbk[x_1, \dots, x_n]$	$\mathbb{A}^n$
maximal ideals	points
radical ideals	affine varieties
prime ideals	irreducible affine varieties

$$\xrightarrow{V}$$
$$\xleftarrow[I]{}$$

Theorem: Each affine variety X can be decomposed as a union $X = X_1 \cup \dots \cup X_m$ where $m \in \mathbb{N}_0$ and $X_1, \dots, X_m$ are non-empty irreducible varieties.
Moreover, if $X_i \subseteq X_j$ whenever $i \neq j$, then the decomposition is unique up to permutation.

Definition: If $X = X_1 \cup \dots \cup X_m$ where $X_1, \dots, X_m$ are irreducible varieties and $X_i \not\subseteq X_j$, whenever $i \neq j$, then $X_1, \dots, X_m$ are called *irreducible components* of X.

slovene: nerazcepne komponente

Proof: If $X \neq \emptyset$, then the decomposition exists (for $m=n$) and it is unique.
Assume now that $X \neq \emptyset$.
Existance of the composition:
Assume that no decomposition $X = X_1 \cup \dots \cup X_m$ where $X_1, \dots, X_n$ are irreducible, exists. Then X is reducible (as otherwise $X = X$ is such decomposition for $m=1$).
$X = X_1 \cup X_1'$ for some varieties X_1 and X_1', and at least one of them is not a union of irr. varieties.
WLOG: this is X_1, X_1 has to be reducible: $X_1 = X_2 \cup X_2'$ for some var. X_2, X_2' and at least one of them is not a union of irr. varieties.

$\vdots$

We get a strictly decreasing chain of varieties:

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \qquad / I$$

$$I(X) \subsetneq I(X_1) \subsetneq I(X_2) \subsetneq \cdots$$

This is a strictly increasing sequence of ideals in $\mathbb{k}[x_1, \dots, x_n]$, which contradicts the noetherian property.
$\Rightarrow X$ can be decomposed into a union of irreducible varieties.

Uniqness: $X = X_1 \cup \dots \cup X_r = X_1' \cup \dots \cup X_s'$ where X_i, X_i' are irreducible, $X_i \not\subseteq X_j$, $X_i' \subseteq X_j'$ whenever $i \neq j$.
Take arbitrary $i \in \{1, \dots, r\}$.

$$X_i = X_i \cap X = X_i \cap (X_1' \cup \dots \cup X_s')$$

$$= \bigcup_{j=1}^{s} (X_i \cap X_j'), \quad X_i \text{ is irreducible}$$

therefore $X_i = X_i \cap X_j'$ for some $j \Rightarrow X_i \subseteq X_j'$
The same argument shows that there exists $l \in \{1, \dots, r\}$ such that $X_j' \subseteq X_l \Rightarrow X_i \subseteq X_j' \subseteq X_l$.

By assumption $l=i \Rightarrow X_j' = X_i$.
$\Rightarrow$ We get uniqness (and in particular, $r=s$). $\blacksquare$

<u>Remark</u>: The crucial part was to show the fact that there does not exist an infinite sequence of closed subsets, each properly contained in the previous one, $X \supsetneq X_1 \supsetneq X_2 \supsetneq$.
We say that varieties are noetherian topological spaces.

<u>Remark</u>: In commutative algebra an importent theorem says that each ideal in a noetherian ring can be written as an intersection of primary ideals. Using this fact we could prove the theorem as follows:

$$I(X) = Q_1 \cap Q_2 \cap \cdots \cap Q_m, \quad Q_i \text{ primary } / V$$

$$X = V(I(X)) = V(Q_1 \cap \cdots \cap Q_m)$$

$$= V(Q_1) \cup \cdots \cup V(Q_m)$$

To show that $V(Q_i)$ are irreducible, apply I:
$I(V(Q_i)) = \sqrt{\underset{\text{primary}}{Q_i}} = \text{prime ideal} \Rightarrow V(Q_i)$ irreducible.

If we use only minimal prime ideals over $I(X)$ then we also get the uniqueness statement.

17. oktober 2025

II PROJECTIVE VARIETIES

1. Projective space and projective varieties

Let V be a finite-dimensional vector space over $\mathbb{k}$ (still alg. closed). On $V \setminus \{0\}$ we define a relation $u \sim v \Leftrightarrow \exists \lambda \in \mathbb{k} \setminus \{0\}$ such that $v = \lambda u$. This is a equivalence relation. The quoetient set $V/_\sim$ is denoted by $\mathbb{P}V$ and called the projective space associated to V. Its elements are equivalence classes in V, so lines through the origin. Dimension of $\mathbb{P}V := \dim V - 1$.

The most common situation is when $V = \mathbb{k}^{n+1}$ for some $n \in \mathbb{N}$. In this case $\dim V = n+1$, so $\dim \mathbb{P}V = n$. Instead of $\mathbb{P}\mathbb{k}^{n+1}$ we write $\mathbb{P}^n_{\mathbb{k}}$ or more usually $\mathbb{P}^n$. We call $\mathbb{P}^n$ the n-dimensional projective space. Its elements (which are lines through 0) are called projective points.

- $\mathbb{P}^1$ is projective line
- $\mathbb{P}^2$ is projective plane

When we work in $\mathbb{P}^n$, we index coordinates in $\mathbb{k}^{n+1}$ from 0 to n: $(x_0, x_1, \dots, x_n)$.
The equivalence class of the point $(x_0, x_1, \dots, x_n)$ is denoted by $(x_0 : x_1 : \dots : x_n)$. In literature there is also notation $[x_0 : x_1 : \dots : x_n]$ or $[x_0, x_1, \dots, x_n]$.
So $(x_0 : x_1 : \dots : x_n)$ is the line in $\mathbb{k}^{n+1}$ through $(x_0, x_1, \dots, x_n)$ and the origin; $x_0, x_1, \dots, x_n$ are called homogeneous coordinates of the point $(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n$. $x_0, \dots, x_n$ are not all zero.

The points $(x_0 : x_1 : \dots : x_n)$ and $(y_0 : y_1 : \dots : y_n)$ are equal $\Leftrightarrow \exists \lambda \in \mathbb{k} \setminus \{0\}$ s.t. $y_i = \lambda x_i \ \forall i = 0, 1, \dots, n$.

We can embed $\mathbb{A}^n$ into $\mathbb{P}^n$. For each $i = 0, 1, \dots, n$, we define $U_i = \{(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n ; \ x_i \neq 0\}$.
If $(x_0 : x_1 : \dots : x_n) = (y_0 : y_1 : \dots : y_n)$, then $y_j = \lambda x_j \ \forall j = 0, 1, \dots, n$ and $\lambda \neq 0$, so $x_i \neq 0 \Leftrightarrow y_i \neq 0 \Rightarrow$ the sets U_i are well defined.
We define a map:

$$\begin{aligned} \mathbb{A}^n &\longrightarrow U_i \\ (x_1, \dots, x_n) &\longmapsto (x_1 : \dots : x_i : 1 : x_{i+1} : \dots : x_n). \end{aligned}$$

This is a bijection with the inverse

$$\begin{aligned} U_i &\longrightarrow \mathbb{A}^n \\ (x_0 : x_1 : \dots : x_n) &\longmapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right). \end{aligned}$$

This map is well defined, because $\frac{\lambda x_j}{\lambda x_i} = \frac{x_j}{x_i}$.

$\Rightarrow$ We can identify $\mathbb{A}^n$ with U_i, (most commonly with U_0) and consider it as a subscape of $\mathbb{P}^n$.

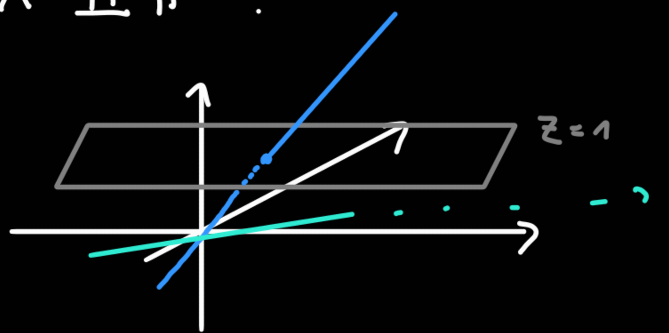
$U_0, U_1, \dots, U_n$ are usually called affine charts of $\mathbb{P}^n$.

$\mathbb{P}^n \setminus U_i$ consists of all points $(x_0 : x_1 : \dots : x_n)$ with $x_i = 0$.
But $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are not all zero, so we have

$$\begin{aligned} \mathbb{P}^n \setminus U_i &\longrightarrow \mathbb{P}^{n-1} \\ (x_0 : \dots : x_{i-1} : 0 : x_{i+1} : \dots : x_n) &\longmapsto (x_0 : \dots : x_{i-1} : x_{i+1} : \dots : x_n) \end{aligned}$$

This is a bijection.
disjoint union
We have $\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{P}^{n-1}$.

<u>Example</u>: $\mathbb{P}^2$



Lines in xy-plane are in bijection with points in $\mathbb{P}^1$.

$\mathbb{A}^n \amalg \mathbb{P}^{n-1}$ $\longrightarrow$ This is usually called the hyperplane at infinity.

We want to study zero loci of polynomials in $\mathbb{P}^n$. In $\mathbb{P}^n$ we have homogeneous coordinates:

$$(x_0 : x_1 : \cdots : x_n) = (\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n) \text{ for } \lambda \neq 0$$

so $f(x_0, x_1, \dots, x_n)$ is not well defined. We restrict to homogeneous polynomials.

<u>Definition</u>: A polynomial $f \in \mathbb{k}[x_0, x_1, \dots, x_n]$ is homogeneous of degree d if

$$f(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_0, x_1, \dots, x_n)$$

for each $\lambda \in \mathbb{k} \setminus \{0\}$.

Since $\mathbb{k}$ is infinite, this is equivalent to that all monomials of f are of degree d.

<u>Example</u>: $x_0^2 x_1 + x_2^3 - x_4 x_5 x_6$ is homogeneous of degree 3

$\underbrace{x_0^2 x_1}_{\deg 3} + \underbrace{x_2^2}_{\deg 2}$ is not homogeneous

<u>Definition</u>: Let $S \subseteq \mathbb{k}[x_0, x_1, \dots, x_n]$ be a set of homogeneous polynomials. The set

$$V(S) := \{ (x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n \mid f(x_0, \dots, x_n) = 0 \text{ for all } f \in S \}$$

is called the projective zero locus of S. A set $X \subseteq \mathbb{P}^n$ is a projective variety if $X = V(S)$ for some set S of homogeneous polynomials.

If $S = \{f_1, \dots, f_m\}$ we write $V(f_1, \dots, f_m)$ instead of $V(\{f_1, \dots, f_m\})$.

Projective zero loci are well defined:
If $F \in S$ is homogeneous of degree d, then
$F(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ for all $\lambda \in \mathbb{k} \setminus \{0\}$.
so we get $f(\lambda x_0, \dots, \lambda x_n) = 0 \Leftrightarrow f(x_0, \dots, x_n) = 0$.

<u>Remark:</u> $V(S)$ can mean affine zero locus or projective zero locus. When there can be confusion, we will write $V_a(S)$ or $V_p(S)$.

<u>Examples of projective varieties</u>

(1) $\mathbb{P}^n = V(0)$

(2) $\emptyset = V(1)$, but also $\emptyset = V(x_0, x_1, \dots, x_n)$.

(3) If V is a vector subspace of $\mathbb{k}^{n+1}$, then $\mathbb{P}V$ is a projective variety in $\mathbb{P}^n$, because vector subspaces are defined by homogeneous linear equations. $\mathbb{P}V$ is called the linear subspace of $\mathbb{P}^n$.

(4) Each point is a projective variety: If $a = (a_0 : a_1 : \dots : a_n)$, then $V(a_i x_j - a_j x_i \mid 0 \le i, j \le n) = \{a\}$. Proof: HW

October 21, 2025

We may define varieties also in product of affine and projective spaces. For example, a variety in $\mathbb{A}^m \times \mathbb{P}^n$ is a zero locus of a set of polynomials in $\mathbb{k}[x_1, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous in the variables $y_0, \dots, y_n$.
Example: $V(x_1^2 y_0^3 - x_2 y_1 y_2^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^2$.
A variety in $\mathbb{P}^m \times \mathbb{P}^n$ is a zero locus of a set of polynomials in $\mathbb{k}[x_0, x_1, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous in $x_0, \dots, x_m$ and (maybe of a different degree) in $y_0, \dots, y_n$.
Example: $V(x_0^2 x_1 y_2^2 - x_2^3 y_0 y_1) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$

$\mathbb{A}^m \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$ similarly

2. Connection between affine and projective varieties

<u>Definition</u>: (1) An affine variety $X \subseteq \mathbb{A}^{n+1}$ is a cone (stožec) if $0 \in X$ and for each $a \in X$ and each $\lambda \in \mathbb{k}$ we have $\lambda a \in X$.

(2) If $X \subseteq \mathbb{A}^{n+1}$ is a cone then the projectiviation (projektivizacija) of X is defined by

$$\mathbb{P}X := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid (a_0, a_1, \dots, a_n) \in X \setminus \{0\}\} \subseteq \mathbb{P}^n.$$

(3) If $X \subseteq \mathbb{P}^n$ is a projective variety, then the cone over X is defined by $C(X) := \{0\} \cup \{(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \mid (a_0 : \cdots : a_n) \in X\} \subseteq \mathbb{A}^{n+1}$.

If X is a cone and $(a_0, a_1, \dots, a_n) \in X$ and $\lambda \in \mathbb{k}$. Since X is a cone, we have $(\lambda a_0, \lambda a_1, \dots, \lambda a_n) \in X \Rightarrow$ $\mathbb{P}X$ is well-defined.

<u>Proposition</u>: If $X \subseteq \mathbb{A}^{n+1}$ is a cone, then $\mathbb{P}X$ is a projective variety.

<u>Proof</u>: X is an affine variety, so $X = V_a(S)$ for some $S \subseteq \mathbb{k}[x_0, \dots, x_n]$. We can assume that $S = I_a(X)$. Let $f \in I_a(X)$ is arbitrary. Write $f = \sum_{i=0}^{d} f_i$, where each f_i is homogeneous of degree i. Take arbitrary $a \in X$ and arbitrary $\lambda \in \mathbb{k}$. Since X is a cone, we have $\lambda a \in X$, so $f(\lambda a) = 0$.

$$0 = f(\lambda a) = \sum_{i=0}^{d} f_i(\lambda a) = \sum_{i=0}^{d} \lambda^i f_i(a)$$

This holds for each $\lambda \in \mathbb{k}$, $\mathbb{k}$ infinite $\Rightarrow f_i(a) = 0 \ \forall i$
This holds for each $a \in X$, so $f_i \in I_a(X)$ for each i.

We showed that $I_a(X)$ can be generated by homogeneous polynomials. Let $I_a(X) = (S')$ where S' is a set of homogeneous polynomials. Then

$$\mathbb{P}X = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid (a_0, \ldots, a_n) \in \overset{V(S')}{\overset{\shortparallel}{X}} \setminus \{0\}\}$$

$$= \{(a_0 : \cdots a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0 \ \ \forall f \in S'\}$$

$$\underset{\substack{\uparrow \\ S' \text{ is a set of homogeneous polynomials}}}{=} V_p(S')$$

∎

We proved two more things:

Corollary: If S is a set of homogeneous polynomials then $\mathbb{P}V_a(S) = V_p(S)$.

Corollary: X cone $\Rightarrow I_a(X)$ generated by hom. polynomials.

Proposition: The cone over a projective variety is a cone.

Proof: If $X \neq \emptyset$, then $C(X) = \{0\}$ which is a cone. Assume that $X \subseteq \mathbb{P}^n$ is a non-empty projective variety. Then $X = V_p(S)$ for some set S of non-constant polynomials. If $f \in S$ is hom. of degree d, then $f(\lambda a) = \lambda^d f(a)$ $\forall \lambda \in \mathbb{k}$ and $\forall a \in X$. $\Rightarrow f(0) = 0$.

$$\begin{aligned} C(X) &= \{0\} \cup \{(a_0, \ldots, a_n) \in \mathbb{A}^{n+1} \mid (a_0 : \cdots : a_n) \in X\} \\ &= \{(a_0, a_1, \ldots, a_n) \in \mathbb{A}^{n+1} \mid f(a_0, a_1, \ldots, a_n) = 0 \ \ \forall f \in S\} \\ &= V_a(S) \end{aligned}$$

We know $0 \in C(X)$ and that if $a \in V_a(S)$, $\lambda \in \mathbb{k}$, then $f(\lambda a) = \lambda^d f(a) = 0$, so $\lambda a = V_a(S)$. ∎

We proved also:

Corollary: If S is a set of homogeneous polynomials then $C V_p(X) = V_a(S)$.

Corollary: The maps

$$X \longmapsto \mathbb{P}X$$
$$C(X) \longleftarrow X$$

give bijective correspondence between the cones in $\mathbb{A}^{n+1}$ and projective varieties in $\mathbb{P}^n$.

Proof: Both corollaries tell that $\mathbb{P}X$ is a projective variety if X is a cone and that $C(X)$ is a cone if X is a projective variety. We have to prove bijectivity.

Suppose $X \subseteq \mathbb{A}^{n+1}$ is a cone. Then we know from one of the corollaries above that $X = V_a(S)$ for a set of hom. poly. $S \subseteq \mathbb{k}[x_0, \dots, x_n]$. By the other two corollaries:

$$C(\mathbb{P}X) = C(\mathbb{P}V_a(S)) = C(V_p(S)) = V_a(S) = X.$$

Similarly $\mathbb{P}(C(X)) = X$ if X is a projective variety. ☒

Corollary: Each projective variety is a zero locus of a finite set of homogeneous polynomials.

Proof: $X = V_p(S)$ for some set S of hom. polynomials. Then $C(X) = V_a(X)$. Let $J = I_a(C(X))$. Then we know that J is finitely generated ($\mathbb{k}[x_0, \dots, x_n]$ noetherian) $J = (f_1, \dots, f_n)$. We proved before that homogeneous parts lie in J, and they obviously generate J. So we have a finite set S' of hom. polynomials that generate J.

$\Rightarrow X = \mathbb{P}(C(X)) = \mathbb{P}(V_a(J)) = \mathbb{P}(V_a(S')) = V_p(S')$ $\blacksquare$

As in the affine case, we can use this corollary to prove:

Lemma: (1) If $\{S_j\}_{j\in J} \subseteq \mathbb{k}[x_0,\dots,x_n]$ is a family of sets of homogeneous polynomials, then

$$V_p\Big(\bigcup_{j\in J} S_j\Big) = \bigcap_{j\in J} V_p(S_j).$$

(2) If $f_1,\dots,f_s, g_1,\dots,g_t \in \mathbb{k}[x_0,\dots,x_n]$ are homogeneous polynomials, then

$$V_p(f_1,\dots,f_s) \cup V_p(g_1,\dots,g_t) = V_p\Big(f_ig_j ;\ \substack{1\le i\le s\\ 1\le j\le t}\Big).$$

Corollary: (1) $\emptyset$ and $\mathbb{P}^n$ are projective varieties.
(2) The intersection of any family of projective varieties is a projective variety.
(3) The union of finitely many projective varieties is a projective variety.

Projective varieties are therefore exactly the closed sets on some topology on $\mathbb{P}^n$ – Zariski topology on $\mathbb{P}^n$.
As in the affine case, the Zariski topology on subsets of $\mathbb{P}^n$ is the relative topology:
Let $X \subseteq \mathbb{P}^n$. A set $Z \subseteq X$ is Zariski-closed if there exists a proj. var. $Y \subseteq \mathbb{P}^n$ s.t. $Z = X \cap Y$. If X is a proj. var., then it's closed subsets are exactly the subvarieties.

As in the affine case, we also define distinguished open subsets: $D(f) = \mathbb{P}^n \setminus V_p(f)$ where f is a hom. poly.

In a similar way we can define the Zariski topology in any product $\mathbb{A}^m \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$.

<u>Definition</u>: Let $X \subseteq \mathbb{P}^n$. X is reducible if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subsetneq X$, irreducible otherwise.

If X is a proj. var., the X is reducible $\Leftrightarrow$ X is a union of two proper subvarieties.

<u>Theorem</u>: Each proj. var. $X \subseteq \mathbb{P}^n$ can be written as $X = X_1 \cup X_2 \cup \dots \cup X_m$ where $m \in \mathbb{N}_0$ and $X_1, \dots, X_m$ are irreducible proj. var. Moreover, if $X_i \nsubseteq X_j$ whenever $i \neq j$, then this decomposition is unique up to an order. In this case $X_1, \dots, X_m$ are called irreducible components of X.

$\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{P}^{n-1}$

Recall: $U_i = \{(x_0 : \dots : x_n) \in \mathbb{P}^n;\ x_i \neq 0\} = D(x_i)$
We identified U_i with $\mathbb{A}^n$.
$U_i = D(x_i)$ are open in Zariski topology, moreover, $\{U_0, \dots, U_n\}$ is an open cover of $\mathbb{P}^n$.
$\Rightarrow$ We can consider $\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$.
We have 2 Zariski topologies on $\mathbb{A}^n$: one defined by affine varieties and one as a relative topology in $\mathbb{P}^n$.
Are they equal?

$\boxed{\text{We will identify } \mathbb{A}^n \text{ with } U_0.}$

<u>Definition</u>: Let $f \in \mathbb{k}[x_0, \dots, x_n]$ be a homogeneous polynomial. Dehomogenization of f is the polynomial

$$f^{(d)} := f(1, x_1, \dots, x_n) \in \mathbb{k}[x_1, \dots, x_n].$$

Dehomogenization is evaluation $x_0 = 1$, so it is a ring homomorphism, so: $(fg)^{(d)} = f^{(d)} \cdot g^{(d)}$,

$$(f+g)^{(d)} = f^{(d)} + g^{(d)}.$$

<u>Definition</u>: Let $f \in \mathbb{k}[x_1, \dots, x_n]$ be a non-zero polynomial of degree d. Then

$$f^{(h)} := x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

is a polynomial, called the homogenization of f.

<u>Example</u>: $f(x_1, x_2, x_3) = x_1^3 - x_2 x_3 + 2x_2^4$

$\deg f = 4 \Rightarrow f^{(h)}(x_0, x_1, x_2, x_3) = x_0 x_1^3 - x_0^2 x_2 x_3 + 2x_2^4$

We have $(fg)^{(h)} = f^{(h)} g^{(h)}$, but $(f+g)^{(h)} \neq \underbrace{f^{(h)} + g^{(h)}}_{\text{this polynomial may not be homogeneous}}$

<u>Proposition</u>: Each affine variety $X \subseteq \mathbb{A}^n \equiv U_0 \subseteq \mathbb{P}^n$ is of the form $X = Z \cap U_0$ for some projective variety Z. More precisely, if $X = V_a(f_1, \dots, f_m)$, we may take $Z = V_p(f_1^{(h)}, \dots, f_m^{(h)})$.

<u>Proof</u>: Let f_i be of degree d_i for each i.

$$Z \cap U_0 = V_p(f_1^{(h)}, \dots, f_m^{(h)}) \cap U_0$$

$$= \{ (a_0 : a_1 : \dots : a_n) \in \mathbb{P}^n \mid a_0 \neq 0 \ \forall i : f_i^{(h)}(a_0, \dots, a_n) = 0 \}$$
$$= \{ (1 : \tfrac{a_1}{a_0} : \dots : \tfrac{a_n}{a_0}) \in \mathbb{P}^n \mid a_0 \neq 0, \forall i : f_i^{(h)}(1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}) = 0 \}$$
$$= \{ (1 : \tfrac{a_1}{a_0} : \dots : \tfrac{a_n}{a_0}) \in \mathbb{P} \mid a_0 \neq 0, \forall i : \underbrace{a_0^{d_i}}_{\neq 0} f_i(1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}) = 0 \}$$
$$= \{ (1 : \tfrac{a_1}{a_0} : \dots : \tfrac{a_n}{a_0}) \in \mathbb{P} \mid a_0 \neq 0, \forall i : f_i(1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}) = 0 \}$$
$$\cong \{ (b_1, \dots, b_n) \in \mathbb{A}^n \mid \forall i. \ f_i(b_1, \dots, b_n) = 0 \}$$
$$= V_a(f_1, \dots, f_n) = X$$

$\square$

Corollary: Both Zariski topologies on $\mathbb{A}^n$ coincide.

We often study open subsets of projective varieties. Such sets are called *quasiprojective* varieties. Important examples of quasiprojective varieties are proj. varieties and affine varieties.

Definition: Let $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be an affine variety. The *projective closure* of X is the smallest projective variety that contains X. Notation: $\overline{X}$

In general $\overline{X} \neq V_p(f_1^{(h)}, \dots, f_m^{(h)})$ if $X = V_a(f_1, \dots, f_m)$.

Example: $X = V_a(x_1, x_2 - x_1^2) = \{(0,0)\}$. $\overline{X}$ also has to be one point: $\overline{X} = \{[1:0:0]\}$, but
$V_p(x_1, x_0 x_2 - x_1^2) = \{[1:0:0], [0:0:1]\} \neq \overline{X}$.

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Proposition: Let $X \subseteq \mathbb{A}^n$ ($= U_0 \subseteq \mathbb{P}^n$) be a an affine variety and $\overline{X} \subseteq \mathbb{P}^n$ its projective closure. Then:

(1) $\overline{X} \cap U_0 = X$ (2) X irreducible $\Rightarrow$ $\overline{X}$ irreducible

(3) No irreducible component of $\overline{X}$ lies in $V_p(x_0)$ (= hyperplane at infinity).

Proof: (1) We know that $X = Z \cap U_0$ for some projective variety Z. Z is a projective variety that contains X, so it also contains $\bar{X}$.

(2) Suppose that $\bar{X} = Z_1 \cup Z_2$ for some projective varieties Z_1, Z_2. By (1) we get

$$X = \bar{X} \cap U_0 = (Z_1 \cap U_0) \cup (Z_2 U_0)$$

$Z_1 \cap U_0$ and $Z_2 \cap U_0$ are affine varieties, so irreducibility of X implies $X = Z_i \cap U_0$ for some $i = 1, 2$. Z_i is a projective variety that contains X, so $\bar{X} \subseteq Z_i \Rightarrow \bar{X}$ irreducible.

(3) Let $\bar{X} = Z_1 \cup \cdots \cup Z_m$ be the decomposition into irreducible components. Suppose that $Z_1 \subseteq V_p(x_0)$.

$$\begin{aligned} X = \bar{X} \cap U_0 &= (Z_1 \cup \cdots \cup Z_m) \cap U_0 \\ &= \underbrace{(Z_1 \cap U_0)}_{=\emptyset} \cup ((Z_2 \cup \cdots \cup Z_1) \cap U_0) \\ &= (Z_2 \cup \cdots \cup Z_m) \cap U_0 \end{aligned}$$

$Z_2 \cup \cdots \cup Z_m$ is a projective variety that contains X, so it contains $\bar{X}$ and therefore

$$Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_m$$

Z_1 is irreducible, therefore $Z_1 \subseteq Z_i$ for some $i = 2, \ldots, m$. This contradicts the fact that Z_i are components. $\boxtimes$

3. Projective algebra-geometry correspondence

<u>Definition</u>: A ring/algebra R is graded if we can write it as a direct sum of abelian groups/$\mathbb{k}$-vector spaces $R = \bigoplus_{d=0}^{\infty} R_d$ such that $R_d \cdot R_e \subseteq R_{d+e}$ for all $d, e \in \mathbb{N}_0$.

slovene: stopničast kolobar

If $f \in R_d$ and $g \in R_e$, then $fg \in R_{d+e}$.

We say that elements of $R_d \setminus \{0\}$ are homogeneous of degree d.

<u>Example</u>: $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ is a graded $\mathbb{k}$-algebra:
$R = \bigoplus_{d=0}^{\infty} R_d$, where $R_d = \{0\} \cup \{\text{homogeneous polynomials of degree } d\}$.

Let $f \in R$. Then f can be uniquelly decomposed as $f = \sum_{d=0}^{\infty} f_d$ where $f_d \in R_d$ for each d and only finitely many f_d's are nonzero. The decomposition $f = \sum_{d=0}^{\infty} f_d$ is called the homogeneous decomposition of f.
If $f \neq 0$, then the degree of f is the largest d s.t. $f_d \neq 0$.

<u>Definition</u>: Let R be a graded ring/algebra. An ideal $I \trianglelefteq R$ is homogeneous if it can be generated by homogeneous elements.

<u>Example</u>: $I = (x_1, x_2 - x_1^2) = I(x_1, x_2)$ is a homogeneous ideal.

<u>Example</u>: If X is a cone, then we showed that $I_a(X)$ is homogeneous.

Proposition: Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a graded ring and $J, J_1, J_2 \trianglelefteq R$. Then the following holds:

(1) J is homogeneous $\Leftrightarrow$ for each $f \in J$ with homogeneous decomposition $f = \sum_{d=0}^{\infty} f_d$ we have $f_d \in J$ for each d.

(2) J, J_1, J_2 homogeneous $\Rightarrow$ $J_1 + J_2$, $J_1 \cap J_2$, $J_1 J_2$, $\sqrt{J}$ homogeneous

(3) J homogeneous, then R/J is a graded ring with the homogeneous decomposition:

$$R/J = \bigoplus_{d=0}^{\infty} R_d / (R_d \cap J).$$

$$\cong \text{ isomorphism theorem}$$

$$(R_d + J)/J$$

(4) If R is noetherian and J is a homogeneous ideal, then J can be generated by finitely many homogeneous elements.

Proof: Exercise.

Definition: (1) For a homogeneous ideal $J \trianglelefteq \mathbb{k}[x_0, x_1, \dots, x_n]$ we define:
$$V(J) = V_p(J) := \{ x \in \mathbb{P}^n \mid f(x) \ \forall \text{ homogeneous } f \in J \}.$$

(2) For $X \subseteq \mathbb{P}^n$ we define the ideal of X as
$$I(X) = I_p(X) := \{ f \in \mathbb{k}[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \ \forall x \in X \}.$$

Remark: (1) $V(J)$ is well defined, because it is defined only using homogeneous polynomials. If S is a homogeneous set of generators of J, then $V_p(J) = V_p(S)$. $V_p(J)$ is a projective variety.

(2) The set of all homogeneous polynomials vanishing on X is not an ideal. To get $I_p(X)$, we must take the ideal generated by them.

Lemma: (1) If $X \subseteq Y \subseteq \mathbb{P}^n$, then $I_p(Y) \subseteq I_p(X)$.
(2) For homogeneous ideals $I_1 \subseteq I_2$ we have $V_p(I_2) \subseteq V_p(I_1)$.

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Proposition: Let $X \subseteq \mathbb{P}^n$ be an non-empty projective variety. Then $I_p(X) = I_a(C(X))$.

Proof: ($\subseteq$): Let $f \in I_p(X)$ and $f = \sum_{d=0}^{\infty} f_d$ be the homogeneous decomposition of f.
We know that $I_p(X)$ is a homogeneous ideal, so $f_d \in I_p(X)$ for each d.
$\Rightarrow f_d(x_0, x_1, \dots, x_n) = 0 \quad \forall (x_0, \dots, x_n) \in X$
$\Rightarrow f_d(x_0, \dots, x_n) = 0$ for each point in the cone
$\Rightarrow f_d \in I_a(C(X))$ for each $d \Rightarrow f \in I_a(C(X))$.

($\supseteq$): We proved that $I_a(C(X))$ is a homogeneous ideal, so it is generated by homogeneous elements.
$\Rightarrow$ Let $I_a(C(X)) = (g_j \mid j \in J)$. $g_j \in I_a(C(X))$
$\Rightarrow g_j(x_0, x_1, \dots, x_n) = 0 \quad \forall (x_0, x_1, \dots, x_n) \in C(X)$
$\Rightarrow g_j(x_0 : x_1 : \dots : x_n) = 0 \quad \forall (x_0 : x_1 : \dots : x_n) \in X$
$\Rightarrow g_j \in I_p(X) \; \forall j \in J \Rightarrow I_a(C(X)) \subseteq I_p(X)$ ▨

Corollary: $I_p(X)$ is always a radical ideal.

Theorem: If X is a projective variety, then $V_p(I_p(X)) = X$.

Proof: If $X = \emptyset$, then $I_p(X) = \mathbb{k}[x_0, x_1, \dots, x_n]$ and $V_p(I_p(X)) = \emptyset$.
If $X \neq \emptyset$, then

$$V_p(I_p(X)) = \mathbb{P} V_a(I_p(X)) \overset{\text{previous proposition}}{=} \mathbb{P} V_a(I_a(C(X)))$$
$$= \mathbb{P}(C(X)) = X.$$
▨

As in the affine case, we have:

Proposition: If $X \subseteq \mathbb{P}^n$ is any set, then $\bar{X} = V_p(I_p(X))$.

Affine weak Nullstelensatz: If $J \neq (1)$ is a proper ideal, then $V_a(J) \neq \emptyset$.
In projective case: $V_p(x_0, x_1, \dots, x_n) = \emptyset$

Definition: The ideal $I_0 = (x_0, x_1, \dots, x_n) \trianglelefteq \mathbb{k}[x_0, x_1, \dots, x_n]$ is called the irrelevant ideal.

Theorem [projective weak Nullstellensatz]: For a proper homogeneous ideal $J \trianglelefteq \mathbb{k}[x_0, x_1, \dots, x_n]$ we have $V_p(J) = \emptyset \Leftrightarrow \sqrt{J} = I_0$.

Proof: ($\Leftarrow$): Suppose $\sqrt{J} = I_0 = (x_0, x_1, \dots, x_n)$.
$\Rightarrow x_i \in \sqrt{J}$ for each $i \Rightarrow \forall i.\ \exists N_i$ s.t. $x_i^{N_i} \in J$.
Suppose that $(a_0 : a_1 : \dots : a_n) \in V_p(J)$.
$\Rightarrow \forall i.\ a_i^{N_i} = 0 \Rightarrow a_i = 0 \ \forall i \Rightarrow \underbrace{(0 : 0 : \dots : 0)}_{\text{(not a projective point)}} \in V_p(J) \Rightarrow V_p(J) = \emptyset$

($\Rightarrow$): Suppose that $V_p(J) = \emptyset$. $J \neq (1) \Rightarrow V_a(J) \neq \emptyset$
J is a homogeneous ideal, so $J = (S)$ where S is a of homogeneous polynomials. We know $V_p(J) = V_p(S)$, $V_a(J) = V_a(S)$, $V_p(S) = \mathbb{P} V_a(S) \Rightarrow \underbrace{V_p(J)}_{= \emptyset} = \mathbb{P} \underbrace{V_a(J)}_{\text{non-empty cone}}$

The only possibility is $V_a(J) = \{(0, 0, \dots, 0)\}$.
$\Rightarrow I_a(V_a(J)) = \sqrt{J} = (x_0, x_1, \dots, x_n)$. ▨

Theorem [projective Nullstellensatz]: For a homogeneous ideal $J \trianglelefteq \mathbb{k}[x_0, x_1, \dots, x_n]$ with $\sqrt{J} \neq I_0$ we have $I_p(V_p(J)) = \sqrt{J}$.

Proof: If $V_p(J) = \emptyset$, then $J = (1)$ by the projective weak Nullstellensatz. Then $\sqrt{J} = (1) = I_p(V_p(J)) = I_p(\emptyset)$.

Assume now that $V_p(J) \neq \emptyset$. J is a homogeneous ideal, so $J=(S)$ for some set S of homogeneous polynomials.

$$I_p(V_p(J)) = I_p(V_p(S)) = I_a(C(V_p(S))) = I_a(C(\mathbb{P}(V_a(S))))$$
$$= I_a(V_a(S)) = I_a(V_a(J)) = \sqrt{J}$$ ▨

↑ affine Nullstelensatz

The maps I_p and V_p have similar properties as I_a and V_a. In some cases we need to assume that some ideal is not irrelevant.

Corollary: (1) We have a bijection:

$$\left\{\begin{array}{l}\text{projective}\\ \text{varieties}\\ \text{in } \mathbb{P}^n\end{array}\right\} \underset{V_p}{\overset{I_p}{\rightleftarrows}} \left\{\begin{array}{l}\text{homogeneous radical ideals}\\ \text{in } \Bbbk[x_0,\dots,x_n] \text{ different from}\\ I_0=(x_0,\dots,x_n)\end{array}\right\}$$

(2) We have a bijection:

$$\left\{\begin{array}{l}\text{irreducible}\\ \text{projective}\\ \text{varieties}\\ \text{in } \mathbb{P}^n\end{array}\right\} \underset{V_p}{\overset{I_p}{\rightleftarrows}} \left\{\begin{array}{l}\text{homogeneous prime ideals}\\ \text{in } \Bbbk[x_0,\dots,x_n] \text{ different}\\ \text{from } I_0=(x_0,\dots,x_n)\end{array}\right\}$$

Definition: Let $J \trianglelefteq \Bbbk[x_1,\dots,x_n]$. The homogenization of J is the ideal generated by the homogenizations of all elements from J:

$$J^{(h)} = (f^{(h)} \mid f \in J) \trianglelefteq \Bbbk[x_0,x_1,\dots,x_n].$$

Proposition: Let $X \subseteq \mathbb{A}^n$ be an affine variety, $\mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n$. Let $J = I_a(X)^{(h)} \trianglelefteq \Bbbk[x_0,\dots,x_n]$. Then $I_p(X) = J$ and $\overline{X} = V_p(J) = V_p(I_a(X)^{(h)})$.

Zariski closure

Proof: It is enough to prove that $J = I_p(X)$.

($\subseteq$): It is enough to show that $f^{(h)} \in I_p(X)$ $\forall f \in I_a(X)$. Let $f \in I_a(X)$ be arbitrary. $I_a(X) \trianglelefteq \Bbbk[x_1,\dots,x_n] \subseteq \Bbbk[x_0,\dots,x_n]$. Let $a \in X$, $a = (a_0 : a_1 : \dots : a_n)$. Since $X \subseteq U_0$, we may assume that $a_0 = 1$.

$\Rightarrow f(1,a_1,\dots,a_n) = 0$ ($f \in I_a(X)$, $a \in X$)

$$f^{(h)}(a_0,a_1,\dots,a_n) = \underset{1}{a_0}^d f\left(\underset{1}{\frac{a_1}{a_0}},\dots,\underset{1}{\frac{a_n}{a_0}}\right) = 0 \Rightarrow f^{(h)} \in I_p(X)$$

($\supseteq$): It is enough to show that $f \in J$ for each homogeneous f that vanishes on X.
Let $f \in I_p(X)$. If $f = x_0 \cdot g$, then also $g \in I_p(X)$, because $X \subseteq U_0$ and $x_0 \neq 0$ on X. So we assume that f is not divisible by x_0. Let $a = (a_0, a_1, \dots, a_n) \in X$ be arbitrary. We can assume $a_0 = 1$.
$f \in I_p(X) \Rightarrow f(1, a_1, \dots, a_n) = 0$
$\Rightarrow$ The polynomial $f(1, x_1, \dots, x_n)$ vanishes on X.
$\Rightarrow f(1, x_1, \dots, x_n) \in I_a(X)$
$f = f(1, x_1, \dots, x_n)^{(h)} \quad \Rightarrow f \in J$ ▨
(not divisible by x_0)

Example: $X = V(x_1, x_2 - x_1^2)$
The ideal $\underset{J}{\underbrace{(x_1, x_2 - x_1^2)}} = (x_1, x_2)$ is homogeneous.

$\Rightarrow J^{(h)} = (x_1, x_2) \trianglelefteq \mathbb{k}[x_0, x_1, x_2] \Rightarrow \overline{X} = V_p(x_1, x_2) = \{[1:0:0]\}$.

Corollary: Let $X \subseteq \mathbb{P}^n$ be a projective variety and $J_i = I_a(X \cap U_i)$ for $i = 0, \dots, n$; $J_i \trianglelefteq \mathbb{k}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$.
Then $I_p(X) = J_0^{(h)} \cap J_1^{(h)} \cap \dots \cap J_n^{(h)}$.

Proof: $X = \bigcup_{i=0}^{n} (X \cap U_i) \quad | I_p$

$$I_p(X) = I_p\left(\bigcup_{i=0}^{n} (X \cap U_i)\right) = \bigcap_{i=0}^{n} I_p(X \cap U_i) \overset{\text{proposition}}{=} \bigcap_{i=0}^{n} I_a(X \cap U_i)^{(h)}.$$ ▨

III. MAPS BETWEEN VARIETIES

1. Polynomial maps and coordinate ring

⚠ In this section everything will be affine.

Definition: Let $X \subseteq \mathbb{A}^n$ be an affine variety and $f \in \mathbb{k}[x_1, \dots, x_n]$ a polynomial. The map $X \longrightarrow \mathbb{k}$, $a \longmapsto f(a)$ is called a polynomial function on X.
The set of all polynomial functions on X is a ring for point-wise addition and multiplication. We call it the coordinate ring of X. Notation: $\mathbb{k}[X]$

Proposition: Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then $\mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/I(X)$.

Proof: Let $\Phi : \mathbb{k}[x_1, \dots, x_n] \longrightarrow \mathbb{k}[X]$ be the map defined by $f \longmapsto (a \longmapsto f(a))$. Φ is a ring homomorphism and it is clearly surjective, so $\mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/\ker \Phi$.
$f \in \ker \Phi \Leftrightarrow f(a) = 0 \ \forall a \in X \Leftrightarrow f(a) \in I(X)$.
$\Rightarrow \mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/I(X)$. ▨

Corollary: $\mathbb{k}[\mathbb{A}^n] = \mathbb{k}[x_1, \dots, x_n]$ $\qquad I(\mathbb{A}^n) = (0)$

Proposition:
(1) $\mathbb{k}[X]$ is without nilpotents. We say it is reducible.
(2) $\mathbb{k}[X]$ is a domain $\Leftrightarrow$ X is irreducible.

Proof: (1) Some power of a function is $0 \Leftrightarrow$ the function is 0.
(2) $\mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/I(X)$ is a domain $\Leftrightarrow I(X)$ is a prime ideal $\Leftrightarrow X$ is irreducible. ▨

Remark: If $X = X_1 \cup \cdots \cup X_m$ is the decomposition of X into irreducible components, then $\Bbbk[X] \cong \Bbbk[X_1] \times \cdots \times \Bbbk[X_m]$.
Commutative algebra: Chinese Remainder Theorem

Definition: Let $X \subseteq \mathbb{A}^n$ be a variety.
(1) A subvariety of X is any subset of the form
$$V_X(S) := \{ a \in X \mid f(a) = 0 \ \forall f \in S \}$$
where $S \subseteq \Bbbk[X]$.

(2) For any subset $Y \subseteq X$ we define the ideal of Y in $\Bbbk[X]$ by $I_X(Y) = \{ f \in \Bbbk[X] \mid f(a) = 0 \ \forall a \in Y \} \trianglelefteq \Bbbk[X]$.

The maps I_X and V_X have the following properties:
(1) If $S \subseteq \Bbbk[X]$ and $J \trianglelefteq \Bbbk[X]$ is the ideal generated by S, then $V_X(S) = V_X(J)$.
(2) $\Bbbk[X]$ is a quotient of a noetherian ring, so it is noetherian $\Rightarrow$ subvarieties of X are of the form $V_X(S)$ for finite S.
(3) Subvarieties of X are precisely the varieties that are contained in X.
(4) If Y is a subvariety of X, then
$$\Bbbk[X]/I_X(Y) \cong \left(\Bbbk[x_1,\dots,x_n]/I(X)\right) / \left(I(Y)/I(X)\right) \overset{\text{isomorphism theorem}}{\cong} \Bbbk[x_1,\dots,x_n]/I(Y) \cong \Bbbk[Y]$$
(5) $V_X(I_X(Y)) = Y$ if Y is a subvariety of X.
(6) Relative Nullstellensatz:
If $J \trianglelefteq \Bbbk[X]$, then $I_X(V_X(J)) = \sqrt{J}$.
(7) Versions of properties from the proposition with 16 properties for V and I hold.
(8) There is a bijective correspondence between subvarieties of X and radical ideals of $\Bbbk[X]$.

October 28, 2025

Definition: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. A map $\phi: X \longrightarrow Y$ is a **polynomial map** if there exist polynomials $f_1, \ldots, f_m \in \mathbb{k}[x_1, \ldots, x_n]$ such that $\phi(a) = (f_1(a), \ldots, f_m(a))$ for each $a \in X$.

Lemma: Polynomial maps are continuous in the Zariski topology.

Proof: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and $\phi: X \longrightarrow Y$ a polynomial map. Then there exist polynomials $f_1, \ldots, f_m \in \mathbb{k}[x_1, \ldots, x_n]$ s.t. $\phi(a) = (f_1(a), \ldots, f_m(a))$ $\forall a \in X$. Let $Z \subseteq Y$ be a closed subset. We have to prove that $f^{-1}(Z)$ is closed. Z is closed in Y, which is closed in $\mathbb{A}^m$ so Z is closed in $\mathbb{A}^n$, so Z is an affine variety. Therefore there exist $g_1, \ldots, g_\ell \in \mathbb{k}[x_1, \ldots, x_n]$ s.t. $Z = V(g_1, \ldots, g_\ell)$.

$$\begin{aligned}
\phi^{-1}(Z) &= \{a \in X \mid \phi(a) \in Z\} \\
&= \{a \in X \mid g_i(\phi(a)) = 0 \ \forall i = 1, \ldots, \ell\} \\
&= \{a \in X \mid g_i(f_1(a), \ldots, f_m(a)) = 0 \ \forall i\}
\end{aligned}$$

Observe that $g_i(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is a polynomial for each i. So $\phi^{-1}(Z)$ is an affine variety, so closed in X. ▨

Corollary: If $\phi: X \longrightarrow Y$ is a polynomial map and $S \subseteq X$ any subset, then $\phi(\overline{S}) = \overline{\phi(S)}$.

Corollary: If X is an irreducible variety and $\phi: X \longrightarrow Y$ is a polynomial map, then $\overline{\phi(X)}$ is irreducible.

Proof: Assume that $\overline{\phi(X)} = Z_1 \cup Z_2$ for two closed subsets $Z_1, Z_2 \subseteq \overline{\phi(X)}$. If $a \in X$ is arbitrary, then $\phi(a) \in \overline{\phi(X)} = Z_1 \cup Z_2$, so $\phi(a) \in Z_i$ for some $i \in \{1, 2\}$. We showed that $X \subseteq \phi^{-1}(Z_1) \cup \phi^{-1}(Z_2)$.

$a \in \phi^{-1}(Z_i)$

Since ϕ is continuous, $\phi^{-1}(Z_1)$ and $\phi^{-1}(Z_2)$ are closed, and irreducibility of X implies $X \subseteq \phi^{-1}(Z_1)$ for some $i \in \{1,2\}$.

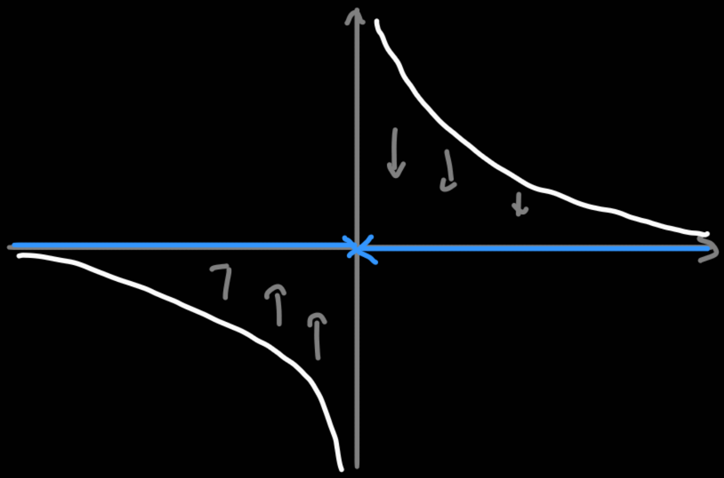
$\Rightarrow \phi(X) \subseteq \phi(\phi^{-1}(Z_i)) \subseteq Z_i$

Z_i is closed, so $\overline{\phi(X)} \subseteq Z_i \Rightarrow Z_i$ is not a proper subset of $\overline{\phi(X)} \Rightarrow \overline{\phi(X)}$ is irreducible $\square$

Corollary: If $\phi: \mathbb{A}^n \longrightarrow X$ is a polynomial map, then $\overline{\phi(\mathbb{A}^n)}$ is irreducible.

The image of a polynomial map is not necessarily closed.

Example: $X = V(xy-1)$, $\phi: X \longrightarrow \mathbb{A}^1$ projection



$\phi(X) = \mathbb{A}^1 \setminus \{0\}$

The image of a polynomial map is also not necessarily open.

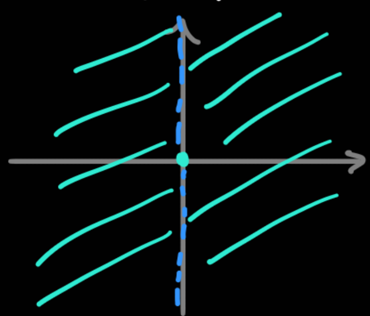
Example: $\phi: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$
$(x,y) \longmapsto (x, xy)$

What is $\phi(\mathbb{A}^2)$?

If $x \neq 0$, we can get all pairs of the form (x,z), $z \in \mathbb{A}^1$.

If $x = 0$, we get only $(0,0)$.

$\Rightarrow \phi(\mathbb{A}^2) = (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 \cup \{(0,0)\}$.



This is not open in $\mathbb{A}^2$.

Definition: Affine varieties X and Y are isomorphic if there exist polynomial maps $\phi: X \longrightarrow Y$ and $\psi: Y \longrightarrow X$ s.t. $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$.

Bijective polynomial maps are not necessarilly isomorphisms.

Example: $X = V(x^2 - y^3)$, $\quad \phi: \mathbb{A}^1 \longrightarrow X$
$\qquad\qquad\qquad\qquad\qquad t \longmapsto (t^3, t^2)$

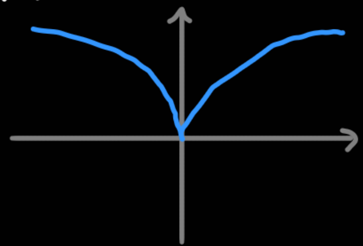
$\phi(\mathbb{A}^1)$ indeed lies in X: $(t^3)^2 - (t^2)^3 = 0$.

ϕ is a polynomial map and it is injective:
If $(t^3, t^2) = (u^3, u^2)$ and $t \neq 0$, then $\left.\begin{matrix} t^3 = u^3 \\ t^2 = u^2 \end{matrix}\right\} \Rightarrow t = u$

The map ϕ is also surjective: Let $(a,b) \in X$ and suppose $a, b \neq 0$.

$$\phi\left(\frac{a}{b}\right) = \left(\frac{a^3}{b^3}, \frac{a^2}{b^2}\right) \underset{(a,b)\in X \Rightarrow a^2 = b^3}{=} \left(\frac{a^3}{a^2}, \frac{b^3}{b^2}\right) = (a, b)$$

$\Rightarrow \phi$ is a bijective polynomial map, but it is not an isomorphism (proof later).
X has a cusp in $(0,0)$,
but $\mathbb{A}^1$ is smooth



Lemma: Composition of polynomial maps is a polynomial map.

Proof: $\phi: X \rightarrow Y$, $\psi: Y \longrightarrow Z$, $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $Z \subseteq \mathbb{A}^\ell$,
ϕ, ψ polynomial maps.
$\Rightarrow \exists f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a))$ $\forall a \in X$.
$\exists g_1, \dots, g_\ell \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\psi(a) = (g_1(a), \dots, g_\ell(a))$ $\forall a \in Y$.

$\forall a \in X$: $(\psi \circ \phi) = \psi(f_1(a), \dots, f_m(a))$
$\qquad\qquad\qquad = (g_1(f_1(a), \dots, f_m(a)), \dots, g_\ell(f_1(a), \dots, f_m(a)))$

Components are polynomials, as they are compositions of polynomials. $\blacksquare$

Lemma: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $\phi: X \to Y$ a map and $\pi_i: Y \to \mathbb{A}^1$ be the projection to the i-th component. ϕ is a polynomial map $\Leftrightarrow$ all compositions $\pi_i \circ \phi$ are polynomial functions.

Proof: ($\Rightarrow$): Follows from the previous lemma.
($\Leftarrow$): Suppose that $\pi_i \circ \phi$ is a polynomial function for each i. Then $\forall i$ $\exists f_i \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\pi_i(\phi(a)) = f_i(a)$ $\forall a \in X$.
$\Rightarrow \phi(a) = (f_1(a), \dots, f_m(a)) \Rightarrow \phi$ is a polynomial map. $\blacksquare$

Corollary: Let $\phi: X \to Y$ be a polynomial map and $g \in \mathbb{k}[Y]$. Then $g \circ \phi \in \mathbb{k}[X]$. $X \xrightarrow{\phi} Y \xrightarrow{g} \mathbb{k}$

Definition: Let ϕ and g be as in the corollary. The element $g \circ \phi \in \mathbb{k}[X]$ is called the pullback (slovene: povlek) of g under ϕ. We will denote it by $\phi^*(g)$.

Let $\phi: X \to Y$ be a polynomial map. Then we have a map $\phi^*: \mathbb{k}[Y] \to \mathbb{k}[X]$.

Lemma: ϕ^* is a homomorphism of $\mathbb{k}$-algebras.

Lemma: If $\phi: X \to Y$ and $\tau: Y \to Z$ are polynomial maps, then $(\tau \circ \phi)^* = \phi^* \circ \tau^*$.

Theorem: The map $\phi \longrightarrow \phi^*$ gives a bijection between the set of polynomial maps $X \to Y$ and the set of $\mathbb{k}$-algebra homomorphisms $\mathbb{k}[Y] \longrightarrow \mathbb{k}[X]$.

Proof: $\phi^*: \mathbb{k}[Y] \longrightarrow \mathbb{k}[X]$ is an algebra homomorphism by the lemma. We have to prove bijectivity.

<u>Injectivity</u>: Suppose $\phi, \psi : X \to Y$ are polynomial maps s.t. $\phi^* = \psi^* : \mathbb{k}[Y] \to \mathbb{k}[X]$.

$$\phi^*(g) = \psi^*(g) \quad \forall g \in \mathbb{k}[Y]$$
$$g \circ \phi = g \circ \psi \quad \forall g \in \mathbb{k}[Y]$$
$$g(\phi(a)) = g(\psi(a)) \quad \forall g \in \mathbb{k}[Y],\ \forall a \in X$$

Let $f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a))\ \forall a \in X$
$h_1, \dots, h_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\psi(a) = (h_1(a), \dots, h_m(a))\ \forall a \in X$

$\Rightarrow g(f_1(a), \dots, f_m(a)) = g(h_1(a), \dots, h_m(a))\ \forall g \in \mathbb{k}[Y]\ \forall a \in X$

For g we take projection to the i-th component:
$$f_i(a) = h_i(a) \quad \forall a \in X\ \forall i$$
$\Rightarrow (f_1(a), \dots, f_m(a)) = (g_1(a), \dots, g_m(a))\ \forall a \in X \Rightarrow \phi = \psi$

<u>Surjectivity</u>: Let $F : \mathbb{k}[Y] \to \mathbb{k}[X]$ be a homomorphism of k-algebras. We have to show that there is a polynomial map $\phi : X \to Y$ s.t. $F = \phi^*$.
$X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$
Then $\mathbb{k}[Y] = \mathbb{k}[x_1, \dots, x_m]/I(Y)$.
For $i = 1, \dots, m$ denote $\overline{x_i} = x_i + I(Y) \in \mathbb{k}[Y]$. As a function on Y, $\overline{x_i}$ is the projection to the i-th component.
Define $g_i := F(\overline{x_i}) \in \mathbb{k}[X]$ for $i = 1, \dots, m$. Consider g_i as polynomial functions on X
$$g : X \to \mathbb{k}$$
$$a \mapsto g_i(a)$$
Define $\phi : X \to \mathbb{A}^m$
$$a \mapsto (g_1(a), \dots, g_m(a))$$

<u>$\phi(X) \subseteq Y$</u>
Let $h \in I(Y)$ and $a \in X$ arbitrary. We have to show that $h(\phi(a)) = 0$.

$$h(\phi(a)) = h(g_1(a),\dots,g_m(a)) = h(g_1,\dots,g_m)(a) = 0$$

$$h(g_1,\dots,g_m) = h(F(\bar{x}_1),\dots,F(\bar{x}_m))$$

F homomorphism of algebras, h is a polynomial

$$= F(h(\bar{x}_1,\dots,\bar{x}_m))$$

$$= F(\underbrace{\underbrace{h(x_1,\dots,x_m)}_{\in I(Y)} + I(Y)}_{=0 \text{ in } \mathbb{k}[x_1,\dots,x_m]/I(Y) = \mathbb{k}[Y]}) = 0$$

$F = \phi^* \iff F(f) = \phi^*(f) \quad \forall f \in \mathbb{k}[Y]$.

F and ϕ^* are algebra homomorphisms, therefore it is enough to check the equality $F(f) = \phi^*(f)$ on the generators o $\mathbb{k}[Y]$, i.e. on $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$.

$F(\bar{x}_i) = \phi^*(\bar{x}_i)$

For each $a \in X$ we have $\phi^*(\bar{x}_i)(a) = \bar{x}_i(\phi(a))$

$\Rightarrow \bar{x}_i(g_1(a),\dots,g_m(a)) = g_i(a) = F(\bar{x}_i)(a)$.

projection to ith component

$\Rightarrow \phi^*(\bar{x}_i) = F(\bar{x}_i)$. ▨

Corollary: There is a contravariant functor

$\{\text{affine varieties}\} \longrightarrow \{\text{Finitely generated reduced } \mathbb{k}\text{-algebras}\}$

morphisms: polynomial maps

On objects: $X \longmapsto \mathbb{k}[X]$

On morphisms: $(\phi: X \to Y) \longrightarrow (\phi^*: \mathbb{k}[Y] \longrightarrow \mathbb{k}[X])$.

Moreover, the following version of surjectivity holds: If A is any finitely generated reduced $\mathbb{k}$-algebra, then there exists an affine variety X. s.t. $\mathbb{k}[X] \cong A$.

Proof: We already proved all properties of the functor. Let A be a finitely generated reduced $\mathbb{k}$-algebra. Let A be generated by $a_1, \dots, a_n$. Then we have an algebra homomorphism $\Bbbk[x_1, \dots, x_n] \longrightarrow A$, $x_i \mapsto a_i$ $\forall i$.

$\Rightarrow A \cong \Bbbk[x_1, \dots, x_n]/I$ for some ideal I

A is reduced $\Rightarrow I$ is a radical ideal

By the Nullstellensatz: $I = I_a(X)$ for some affine variety $X \subseteq \mathbb{A}^n$

$\overset{X}{\underset{V(I)}{\shortparallel}}$ $\Rightarrow A \cong \Bbbk[x_1, \dots, x_n]/I(X) = \Bbbk[X]$ ▨

Remark: The above functor induces a functor:

$$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of affine varieties} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{finitely generated reduced} \\ \Bbbk\text{-algebras} \end{array} \right\}$$

which is bijective on objects.

Another corollary of the theorem:

Corollary: Let $\phi: X \to Y$ be a polynomial map. Then ϕ is an isomorphism $\Leftrightarrow \phi^*: \Bbbk[Y] \to \Bbbk[X]$ is an isomorphism.
of affine varieties of algebras

Proof: ($\Rightarrow$): Suppose ϕ is an isomorphism. Then $\exists\, \psi: Y \to X$ polynomial map s.t. $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$.

By one of the lemmas before the theorem:

$$\phi^* \circ \psi^* = (\psi \circ \phi)^* = id_X^* = id_{\Bbbk[X]}$$

The same argument shows $\psi^* \circ \phi^* = id_{\Bbbk[Y]} \Rightarrow \phi^*$ is an isomorphism of algebras.

($\Leftarrow$): Assume that $\phi^*: \Bbbk[Y] \longrightarrow \Bbbk[X]$ is an isomorphism. Then $\exists$ an algebra homomorphism $F: \Bbbk[X] \longrightarrow \Bbbk[Y]$ s.t. $\phi^* \circ F = id_{\Bbbk[X]}$, $F \circ \phi^* = id_{\Bbbk[Y]}$.

By surjectivity in the theorem $\exists$ a polynomial map

$\psi : Y \longrightarrow X$ s.t. $F = \psi^*$.

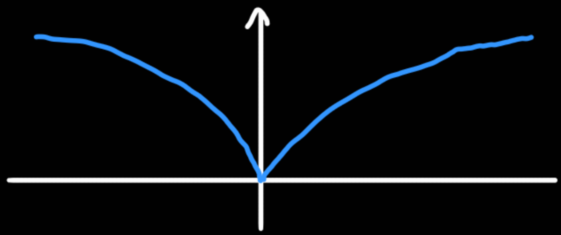
$$(\phi\circ\psi)^* = \psi^*\circ\phi^* = F\circ\phi^* = \mathrm{id}_{\Bbbk[Y]}$$

Also: $\mathrm{id}_Y^* = \mathrm{id}_{\Bbbk[Y]}$. By the injectivity of the theorem $\phi\circ\psi = \mathrm{id}_Y$. The same argument gives $\psi\circ\phi = \mathrm{id}_X$
$\Rightarrow \phi$ is an isomorphism of varieties. ∎

7. november 2025

<u>Example</u>: $X = V(x^2 - y^3) \subseteq \mathbb{A}^2$ $\quad$ $\phi : \mathbb{A}^1 \longrightarrow X$

$$t \longmapsto (t^3, t^2)$$



We will show that ϕ is not an isomorphism by showing that ϕ^* is not an isomorphism.

$$\phi^* : \Bbbk[X] \longrightarrow \Bbbk[\mathbb{A}^1] = \Bbbk[t]$$

We will show that it is not surjective. The image of ϕ^* is generated by the images of generators of $\Bbbk[X]$.

$\Bbbk[X] = \Bbbk[x,y]/_{(x^2,y^3)}$ is generated by $x + (x^2-y^3)$ and $y + (x^2-y^3)$.

What is $\phi^*(x + (x^2-y^3))$?

Geometrically, $x + (x^2-y^3)$ is a projection to the first component, so $\phi^*(x+(x^2-y^3)) = \phi^*(\pi_1) = \pi_1\circ\phi$.

For each $a \in \Bbbk$ we have $\pi_1(\phi(a)) = \pi_1(a^3, a^2) = a^3$

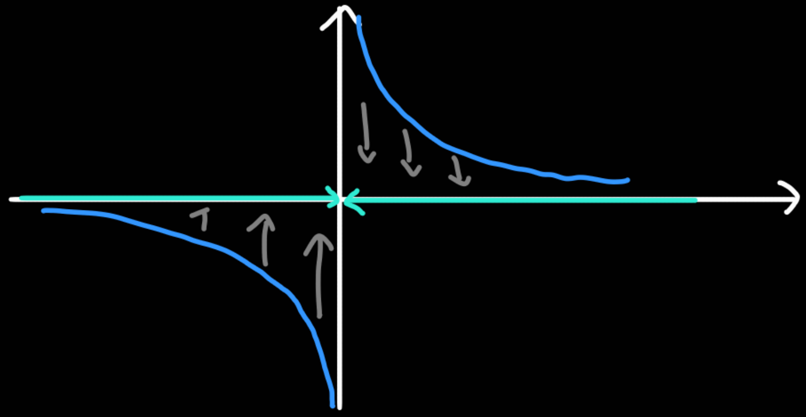
$\Rightarrow \phi^*(\pi_1) = \phi^*(x + (x^2-y^3)) = t^3$

Similarly $\phi^*(y + (x^2-y^3)) = t^2$

$\Rightarrow$ The image of ϕ^* is generated by t^3 and t^2, so it does not contain t. $\Rightarrow$ ϕ^* is not surjective $\Rightarrow \phi^*$ is not an isomorphism

Definition: A map $\phi: X \longrightarrow Y$ is *dominant* if the image $\phi(X)$ is dense in Y.

We can have dominant polynomial maps that are not surjective. For example, projection $V(xy-1) \longrightarrow \mathbb{A}^1$



Proposition: Let $\phi: X \longrightarrow Y$ be a polynomial map. Then
(1) ϕ^* is injective $\iff$ ϕ is dominant
(2) If ϕ^* is surjective, then ϕ is injective.

The previous example shows that the converse of (2) does not hold.

Proof: (1) ϕ^* is a ring homomomorphism, so ϕ^* is injective $\iff \ker\phi^* = \underline{0}$.
$g \in \ker\phi^* \iff \phi^*(g) = 0 \iff g \circ \phi = 0 \iff g(\phi(x)) = 0 \quad \forall x \in X$
$\iff g|_{\phi(X)} = 0$

($\Leftarrow$): Suppose that ϕ is dominant. Then $\overline{\phi(X)} = Y$. So, if g is zero on $\phi(X)$, then it is zero also on $\overline{\phi(X)} = Y$, because g is continuous $\Rightarrow g = 0$, so $\ker\phi^* = 0$

($\Rightarrow$): Suppose ϕ is not dominant. Then $\overline{\phi(X)}$ is a proper subvariety of Y, so we can write it as $\overline{\phi(X)} = V_Y(f_1, \dots, f_r)$

For some $f_1, \dots, f_r \in \mathbb{k}[Y]$. Then f_1 vanishes on $\overline{\phi(X)}$, so $f_1|_{\phi(X)} = 0$
$\Rightarrow f_1 \in \ker \phi^* \Rightarrow \phi^*$ is not injective.

(2) Let $a = (a_1, \dots, a_n)$, $a' = (a_1', \dots, a_n') \in X$, $a \neq a'$. Then $\exists i.\ a_i \neq a_i'$.
If π_i is the projection to the i-th component, then
$\pi_i : X \longrightarrow \mathbb{k}$, so $\pi_i \in \mathbb{k}[X]$ and $\pi_i(a) \neq \pi_i(a')$. By assumption, ϕ^* is surjective, so there exists $g \in \mathbb{k}[Y]$ s.t. $\pi_i = \phi^*(g) = g \circ \phi$
$g(\phi(a)) = \pi_i(a) \neq \pi_i(a') = g(\phi(a')) \Rightarrow \phi(a) \neq \phi(a')$. ▨

<u>Definition</u>: Let $X \subseteq \mathbb{P}^n$ be a projective variety. The homogeneous coordinate ring of X is $S(X) = \mathbb{k}[x_0, x_1, \dots, x_n] / I_p(X)$.

Two properties of $S[X]$:
- $\mathbb{k}[x_0, x_1, \dots, x_n]$ is noetherian, so $S[X]$ is noetherian.
- $\mathbb{k}[x_0, x_1, \dots, x_n]$ is a graded ring and $I_p(X)$ is a homogeneous ideal, so $S[X]$ is a graded ring.

<u>Definition</u>: Let $X \subseteq \mathbb{P}^n$ be a projective variety.
(1) For a homogeneous ideal $J \trianglelefteq S[X]$ we define
$V_X(J) = \{x \in X \mid f(x) = 0 \text{ for each homogeneous } f \in J\}$.
This is a projective subvariety of X.
(2) For each subset $Y \subseteq X$ we define the ideal of Y in $S[X]$ by $I_X(Y) = (f \in S[X] \text{ homogeneous} \mid f(x) = 0\ \forall x \in Y)$.

As in the affine case, we have:
- If Y is a subvariety, then $V_X(I_X(Y)) = Y$
- If $J \trianglelefteq S[X]$ is homogeneous and the radical of J is not the irrelevant ideal of $S[X]$, then $I_X(V_X(J)) = \sqrt{J}$

[Projective Relative Nullstellensatz]

2. Regular Functions

<u>Definition</u>: Let $X \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ an open subset. A regular function on U is a map $\phi: U \to \mathbb{k}$ such that for each $a \in U$ there exists an open neighbourhood U_a of a in U and there exist $p_a, g_a \in \mathbb{k}[x]$ such that $g_a(x) \neq 0$ for $x \in U_a$ and $\phi(x) = \frac{p_a(x)}{g_a(x)}$ for each $x \in U_a$.

The quotient $\frac{p_a(x)}{g_a(x)}$ is not necessarily globaly defined on U.

<u>Example</u>: $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$. This is an irreducible hypersurface in $\mathbb{A}^4$. This is a set of all 2×2 singular matrices. Let $U = X \setminus V(x_2, x_4) = \{(a_1,a_2,a_3,a_4) \in \mathbb{A}^4 \mid x_1x_4 = x_2x_3,\ a_2 \neq 0 \text{ or } a_4 \neq 0\}$ be the set of all singular matrices with the second column nonzero.

$$\phi: U \longrightarrow \mathbb{k}$$
$$(a_1,a_2,a_3,a_4) \longmapsto \begin{cases} \frac{a_1}{a_2}; & a_2 \neq 0 \\ \frac{a_3}{a_4}; & a_4 \neq 0 \end{cases}$$

This is a well defined map, because $\frac{a_1}{a_2} = \frac{a_3}{a_4}$ if $(a_1,a_2,a_3,a_4) \in X$ $a_2 \neq 0$ and $a_4 \neq 0$. It is a regular function, but neither $\frac{a_1}{a_2}$ nor $\frac{a_3}{a_4}$ is defined everywhere on U.

November 11, 2025

<u>Definition</u>: Let $X \subseteq \mathbb{P}^n$ be a projective variety and $U \subseteq X$ an open subset. A regular function on U is a map $\phi: U \to \mathbb{k}$ satisfying the following property: For each $a \in U$ there exists an open neighbourhood U_a of a in U and there exist homogeneous polynomials $f_a, g_a \in \mathbb{k}[x_0, x_1, \dots, x_n]$ of the same degree such that $g_a(x) \neq 0$ for each $x \in U_a$ and $\phi(x) = \frac{f_a(x)}{g_a(x)}$.

Definition is well-defined: If f_a, g_a are of degree d, then for each $\lambda \neq 0$ we have $\frac{f_a(\lambda x_0, \dots, \lambda x_n)}{g_a(\lambda x_0, \dots, \lambda x_n)} = \frac{\lambda^d f_a(x_0, \dots, x_n)}{\lambda^d g_a(x_0, \dots, x_n)}$, so the map $x \mapsto \frac{f_a(x)}{g_a(x)}$ is well defined.

Each affine variety $X \subseteq \mathbb{A}^n$ is an open subset of a projective variety $\overline{X} \subseteq \mathbb{P}^n$. Last time we had a definition of a regular function on an (open subset of an) affine variety. Is this definition equivalent to the above? Yes.

Suppose $X \subseteq \mathbb{A}^n \cong V_0 \subseteq \mathbb{P}^n$ is an affine variety, and $U \subseteq X$ is an open subset. Assume that $\phi : U \longrightarrow \mathbb{k}$ is a regular map according to the definition from last time. For each $a \in V$ there exists an open neighbourhood V_a of a in $U \subseteq V_0 \cong \mathbb{A}^n$ and there exist polynomials $f_a, g_a \in \mathbb{k}[x_1, \dots, x_n]$ such that $\phi(x) = \frac{f_a(x)}{g_a(x)}$ for each $x \in V_a$. Let $d = \max\{\deg f_a, \deg g_a\}$. Define $F(x_0, \dots, x_n) = x_0^d f_a\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$, $G(x_0, \dots, x_n) = x_0^d g_a\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$. Then F and G are homogeneous polynomials of the same degree d and for each $x = (1 : x_1 : \dots : x_n)$ we have

$$\frac{F(1, x_1, \dots, x_n)}{G(1, x_1, \dots, x_n)} = \frac{f_a(x_1, \dots, x_n)}{g_a(x_1, \dots, x_n)} = \phi(x)$$

$\Rightarrow \phi$ is regular according to the new definition.
The converse can be shown the same way, using dehomogenization.

We will often use only the second definition for quasiprojective varieties (= open subsets of projective varieties), this includes open subsets of affine varieties.

Sometimes we will also need regular functions on open subsets of varieties in $\mathbb{P}^n \times \mathbb{P}^m$. Let X be a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$ and V an open subset of X. A regular function on U is a map $\phi : U \longrightarrow \mathbb{k}$ such that for each $a \in V$ there exist an open neighbourhood V_a of a in U and polynomials $f_a, g_a \in \mathbb{k}[x_0, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous of the same degree in $x_0, \dots, x_m$ and homogeneous of the same degree in $y_0, \dots, y_n$ s.t. $\forall x \in U_a.\ g_a(x) \neq 0$ and $\phi(x) = \frac{f_a(x)}{g_a(x)}$.

Lemma: Let X be a quasiprojective variety and U an open subset of X. Then the set $\mathcal{O}_X(U)$ of all regular functions on U is a $\mathbb{k}$-algebra for point-wise operations.

Proof: The only question is why the sum and the product of regular functions is a regular function. We show this for the sum. Let $\phi_1, \phi_2 : U \longrightarrow \mathbb{k}$ be two regular functions. Let $a \in U$ be arbitrary. Then there exist open neighbourhoods V_1, V_2 of a in U and homogeneous polynomials of the same degree d_1, $f_1, g_1 \in \mathbb{k}[x_0, \dots, x_n]$ and homogeneous polynomials of the same degree d_2, $f_2, g_2 \in \mathbb{k}[x_0, \dots, x_n]$ s.t.

$$\phi(x) = \frac{f_1(x)}{\underbrace{g_1(x)}_{\neq 0}} \ \forall x \in V_1, \qquad \phi(x) = \frac{f_2(x)}{\underbrace{g_2(x)}_{\neq 0}} \ \forall x \in V_2.$$

Let $U_a = V_1 \cap V_2$. Then $g_1(x) g_2(x) \neq 0$ on U_a, and

$$\phi_1(x) + \phi_2(x) = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} = \frac{f_1(x) g_2(x) + f_2(x) g_1(x)}{g_1(x) g_2(x)},$$

and the numerator and denominator are homogeneous of the same degree $d_1 + d_2$.

Remark: In the definitions of regular maps, we allow that the numerator is 0.

$\Rightarrow \phi_1 + \phi_2$ is a regular map on U.
The same for $\phi_1 \phi_2$. ▨

Remark: $\mathcal{O}_X(U)$ is not necessarily finitely generated. The first counterexamples were constructed by Rees and Nagata. These counterexamples are among the few non-noetherian rings that we will consider, but we will never work with ideals of $\mathcal{O}_X(U)$.

Lemma: Let X be a quasiprojective variety and ϕ a regular function on X. Then the set $V(\phi) := \{x \in X \mid \phi(x) = 0\}$ is closed in X.

Proof: For each $a \in X$ there exists an open neighborhood V_a of a in X and homogeneous polynomials of the same degree f_a, g_a s.t. $g_a(x) \neq 0 \ \forall x \in V_a$ and $\phi(x) = \frac{f_a(x)}{g_a(x)} \ \forall x \in X$.

$$V_a \setminus V(\phi) = V_a \setminus V(f_a) = \underbrace{V_a}_{\text{open in } X} \cap \underbrace{(X \setminus V(f_a))}_{\text{open in } X}$$

$\Rightarrow V_a \setminus V(\phi)$ is open in X for each $a \in X$

$\Rightarrow \bigcup_{a \in X} (V_a \setminus V(\phi)) = X \setminus V(\phi)$ is open in X. $\Rightarrow V(\phi)$ is closed in X. ∎

Corollary: Let X be an irreducible quasiprojective variety, U an open subset of X and $\phi, \psi : X \longrightarrow \mathbb{k}$ regular functions that agree on U. Then $\phi = \psi$ on X.

Proof: $\phi(x) = \psi(x) \ \forall x \in U \Rightarrow U \subseteq V(\phi - \psi)$. By the lemma $V(\phi - \psi)$ is closed in X, so $\bar{U} \subseteq V(\phi - \psi)$. X is irreducible, so $\bar{U} = X$ and $X \subseteq V(\phi - \psi)$, which means $\phi(x) = \psi(x) \ \forall x \in X$. ∎

Let X be a quasiprojective variety and $U \subseteq V$ be an open subset of X. For each $\phi \in \mathcal{O}_X(V)$ the restriction $\phi|_U$ is a regular function on U, so $\phi|_U \in \mathcal{O}_X(U)$. So we have a restriction map $res_{V|U} : \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$

$$\phi \longmapsto \phi|_U$$

This map satisfies the following two properties:

- $res_{V|V} = id_{\mathcal{O}_X(V)}$
- If $U \subseteq V \subseteq W$ are open subsets of X, then $res_{U|V} \circ res_{W|V} = res_{W|U}$.

<u>Definition</u>: Let X be a topological space. A presheaf (slovene: predsnop) $\mathcal{F}$ on X consists of the following data:

(1) For each open subset $U \subseteq X$ we have a set $\mathcal{F}(U)$. Its elements are called sections on U. If $U = X$, they are called global sections.

(2) For each pair of open subsets $U, V \subseteq X$ satisfying $U \subseteq V$ we have a map $res_{V|U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$, called restriction, which satisfies the following two properties:

(a) $res_{U|U} = id_{\mathcal{F}(U)}$ for each open set U in X.

(b) If $U \subseteq V \subseteq W$ are open subsets of X, then
$$res_{W|U} = res_{V|U} \circ res_{W|U}.$$

We showed that $\mathcal{O}_X$ is a presheaf on X: For each open U we defined the set $\mathcal{O}_X(U)$ of regular functions on U, res is the usual restriction and (a), (b) are satisfied.

<u>Definition</u>: Let X be a topological space. A sheaf (slovene: snop) is a presheaf $\mathcal{F}$ on X satisfying the gluing property (lastnost lepljenja): For each open subset $U \subseteq X$ and each open cover $\{U_i \mid i \in I\}$ of U and each collection of sections $\phi_i \in \mathcal{F}(U_i)$ $(i \in I)$ satisfying
$$res_{U_i|U_i \cap U_j}(\phi_i) = res_{U_j|U_i \cap U_j} \qquad \forall i, j$$
there exists $\phi \in \mathcal{F}(U)$ s.t. $res_{U|U_i}(\phi) = \phi_i$.

Let's check that $\mathcal{O}_X$ is a sheaf. Let $U \subseteq X$ be an open subset, $\{U_i \mid i \in I\}$ open cover of U and for each i let ϕ_i be a regular function on U_i ($\phi_i \in \mathcal{O}_X(U_i)$), such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ $\forall i,j$. Because of the last equality we have a well defined function $\phi : U \longrightarrow \mathbb{k}$, $x \in U_i$, $x \longmapsto \phi_i(x)$. We have to show that ϕ is a regular function. Let $a \in U$ be an arbitrary point. Then $a \in U_i$ for some i. ϕ_i is a regular function on U_i, so there exists an open neighborhood $V_a \subseteq U_i$ and homogeneous polynomials of the same degree f_a, g_a s.t. $g_a(x) \neq 0$ for $x \in V_a$ and $\phi_i(x) = \frac{f_a(x)}{g_a(x)}$ $\forall x \in V_a$. Then V_a is also open in U and $\phi(x) = \frac{f_a(x)}{g_a(x)}$ $\forall x \in V_a$.
$\Rightarrow \phi$ is a regular function on U. It is also clear $\phi|_{U_i} = \phi_i$ $\forall i$.
$\Rightarrow \mathcal{O}_X$ is a sheaf on X. We call it the structure sheaf of X.
strukturni snop

We can define sheaves and presheaves in a categorical way. Let X be a topological space. Consider the category $\mathcal{C}$ of all open subsets of X. If $U \subseteq V$ are open subsets of X, then there is a unique morphism from U to V: $U \hookrightarrow V$. If $U \not\subseteq V$, then the set of morphisms from U to V is empty. The compositions are defined in the obvious way.
Then a presheaf on X is a contravariant functor from $\mathcal{C}$ to Set, and a sheaf is a contravariant functor satisfying an additional (gluing) property.
We could also look at contravariant functors from $\mathcal{C}$ to categories of groups, rings, modules,... We get (pre)sheaves of groups, rings, modules,... This means that $\mathcal{F}(U)$ is a group/ring/module... for each open subset U of X and that the restriction maps are homomorphisms of groups/rings/modules...
We showed that $\mathcal{O}_X(U)$ is a ring for each open subset

U of a quasiprojective variety X. The restrictions are ring homomorphisms $\Rightarrow \mathcal{O}_X$ is a sheaf of rings on the quasi-projective variety X.

This will be important later, when we will consider schemes.

Recall that a distinguished open subset of an affine variety $X \subseteq \mathbb{A}^n$ is a set of the form $D(f) := \{x \in X \mid f(x) \neq 0\}$ where $f \in \Bbbk[X]$

Theorem: Let X be an affine variety and $f \in \Bbbk[X]$. Then:

(1) $\mathcal{O}_X(X) = \Bbbk[X]$.

(2) $\mathcal{O}_X(D(f)) = \{ \frac{g}{f^m} \mid g \in \Bbbk[X],\ m \in \mathbb{N}_0 \}$.

In particular, regular functions on a distinguished open set an affine variety are everywhere defined quotients of two polynomial functions.

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Proof: (1) is a special case of (2) if we take $f=1$. We have to prove only (2).

$(\supseteq)$: If $x \in D(f)$, then $f(x)^m \neq 0$, so $\frac{g(x)}{f(x)^m}$ is well defined, and $x \mapsto \frac{g(x)}{f^m(x)}$ is clearly a regular function on $D(f)$. This proves the second part of (2): $x \longmapsto \frac{g(x)}{f(x)^m}$ is everywhere defined on $D(f)$.

$(\subseteq)$: Let $\phi : D(f) \longrightarrow \Bbbk$ be a regular function. For each $a \in D(f)$ there exists an open neighborhood V_a of a in $D(f)$ and there exist polynomial functions $p_a, q_a \in \Bbbk[X]$ such that for each $x \in V_a$ we have $q_a(x) \neq 0$ and $\phi(x) = \frac{p_a(x)}{q_a(x)}$.

We first make some reductions so that we will get nicer V_a, p_a, q_a.

The assumptions do not change if we take a smaller neighbourhood. Each open set is a union of distinguished open sets, therefore we may assume that V_a is a distinguished open set, so of the form $V_a = D(r_a)$ for some $r_a \in \mathbb{k}[X]$.

On $D(r_a)$ we have $r_a(x) \neq 0$, so $\phi(x) = \frac{p_a(x)}{q_a(x)} = \frac{p_a(x) r_a(x)}{q_a(x) r_a(x)}$ for all $x \in D(r_a)$. We can replace p_a by $p(a)r(a)$ and $q(a)$ by $q(a)r(a)$ and the assumptions still hold, so we may assume that $p_a(x) = q_a(x) = 0$ for $x \in V_X(r_a) = X \setminus D(r_a)$.

$\Rightarrow D(r_a) = D(q_a)$, $V_X(r_a) = V_X(q_a)$.

Let $a, b \in D_f$ be different points. We decompose $D(f)$ as a union $D(f) = (D(r_a) \cap D(r_b)) \cup (D(r_a) \cap V_X(r_b)) \cup (V_X(r_a) \cap D(r_b)) \cup$

$$\cup \left(D(f) \cap V_X(r_a) \cap V_X(r_b)\right)$$

If $x \in D(r_a) \cap D(r_b)$, then

$$\phi(x) = \frac{p_a(x)}{q_a(x)} = \frac{p_b(x)}{q_b(x)} \Rightarrow p_a(x) q_b(x) = p_b(x) q_a(x).$$

The equality also holds on $V_X(r_a)$ and on $V_X(r_b)$, since by the last reduction we have $p_a(x) = q_a(x) = 0$ on $V_X(r_a)$ and $p_b(x) = q_b(x) = 0$ on $V_X(r_b)$. So we have $p_a(x) q_b(x) = p_b(x) q_a(x)$ for each $x \in D(f)$. For each $a \in D(f)$, $U_a = D(r_a)$ is a neighbourhood of a in $D(f)$, so $D(f) = \bigcup_{a \in D(f)} D(r_a)$ $/^c$

$$V_X(f) = \bigcap_{a \in D(f)} V_X(r_a) = \bigcap_{a \in D(f)} V_X(q_a) = V_X\Big(\bigcup_{a \in D(f)} \{q_a\}\Big) = V_X(\{q_a \mid a \in D(f)\}).$$

We apply $I_X(\cdot)$ to this equality:

$$I_X(V_X(f)) = I_X(V_X(\{q_a \mid a \in D(f)\})) = \sqrt{(q_a \mid a \in D(f))}$$

relative Nullstellensatz

$f \in I_X(V_X(f)) \Rightarrow \exists m \in \mathbb{N}.\ f^m \in (q_a \mid a \in D(f))$.

By the definition of an ideal generated by some set there exist finitely many elements q_a and $h_a \in \mathbb{k}[X]$ such that $f^m = \sum_{\substack{\text{finite} \\ a \in A}} h_a q_a$. We define $g = \sum_{a \in A} h_a p_a$.

So we have $p_a(x) g_b(x) = p_b(x) g_a(x)$ for each $x \in D(f)$. We have to show that $\phi(x) = \frac{g(x)}{f(x)^m}$ for each $x \in D(f)$. Let $b \in D(f)$ be arbitrary. For all $x \in V_b = D(r_b)$ we have $\phi(x) = \frac{p_b(x)}{g_b(x)}$.

$$g(x) g_b(x) = \sum_{a \in A} h_a(x) p_a(x) g_b(x) = \sum_{a \in A} h_a(x) g_a(x) p_b(x) = f(x)^m p_b(x)$$

On $D(r_b)$ we have $g_b(x) \neq 0$ and $f(x) \neq 0$, so $\frac{g(x)}{f(x)^m} = \frac{p_b(x)}{g_b(x)} = \phi(x)$ for all $x \in V_b$. Since $b \in D(f)$ was arbitrary, we get $\phi(x) = \frac{g(x)}{f(x)^m}$ for all $x \in D(f)$. $\blacksquare$

<u>Definition</u>: Let R a (commutative) ring. Let S be a set that is multiplicatively closed ($a \in S, b \in S \Rightarrow ab \in S$) and contains 1. On $R \times S$ we define a relation

$$(a, b) \sim (b, t) :\Longleftrightarrow \exists u \in S.\ u(at - bs) = 0$$

This is an equivalence relation. We denote the equivalence class $[(a,s)]$ with $\frac{a}{s}$, and we denote the quotient set $(R \times S)/_\sim$ by $S^{-1}R$.

We define adition and multiplication on $S^{-1}R$ by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

We can check that these operations are well defined and that $S^{-1}R$ is a ring for these operations.
We call it the ring of fractions of R.

<u>Remark</u>: If R is noetherian, then $S^{-1}R$ is noetherian.

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<u>Examples</u>: 1) If R is a domain (without zero-divisors) and $S = R \setminus \{0\}$, then $S^{-1}R$ is the field of fractions of R.
2) Let $a \in R$ be an element that is not nilpotent and $S = \{1, a, a^2, \dots\}$.
Then S is multiplicatively closed and we can define $S^{-1}R$.

$$S^{-1}R := \left\{ \frac{x}{a^n} \mid x \in R,\ n \in \mathbb{N}_0 \right\}.$$

This ring is usually denoted by R_a or $R[\frac{1}{a}]$. We will use the notation $R[\frac{1}{a}]$.
Concrete example: $R=\mathbb{Z}$, $a=2 \Rightarrow S^{-1}R$ are all fractions where the denominator is a power of 2.
Remark: We assume that a is not nilpotent, as otherwise $S^{-1}R$ is a trivial ring.
3) If $p \trianglelefteq R$ is a prime ideal, then $S=R\setminus p$ is a multiplicatively closed set with 1. The ring of fractions $S^{-1}R$ is in this case denoted by R_p and called the localization of R at p.
Prime ideals in R_p are of the form $S^{-1}Q = \{\frac{a}{s} \mid a\in Q, s\in S=R\setminus p\}$ where Q is a prime ideal of R and contained in P.
$\Rightarrow S^{-1}P$ is a maximal ideal.
The set of all elements from S are invertible in $S^{-1}R$
$\Rightarrow S^{-1}P = P_p$ is the unique maximal ideal of R_p.

Definition A ring is local if it has a unique maximal ideal.

Example: Localizations of R at prime ideals of R are local rings.

Corollary: Let X be an affine variety and $f\in \mathbb{k}[X]$. Then $\mathcal{O}_X(D(f)) \cong \mathbb{k}[X][\frac{1}{f}]$.

Proof: We define a map
$$\Phi: \mathbb{k}[X][\tfrac{1}{f}] \longrightarrow \mathcal{O}_X(D(f))$$
$$\frac{g}{f^m} \longmapsto \left(x \longmapsto \frac{g(x)}{f(x)^m}\right)$$

If $x\in D(f)$, then $f(x)^m \neq 0$, so $\frac{g(x)}{f(x)^m}$ is defined, so $x \mapsto \frac{g(x)}{f(x)^m}$ is a regular function on $D(f)$.
We have to show that Φ is well defined. Suppose we have $\frac{g}{f^m} = \frac{h}{f^n}$ in $\mathbb{k}[X][\frac{1}{f}]$. $S=\{1, f, f^2, \dots\}$
By definition then there exists $k\in\mathbb{N}_0$ such that $(gf^n - hf^m)f^k = 0$

in $\Bbbk[X] \Rightarrow (g(x)f(x)^n - h(x)f^m(x))\cdot f(x)^k = 0 \quad \forall x \in X$.
If $x \in D(f)$, then $f(x) \neq 0$ and we get

$$g(x)f(x)^n = h(x)f^m(x) \quad \Rightarrow \quad \frac{g(x)}{f(x)^m} = \frac{f(x)}{f(x)^n} \quad \forall x \in D(f)$$

$\Rightarrow \Phi$ is well defined.
Clearly Φ is a homomorphism of $\Bbbk$-algebras. It is surjective by the theorem from last time.
Injectivity of Φ:
Assume that $\frac{g}{f^m} \in \ker \Phi$. This means $\frac{g(x)}{f(x)^m} = 0 \ \forall x \in D(f) \Rightarrow$
$\Rightarrow g(x) = 0 \ \forall x \in D(f) \Rightarrow g(x)\cdot f(x) = 0 \ \forall x \in X \Rightarrow gf = 0$ in $\Bbbk[X]$.

$$(g\cdot 1 - 0 \cdot f^m)\cdot f = 0 \quad \text{in } \Bbbk[X].$$

By the definition of fractions we get $\frac{g}{f^m} = \frac{0}{1}$ in $\Bbbk[X][\frac{1}{f}]$.
$\Rightarrow$ The kernel is trivial $\Rightarrow \Phi$ is injective. ∎

3. Regular maps

<u>Definition 1</u>: Let X be a quasiprojective variety, $Y \subseteq \mathbb{A}^n$ an affine variety and $U \subseteq Y$ an open subset. The maps $\phi: X \to U$ is a regular map if there exist regular functions $\phi_1, \dots, \phi_n$ on X such that $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$ for each $x \in X$.

<u>Lemma</u>: Let X be a quasiprojective variety, $Y \subseteq \mathbb{A}^n$ an affine variety, $U \subseteq Y$ an open subset and $\phi: X \to U$ a regular map. Then ϕ is continuous in the Zariski topology.

The proof is the same as in the case of a polynomial, the only difference is that we use the fact that $V(\psi) = \{x \in X \mid \psi(x) = 0\}$ is closed in X if ψ is a regular function on X.

In the case when $Y \subseteq \mathbb{P}^n$ is a quasiprojective variety we cannot define a regular map $X \to Y$ as an $(n+1)$-tuple of regular functions, because we have to be carefull about common zeros.

<u>Definition 2:</u> Let X, Y be quasiprojective varieties, $Y \subseteq \mathbb{P}^n$. A map $\phi: X \longrightarrow Y$ is called a regular map if for each $a \in X$ the following holds: For some index $i \in \{0, 1, \dots, n\}$ satisfying $\phi(a) \in U_i = \mathbb{P}^n \setminus V_p(x_i)$ there exists an open neighbourhood V of a in X such that $\phi(V) \subseteq U_i$ and the restriction $\phi|_V: V \longrightarrow U_i$ is regular according to the previous definition.

<u>Remark 1:</u> The definition is independent of the chosen index i: Let $a \in X$ be such that $\phi(a) \in U_i \cap U_j$. We use the definition for i: there exists an open neighbourhood U of a in X such that $\phi(U) \subseteq U_i$ and $\phi|_U: U \longrightarrow U_i \cong A_n$ is according to Def. 1. By the previous lemma this restriction is continuous, so $U' = (\phi|_U)^{-1}(U_i \cap U_j)$ is open in U. $\phi(U') = (\phi|_U)(U') \subseteq U_j$. We want to show that $\phi|_{U'}: U' \longrightarrow U_j \cong A^n$ is a regular map according to Definition 1. $\phi|_U$ is regular according to Definition 1, so there exist regular functions $\phi_0, \dots, \phi_n$ on U such that ϕ_i constantly equal to 1 and $\phi(x) = (\phi_0(x) : \dots : \phi_n(x))$ for each $x \in U$. $\phi(U') \subseteq U_j$, so $\phi_j(x) \neq 0 \; \forall x \in U'$, and for each $x \in U'$ we have

$$\phi|_{U'}(x) = \left(\frac{\phi_0(x)}{\phi_j(x)} : \dots : \frac{\phi_n(x)}{\phi_j(x)}\right),$$

$\frac{\phi_j(x)}{\phi_j(x)} = 1$, so $\phi|_{U'}: U' \longrightarrow U_j \cong A^n$ is regular according to Def. 1.
$\Rightarrow$ Definition 2 is independent of i.

<u>Remark 2:</u> If Y is an (open subset of an) affine variety, then Definition 2 is equivalent to Definition 1.

Lemma: Regular maps are continuous in the Zariski topology.

Proof: Let X and $Y \subseteq \mathbb{P}^n$ be quasiprojective varieties and $\phi: X \to Y$ a regular map. Let Z be a closed subset of Y. We have to show that $\phi^{-1}(Z)$ is closed in X.
By the definition of a regular map for each $x \in X$ there exists an index i and an open neighborhood U_x of x in X such that $\phi(U_x) \subseteq U_i$ and $\phi|_{U_x}: U_x \longrightarrow U_i \cong \mathbb{A}^n$ is regular according to Definition 1. For this restriction we can use the result that a regular map to an affine variety is continuous, so
$(\phi|_{U_x})^{-1}(U_i \cap Z) = U_x \cap \phi^{-1}(Z)$ is closed in U_x.
$\Rightarrow U_x \setminus \phi^{-1}(Z)$ is open in U_x, so also open in X.
$\Rightarrow \bigcup_{x \in X} (U_x \setminus \phi^{-1}(Z)) = X \setminus \phi^{-1}(Z)$ is open in X
$\Rightarrow \phi^{-1}(Z)$ is closed in X. ▨

Corollary: Let X, Y be quasiprojective varieties, $Y \subseteq \mathbb{P}^n$. Then a map $\phi: X \to Y$ is regular $\Leftrightarrow$ it is continuous and the restriction $\phi^{-1}(U_i) \longrightarrow U_i \cong \mathbb{A}^n$ is regular by Def. 1 for each $i = 0, 1, \dots, n$.

Proposition: Let $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ be quasiprojective varieties. A map $\phi: X \longrightarrow Y$ is regular $\Leftrightarrow$ for each $a \in X$ there exists an open neighbourhood U_a of a in X and polynomials $f_0, \dots, f_n \in \mathbb{k}[x_0, \dots, x_n]$ that are homogeneous of the same degree such that for each $x \in U_a$ we have $f_i(x) \neq 0$ for at least one i and $\phi(x) = (f_0(x) : \dots : f_n(x))$.

Proof: ($\Rightarrow$): Let ϕ be a regular map and $a \in X$ arbitrary. Then there exists i s.t. $\phi(a) \in U_i$. By the definition of a regular map there exists an open neighbourhood U_a of a in X and regular functions $\phi_0, \dots, \phi_n$ on U_a with ϕ_i constantly 1 such that $\phi(x) = (\phi_0(x) : \dots : \phi_{i-1}(x) : 1 : \phi_{i+1}(x) : \dots : \phi_n(x))$ for each $x \in U_a$.

By the definition of regular functions there exists an open neighbourhood U_a' of a in U_a and there exist homogeneous polynomials $g_0, \dots, g_n, h_0, \dots, h_n \in \mathbb{k}[x_0, \dots, x_m]$, g_j and h_j of the same degree $\forall j$ such that $\phi_j = \frac{g_j(x)}{h_j(x)}$ $\forall x \in U_a$.

$$\Rightarrow \phi(x) = \left(\frac{g_0(x)}{h_0(x)} : \dots : \frac{g_{i-1}(x)}{h_{i-1}(x)} : 1 : \frac{g_{i+1}(x)}{h_{i+1}(x)} : \dots : \frac{g_n(x)}{h_n(x)}\right) \quad \forall x \in U_a'.$$

We clear the denominators and get polynomials $f_0, \dots, f_n \in \mathbb{k}[x_0, \dots, x_n]$ such that $\phi(x) = (f_0(x) : \dots : f_n(x))$ $\forall x \in U_a'$.

One can also check that $f_0, \dots, f_n$ don't have common zeroes on U_a'.

$\Leftarrow$: Similarly. $\blacksquare$

Corollary: Let $X \subseteq \mathbb{P}^n$ be a projective variety, $f_0, \dots, f_n \in \mathbb{k}[x_0, \dots, x_n]$, $U = X \setminus V_p(f_0, \dots, f_n)$ and $\phi(x) = (f_0(x) : \dots : f_n(x))$ for $x \in U$. Then $\phi : U \to \mathbb{P}^n$ is a regular map.

In general we cannot assume that polynomials $f_0, \dots, f_n$ from the proposition are defined globally on X, even if X is a projective variety.

<u>Example:</u> $X = \{(x:y:z) \in \mathbb{P}^2 \mid x^2 + y^2 = z^2\} \subseteq \mathbb{P}^2$

$$\phi : X \longrightarrow \mathbb{P}^1$$

$$(x:y:z) \longmapsto \begin{cases} (x : y-z) & \text{if } (x:y:z) \neq (0:1:1) \\ (y+z : x) & \text{if } (x:y:z) \neq (0:1:-1) \end{cases}$$

The definitions agree on the intersection, so ϕ is a regular map. Neither of the expressions is defined on the entire X.