Operator theory

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PART I

Compact and Fredholm operators

Preliminaries

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Def: (X, ρ) - metric space if X - set, and ρ is a metric:

i) $\rho(x,y) \geq 0 \quad \forall x,y \in X$. $\rho(x,y) = 0 \Leftrightarrow x = y$

ii) $\rho(x,y) = \rho(y,x) \quad \forall x,y \in X$

iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \forall x,y,z \in X$

Def: $U \subseteq X$ is open if $\forall x \in U$. $\exists \delta > 0$. s.t. $B(x,\delta) \subset U$ $\quad (B(x,\delta) = \{y \in X \mid \rho(x,y) < \delta\})$

Def: $K \subset X$ is compact if every open cover $\{U_\alpha\}_{\alpha \in I}$ of K has a finite subcover.

cover: $\{U_\alpha\}_\alpha$ is a cover of K if $\bigcup_{\alpha \in I} U_\alpha \supset K$)

Def: A precompact set $A \subset X$ is a set $A \subset X$ s.t. $\underset{\text{closure of } A \text{ in } X}{\overline{A}}$ is compact.

Def: $\{x_j\}_{j\geq 1}$ is **Cauchy sequence** in X if
$\forall \varepsilon > 0.\ \exists N = N(\varepsilon).\ \rho(x_j, x_k) < \varepsilon\ \forall j,k \geq N(\varepsilon).$

Def: X is **complete** if $\forall$ Cauchy $\{x_j\}_{j\geq 1} \subset X.$
$\exists x \in X.\ \rho(x_j, x) \to 0$ as $j \to \infty$
(Every Cauchy sequence converges.)

Ex. $\left(\mathbb{R}^n,\ \rho_{\mathbb{R}^n}(\{x_i\},\{y_j\}) := \sqrt{\sum |x_i - y_i|}\right)$ – complete metric space

Ex. $(\mathbb{Q},\ \rho_{\mathbb{R}^n})$ – metric space but non-complete

Ex. $[0,1]$ is a compact-subset of $(\mathbb{R}, \rho_{\mathbb{R}})$

Th: $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ closed and bounded

Def: $A \subset (X,\rho)$ is bounded if $\exists x \in X.\ \exists R > 0.\ A \subset B(x,R)$

Ex.: $(X,\rho) = \ell^2(\mathbb{Z}) = \left(\left\{\{\underset{\mathbb{C}}{x_j}\}_{j\in\mathbb{Z}} \mid \sum_{j\in\mathbb{Z}} |x_j|^2 < \infty\right\},\right.$

$$\rho(\{x_j\},\{y_j\}) = \sqrt{\sum_{j\in\mathbb{Z}} |x_j - y_j|^2}$$

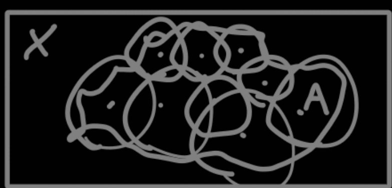
$$B[0,1] = \{y \in \ell^2(\mathbb{Z}) : \rho(0,y) \leq 1\}$$

– bounded, closed but not compact HW

Theorem: Let (X,ρ) be a complete metric space, $A \subset X$. The following assertions are equivalent:

i) A is precompact

ii) $\forall \varepsilon > 0.\ \exists_N$ a finite ε-net $\{x_j\}_{j=1}^{N_\varepsilon}$ in A, that is, $\bigcup_{j=1}^{N} B(x_j, \varepsilon) \supset A.$



iii) $\forall \{x_j\}_{j\geq 1} \subset A$ there is a converging subsequence to some element $x \in X$.

<u>Proof</u>: i) $\Rightarrow$ ii) $\{U_x\}_{x\in A} = \{B(x,\varepsilon)\}_{x \in A}$ - open cover of A.

- is an open cover of $\bar{A}$:

$(\forall y \in \bar{A}.\ \exists x \in A.\ \rho(x,y) < \varepsilon/2 \Rightarrow y \in B(x,\varepsilon))$

definition of closure

$\Rightarrow \exists \ \{U_{x_j}\}_{j=1}^{N}$ - finite subcover of $\bar{A}$

$\Rightarrow \ \{U_{x_j}\}_j^N$ is a ε-net in A

ii) $\Rightarrow$ iii) Observe that $\forall \varepsilon > 0$. any sequence $\{y_i\} \subset A$ has an infinite subsequence that is contained in some $B(x,\varepsilon)$. (we have a finite ε-net)

Assume that $\{y_k\}$ is arbitrary in A.

Cantor diagonal process

$y_1, \ y_2, \ y_3, \ y_4, \dots$

$y_1, \ y_{k_2}, \ y_{k_3}, \dots$ where $\{y_{k_j}\} \subset B(x_2, \tfrac{1}{2})$

$y_1, \ y_{k_2}, \ y_{kj_3}, \ y_{kj_4}$ where $\{y_{kj_s} \subset B(x_3, 1/3)$ $\in X$

$y_1, \ y_{k_2}, \ y_{kj_3}, \ y_{kjs_4}$ ...

Consider $z_1 = y_1$

$z_2 = y_{k_2}$

$z_3 = y_{kj_3}$

$z_4 = y_{kjs_4}$

$\vdots$

<u>Claim</u>: $\{z_j\}$ is a Cauchy sequence.

Indeed $\underset{k<j}{}$ $\rho(z_j, z_k) < \frac{1}{k}$

because $z_j, z_k \in B(x_k, \frac{1}{k})$ $\rho(z_j, z_k) < \frac{2}{K} \Rightarrow$ ok
$\to \infty$

X is complete $\Rightarrow \{z_j\}$ converges

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<u>iii) ⇒ i)</u>:

Plan: a) A contains a dense countable subset

a)+b)+c) under assumption 3)

b) If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $\bar{A}$
$\Rightarrow \exists \{U_{\alpha_j}\}_{j \in J}$ - an open coutable subcover of $\bar{A}$

c) $\Rightarrow \{U_{\alpha_j}\}_{j=1}^{r}$ is a cover of $\bar{A}$

a) Observe that $\forall \varepsilon > 0$ there exists at most $N(\varepsilon)$ points
$A \supset \{y_j(\varepsilon)\}_{1 \le j \le N(\varepsilon)}$ s.t. $\rho(y_k(\varepsilon), y_j(\varepsilon)) > \varepsilon \;\; \forall k \neq j$.
(If this is not true, then $\exists \{y_j(\varepsilon)\}_{j=1}^{\infty}$ such that
$\rho(y_j(\varepsilon), y_k(\varepsilon)) > \varepsilon$ and it cannot contain a convergent subsequence by Cauchy criterion.)

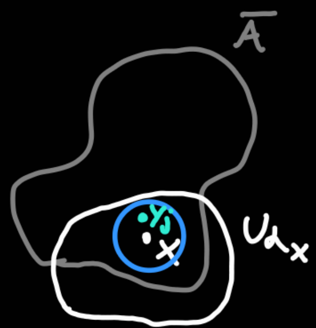
Now $E = \left\{ y_k\{\frac{1}{n}\} \;\middle|\; 1 \le k \le N(\frac{1}{n}),\; n \ge 1 \right\}$ is a dense countable subset.
(E is dense since $\forall n . \forall x \in A. \; \min\limits_{1 \le k \le N(\frac{1}{n})} (\rho(x, y_k(\frac{1}{n})) \le \frac{1}{n}$ by construction)

b) Assume that $\{U_\alpha\}_{\alpha \in I}$ is some open cover of $\bar{A}$.
For every $x \in \bar{A}$ define

$$\varepsilon(x) \overset{(*)}{:=} \frac{\sup\{\varepsilon > 0 \mid B(x, \varepsilon) \subset U_\alpha \text{ for some } \alpha\}}{2} > 0$$

<u>Claim</u>: if $\{y_j\}_{j=1}^{\infty}$ is a countable dense subset in A, then

$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$.



Take $x \in \bar{A}$ $\exists d_x.\ x \in U_{d_x}$, and let $\eta(x) > 0$ such that $B(x, \eta(x)) \subset U_{d_x}$ (U_{d_x} is open $\Rightarrow \eta(\varepsilon)\ \exists$)
Find y_j such that $\rho(x, y_j) < \frac{\eta(x)}{10}$.
Then, since $\eta(x) \leq 2\varepsilon(x)$, $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow \rho(x, y_j) < \varepsilon(y_j)$

$\varepsilon(y_j) \geq \frac{\eta(x)}{10}$ - Because $B(y_j, \frac{\eta(x)}{5}) \subset U_{d_x}$ by triangle inequality and $\varepsilon(y_j)$ satisfies $(*)$.

Since $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, then $\{U_{d_{y_j}}\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, where $U_{d_{y_j}}$ is the set U_d from the definition of $\varepsilon(y_j)$
(that is, $U_{d(y_j)} \supseteq B(y_j, \varepsilon(y_j))$).
Since we have $(**)$, $\bigcup_{j=1}^{\infty} U_{y_{d_i}} \supset \bar{A}$.

c) <u>Claim</u>: $\exists N.\ \{U_{d_{y_j}}\}_{j=1}^{N}$ is a cover of $\bar{A}$.

Suppose this is not the case $\Rightarrow \forall j \geq 1.\ \exists x_j \in A_j \setminus \bigcup_{k=1}^{j} U_{y_j}$.
Consider $\{x_j\}_{j=1}^{\infty}$, and assume that the sequence $\{x_{j_k}\}$ - convergent to some $x \in X$. Note that $x \in \bar{A}$ $(x_j \in \bar{A})$.
$\Rightarrow \exists j_*.\ x \in U_{d_{j_*}} \Rightarrow \exists \delta > 0$. s.t. $B(x, \delta) \subset U_{d_{j_*}}$, but $x_{j_k} \notin U_{d_{j_*}}$ for large k by construction.
(in particular, $x_{j_k} \notin B(x, \delta)$, hence $\rho(x, x_{j_k}) > \delta$, but this contradicts the fact that $x_{j_k} \longrightarrow x$).
We have shown that $\{x_j\} \subset \bar{A}$ cannot have a convergent subsequence.
Then if $\tilde{x}_j \in A.\ \rho(\tilde{x}_j, x_j) < \frac{1}{j}$, then $\{\tilde{x}_j$ also has no convergent subsequence. So, we assumed there is no

finite subcover $\{U_{\alpha_{y_j}}\}$ and found a sequence $\{\tilde{x}_j\} \subset A$ that has no converging subsequence, a contradiction with 3). Therefore 3) $\Rightarrow$ 1). $\blacksquare$

Examples of compact sets and their properties:

1) $K \subset (X, \rho)$ is compact $\Rightarrow$ K is bounded

Indeed, if K is not bounded, then $\{B(x, n)\}_{n \geq 1}$ is an open cover without a finite subcover.

2) $K \subset (X, \rho)$ is compact, then it is closed
($\Leftrightarrow$ $\{x_j\} \subset K$ such that $x_j \to x$ in (X, ρ) we also have $x \in K$)

Let's check that $X \setminus K$ is open. Take $y \in X \setminus K$, take $x \in K$, let $\delta(x) > 0$. $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$
$\{B(x, \delta(x))\}_{x \in K}$ is an open cover, let $\{B(x_j, \delta(x_j))\}_{j=1}^{N}$ be a finite subcover, then $\delta := \min_{1 \leq j \leq K} \delta(x_j)$, $B(y, \delta) \cap K = \emptyset$
$\Rightarrow$ $X \setminus K$ is open.

Another proof: Suppose $\{y_j\} \subset K$ s.t. $y_j \to y$, $y \notin K$.

$$U_j = \{x \in X \mid \rho(x, y) > \frac{1}{j}\}$$

$\{U_j\}_{j=1}^{\infty}$ open cover, $\{U_{j_k}\}_{k=1}^{N}$ finite subcover

$$\varepsilon := \min_{1 \leq k \leq N} \left(\frac{1}{j_k}\right), \quad \rho(x, y) > \varepsilon \quad \forall x \in K \quad \text{contradiction} \quad \blacksquare$$

linear space = vector space

3) Let X be a finite-dimensional complete linear normed space. Then $E \subset X$ is compact $\Leftrightarrow$ E is closed and bounded.

($\Rightarrow$): Examples 1 + 2.

($\Leftarrow$): $X = \{ \sum_1^N a_k e_k \mid c_k \in \mathbb{C} \}$, $N = \dim X$, $\{e_k\}_1^N$ - basis

$$\varphi\left(\sum a_k e_k\right) = \max_{1 \le k \le N} |c_k| \quad \text{- norm on } X$$

Since all norms on finite dimensional vec. spaces are equivalent.

$\exists A, B > 0.\quad A\|x\| \le \varphi(x) \le B\|x\| \quad \forall x \in X.$

in particular the set $\left\{ \{c_k(x)\}_1^N, \; \begin{array}{l} x \in E \\ x = \sum c_k(x) e_k \end{array} \right\}$

is bounded (= bdd) in $\mathbb{C}^n$ For every bounded $E \subset X$.

$$\sup_{x \in E} \varphi(x) \le B \cdot \sup_{x \in E} \|x\| < \infty$$

$\Rightarrow$ for any sequence $\{ \{c_k(x_j)\}_{k=1}^N \}_{j=1}^\infty$ one can extract a converging subsequence in $\mathbb{C}^n$, i.e.

$$c_k(x_{j_n}) \longrightarrow c_k \qquad n \to \infty.$$

But then $\sum_1^N c_k(x_{j_n}) e_k \longrightarrow \sum c_k e_k$ in X

$\Rightarrow$ bounded subsets are precompact in X

$\Rightarrow$ bdd + closed sets are compact

4) $\ell^2(\mathbb{Z}) = \{ \{c_k\}_{k \in \mathbb{Z}} \mid \sum |c_k|^2 < \infty \}$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^\infty |c_k|^2}$$

$$B[0,1] = \{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \|\{c_k\}\| < 1 \}$$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite $\frac{1}{2}$-net in $B[0,1]$, beca $\rho(e_k, e_j) > 1/2$ for $e_k = (0,\dots,0,\underset{\substack{\wedge \\ k\text{th place}}}{1},0,\dots,0)$.

Definition: X is a Banach space if it is a linear normed space such that X is complete with respect to this norm.

Example: Let (K,ρ) be a metric compact space.

$$C(K) := \{f : K \to \mathbb{C} \mid \text{cont. in the metric } \rho\}$$
$$(\Leftrightarrow f(x_j) \to f(x) \quad \forall x_j \to x \quad \text{in } (K,\rho))$$

$$\|f\|_{C(K)} = \|f\| := \max_{x\in K} |f(x)| \quad \text{- norm in } C(K)$$

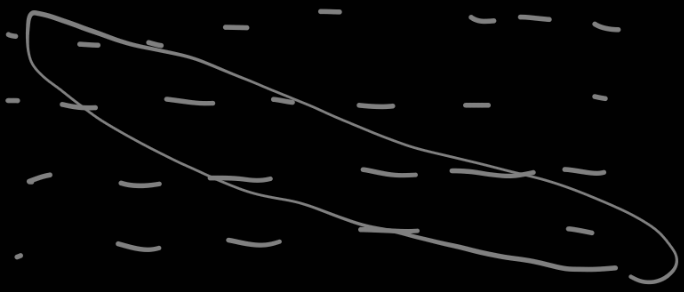
Theorem: [Arzela-Ascoli]: Assume that K is a complete compact metric space. $E \subset C(K)$ is precompact $\Leftrightarrow$

$\Leftrightarrow$ 1) E is bounded in C

2) Functions in E are equicontinuous, that is,
$\forall \varepsilon > 0, \ \exists \delta_\varepsilon > 0. \ |f(x) - f(y)| < \varepsilon \quad \forall x,y \in K. \ \rho(x,y) < \delta_\varepsilon$
$\forall f \in E$

We will need 1)+2) $\Rightarrow$ precompactness.

Proof: Find a dense sequence $\{x_j\}$ in K.
(such sequence exists because K is compact)
Then take $\{f_n\}$ arbitrary sequence in E.
We want to find a converging subsequence of $\{f_n\}$
(then E - precompact)
For this find a subsequence $\{f_{n_k}\}$ such that
$$f_{n_k}(x_j) \longrightarrow f(x_j) \quad \text{for every } j$$
(cantor diagonalization process + uniform boundedness)



look at the proof from the 1st lecture

Claim: f_{n_k} is Cauchy sequence in $C(K)$.

$\boxed{\text{October 8, 2025}}$

Aim: $\|f_{n_s} - f_{n_m}\|_{C(K)} \longrightarrow 0$ as $s, m \to \infty$

For simplicity let $g_s = f_{n_s}$ $\quad s \geq 1$.

Idea:

$$|g_s(x) - g_m(x)| \leq |g_s(x) - g_s(x_j)| + \underbrace{|g_s(x_j) - g_m(x_j)|}_{\substack{\text{take } s,m \text{ large} \\ \text{enough}: \\ \leq \varepsilon/3}} + \underbrace{|g_n(x_j) - g_m(x)|}_{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x \\ \text{(equicontinuity)}}}$$

$\leq \varepsilon/3$ For all s if x_j is close to x

To make the idea work we need to check that in this construction we can deal only with finite number of points x_j, $j = 1, \dots, N(\varepsilon)$.

For this it suffices to find $N(\delta_\varepsilon)$ such that $\rho(x, x_j) < \delta_\varepsilon$ for every $x \in K$ and x_j, $j = 1 \dots N_\varepsilon$.
($\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is δ_ε-net).

So, it remains to show that if $\{x_j\}_{j=1}^{\infty}$ is dense then $\forall \delta_\varepsilon > 0 . \exists N(\delta_\varepsilon) . \{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net.
To this end, let $\{y_k\}_1^N$ is a $\delta_\varepsilon/2$-net in K (K is compact)
Let $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ be the part of $\{x_j\}$ such that

$$\operatorname{dist}\left(\{x_j\}_{j=1}^{N(\delta_\varepsilon)}, y_k\right) \leq \delta_\varepsilon/2 \quad \forall 1 \leq k \leq N.$$

$\Rightarrow$ then $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net by triangle inequality. $\blacksquare$
($\|g_s - g_m\|_{C(K)} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ for s, m large enough)

Example: $K=[0,1]$, $E_A = \left\{ f \in C[0,1],\ f(0)=0,\ f \text{ is Lipshitz with constant at most } A \right\}$, $A>0$

E_A is compact

i) E_A is bounded in $C[0,1]$:

$$|f(x)| \leq |f(x)-f(0)| \leq A|x| \leq A$$

$$E_A \subset B(0,A)$$

ii) $|f(x)-f(y)| \leq A|x-y| \leq A\delta = \varepsilon$ if $\varepsilon>0$, $\delta := \frac{\varepsilon}{A}$, $x,y \in [0,1]: |x-y| \leq \delta$

i + ii + AA theorem $\Rightarrow$ E_A is precompact

iii) E_A is closed

if $f_n \rightrightarrows f$ in $C(K)$ then $f_n(0) \to f(0) \Rightarrow f(0)=0$

$$|f_n(x)-f_n(y)| \leq A|x-y|$$
$$\downarrow$$
$$|f(x)-f(y)| \Rightarrow f \text{ is } Lip(A)$$

Example: Let $E = \left\{ \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}, \text{ where } c_k \in \mathbb{C} : |c_k| \leq \frac{1}{k^2+1} \right\}$

Then E is compact as well in $C[0,1]$.

i) bbs: $f \in E$. $\|f\| \leq \sum_{k\in\mathbb{Z}} \frac{1}{k^2+1}$

Details: exercise

ii) equicontinuity $f = \sum_{|k|\leq N} + \sum_{|k|>N}$ small if N large

Lipshitz with some constant A_N ~ does not depend on f

Compact operators: basic properties

<u>Definition</u>: Let X, Y be Banach spaces, $T: X \to Y$ a linear map. T is called bounded if $T(B(0,1))$ is a bounded in Y set in Y. T is called compact if $T(B(0,1))$ is a precompact set in Y. ($B(0,1) = \{\|x\|_X < 1\}$) bounded linear operator

<u>Some observations</u>:

1) If $S \subset X$ is bdd then $T(S)$ is bdd for any bounded operator
precompact compact

2) T is compact $\Rightarrow$ T is bounded
(precompact sets are bounded)

3) with the norm $\|T\| = \sup\limits_{x \in B(0,1)} \|Tx\|_Y$, the set of bdd linear operators becomes a linear normed space, to be denoted by $B(X,Y)$ or $B(X)$ if $X = Y$.

4) A linear map between Banach spaces X, Y is continuous if and only if it is bounded.
Hint: $\|Tx - Ty\| \le \|T\| \cdot \|x-y\|$, so bounded operators are Lipshitz.

$\boxed{\text{October 14, 2025}}$

<u>Definition</u>: Banach algebra is an associative algebra which is a linear space with a norm $\|\cdot\|$ such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and $\|T_1 T_2\| \le \|T_1\| \|t_2\|$ for any elements in this algebra.

<u>Proposition</u>: Let X be a Banach space, then $B(X,X)$ is a Banach algebra.

<u>Proof</u>: $\alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{B}(X,X)$ $\forall T_1 T_2 \in \mathcal{B}(X,X)$ (proved)

$T_1 \cdot T_2 \in \mathcal{B}(X,X)$, since $\forall x \in X$. $\|T_1 T_2 x\| \leq \|T_1\| \cdot \|T_2 x\|$

since $T_1 \in \mathcal{B}(X,X)$ we have $\|T_1 y\| \leq \|T_1\| \|y\|$ $\forall y \in X$

$\Rightarrow \underbrace{\|T_1\| \cdot \|T_2\|}_{\sup_{y \in X, \|y\| \leq 1} \|T_1 T_2 x\|} \leq \|T_1\| \cdot \|T_2\|$

We see that $T_1 T_2 \in \mathcal{B}(X,X)$ and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

Now let us prove that $\mathcal{B}(X,X)$ is Banach.

Let us show that $\sum_{n=1}^{\infty} B_k$ converges in $\mathcal{B}(X,X)$ if $\sum \|B_k\| \overset{(*)}{<} \infty$.

$T_n = \sum_{k=1}^{N} B_k$, $x \in X$, $\|T_N x - T_M x\|_{N>M} = \|\sum_{M+1}^{N} B_k x\| \leq \sum_{M+1}^{N} \|B_k\| \cdot \|x\| \xrightarrow[M,N\to\infty]{} 0$, because of $(*)$.

$\Rightarrow \{T_N x\}_N$ Cauchy in X, but X - banach $\Rightarrow \exists Tx = \lim_{X\to\infty} T_N x$

Moreover, $\|Tx\| = \lim \|T_N x\| \leq \lim_{N\to\infty} \sum_1^N \|B_k\| \|x\| \leq \left(\sum_1^\infty \|B_k\|\right) \cdot \|x\|$.

$\Rightarrow T \in \mathcal{B}(X,X)$, $\|T\| \leq \sum_1^\infty \|B_k\|$

$$\sup_{\substack{x\in X\\ \|x\|<1}} \|Tx - T_N x\| = \lim_{M\to\infty} \|T_N x - T_M x\| \leq \sup_{\|x\|<1} \lim_{N\to\infty} \sum_{M+1}^{N} \|B_k\| \cdot \|x\| = \sum_{M+1}^{\infty} \|B_k\| \xrightarrow[(*)]{M\to\infty} 0$$ ▨

<u>Proposition</u>: The set $S_\infty(X)$ $\left(= S_\infty(X,X)\right.$ (index ∞ will be explained later)$\left.\right)$ of all compact operators on X is a two-sided ideal in $\mathcal{B}(X) = \mathcal{B}(X,X)$: $\forall T_1 \in S_\infty(X)$. $\forall T_2 \in \mathcal{B}(X)$. $T_1 T_2 \in S_\infty(X)$ and $T_2 T_1 \in S_\infty(X)$.

<u>Proof</u>: Take $\{x_n\}_1^\infty$ s.t. $\|x_n\| \leq 1$, and let us check that there is a subsequence $\{x_{n_k}\}$: $T_1 T_2 x_{n_k}$ converges.

Note that $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$. T_1 takes $B_X(0, \|T_2\|)$ into a precompact subset of $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$ - convergent subsequence

Now let's consider $\{T_2 T_1 x_n\}$. Note that $\{T_1 x_{n_k}\}$ - convergent subsequence ($T_1 \in S_\infty(X)$). Then $\{T_2 T_1 x_{n_k}\}$ converges, since T_2 is continuous. ▨

Proposition: $S_\infty(X,Y)$ is a closed subset in $\mathcal{B}(X,Y)$, i.e. $T_n(X,Y)$, $T_n \longrightarrow T$ in $\mathcal{B}(X,Y) \Rightarrow T \in S_\infty(X,Y)$?

Proof: Let's find a finite ε-net in $T(B_X[0,1])$.
Take finite $\varepsilon/3$-net for $T_n B_X(0,1)$ for $n: \|T-T_n\| \le \varepsilon/2$; denote it by $\{y_k\}_{k=1}^N$, then

$$\|Tx - Tx_k\| \le \underbrace{\|Tx - T_n x\|}_{A} + \underbrace{\|T_n x - T_n x_k\|}_{B} + \underbrace{\|T_n x_k - Tx_k\|}_{C} \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \le \varepsilon.$$

for any $1 \le k \le N$
so choose $k: B \le \varepsilon/2$
and note $A \le \|T-T_n\| \le \varepsilon/3$
for every $x \in B_X(0,1)$
$C \le \varepsilon/3$

Corollary: If T is a limit of finite-rank operators in $\mathcal{B}(X,Y)$, then $T \in S_\infty(X,Y)$.

Proof: Since finite-rank operators are in $S_\infty(X,Y)$, we have $T \in S_\infty(X,Y)$ by the previous proposition. ☒

Remark: At a general Banach space $\exists T \in S_\infty(X,Y)$ such that $\nexists (T_n)_n : \operatorname{rank} T_n < \infty$ and $\|T-T_n\| \longrightarrow 0$.

Definition: Let X be a Banach space. $\{e_k\}_{k=1}^\infty$ is a Schauder basis if $\forall x \in X.\ \exists! \{c_k(x)\}_{k=1}^\infty$ such that $x = \sum_{k=1}^\infty c_k(x) e_k$, where the series converges in X.

Theorem: Let X be a Banach space with Schauder basis, then $T \in S_\infty(X) \Leftrightarrow \exists T_n .\ \operatorname{rank} T_n \le n$ and $\|T-T_n\| \to 0$.
(here $\operatorname{rank} S = \dim S(X)$ $\forall S \in \mathcal{B}(X)$)

Proof: ($\Leftarrow$): we already know

($\Leftarrow$): Let $T \in S_\infty(X)$, and let $P_n : x \longmapsto \sum_{k=1}^{\infty} c_k(x) e_k$.

<u>P is linear</u>: $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X.\ P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$?

$$\left.\begin{aligned} \text{If } x &= \sum c_k(x) e_k \\ y &= \sum c_k(y) e_k \end{aligned}\right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha c_k(x) + \beta c_k(y)) e_k$$

$$\alpha x + \beta y \underset{\text{by def. of Schauder basis}}{=} \sum_{k=1}^{\infty} c_k(\alpha x + \beta y) e_k$$

$$c_k(\alpha x + \beta y) \overset{\text{by uniqueness}}{=} \alpha c_k(x) + \beta c_k(y) \quad \forall k$$

$$\text{Then } P_n(\alpha x + \beta y) = \sum_{k=1}^{n} c_k(\alpha x + \beta y) e_k = \sum_{k=1}^{n} \alpha c_k(x) e_k + \sum_{k=1}^{n} \beta c_k(y) e_k$$
$$= \alpha P_n(x) + \beta P_n(y) \quad \Rightarrow P_n \text{ linear}$$

Note that $T_n := P_n T$ are such that $\operatorname{rank}(T_n) \leq n$ because $\dim P_n T(X) \leq \dim P_n(X) \leq n$.

It remains to show that $T_n \longrightarrow T$ in $\mathcal{B}(X)$. Since T is compact, $\forall \varepsilon > 0.\ \exists \{x_k\}_{k=1}^{N}$ such that $\|x_k\| \leq 1$ $\forall k$ and $\{T x_k\}_{k=1}^{N}$ is a ε-net in $T(B_X(0,1))$. Now take $x \in B_X(0,1)$ and write

$$\|Tx - T_n x\| \leq \|Tx - Tx_k\| + \|Tx_k - T_n x\| + \|T_n x_k - T_n x\|$$
$$\leq \underbrace{\|Tx - Tx_k\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_k - P_n T x_k\|}_{\substack{\leq \varepsilon \text{ if } n \\ \text{large enough} \\ \text{for any fixed } k}} + \underbrace{\|P_n T x_k - P_n T x\|}_{\leq \|P_n\| \cdot \|Tx_k - Tx\|}$$
$$\leq \varepsilon + \varepsilon + \underbrace{\sup \|P_n\|}_{< \infty} \varepsilon$$

$\sup \|P_n\| < \infty$ by Banach-Schteinhaus theorem theorem on uniform point-wise convergence.

$\|T - T_n\| \leq \varepsilon (2 + \sup \|P_n\|)$ for n large enough $\blacksquare$

Theorem [Banach–Schteinhaus]: Assume that $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ where X is a Banach space, such that

$$\sup_n \|T_n x\| \le C(x) < \infty$$

local information ~ uniform estimate

For every $x \in X$. Then $\sup_n \|T_n\| < \infty$. In particular, one can take C in place of $C(x)$.

Remark: In our situation $\sup_{1 \le n < \infty} \|P_n\| < C(x) < \infty$ because $P_n x \longrightarrow x$ in X.

Banach adjoint operators

October 15, 2025

Definition: Let X be a Banach space, then $X^* = \mathcal{B}(X, \mathbb{C})$ is called the dual space to X. The elements of X^* are called functionals.

$= \{\phi : X \to \mathbb{C} \mid \phi \text{ linear}, \|\phi\| < \infty\}$

Examples: (can ignore, if one does not know measure theory)

i) $L^p(\mu) = \{ f : S \longrightarrow \mathbb{C} \mid f \text{ is measurable with respect to } \sigma\text{-algebra of } \mu, \int_S |f|^p \, d\mu < \infty \} / f = g$ if $f(x) = g(x)$ for μ-a.e. $x \in S$, $1 \le p < \infty$

$$\|f\|_{L^p(\mu)} = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

ii) $\ell^p(\mathbb{Z}) = \left(\{ \{x_k\}_{k \in \mathbb{Z}} \mid \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \}, \|x\|_{\ell^p(\mathbb{Z})} = \left(\sum |x|^p \right)^{1/p} \right)$, $1 \le p < \infty$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

i) $g \in L^q(\mu) \leftrightsquigarrow \Phi_g : f \longmapsto \int_S f g \, d\mu$, $\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$

ii) $\{y_k\}_{k=1}^{\infty}$ in $\ell^q(\mathbb{Z}) \leftrightsquigarrow \Phi_{\{y_k\}} : \{x_k\} \longmapsto \sum_{k \in \mathbb{Z}} x_k y_k$

$\Phi_{\{y_k\}} : \ell^p(\mathbb{Z}) \to \mathbb{C}$

Remark: i) is non-trivial measure theory

More examples:

iii) $c_0(\mathbb{Z}) = \left\{ \{x_k\}_{k \in \mathbb{Z}} \mid x_k \to 0 \text{ as } |k| \to \infty \right\}$

$c_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$ same identification

(Hausdorff is actually sufficient ~ hard)

iv) Let K be a compact metric space, and $X = C(K)$.
Then $X^* = \mathcal{M}(K)$.

$C(K)$: set of cont. maps $(K, \mathcal{S}) \to \mathbb{C}$

$$\mathcal{M}(K) = \left\{ \begin{array}{l} \text{the set of Borel} \\ \text{measures on } K \\ \text{(complex valued)} \end{array} \;\middle|\; |\mu|(K) = \sup_{\substack{K = \cup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n \in \mathbb{Z}} |\mu(E_n)| < \infty \right\}$$

$\|\mu\| = |\mu|(K)$

Riezs - Markov representation theorem

Here $\mu \in \mathcal{M} \overset{(*)}{\leftrightsquigarrow} \Phi_\mu : f \longrightarrow \int_K f \, d\mu$

We can also define ℓ^p, L^p for $p = \infty$:

- $\ell^\infty(\mathbb{Z}) := \left\{ \{x_k\} \subset \mathbb{C} : \sup_{k \in \mathbb{Z}} |x_k| < \infty \right\}$
- $L^\infty(\mu) := \left\{ f : \text{—} , \text{——} : \operatorname{ess\,sup} |f| < \infty \right\}$

Remark: If $1 < p < \infty$ then $(L^p)^* = L^q$, $(L^q)^* = L^p$

But for $p=1$ $(L^1)^* = L^\infty$, $(L^\infty)^* \neq L^1$
$\ell^1(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})$, but $(\ell^\infty(\mathbb{Z}))^* \neq \ell^1(\mathbb{Z})$

<u>Definition</u>: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ is defined by

$$T^*: \underset{\in Y^*}{\Psi} \longmapsto \Big(\underset{\in X^*}{(T^*\Psi)}: x \mapsto \langle Tx, \underset{\Psi(Tx)}{T^*\Psi} \rangle\Big),$$

where $\langle x, \phi \rangle = \phi(x)$ for $x \in X$, $\phi \in X^*$.

<u>Remark</u>: $\langle Tx, \Psi \rangle = \langle x, T^*\Psi \rangle$ → this formula is equivalent to the definition of T^*

<u>Remark</u>: Operation that sends x, ϕ into $\phi(x) = \langle x, \phi \rangle$ for $x \in X$, $\phi \in X^*$ is called a pairing of Banach spaces X, X^*.

<u>Example</u>: For $\varphi \in C[0,1]$, μ on $[0,1]$ then the pairing is $\langle \varphi, \mu \rangle = \int_0^1 \varphi \, d\mu$, see (✱).

<u>Theorem</u>: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then the map $T^*: Y^* \longrightarrow X^*$ defined by $\langle x, T^*\Psi \rangle := \langle Tx, \Psi \rangle$, $x \in X$, is an element of $\mathcal{B}(Y^*, X^*)$.
$(\Leftrightarrow (T^*\Psi)(x) = \Psi(Tx))$

<u>Lemma 1 [Hahn-Banach theorem]</u>: Let X be a Banach space, $E \subset X$ - subspace in X, $\phi_0 : E \to \mathbb{C}$ is linear and bdd ($\Leftrightarrow \phi_0 \in E^*$). Then $\exists\, \phi \in X^*$ such that $\phi|_E = \phi_0$ and $\|\phi\| = \|\phi_0\|$.

Lemma 2 ["Sufficient amount of functionals"]:

Let $x \in X$, then $\|x\| = \sup\limits_{\|\phi\|\le 1} |\phi(x)|$.

Proof: $|\phi(x)| \le \|\phi\| \cdot \|x\| \le \|x\|$, so $\|x\| \ge \sup\limits_{\|\phi\|\le 1} |\phi(x)|$

To prove "$\le$", define $E = \operatorname{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$,
$\phi_0 : y \longrightarrow c_y$ if $y = c_y x \in E$.
Assume that $\|x\| = 1$, then $\|\phi_0\|_{E^*} = \sup\limits_{\|y\|\le 1} |c_y| =$ (from (**),
$|c_y| = \|y\|$ if $\|x\| = 1$) $= \sup\limits_{\|y\|\le 1} \|y\| = 1$.

Hahn-Banach theorem $\Rightarrow \exists\, \tilde{\phi}_0 \in X^*$: $\|\tilde{\phi}_0\| = 1$, $\tilde{\phi}|_E = \phi_0$.
In particular $\sup\limits_{\|\phi\|\le 1} |\phi(x)| \ge |\tilde{\phi}| = |\tilde{\phi}_0(x)| = 1 = \|x\|$.

We have proved "$\le$" in the case where $\|x\| = 1$.
The general case follows from consideration of $\frac{x}{\|x\|}$ in place of x.

October 21, 2025

We are proving that $T \in \mathcal{B}(X,Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.

Let $T \in \mathcal{B}(X,Y)$, consider

$$\|T^*\| = \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} \|T^*\psi\|_{X^*} = \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} \sup_{\substack{x \in X \\ \|x\|\le 1}} |(T^*\psi)(x)|$$

$$= \sup_{-\shortparallel-}\sup |\langle x, T^*\psi\rangle|$$

$$= \sup\sup |\langle Tx, \psi\rangle|$$

$$= \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} \sup_{\substack{x \in X \\ \|x\|\le 1}} |\psi(Tx)|$$

$$= \sup_{\substack{x \in X \\ \|x\|\le 1}} \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} |\psi(Tx)|$$

Lemma "sufficient amount of functionals"
$= \|Tx\|$

$$= \sup_{\|x\|\le 1} \|Tx\| = \|T\| < \infty$$

The claim follows. ▨

Corollary: $T \in \mathcal{B}(X,Y)$ is invertible $(\exists T^{-1} \in \mathcal{B}(Y,X))$ iff $T^* \in \mathcal{B}(Y^*, X^*)$ is invertible $(\exists (T^*)^{-1} \in \mathcal{B}(X^*, Y^*))$.

We prove just $\Rightarrow$.

Proof: Assume that T is invertible $\Leftrightarrow$ $T^{-1}T = I_X$
$TT^{-1} = I_Y$

Let's take adjoint operators and see:

$$\left.\begin{array}{l} (T^{-1}T)^* = (I_X)^* \\ (TT^{-1})^* = (I_Y)^* \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} T^*(T^{-1})^* = I_X^* \\ (T^{-1})^* T^* = I_Y^* \end{array}\right.$$

Exercise: $(AB)^* = B^*A^*$

It remains to check that $I_X^* = I_{X^*}$, $I_Y^* = I_{Y^*}$. Then, by the previous theorem, $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$, hence T^* is invertible and its bounded inverse is $(T^*)^{-1} = (T^{-1})^*$.

Let's check that $I_X^* = I_{X^*}$. Take $\Phi \in X^*$, $x \in X$.

$$(I_X^* \Phi)(x) = \langle x, I_X^* \Phi \rangle = \langle I_X x, \Phi \rangle = \langle x, \Phi \rangle = \Phi x$$

$$(I_{X^*} \Phi(x)) \underset{def}{=} (\Phi)(x) = \Phi(x).$$

Similarly, $I_Y^* = I_{Y^*}$. ▨

The „pairing notation" is often not used in literature, but it is very usefull to not make mistakes.

Theorem [Schauder]: We have $T \in S_\infty(X,Y) \Leftrightarrow T^*_\infty(Y^*, X^*)$.

Proof: We will prove just "$\Rightarrow$".

Consider $K = \overline{TB_X(0,1)}$ – a compact set. Let $C(K)$ be the Banach space of continuous functions on K with

$$\|f\|_{C(K)} = \max_{s\in K}|f(s)|, \quad f: K \to \mathbb{C}$$

Let $E := \left\{ \psi \in Y^* \mid \|\psi\|_{Y^*} \le 1, \ \substack{\psi \text{ is considered as a} \\ \text{function on } K} \right\}$

$K \subset Y$, K metric space with respect to the metric $\rho(y_1,y_2) = \|y_2 - y_1\|_Y$

So, $E \subset C(K)$ and we claim that E is precompact.

1) uniform boundedness:

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s\in K}|\psi(s)| = \max_{s\in\overline{TB_X(0,1)}}|\psi(s)| \overset{\psi \text{ cont.}}{=} \sup_{x\in B_X(0,1)}|\psi(Ts)|$$

$$\le \|\psi\| \sup_{\|x\|\le 1}\|Tx\| \le \|\psi\|\cdot\|T\| \le \underbrace{\|T\| < \infty}_{\text{does not dependt on } \psi}$$

2) Equicontinuity: take $s_1, s_2 \in K$, let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \le \|\psi\|\cdot\|s_1 - s_2\| \le \|s_1 - s_2\|,$$

so maps from E are Lipschitz with constant 1, hence equicontinuous.

$\Rightarrow$ By Arzela-Askoli theorem, E is precompact.

We are now ready to prove $T^* \in S_\infty(Y^*, X^*)$. For this, we need to check that if $\{\psi_n\}$ is a sequence in $B_{Y^*}(0,1)$, then $\exists \{\psi_{n_k}\}$ such that $T^*\psi_{n_k}$ converges in X^*. So, take $\{\psi_n\} \subset B_{Y^*}(0,1)$ and consider it as elements $E \subset C(K)$.

Let $\{\psi_{n_k}\}$ be such that $\psi_{n_k} \xrightarrow{(*)} \psi$ in $C(K)$. $\left(\substack{E \text{ is} \\ \text{precompact!}}\right)$

Let's prove that $\{T^*\psi_{n_k}\}$ is Cauchy in X^*, then the theorem will follow.

Take $x \in X$, $\|x\| \le 1$, and consider

$$\|(T^*\psi_{n_k})(x) - (T^*\psi_{n_j})(x)\| = \|\langle x, T^*\psi_{n_k}\rangle - \langle x, T^*\psi_{n_j}\rangle\|$$
$$= \|\langle Tx, \psi_{n_k}\rangle - \langle Tx, \psi_{n_j}\rangle\|$$
$$= \|\psi_{n_k}(Tx) - \psi_{n_j}(Tx)\|$$
$$\le \sup_{s \in K} |\psi_{n_k}(s) - \psi_{n_j}(s)|$$
$$= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|_{C(K)}}_{\varepsilon_{kj} \text{ - does not depend on } x} \longrightarrow 0 \quad \text{by } (*).$$

$$\Rightarrow \|T^*\psi_{n_k} - T^*\psi_{n_j}\| \le \varepsilon_{k,j} \longrightarrow 0.$$

∎

Fredholm alternative

<u>Example</u>: Consider the equation $f(t) - \int_0^1 e^{t-s} f(s)\,ds \overset{(**)}{=} g(t)$ in $L^2[0,1]$.

Question: For which $g \in L^2[0,1]$ do we have a solution $f \in L^2[0,1]$?

Observation: g has to satisfy $\int_0^1 e^{-t} g(t)\,dt \overset{(***)}{=} 0$

$\int_0^1 c\,dt = c$

Indeed, $\int_0^1 e^{-t} g(t)\,dt = \int_0^1 e^{-t} f(t)\,dt - \int_0^1 \cancel{e^{-t}} \left(\int_0^1 \cancel{e^{t}}e^{-s} f(s)\,ds \right) dt = 0$

It is not clear so far if there are other restrictions.

<u>Theorem [Fredholm alternative]</u>: Let X be a Banach space, $T = I - K$ for $K \in S^\infty(X,X)$. Then

$$\operatorname{Ran} T = \{x \in X \mid \langle x, \Phi\rangle = 0 \ \forall \Phi \in \ker T^*\}.$$

In other words, either:

(1) $\ker T^* = \{0\}$ and the equation $Tf = g$ has solution $\forall g \in X$.

or

(2) $\ker T^* \neq \{0\}$ and the equation $Tf = g$ has solutions only for g s.t. $\langle g, \Phi\rangle = 0 \ \forall \Phi \in \ker T^*$.

Let's complete the consideration of the example: we need to check that $K: f \longrightarrow \int_0^1 e^{t-s} f(s)ds$ is compact (exercise) and find $\operatorname{Ker} T^*$.

$$\phi \in \operatorname{Ker} T^* \iff T^* \varphi = 0$$

Adjoint operator T^* is defined by

$$\langle Tf, g\rangle = \langle f, T^* g\rangle \qquad f, g \in L^2[0,1]$$

$(L^2[0,1]^* = L^2[0,1])$

$$\iff \langle f - \int_0^1 e^{t-s} f, g\rangle = \int_0^1 f(t) g(t) dt - \int_0^1 \int_0^1 e^{t-s} f(s) ds\, g(t) dt$$

$$= \int_0^1 f(t) g(t) dt - \int_0^1 f(s) \left(\int_0^1 e^{t-s} g(t) dt \right) ds$$

$$= \langle f, g - \int_0^1 e^{t-s} g(t) dt \rangle_{L^2[0,1]}$$

$$(T^* g): s \longmapsto g(s) - \int_0^1 e^{t-s} g(t) dt, \quad s \in [0,1].$$

$$T^* g = 0 \iff g(s) = \int_0^1 e^{t-s} g(t) dt \quad \text{a.e. on } [0,1]$$

$$\iff e^s g(s) = \underbrace{\int_0^1 e^t g(t) dt}_{\text{constant}} \quad \text{for almost every } s \in [0,1]$$

$\Rightarrow$ So, $\ker T^* = \{ c \cdot e^{-s}, c \in \mathbb{C} \}$, $\dim(\operatorname{Ker} T^*) = 1$.
By Fredholm theorem, equation (**) is solvable $\iff$ $\forall c \in \mathbb{C}.\ \langle g, c \cdot e^{-s}\rangle = 0 \iff \int_0^1 g(s) e^{-s} ds = 0$, which is (***).

Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let X be a Banach space, $E \subsetneq X$ - a linear closed subspace, $\varepsilon > 0$. Then $\exists x_0 \in X$ such that $\|x_0\| = 1$, $\operatorname{dist}(x_0, E) \geq 1 - \varepsilon$.

Proof: Since $E \neq X$, then $\exists\, \tilde{x}_0 \in X \setminus E$. Since E is closed, we have $\operatorname{dist}(\tilde{x}_0, E) = \delta > 0$ for some $\delta > 0$. Now consider $\tilde{y}_0 \in E$ such that $\delta \le \|\tilde{x}_0 - \tilde{y}_0\| \le (1+\eta)\delta$ for some $\eta \in (0,1)$.

Now let $\quad x_\eta := \dfrac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}$, $\|x_\eta\| = 1$.

$$\operatorname{dist}(x_\eta, E) = \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \operatorname{dist}(\tilde{x}_0 - \tilde{y}_0, E)$$

$$= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \operatorname{dist}(\tilde{x}_0, E)$$

$$= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta}$$

Choosing η so that $\frac{1}{1+\eta} = 1 - \varepsilon$, we are done.

October 22, 2025

Lemma: Let X be a Banach space. Then $I : x \mapsto x$ is compact on $X \iff \dim X < \infty$.

Proof: $\dim X < \infty \Rightarrow I \in S_\infty(X)$ – we already know

$I \in S_\infty(X) \Rightarrow \dim X < \infty$:

Suppose $\dim X = +\infty$, find a sequence $\{e_n\}$: $\|e_n\| = 1$ $\forall n \in \mathbb{N}$

$e_1 \in X$ – arbitrary

e_2: $\operatorname{dist}(e_2, \operatorname{span}\{e_1\}) \geq 1/2$

e_3: $\operatorname{dist}(e_3, \operatorname{span}\{e_1, e_2\}) \geq 1/2$

e_4: etc

existence of $\{e_n\}$ follows from previous lemma, because $\operatorname{span}\{e_1, \dots, e_k\} \neq X$ $\forall k \in \mathbb{N}$

Then $\{e_n\} \subset B_X[0,1] = I(B_X[0,1])$ but there is no convergent subsequence, because $\|e_k - e_j\| \geq 1/2$ $\forall k, j$. ▨

Lemma: Let X be a Banach space, $K \in S_\infty(X)$, $T = I - K$.
Then: 1) $\dim(\operatorname{Ker} T) < \infty$.
2) $\operatorname{Ran} T$ is closed in X. [closed range Lemma]

Proof: 1): We have $I|_{\operatorname{Ker}T} = \underbrace{(I-K)|_{\operatorname{Ker}T}}_{0} + \underbrace{K|_{\operatorname{Ker}T}}_{\in S_\infty(\operatorname{Ker}T, S)}$

$$I|_{\operatorname{Ker}T} \in S_\infty(\operatorname{Ker}T, X) \Rightarrow I \in S^\infty(\operatorname{Ker}T) \Rightarrow \dim(\operatorname{Ker}T) < \infty$$

2): The statement is equivalent to the fact that if $\{x_n\} \subset X$ s.t. $Tx_n \longrightarrow y$ in X then $\exists x \in X.\ Tx = y$.

2.a) Let $\{x_n\}: \|x_n\| \leq C \ \forall n$.
Then $(I-K)(x_n) \longrightarrow y$, $(I-K)(x_{n_k}) \xrightarrow{(*)} y$
For every subsequence x_{n_k}
Let's choose x_{n_k}: Kx_{n_k} converges to $z \in X$
(use $K \in S_\infty(X)$)

Then $x_{n_k} \longrightarrow y + z$ by $(*)$, take $x = y + z$:

$T(y+z) = \lim_{k\to\infty} Tx_{n_k} = y$, so $Tx = y$. ✓

2.b) $\operatorname{dist}(x_n, \operatorname{Ker}T) \leq C \ \forall n \in \mathbb{Z}$
Take $\tilde{x}_n := x_n - w_n$, where $w_n \in \operatorname{Ker}T: \|\tilde{x}_n\| \leq 2C$.
We have $\lim_{n\to\infty} T\tilde{x}_n = \lim_{n\to\infty} Tx_n$ by step 2a) $\exists \tilde{x}: T\tilde{x} = y$ ✓

2.c) $\operatorname{dist}(x_n, \operatorname{Ker}T) \longrightarrow +\infty$. Let us show that this situation does not occur. Suppose the converse:
Consider $\tilde{x}_n = x_n - \underset{\operatorname{Ker}T}{w_n}$: $\underbrace{\operatorname{dist}(x_n, \operatorname{Ker}T) \leq \|\tilde{x}_n\| \leq 2\operatorname{dist}(x_n, \operatorname{Ker}T)}_{(**)}$

For $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ we have $Tz \longrightarrow 0$.

$$Tz_n = T\frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T\frac{x_n}{\|\tilde{x}_n\|} = \frac{Tx_n}{\|\tilde{x}_n\|}$$ $Tx_n \to y \Rightarrow Tx_n$ is bdd in X, $\|\tilde{x}_n\| \to +\infty$

$$\Rightarrow \|Tz_n\| \leq \frac{2\|y\|}{\|\tilde{x}_n\|} \longrightarrow 0$$ For n large enough

At the same time, $Tz_n = z_n - Kz_n$

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $\{Kz_{n_k}\}$ converges to some $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{Tz_{n_k}}_{\to 0} + \underbrace{Kz_{n_k}}_{\to z} \longrightarrow z$$

We have $Tz = 0$ $\left(= \lim Tz_{n_k} = \lim Tz_n = 0\right)$

$\Leftrightarrow z \in \operatorname{Ker}T$, $0 = \operatorname{dist}(z, \operatorname{Ker}T) =$

$$= \lim_{k\to\infty} \operatorname{dist}(z_{n_k}, \operatorname{Ker}T)$$

$$= \lim_{k\to\infty} \operatorname{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \operatorname{Ker}T\right)$$

$$= \lim \frac{\operatorname{dist}(\tilde{x}_{n_k}, \operatorname{Ker}T)}{\|\tilde{x}_{n_k}\|}$$

$$= \lim \frac{\operatorname{dist}(x_{n_k}, \operatorname{Ker}T)}{\|\tilde{x}_{n_k}\|}$$

$$\overset{(**)}{\geq} \frac{1}{2} \rightsquigarrow \text{contradiction}$$ ▨

Lemma: Let X be a Banach space, $T \in \mathcal{B}(X)$: $\operatorname{Ker}(T) = \{0\}$ and $T^{k+1}X = T^kX$ for some $k \geq 0$. Then $\operatorname{Ran}T = X$.

Proof: We need to prove that $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$.
We know that: $T^{k+1}a = T^k\tilde{a}$ for every a and some $\tilde{a}$ depending on a.

$\operatorname{Ker}T^k \neq \{0\} \Rightarrow \operatorname{Ker}T \neq \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}.$$ ▨

Theorem [Fredholm]: Let X be a Banach space. $K \in S_\infty(X)$, $T = 1 - K$. Then TFAE:

1) T is invertible in $\mathcal{B}(X)$ 2) $\operatorname{Ker} T = \{0\}$ 3) $\operatorname{Ran} T = X$

1') T^* is invertible in $\mathcal{B}(X)$ 2') $\operatorname{Ker} T^* = \{0\}$ 3') $\operatorname{Ran} T^* = X^*$

Proof: We will prove $2 \Rightarrow 3 \Rightarrow 2' \Rightarrow 3' \Rightarrow 2$, $1 \Rightarrow 1' \Rightarrow 2'$, $2 \& 3 \Rightarrow 1$

(2) $\Rightarrow$ (3): If $T^k X = T^{k+1} X$ for some k, we are done by the lemma. Define $X_k := T^k X$, $k \geq 0$, and note that $X_0 \supset X_1 \supset X_2 \supset X_3 \cdots$ Assume that all inclusions are strict, i.e. $X_k \supsetneq X_{k+1}$ $\forall k$. The subspaces X_k are closed by the closed range lemma (by induction). By the almost orthogonality lemma: $\exists \{y_k\}_0^\infty$ s.t.

i) $y_k \in X_k$ $\forall k$ ii) $\|y_k\| = 1$ $\forall k$ iii) $\operatorname{dist}(y_k, X_{k+1}) \geq 1/2$

Since K is compact $\{Ky_k\}$ contains a convergent subsequence $\{Ky_{k_j}\}_{j=1}^\infty$. On the other hand, if $j < m$

$$Ky_{k_j} - Ky_{k_m} = (Ky_{k_j} - y_{k_j}) - (Ky_{k_m} - y_{k_m}) + y_{k_j} - y_{k_m}$$
$$= y_{k_j} - \underbrace{y_{k_m} - (Ty_{k_j} - Ty_{k_m})}_{\in X_{k_j+1}:}$$

Since $y_{k_m} \in X_{k_m}$, $j < m$, $k_j < k_m$, $k_j + 1 \leq k_m$, $X_{k_m} \subset X_{k_j+1}$

$\Rightarrow y_{k_m} \in X_{k_j+1}$; $Ty_{k_m} \in TX_{k_j+1} \subset X_{k_j+1}$; $Ty_{k_j} \in T(X_{k_j}) = X_{k_j+1}$

$\Rightarrow Ky_{k_j} - Ky_{k_m} = y_{k_j} + R$, $R \in X_{k_j+1}$. Since $\|y_{k_j} + R\| \geq 1/2$ by (iii), we get a contradiction ($\{Ky_{k_j}\}$ is not Cauchy).

(3) $\Rightarrow$ (2'): Take $\phi \in \operatorname{Ker} T^*$. We have

$T^*\phi = 0 \Leftrightarrow \langle x, T^*\phi \rangle = 0$ $\forall x \in X \Leftrightarrow \langle Tx, \phi \rangle = 0$ $\forall x \in X \Leftrightarrow$
$\Leftrightarrow \langle y, \phi \rangle = 0$ $\forall y \in X$ (by 3) $\Leftrightarrow \phi = 0$

(2') $\Rightarrow$ (3'): (the same as $2 \Rightarrow 3$ using Schauder's theorem)

$\underline{(3') \Rightarrow (2)}$: Assume that $\operatorname{Ran} T^* = X^*$ and take $x \in \operatorname{Ker} T$.
We have $Tx = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \ \forall \phi \in X^* \Leftrightarrow \langle x, T^*\phi \rangle \ \forall \phi \in X^*$
$\Leftrightarrow \langle x, \psi \rangle = 0 \ \forall \psi \in X^*$ (by 3') $\Leftrightarrow x = 0$ lemma on sufficient amount of functionals

Conclusion: $(2) \Leftrightarrow (3) \Leftrightarrow (2') \Leftrightarrow (3')$

$\underline{(1) \Rightarrow (1')}$: We already know for arbitrary $T \in \mathcal{B}(X)$.

$\underline{(1') \Rightarrow (2')}$: Invertible operators are injective.
$(2') \Leftrightarrow (2), (3)$

$\underline{(2 \,\&\, 3) \Rightarrow (1)}$: This holds for every $T \in \mathcal{B}(X)$ by the following Fundamental theorem from general F.A.:

Theorem [linear mapping theorem]: Let X, Y be a Banach space, $T: X \to Y$ - linear bijection. Then $T \in \mathcal{B}(X,Y) \Leftrightarrow T^{-1} \in \mathcal{B}(Y, X)$.

Check injectivity: $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0$ (by 2) $\Leftrightarrow x_1 = x_2$
surjectivity: $\operatorname{Ran} T = X$ (by 3) ▨

Remark: If $T = 1 - K$, $K \in S_\infty(X)$, and $\operatorname{Ker} T = \{0\}$ then the equation $Tx = y$ has a unique solution for every $y \in X$.

Remark: $\operatorname{Ker} T = \{0\} \Leftrightarrow \operatorname{Ker} T^* = \{0\}$, so we have proved half of Fredholm Alternative.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{TX} = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*\}$.

Remark: This implies the other half of Fredholm alternative since $\overline{TX} = TX$ for $T = 1 - K$, $K \in S_\infty(X)$.

Lemma [seperation lemma]: Let Y be a Banach space, $Y_0 \subsetneq Y$ - a closed subspace, then $\exists \phi \in Y^*: \phi|_{Y_0} = 0$, $\phi(y) \neq 0$ for some $y \in Y \setminus Y_0$.

Proof: Take $y \in Y \setminus Y_0$, define $\phi_0: \operatorname{span}\{y, Y_0\} \longrightarrow \mathbb{C}$ by $\phi_0(cy + y_0) \longmapsto c$, for $c \in \mathbb{C}$, $y_0 \in Y_0$.

1) $\{cy, y_0 \mid c \in \mathbb{C}, y_0 \in Y_0\} = \operatorname{span}\{y, Y_0\}$ - clear ✓

2) $\underbrace{cy + y_0}_{e} = \underbrace{\tilde{c}y + \tilde{y}_0}_{\tilde{e}} \Leftrightarrow (c - \tilde{c})y = y_0 - \tilde{y}_0 \in Y_0 \Leftrightarrow c = \tilde{c} \Rightarrow \phi(e) = \phi(\tilde{e})$

$\Rightarrow$ correctness ok

3) ϕ_0 is linear - clear

4) $\phi_0|_{Y_0} = 0$ $\quad$ ($c = 0$ on Y)

5) $|\phi_0(cy + y_0)| \overset{?}{\leq} A\|cy + y_0\| \quad \forall c, y_0$

$$|\phi_0(cy + y_0)| = |c| = \big(\operatorname{dist}(y, \underbrace{Y_0}_{\text{closed}})\big)^{-1} \cdot |c| \cdot \operatorname{dist}(y, Y_0)$$

$$= \operatorname{dist}(y, Y_0)^{-1} \cdot \operatorname{dist}(|c|y, Y_0) = \operatorname{dist}(y, Y_0)^{-1} \cdot \operatorname{dist}(cy, Y_0)$$

$|c|y = \alpha \cdot e \cdot y_0$, $|\alpha| = 1$

$\operatorname{dist}(|c|y, Y_0) = \operatorname{dist}(\underbrace{\alpha|c|}_{c} y_0, \underbrace{y_0 Y}_{Y})$

$$\leq \operatorname{dist}(y, Y_0)^{-1} \|cy + y_0\|, \qquad \text{so } A = \operatorname{dist}(y, Y_0)^{-1} \text{ works}$$

6) $\phi_0(y) = 1 \neq 0$

7) Use the Hahn-Banach theorem and extend ϕ_0 to the whole Y. $\square$

We actually proved that the lemma holds $\forall y \in Y \setminus Y_0$.

We can now prove the theorem from above.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{TX} = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*\}$.

Proof: We have $TX \subset E$, $E = \{x \mid \langle x, \phi\rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*\}$, because $\langle Tx, \phi\rangle = \langle x, T^*\phi\rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*$.
Then $\overline{TX} \subset \overline{E} = E$ since E is closed.
$\left(\begin{array}{l} x_n \to x \text{ and } \langle x_n, \phi\rangle = 0 \text{ for some } \phi \in X^* \\ \text{then } \langle x, \phi\rangle = \lim \langle x_n, \phi\rangle = 0 \end{array} \right)$

We now need to check $\overline{TX} \supset X$. If not, the inclusion $\overline{TX} \subset E$ is proper and by the seperation lemma $\exists \phi : \phi|_{\overline{TX}} \overset{(*)}{=} 0$ but $\phi(e) \neq 0$ for some $e \in E$.
$(*) \Rightarrow \phi|_{TX} = 0 \Leftrightarrow \langle Tx, \phi\rangle = 0 \ \forall x \in X \Leftrightarrow \langle x, T^*\phi\rangle = 0 \ \forall x \in X$
$\Leftrightarrow \phi \in \operatorname{Ker} T^* \Rightarrow \phi(e) = 0 \ \forall e \in E$ by definition of E, contradiction. ∎

Classical Form of Fredholm alternative for integral equations

Theorem: Let (S, μ) be a space with measure μ, and let $K(x,y) : S \times S \longrightarrow \mathbb{C}$:

$$\iint_{S \times S} |K(x,y)|^2 \, d\mu(x) \, d\mu(y) < \infty.$$

Then either equation $f(y) + \int_S K(x,y) f(x) \, d\mu(x) \overset{(**)}{=} 0$ has only the trivial solution $f = 0$ and the equation

$$f(y) + \int_S K(x,y) f(x) \, d\mu(x) = g(y)$$

is solvable for every $g \in L^2(S, \mu)$ or the equation $(**)$ has a non-trivial solution in $L^2(S, \mu)$.

Remark: Uniqueness implies existance.

Proof: Let's define $T = I - K$, $(Kf)(y) = \int_S K(x,y) f(x) \, d\mu(x)$.
Consider T as an operator on $L^2(S, \mu)$.

Then $(**) \Leftrightarrow \operatorname{Ker}T = \{0\} \Leftrightarrow TL^2(S,\mu) = L^2(S,\mu)$ by Fredholm alternative, modulo the fact that $K \in S_\infty(L^2(S,\mu))$. $\boxtimes$

↑ This we pospone until Hilbert spaces theory.

Remarks: Further reading:

1) $\dim \operatorname{Ker}T = \dim \operatorname{Ker}T^*$ if $T = I - K$, $K \in S_\infty$, it coincides with the dimension of the space of solutions

2) There is a version of Fredholm theory for general operators $T \in \mathcal{B}(X)$: $\dim(\operatorname{Ker}T) < \infty$, $\dim(X/\operatorname{Ran}T) < \infty$.

Spectrum of compact operators

October 29, 2025

Definition: Let $T \in \mathcal{B}(X)$, X Banach space.

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$$

is called the spectrum of T.

Definition: A number $\lambda \in \mathbb{C}$ is called an eigenvalue of T if $\exists e \in X \setminus \{0\}.\ Te = \lambda e$.

Definition: $\sigma_p(T) := \{\lambda \text{ eigenvalue of } T\}$ ... point spectrum of T.

Remark: We have $\sigma_p(T) \subset \sigma(T)$ for every $T \in \mathcal{B}(X)$.

Proof: $\lambda \in \sigma_p(T) \Rightarrow \lambda I - T$ is not injective because $\operatorname{Ker}(\lambda I - T) \neq \{0\}$.

Remark: In general, we might have $\sigma_p(T) \neq \sigma(T)$ and even $\sigma_p(T) = \emptyset$.

Theorem: Let X be a Banach space, $\dim X = +\infty$, and let $K \in S_\infty(X)$. Then $0 \in \sigma(K)$, $\sigma(K) \setminus \{0\} \subset \sigma_p(T)$, moreover, each eigenvalue has a finite multiplicity, and
$\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$.

Proof: $0 \in \sigma(K)$ since $0 \notin \sigma(K)$, then $\exists K^{-1} \in \mathcal{B}(X)$:
$I = \underset{\underset{S_\infty(X)}{\wedge}}{K}\underset{\underset{\mathcal{B}(X)}{\wedge}}{K^{-1}} = K^{-1}K$, but then $I \in S_\infty(X) \Rightarrow \dim X < \infty$, contradiction.

Now let's prove that $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$.
Take $\lambda \in \sigma(K)$ and assume that $\lambda \in \sigma_p(K)$. Then

$$\operatorname{Ker}(\lambda I - K) = \{0\} \Leftrightarrow \operatorname{Ker}(I - \tfrac{1}{\lambda}K) = \{0\}$$

$\Rightarrow I - \frac{1}{\lambda}K$ is invertible by Fredholm theorem - contradiction.

Now let's prove that $\dim E_\lambda < \infty$.

$$E_\lambda := \{e \in X \mid Ke = \lambda e\} \quad \left(\begin{array}{l}\text{definition for:}\\ \lambda \text{ has finite multiplicity}\end{array}\right)$$

If this is not the case, there is a sequence $\{e_n\}_{n=1}^\infty$ such that $\operatorname{dist}(e_n, \operatorname{span}\{e_1, \dots, e_{n-1}\}) \geq 1/2$, $\|e_n\| = 1$.
Consider $\{Ke_n\} = \{\lambda e_n\}$: we cannot choose a convergent subsequence from this sequence - contradiction with $K \in S_\infty(X)$.

To prove $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$, assume the converse and let $e_n \in X$: $Ke_n = \lambda_n e_n$, $\|e_n\| = 1$, $\lambda_n \in \sigma_p(K)$, $\lambda_n \neq \lambda_k$ for $k \neq n$.
Define $E_n := \operatorname{span}\{e_1, \dots, e_n\}$ $\forall n$.
Observation: $E_{n+1} \supsetneq E_n$ $\forall n$.
Clearly $E_{n+1} \supset E_n$. If $E_{n+1} = E_n$ for some n, there exists the first such n. Then

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^{n} \lambda_{n+1} \alpha_k e_k$$

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^{n} \alpha_k \lambda_k e_k$$

$\Rightarrow 0 = \sum_{k=1}^{n} \alpha_k (\lambda_{n+1} - \lambda_k) e_k \quad \Rightarrow \alpha_{n+1} - \lambda_n = 0$

$\Rightarrow \lambda_{n+1} = \lambda_n$ ... contradiction

$\Rightarrow$ The observation is true.

Let's choose $y_{n+1} \in E_{n+1}$ such that $\|y_{n+1}\| = 1$, $\operatorname{dist}(y_{n+1}, E_n) \geq \frac{1}{2}$
It remains to prove that $\{Ky_n\}$ does not have a convergent subsequence.

$$y_{n+1} = \alpha_{n+1} e_{n+1} + R_n, \quad \text{where } R_n \in E_n$$

$$Ky_{n+1} - Ky_{m+1} = \lambda_{n+1}\alpha_{n+1} e_{n+1} + \tilde{R}_n - \lambda_{m+1}\alpha_{m+1} e_{m+1} - \tilde{R}_m$$

Assume that $n \geq m+1$, then

$$\underset{E_n}{\underset{\wedge}{\tilde{R}_n}} - \underset{E_{m+1} \subset E_n}{\underset{\wedge}{\underbrace{\lambda_{m+1}\alpha_{m+1} e_{m+1}}}} + \underset{E_m \subset E_n}{\underset{\wedge}{\tilde{R}_m}} \quad \in E_n$$

$$\begin{aligned}
\Rightarrow \|Ky_{n+1} - Ky_{m+1}\| &\geq \operatorname{dist}(\lambda_{n+1}\alpha_{n+1} e_{n+1}, E_n) \\
&= |\lambda_{n+1}| \operatorname{dist}(\alpha_{n+1} e_{n+1}, E_n) \\
&= |\lambda_{n+1}| \operatorname{dist}(y_{n+1}, E_n) \\
&\geq r \cdot \tfrac{1}{2} > 0
\end{aligned}$$

Scheme of the solution of Dirichlet problem in $\mathbb{R}^n$, $n \geq 3$, by means of Fredholm theory

Dirichlet problem: Find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ $\left(\begin{array}{c} \Omega \text{ is a domain in } \mathbb{R}^n \\ \partial\Omega \in C^2 \end{array} \right)$
such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} \overset{(*)}{=} f \end{cases} \quad \text{where } f \in C(\partial\Omega)$$

Physics interpretation: heat

Scheme for solution: we search for a solution of the form

$$u(x) = \int_{\partial\Omega} K(x,y)\varphi(y)\,dy, \quad K(x,y) = c_n \frac{(x-y,\ n_y)}{\|x-y\|^n_{\mathbb{R}^n}}$$

n_y is the ouward unit normal, $c_n \in \mathbb{R}$

$\Delta u = 0$ for every $\varphi \in C(\partial\Omega)$, we only need to find good φ (such that $(*)$ will hold)

$$u_\varphi(x) = -\varphi(x) + \int_{\partial\Omega} K(x,y)\varphi(y)\,dy \quad \text{if} \quad x \in \partial\Omega$$

$$u_\varphi = (-I + K)\varphi, \quad K\varphi = \int K(x,y)\varphi(y)\,dy$$

To check that $\exists\varphi$: $u_\varphi = f$ on $\partial\Omega$ we just check that $K \in S_\infty((C(\partial\Omega))$ and $\operatorname{Ker}(-I+K) = \{0\}$
$\Rightarrow$ we are done by Fredholm alternative.

This is hard to prove (course on PDEs)

PART II Banach Algebras

November 4, 2025

<u>Definition</u>: A is a Banach algebra if A is a Banach space with the operation of multiplication such that:

1) $(xy)z = x(yz)$; $x, y, z \in A$
2) $(x+y)z = xz + yz$
   $x(y+z) = xy + xz$
3) $(\alpha x)y = x(\alpha y)$; $\alpha \in \mathbb{C}$
4) $\exists e \in A.\ xe = ex = x \quad \forall x \in A.$
5) $\|xy\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A.$

<u>Examples</u>: 1) X Banach space $\Rightarrow \mathcal{B}(X)$ is a Banach algebra with unity $e = I$ and norm $\|T\| = \sup\limits_{\|x\| \leq 1} \|Tx\|$

2) Calkin algebra: $\mathcal{B}(X)/S_\infty(X)$, $e = I + S_\infty(X)$

$$\|T\|_{\mathcal{B}(X)/S_\infty(X)} = \inf_{K \in S_\infty(X)} \|T - K\| = \operatorname{dist}(T, S_\infty(X))$$

3) K - compact Hausdorff space, $C(K)$ is a Banach algebra, $e = 1$, $\|f\|_{C(K)} = \max\limits_{x \in K} |f(x)|$

$f: K \to \mathbb{C}$

4) $W(\mathbb{T})$ - Wiener algebra on $\mathbb{T} := \{|z| = 1\}$.

$$W(\mathbb{T}) = \left\{ f = \sum_{k \in \mathbb{Z}} c_k z^k \,\middle|\, c_k \in \mathbb{C}, \sum |c_k| < \infty \right\}$$

$\|f\|_{W(\mathbb{T})} = \sum\limits_{k \in \mathbb{Z}} |c_k|$, $e = 1$, multiplication is the usual multiplication of functions

$$\|f \cdot g\|_{W(\mathbb{T})} \leq \|f\|_{W(\mathbb{T})} \cdot \|g\|_{W(\mathbb{T})}$$

exercise

5) $L^\infty(\mathbb{R})$, $\|f\| = \operatorname{esssup}_{\mathbb{R}} |f|$

6) $H^\infty(\mathbb{D})$ - set of bounded analytic functions on $\mathbb{D} := \{|z|<1\}$

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{|z|<1} |f(z)|$$

Remark: 1) & 2) are noncomutattative Banach algebras, others are commutative.

Remark: If A is a Banach space with multiplication and properties 1), 2), 3), 5), then we can always add the identity to convert A to a Banach algebra as follows:

$\mathcal{A} = A \times \mathbb{C}$, $\quad (x, \alpha) + (y, \beta) = (x+y, \alpha+\beta)$

$$\gamma(x,\alpha) = (\gamma x, \gamma \alpha)$$

$$(x,\alpha)\cdot(y,\beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

$$\left(\text{"}(x+\alpha e)(y+\beta e)\text{"} = xy + \alpha y + \beta z + \alpha\beta e\right) \qquad e=(0,1)$$

$$\|(x,\alpha)\| = \|x\| + |\alpha| \quad \leftarrow ! \text{ corrected on Nov. 5th}$$

$\Rightarrow \mathcal{A}$ is a Banach algebra with identity and $(A,0) \subset \mathcal{A}$.

Example: $L^1(\mathbb{R})$, multiplication $f * g = \int_{\mathbb{R}} f(y) g(x-y)\,dy$

$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f|\,dx$ - this is a Banach algebra without identity, and the above construction is equivalent to consideration of $\mathcal{A} = \mathbb{C}\delta_0 + L^1(\mathbb{R})$

$$\left(\overset{\text{measure}}{\delta_0(S)} = \begin{cases} 1; & 0 \in S \\ 0; & 0 \notin S \end{cases}, \quad \delta_0 * f = f * \delta_0 = \int f(y)\delta_0(x-y) = f(x) \right)$$

measures $\rightsquigarrow$ no dy

$$\langle \varphi, \delta_0 \rangle = \varphi(0) = \int_{\mathbb{R}} \varphi\, d\delta_0, \quad \varphi \in C(\mathbb{R})$$

Definition: A Banach algebra, $x \in A$. We say $x \in A$ is invertible if $\exists x^{-1} \in A.\ xx^{-1} = x^{-1}x = e$.

If x is invertible, then x^{-1} is unique.

Definition: $G(A) := \{x \in A \mid x \text{ is invertible}\}$.
we use G, because G(A) is a group

Definition: $x \in A$. $\sigma(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\}$.

Example: If A is the set of n×n matrices with complex coefficients, then for $T \in A$, $\sigma(T)$ is the set of eigenvalues (the usual spectrum of the matrix).

Example: $A = \mathbb{C}(K)$, $\sigma(f) = ?$
$\lambda \in \mathbb{C}$: $\lambda - f$ is not invertible in $C(K)$.
Since $g \in G(C(K)) \Leftrightarrow \frac{1}{g} \in C(K)$, we have $\lambda - f \in G(C(K)) \Leftrightarrow \frac{1}{\lambda - f} \in C(K)$
$\Leftrightarrow \lambda \notin f(K)$

$\Rightarrow \sigma(f) = f(K)$ ← compact non-empty subset of $\mathbb{C}$

Basic properties

Proposition 1: Let A be a Banach algebra. Then $\|e\| \geq 1$.

Proof: $\|e\| = \|e \cdot e\| \leq \|e\| \cdot \|e\| \Rightarrow \cancel{\|e\| = 0}$ or $\|e\| \geq 1$
$\Updownarrow$
$e = 0$ ✗

Definition: A Banach algebra is called unital if $\|e\| = 1$.

Proposition 2: If A is an arbitrary Banach algebra, then the algebra $A \times \mathbb{C}$ is unital.

Proof: $\|(0,1)\| = \|0\| + |1| = 1$

!**Note**: This was wrong previously and corrected on November 5th.

⚠ From now on, all Banach algebras are unital.

Proposition 3: If $x_n \longrightarrow x$, $y_n \longrightarrow y$ in A, then $x_n \cdot y_n \longrightarrow xy$.

Proof: $\|x_n \cdot y_n - xy\| \leq \underbrace{\|x_n - x\|}_{\to 0}\underbrace{\|y_n\|}_{\text{bdd}} + \|x\|\underbrace{\|y_n - y\|}_{\to 0} \longrightarrow 0$ ▨

Proposition 4: Let $a \in A$, $\|a\| < 1$, then $e - a \in G(A)$.

Proof: Define $(e-a)^{-1} = e + a + a^2 + \cdots = \sum_{k=0}^{\infty} a^k$

this series converges because A is a Banach space and $\sum \|a^k\| \leq \sum \|a\|^k < \infty$

Let us check that $(e-a)^{-1}(e-a) = e$

$$\Leftrightarrow \lim_{n\to\infty} \underbrace{\left(\sum_{k=0}^{n} a^k\right)(e-a)}_{e - \underbrace{a^{n+1}}_{\to 0}} = e$$

$$e - a^{n+1} \longrightarrow e, \quad \text{since } \|a^{n+1}\| \leq \|a\|^{n+1} \longrightarrow 0$$

Similarly, $(e-a)(e-a)^{-1} = e$.

Proposition 5: $G(A)$ is open in A.

Proof: Let $a \in G(A)$, $b \in A$, then $a - b = \underbrace{a}_{G(A)}\underbrace{(e - a^{-1}b)}_{\in G(A) \text{ if } \|a^{-1}b\| < 1}$

$\|a^{-1}b\| < 1$ holds for all b: $\|b\| \leq \frac{1}{\|a^{-1}\|}$

$\Rightarrow B\left(a, \frac{1}{\|a^{-1}\|}\right) \subset G(A) \Rightarrow G(A)$ is open.

Proposition 6: Let $x \in G(A)$, $x_n \longrightarrow x$ in A. Then $x_n \in G(A)$ for n large enough, and $x_n^{-1} \longrightarrow x^{-1}$ as $n \longrightarrow \infty$.

Proof: Write $x_n = x + z_n$, we have $x_n \in G(A)$ for n large enough by Proposition 5.

$$(x + z_n)^{-1} - x^{-1} = (x(e + x^{-1}z_n))^{-1} - x^{-1} = \underbrace{(e + x^{-1}z_n)^{-1}}_{y_n} \cdot x^{-1} - x^{-1} \xrightarrow[\text{Prop. 3}]{} ex^{-1} - x^{-1} = 0$$

$y_n \longrightarrow e$, see Prop. 4: $\|y_n - e\| \leq \sum_{k=1}^{\infty} \|x^{-1} \cdot z_n\|^k \longrightarrow 0$ $\quad \|z_n\| \to 0$ ▨

Theorem: Let A be a Banach algebra, $a \in A$. Then $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.

Proof: $\sigma(a) = \mathbb{C} \setminus \rho(a)$, $\quad \underbrace{\rho(a)}_{\text{resolvent set}} := \{\lambda \mid \lambda e - a \text{ is invertible}\}$

$\rho(a)$ is open by Prop. 5 $\Rightarrow \sigma(a)$ is closed.

Let's check that $\sigma(a)$ is bounded: $\sigma(a) \subset \{\lambda \mid |\lambda| \leq \|a\|\}$

$\Leftrightarrow \rho(a) \supset \{\lambda \mid |\lambda| \geq \|a\|\}$.

Take $\lambda : |\lambda| \geq \|a\|$, then $\lambda e - a = \lambda e \underbrace{\left(e - \frac{a}{\lambda}\right)}_{\text{invertible by Prop. 4}}$

It remains to check that $\sigma(a) \neq \emptyset$.

Assume that $\sigma(a) = \emptyset$ and consider the function

$$f_\phi(\lambda) = \phi((\lambda e - a)^{-1}) \text{ for some } \phi \in A^*; \quad \lambda \in \mathbb{C}$$

Let's check that f_ϕ is analytic. Take $\lambda_0 \in \mathbb{C}$, consider

$$\lim_{\lambda \to \lambda_0} \frac{f_\phi(\lambda) - f_\phi(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} \phi\left(\frac{(\lambda e - a)^{-1} - (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0}\right)$$

$$[x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}] \quad = \lim_{\lambda \to \lambda_0} \phi\left(\frac{(\lambda e - a)^{-1}((\lambda_0 e - \cancel{a}) - (\lambda e - \cancel{a}))(\lambda_0 e - a)^{-1}}{\lambda - \lambda_0}\right)$$

$$= \lim_{\lambda \to \lambda_0} \phi(\underbrace{(\lambda e - a)^{-1}}_{\to (\lambda_0 e - a)^{-1} \text{ Prop. 6}}(\lambda_0 e - a)^{-1})$$

ϕ continuous, continuity of multiplication

$$= -\phi((\lambda_0 e - a)^{-2})$$

$\Rightarrow f_\phi$ is analytic $\quad$ (f_ϕ is $\mathrm{Hol}(\mathbb{C})$) holomorphic

Take $\lambda : |\lambda| \geq 2\|a\|$, $\quad |f_\phi(\lambda)| = \|\phi\| \cdot \|(\lambda e - a)^{-1}\| = \|\phi\| \, |\lambda|^{-1} \underbrace{\|(e - \tfrac{a}{\lambda})^{-1}\|}_{\text{uniformly bounded for } \lambda: |\lambda| \geq 2\|a\|}$ (*)

f_ϕ is bdd on $\mathbb{C}$ by the max. principle

$\Rightarrow f_\phi = c_\phi \in \mathbb{C}$, but $c_\phi = 0$ by (*)

$\Rightarrow \phi(\lambda e - a) = 0 \ \forall \lambda \in \mathbb{C}. \ \forall \phi \in A^* \Rightarrow \lambda e - a = 0 \ \forall \lambda \in \mathbb{C}$
$\rightsquigarrow$ contradiction

$\blacksquare$

Definition: Let A_1, A_2 be Banach algebras. November 5, 2025

We say that A_1 is *isomorphic* to A_2 ($A_1 \cong A_2$) if there exists a map $j: A_1 \longrightarrow A_2$:

$$j(\alpha a + \beta b) = \alpha j(a) + \beta j(b) \quad (*)$$
$$j(ab) = j(a) \cdot j(b) \quad (**)$$
$$\|j(a)\|_{A_2} = \|a\|_{A_1}$$
$$j \text{ is bijective}$$

Theorem [Banach-Mazur]: Let A be a Banach algebra s.t. $G(A) = A \setminus \{0\}$. Then $A \cong \mathbb{C}$.

Proof: For every $a \in A$ we have $\sigma(a) \neq \emptyset$. So there is a $\lambda(a) \in \mathbb{C}$: $\lambda(a)e - a$ is not invertible $\Leftrightarrow$ $\lambda(a)e - a = 0$ $\Leftrightarrow$ $a = \lambda(a)e$.
In particular, such $\lambda(a)$ is unique.

Define $j: A \longrightarrow \mathbb{C}$, $a \longmapsto \lambda(a)$. We have:

$$\left.\begin{aligned} a+b &= \lambda(a+b)e \\ a+b &= \lambda(a)e + \lambda(b)e \end{aligned}\right\} \Rightarrow \lambda(a+b) = \lambda(a) + \lambda(b)$$

Similarly, $\lambda(\alpha a) = \alpha\lambda(a)$, $\lambda(ab) = \lambda(a)\cdot\lambda(b)$ $\Rightarrow$ $(*), (**)$ ✓

$\|j(a)\|_{\mathbb{C}} = \|a\|_A \Leftrightarrow |\lambda(a)| = \|a\|_A \Leftrightarrow \|\lambda(a)e\|_A = \|a\|_A$ ✓

$$\left.\begin{aligned} &j(a) = j(b) \Leftrightarrow \lambda(a)e = \lambda(b)e \Leftrightarrow a = b \\ &j(e) = 1 \Rightarrow j(A) \supseteq j(\mathbb{C}\cdot e) = \mathbb{C} \end{aligned}\right\} j \text{ is a bijection}$$

Definition: Let A be a Banach algebra, then $r(a) := \sup\{|\lambda|, \lambda \in \sigma(a)\}$ is called the *spectral radius of a*.

Theorem: Let A be a Banach algebra. Then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

In particular, the limit above exists.

Proof: Step 1: $r(a) \le \|a\|$, because

$$\lambda e - a = \underbrace{\lambda\left(e - \frac{a}{\lambda}\right)}_{\text{invertible element in } A \text{ (Prop. 4)}} \qquad \|b\| < 1 \text{ for } \lambda : |\lambda| > \|a\|$$

with $b = \frac{a}{\lambda}$.

$\Rightarrow \lambda e - a \in G(A) \Rightarrow \lambda \in \rho(a) \Rightarrow \sigma(a) \subset \mathcal{B}[0, \|a\|]$. ✓

Step 2: $r(a) \le \|a^n\|^{1/n}$ for every $n \ge 2$.

Take $\lambda \in \mathbb{C}$, and consider

$$\lambda^n e - a^n = (\lambda e - a) \cdot p_\lambda(a), \quad p_\lambda \text{ is a polynomial}$$

If $\lambda \in \sigma(a)$, then $\lambda^n e - a^n = z_1 \cdot z_2$, where $z_1, z_2 \in A$, $z_1 \notin G(A)$, $z_1 z_2 = z_2 z_1$. If $z = \lambda^n e - a^n$ is invertible, $\exists z^{-1} \in A$.

$$z^{-1} z_1 z_2 = z_1 (z_2 z^{-1}) = e$$

$$z^{-1} z_1 z_2 = (z^{-1} z_2) z_1 \qquad \Rightarrow z_1 \in G(A) \text{ ... contradiction}$$

$\Rightarrow \lambda^n \in \sigma(a^n) \underset{\text{Step 1}}{\Rightarrow} |\lambda^n| \le \|a^n\| \Rightarrow |\lambda| \le \|a^n\|^{1/n}$. ✓

Step 3: $\inf\limits_{n \ge 1} \|a^n\|^{1/n} \le \underline{\lim} \|a^n\|^{1/n} \le \overline{\lim} \|a^n\|^{1/n} \le r(a) \le \inf \|a^n\|^{1/n}$

$\hookrightarrow$ This implies the claim.

Notation note: $\overline{\lim}$ = limsup, $\underline{\lim}$ = liminf.

All that remains is $\overline{\lim} \|a^n\|^{1/n} \le r(a)$. Take $\phi \in A^*$, $\|\phi\| \le 1$,

$f_\phi(\lambda) := \phi((\lambda e - a)^{-1})$ for $\lambda \in \rho(a)$ (this is an analytic function on $\rho(a)$)

$\varepsilon > 0$, $k \in \mathbb{Z}$:

$$\frac{1}{2\pi i} \oint_{|\lambda| = r(a) + \varepsilon} \lambda^k f_\phi(\lambda)\, d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^k f_\phi(\lambda)\, d\lambda =$$

$$= \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^k \phi\left(\lambda \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n}\right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^{k+1} \sum_{n=0}^{\infty} \frac{\phi(a^n)}{\lambda^n}\, d\lambda$$

$$= \phi(a^{k+2}) \qquad \text{(Cauchy formula from complex analysis)}$$

$$\Rightarrow \phi(a^{k+2}) \le \left| \frac{1}{2\pi i} \oint_{|\lambda| = r(a) + \varepsilon} \lambda^k f_\phi(\lambda)\, d\lambda \right|$$

$$\leq \max_{|\lambda|=r(a)+\varepsilon} \left(|\lambda^k| \cdot |f_\phi(\lambda)| \right) \frac{1}{2\pi} (2\pi (r(a)+\varepsilon))$$

$$= (r(a)+\varepsilon)^{k+1} \underbrace{\|\phi\|}_{\leq 1} \cdot \underbrace{\sup_{|\lambda|=r(a)+\varepsilon} \|(\lambda e - a)^{-1}\|}$$

constant depending only on ε, because $|\lambda|=r(a)+\varepsilon$ is a compact set and $\lambda \mapsto \|(\lambda e - a)^{-1}\|$ is continuous

$$\sup_{\|\phi\|\leq 1} |\phi(a^{k+2})| = \|a^{k+2}\|$$

$$\text{I}\wedge$$

$$C_\varepsilon \cdot (r(a)+\varepsilon)^k$$

$\Rightarrow \overline{\lim} \|a^k\|^{1/k} \leq r(a)+\varepsilon$ for every $\varepsilon > 0$. $\blacksquare$

November 11, 2025

<u>Example</u>: $V: f \longmapsto \int_0^x f(s)\,ds$, $V \in \mathcal{B}(L^2[0,1])$.
Let's prove that $r(V) = 0 \Leftrightarrow \sigma(\{V\}) = \{0\}$.

<u>Proof 1</u>: $r(V) = \lim_{r\to\infty} \|V^n\|^{1/n}$, so we need a formula for V^n.
By induction: $(V^n f)(x) \overset{(*)}{=} \int_0^x f(s) \frac{(x-s)^{n-1}}{(n-1)!}\,ds$.

$n=1$: ✓

Assume $(*)$ for some n and compute

$$(V^{n+1}f)(x) = \int_0^x \left(\int_0^t f(s) \frac{(t-s)^{n-1}}{(n-1)!}\,ds \right) dt$$

$$= \int_0^x \left(f(s) \underbrace{\int_s^t \frac{(t-s)^{n-1}}{(n-1)!}\,dt}_{\left.\frac{(t-s)^n}{n!}\right|_s^x = \frac{(x-s)^n}{n!}} \right) ds$$

$$= \int_0^x f(s) \frac{(x-s)^n}{n!}\,ds.$$

$$\|V^n f\| \leq \max_{x\in(0,1)} \max_{s\in(0,x)} \frac{(x-s)^{n-1}}{(n-1)!} \left(\int_0^1 \Big[\int_0^x \underbrace{|f(s)|}_{= 1\cdot|f(s)|}\,ds \Big]^2 dx \right)^{1/2}$$

$$\overset{CS}{\leq} \frac{1}{(n-1)!}\left(\int_0^1 \underbrace{\sqrt{x}}_{\leq 1} \underbrace{\int_0^x |f|^2 ds}_{\leq \|f\|^2} dx\right)^{1/2}$$

$$\leq \frac{\|f\|_{L^2[0,1]}}{(n-1)!}$$

$$\Rightarrow \|V^n\| \leq \frac{1}{(n-1)!}, \quad \lim \|V^n\|^{1/n} \leq \overline{\lim}\left(\frac{1}{(n-1)!}\right)^{1/n}$$

Stirling formula

$$= \overline{\lim} \frac{1}{\left(\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}\right)^{1/n}(1+o(1))^{1/n}}$$

$$= \overline{\lim_{n\to\infty}} \underbrace{\frac{e^{\frac{n-1}{n}}}{(2\pi)^{1/2n}}}_{\to e} \cdot \underbrace{\frac{1}{(n-1)^{3/2\cdot\frac{n-1}{n}}}}_{\to 0}$$

$$= 0$$

$\square$

<u>Proof 2</u>: $Vf = \int_0 K(x,y) f(y)\,dy, \quad K(x,y) = \chi_{\mathbb{R}_+}(x-y)$

$$\int_0^1\int_0^1 |K(x,y)|^2 dx\,dy < \int_0^1\int_0^1 dx\,dy = 1 < \infty$$

$$\Rightarrow V \in S_\infty(L^2[0,1]) \Rightarrow \sigma(V) = \left\{\begin{matrix}\lambda \text{ is an eigenvalue} \\ \text{of } V\end{matrix}\right\} \cup \{0\}$$

theorem about the spectrum of compact operators

$$Vf = \lambda f \iff \underbrace{\int_0^x f(s)ds}_{\in C[0,1]} \overset{(**)}{=} \lambda f(x) \Rightarrow f \in C[0,1]$$

$$\Rightarrow \int_0^x f(s)ds \in C^1[0,1] \quad \Rightarrow \dots \Rightarrow f \in C^\infty[0,1].$$

Differentiating (**), we get $\lambda f' = f$ on $[0,1]$.
Substituting 0 into (**), we get $\lambda f(0) = \lambda$.

So, if $\lambda \neq 0$, then $\begin{cases}\lambda f' = f \\ f(0) = 0\end{cases} \iff \begin{cases} f = c\cdot e^{\lambda x} \\ 0 = c\cdot e^0\end{cases} \iff f = 0$

$\Rightarrow$ any $\lambda \neq 0$ is not an eigenvalue $\Rightarrow \sigma(V) \cap (\mathbb{C}\setminus\{0\}) = \varnothing$
Since $\sigma(V) \neq \varnothing$, we get $\sigma(V) = \{0\}$.

Commutative Banach algebras

<u>Definition</u>: Let A be a commutative Banach algebra and $J \subset A$. J is called a proper ideal in A if J is a linear subspace such that $a \cdot J \subset J$ $\forall a \in A$, and $J \neq \{0\}$, $J \neq A$.

<u>Definition</u>: J is a maximal ideal if J is a proper ideal and there is no proper ideal J' such that $J' \supsetneq J$.

<u>Proposition</u>: Every proper ideal is contained in some maximal ideal. Take some proper ideal J, $J \neq A$, and consider all proper ideals J': $J' \supseteq J$. This set is partially ordered by inclusion, and for every chain $\{J'_\alpha\}_{\alpha \in I}$ of ideals ordered by inclusion, the set $\bigcup_{\alpha \in I} J'_\alpha = J'$ is again a proper ideal.

- linearity: $p \cdot x + q y \in J$ for every $p, q \in \mathbb{C}$ and $x, y \in J'$, because $\exists \alpha_x, \alpha_y$. $x \in J_{\alpha_x}$, $y \in J_{\alpha_y}$ $\Rightarrow$ $x, y \in J_{\alpha_x}$ or $x, y \in J_{\alpha_y}$, then $px + qy$ are in the same J_{α_x} or J_{α_y} ✓
- ideal property $aJ' = \bigcup a \cdot J'_\alpha \subset \bigcup J'_\alpha$ $\forall a \in A$.
- properness: $J' \neq A$ (If $J' = A$, then $e \in J'$, then $e \in J'_\alpha \Rightarrow \overset{A}{\overset{\shortparallel}{eA}} \subseteq J'_\alpha \subsetneq A$.)

By Zorn's lemma, the set of all proper J': $J' \supset J$ has a maximal element. ▨

<u>Proposition</u>: If M is a maximal ideal in A, then M is closed.

<u>Proof</u>: Let's prove that $\overline{M}$ is a proper ideal.

- $\overline{M}$ is linear ✓
- $\overline{M}A \subseteq \overline{M}$: true by continuity of multiplication ✓
- $\overline{M} \neq A$ (If $\overline{M} = A$, then $e \in \overline{M} \Rightarrow \exists x \in M.\ \mathrm{dist}(x, e) < 1 \Rightarrow x \in G(A)$
  $\Rightarrow e = x \cdot x^{-1} \in xA \subset M \Rightarrow M = A$, contradiction.) ▨

Example: $A = C(K)$, $M_{x_0} = \{f \in A.\ f(x_0) = 0\}$. $\neq\{x_0\}$

Then M_{x_0} is a maximal ideal for every $x_0 \in K$.

- M_{x_0} is linear ✓
- $M_{x_0} \cdot A \subseteq M_{x_0}$ ✓
- $M_{x_0} \neq A$, because $1 \notin M_{x_0}$ ✓
- M_{x_0} is maximal: If $\exists J$-proper: $J \supsetneq M_{x_0}$ then $\exists f \in J.\ f(x_0) \neq 0$.

But then $\forall g \in A$ we have $g \overset{(***)}{=} c \cdot f + h$ for $c \in \mathbb{C}$ and $h \in M$, where c is such that $(g - cf)(x_0) = 0$, i.e. $c := \frac{g(x_0)}{f(x_0)}$.

So $A \subset \mathbb{C} \cdot f + M \subset J$, contradiction.

Observation: $M_{x_0} = \operatorname{Ker} \phi_{x_0}$, $\phi_{x_0} : f \longrightarrow f(x_0)$

ϕ_{x_0} is a multiplicative functional: $\phi_{x_0}(fg) = \phi_{x_0}(f) \cdot \phi_{x_0}(g)$

Definition: Let $\phi \in A^*$. We say ϕ is a multiplicative functional if $\phi(fg) = \phi(f) \cdot \phi(g)$ $\forall f, g \in A$, and $\phi \neq 0$.

Theorem: Let A be a commutative Banach algebra. TFAE

1) M is a maximal ideal in A.
2) $M = \operatorname{Ker} \phi$ for some mutiplicative functional $\phi \in A^*$.

Proof: 2) $\Rightarrow$ 1) obvious: i) $\operatorname{Ker} \varphi$ is linear

ii) $x \in \operatorname{Ker} \varphi, a \in A$, $\varphi(xa) = \varphi(x)\varphi(a) = 0 \Rightarrow xa \in \operatorname{Ker} \varphi$

iii) $\operatorname{Ker} \varphi \neq A$ because $\varphi \neq 0$

iv) $\operatorname{Ker} \varphi$ is maximal, because $\operatorname{Ker} \phi + \mathbb{C}a = A$ $\forall a: \phi(a) \neq 0$ (see $***$)

1) $\Rightarrow$ 2) Note that A/M is a Banach algebra (M is closed) in which every nonzero element is invertible. If $[a] \in A/M$, $\neq[0]$ then $a \cdot A$ is an ideal in A containing M, but not equal to $M \Rightarrow aA + M = A$

$aA + M \ni e \Rightarrow ab + M \ni e$ for some $b \in A \Rightarrow [a][b] = [e]$.

By Banach-Mazur theorem, there is an isomorphism j of Banach algebras A/M and $\mathbb{C}$. Let $\phi(x) := j([x])$.

i) Φ is linear, because j is linear.
ii) Φ is multiplicative, because j is multiplicative.
iii) $\operatorname{Ker}\Phi = \{x \mid j([x]) = 0\} \Leftrightarrow [x]=0 \Leftrightarrow x \in M$.

November 12, 2025

Notation: $\mathcal{M}(A)$ - set of all maximal ideals in A
A^*_{mult} - set of multiplicative functionals on A

Last time: $\mathcal{J}: \phi \longmapsto \operatorname{Ker}\phi$ maps A^*_{mult} onto $\mathcal{M}(A)$.

Proposition: $\mathcal{J}$ is a bijection.

Proof: We need to check that $\mathcal{J}(\phi_1) = \mathcal{J}(\phi_2) \Rightarrow \phi_1 = \phi_2$.
For this, note that $\phi(e) = 1$ $\forall \phi \in A^*_{mult}$, because $\begin{cases} \phi(e) = \phi(e)\,\phi(e) \\ \phi(e) \neq 0 \end{cases}$
for every $\phi \in A^*_{mult}$. So, if $\mathcal{J}(\phi_1) = \mathcal{J}(\phi)_2$, we have
$0 = \phi_1(y - \phi_1(y)e) \Rightarrow \phi_2(y - \phi_1(y)e) = 0 \quad \forall y \in A$
$\Rightarrow \phi_2(y) = \phi_1(y)\,\phi_2(e) = \phi_1(y)$, so $\phi_1 = \phi_2$.

Theorem: Let A be a commutative Banach algebra, and $a \in A$.
TFAE:
1) $a \in A \setminus G(A)$
2) $a \in M$ for some $M \in \mathcal{M}(A)$
3) $\exists \phi \in A^*_{mult}: \phi(a) = 0$

Proof: 1) $\Rightarrow$ 2): $J = aA$ - a proper ideal in A $(e \notin J)$
$\Rightarrow \exists M \in \mathcal{M}(A).\ M \subset J$
2) $\Rightarrow$ 3): Take $\phi: \operatorname{Ker}\phi = M \Rightarrow \phi(a) = 0$.
3) $\Rightarrow$ 1): If $\exists b \in A.\ ab = e \Rightarrow \phi(a)\cdot\phi(b) = \phi(e) = 1$, but $\phi(a) = 0$.

Corollary: $\sigma(a) = \{\phi(a) \mid \phi \in A^*_{mult}\}$

Proof: $\sigma(a) = \{\lambda \mid a - \lambda e \in A \setminus G(A)\} = \{\lambda \mid \exists \phi \in A^*_{mult}.\ \phi(a-\lambda e) = 0\}$
$= \{\lambda \mid \lambda \in \phi(a) \text{ for some } \phi \in A^*_{mult}\}.$ ▨

Remark: $\forall \phi \in A^*_{mult}$, $\|\phi\| = 1$, because $\phi(e) = 1$ and $|\phi(a^k)|$ is uniformly bounded for every $a \in B_A(0,1)$.

Applications

Theorem [Wiener]: Let $f = \sum_{k\in\mathbb{Z}} c_k z^k$, and $\sum_{k\in\mathbb{Z}} |c_k| < \infty$. Assume that $f(z) \neq 0$ for every $z \in \mathbb{T} = \{|z| = 1\}$. Then $\frac{1}{f} = \sum_{k\in\mathbb{Z}} b_k z^k$ where $\sum_{k\in\mathbb{Z}} |b_k| < \infty$.

Proof: 1. $W^1(\mathbb{T}) = \{\sum c_k z^k \mid \sum |c_k| < \infty\}$ is a Banach algebra:

Indeed, $W(\mathbb{T})$ is a Banach space with respect to the norm $\|\sum c_k z^k\| = \sum |c_k|$, and

$$\begin{aligned}
\|(\textstyle\sum c_k z^k)(\sum b_k z^k)\| &= \sum_{n\in\mathbb{Z}} \Big| \Big(\sum_{k\in\mathbb{Z}} c_k b_{n-k}\Big)\Big| \\
&\leq \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |c_k||b_{n-k}| \\
&= \sum_{k\in\mathbb{Z}} |c_k| \sum_{j\in\mathbb{Z}} |b_j| \\
&= \|\textstyle\sum c_k z^k\| \cdot \|\sum b_k z^k\|
\end{aligned}$$

λ.

2. Identification of $W^1(\mathbb{T})^*_{mult}$:

Let $\phi \in W^1(\mathbb{T})^*_{mult}$, $\lambda = \phi(z)$, then $\phi(\frac{1}{z}) \cdot \phi(z) = 1$, $\phi(\frac{1}{z}) = \frac{1}{\lambda}$
$|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot \|\frac{1}{z}\| = 1$
$|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot |\frac{1}{z}| = 1 \Rightarrow |\lambda| = 1$

$\phi(\sum_{-N}^{N} c_k z^k) = \sum_{-N}^{N} c_k \lambda^k$, and hence $\phi(f) = f(\lambda)$ $\forall f \in W^1(\mathbb{T})$, because $\{\sum_{-N}^{N} c_k z^k\}$ is dense in $W^1(\mathbb{T})$ and ϕ is continuous.

3. Application of invertibility criterion:

$f \in W^1(\mathbb{T})$ is invertible $\Leftrightarrow \nexists\, \phi \in W^1(\mathbb{T})^*_{mult}.\ \phi(f)=0 \Leftrightarrow f(z)=0\ \forall z \in \mathbb{T}$

This is the case in our case.

$\Rightarrow fg=1,\ g \in W^1(\mathbb{T}) \Rightarrow g = \frac{1}{f},\ g = \sum b_k z^k,\ \sum |b_k| < \infty.$ ▨

Bezout equation: Let $\{f_k\}_1^N \subset A(\overline{\mathbb{D}})$. We are interested if there exists $\{g_k\}_{k=1}^N \subset A(\overline{\mathbb{D}}) : \sum_1^N f_k g_k = 1.$

Necessary condition: $\nexists z_0 \in \overline{\mathbb{D}}.\ f_k(z_0)=0$ for every $1 \le k \le N$.

Theorem: Necessary condition is also sufficient.

Proof: 1. $A(\mathbb{D})$ is a Banach algebra with respect to the norm $\|f\| = \max_{z \in \mathbb{D}} |f(z)|$ ✓

2. Identification of $A(\overline{\mathbb{D}})^*_{mult}$:

$\phi \in A(\overline{\mathbb{D}})^*_{mult} \Leftrightarrow \phi(f) = f(\lambda)$ for some $\lambda \in \overline{\mathbb{D}}$ [exercise]

3. $J = \{ \sum_1^N f_k g_k \mid g_k \in A(\overline{\mathbb{D}}) \}$ is a proper ideal in $A(\overline{\mathbb{D}})$

$\Leftrightarrow J \subset M,\ M \in \mathcal{M}(A(\overline{\mathbb{D}})) \Leftrightarrow \sum_1^N (f_k g_k)(z_0) = 0 \quad \forall g_k \in A(\overline{\mathbb{D}})$ for some $z_0 \in \overline{\mathbb{D}}$

$\Leftrightarrow f_k(z_0) = 0 \quad \forall 1 \le k \le N$

At the same time: J is proper $\Leftrightarrow e \notin J$. ▨

November 18, 2025

Stone-Weierstrass Theorem

Definition: $C_{\mathbb{R}}(K)$ - the real Banach space of continous functions on a compact Hausdorff space K.

Definition: $A \subset C_{\mathbb{R}}(K)$ is a (real) Stone-Weierstrass algebra, if it is an algebra, and

(1) A does not vanish at any point $x \in K$, that is $\forall x \in K. \exists f \in A.$ $f(x) \neq 0$.

(2) A separates points in K, that is, $\forall x, y \in K. \exists f \in A. f(x) \neq f(y)$.

Theorem [Stone-Weierstrass]: Let $A \subset C_{\mathbb{R}}(K)$ be an algebra. Then A is dense in $C_{\mathbb{R}}(K) \Leftrightarrow A$ is a Stone-Weierstrass algebra.

Example: $K = [0,1]$, $A = \mathcal{P}$ ... the set of polynomials. Indeed, $\mathcal{P}$ is an algebra, and

$$\left.\begin{array}{l} \cdot\ 1 \in \mathcal{P} \Rightarrow (1) \text{ is satisfied} \\ \cdot\ x \in \mathcal{P} \Rightarrow (2) \text{ is satisfied} \end{array}\right\} \Rightarrow \begin{array}{l} \text{polynomials are dense in } [0,1] \\ \text{(Classical Weierstrass theorem)} \end{array}$$

Example: $K = [0,1]$, $A = \operatorname{span}\{\sin(2\pi kx), \cos(2\pi kx) \mid k \in \mathbb{Z}\}$

A is not dense in $C[0,1]$, because 0 and 1 are not seperated. On the other hand, A is dense in $C[0,a]$ for every $0 < a < 1$.

For the proof we need some lemmas:

Lemma: Let $a > 0$. Then $\exists \{p_k\} \in \mathcal{P}$ such that $\| p_k - |x| \|_{C[-a,a]} \longrightarrow 0$.

Proof: By scaling, we can assume that $a = 1$. Then we need to approximate $|x| = \sqrt{x^2} = \sqrt{1-y}$, $y = 1 - x^2$ by polynomials. Thus, it suffices to approximate $y \mapsto \sqrt{1-y}$ by polynomials uniformly on $[0,1]$.

Taylor: $\sqrt{1-y} = \sum_{k=0}^{\infty} c_k y^k$ for $c_k = (-1)^k \binom{1/2}{k} = (-1)^k \frac{1/2\,(1/2 - 1) \cdots (1/2 - k + 1)}{k!}$.

Observation: $c_0 = 1$, $c_k < 0$ for $k \geq 1$. (e.g. $c_1 = -\frac{1}{2}$, $c_2 = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} < 0$

$c_3 = -1 \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} < 0$)

In particular, we have $\sum_{1}^{\infty} |c_k| = \sup_{0 < y < 1} \left(- \sum_{k=1}^{\infty} c_k y^k \right) = \sup_{0<y<1} (\underbrace{c_0}_{= 1} - \sqrt{1-y}) = 1$.

So $\sum_0^\infty |c_k| < \infty \Rightarrow p_k = \sum_{j=0}^{k} c_j y^j$ are such that

$$\|p_k - \sqrt{1-y}\|_{C[0,1]} \leq \sum_{j=k+1}^{\infty} |c_j| \cdot \underbrace{\|y^j\|_{C[0,1]}}_{1} \longrightarrow 0. \quad ▨$$

Lemma: If A is a SW algebra, then $\forall x,y \in K.\ \exists h \in A.\ h(x)=1, h(y)=0.$

Proof: We know that $\exists f, g \in A.\ f(x) \neq 0,\ g(x) \neq g(y)$.
1) If $f(y)=0$, then $h := \frac{f}{f(x)}$.
2) If $f(y) \neq 0$, and $f(y) \neq f(x)$, then $h := \frac{f^2 - \overbrace{f(y) f}^{\in A}}{\underbrace{f^2(x) - f(x) f(y)}_{\neq 0}}$, $f^2 \in A$

$h = \frac{f - f(y)}{f(x) - f(y)}$ is not okay $\frac{f}{f(x)-f(y)} \in A$, $\frac{f(y)}{f(x)-f(y)}$ might not be in A, if A does not contain 1

3) If $f(y) \neq 0$, and $f(y) \neq f(x)$, then $\begin{pmatrix} f(x) \\ f(y) \end{pmatrix}, \begin{pmatrix} g(x) \\ g(y) \end{pmatrix}$ are not collinear in $\mathbb{R}^2 \Rightarrow \exists\, \alpha, \beta \in \mathbb{R}$.

$$\alpha \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} + \beta \begin{pmatrix} g(x) \\ g(y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow h := \alpha f + \beta g \quad ▨$$

Lemma: Let A be a SW algebra in $C_{\mathbb{R}}(K)$. Then $\forall f, g \in A$ we have $\min(f,g) \in \bar{A}$, $\max(f,g) \in \bar{A}$.

Proof: $\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$, $\max(f,g) = \frac{f+g}{2} + \frac{|f+g|}{2}$
So we only need to show that $|h| \in \bar{A}$ for every $h \in A$.
Let's approximate $|x|$ by polynomials $p_k \in \mathcal{P}$ on $[-a,a]$ for $a = \|h\|_{C_{\mathbb{R}}(K)}$. Then $p_k \circ h \in A$, $p_k(h) \rightrightarrows |h|$ on K. $\Rightarrow |h| \in \bar{A}$.

Lemma: Let $x \in K$, $\varepsilon > 0$, $\varphi \in C_{\mathbb{R}}(K)$. Then $\exists g_x \in \bar{A}$ s.t. $g_x(x) = \varphi(x)$ and $g_x(z) \geq \varphi(z) - \varepsilon$ for every $z \in K$.

Proof: For every $y \in K$ define $g_{x,y}$ to be a function in A: $\begin{array}{l} g_{x,y}(x) = \varphi(x) \\ g_{x,y}(y) = \varphi(y) \end{array}$
Such a function is a linear combination of functions

taking values 0 and 1 at points x, y.

Let $\mathcal{U} = \{z \in K \mid g_{x,y}(z) > f(z) - \varepsilon\}$, $y \in K$.

- For each $x \in K$ we have $K = \bigcup_{y \in K} \mathcal{U}_y$, because $y \in \mathcal{U}_y$.
- Each $\mathcal{U}_y$ is open, because $g_{x,y} - f$ is continuous, and $\mathcal{U}_y$ is the preimage of $(-\varepsilon, +\infty)$ - an open set.

$\Rightarrow$ By compactness, $\exists \{y_k\}_{k=1}^{N}$. $K = \bigcup_{k=1}^{N} \mathcal{U}_{y_k}$.

Now $g_x := \max_{1 \le k \le N} g_{x, y_k} \in \bar{A}$ works.

<u>Remark</u>: A SW algebra $\Rightarrow$ $\bar{A}$ SW algebra

<u>Proof of SW theorem</u>: Assume that A is a SW-algebra, and take $f \in C_{\mathbb{R}}(K)$, $\varepsilon > 0$. For every $x \in K$ construct g_x: $g_x(x) = f(x)$, $g_x(z) \ge f(z) - \varepsilon$ $\forall z \in K$. Consider $V_x = \{z \in K \mid g_x(z) < f(z) + \varepsilon\} \ni x$. Thus, V_x is open $\forall x \in K$, and $K = \bigcup_{x \in K} V_x \Rightarrow \exists \{x_k\}_{k=1}^{M}$. $K = \bigcup_{k=1}^{M} V_{x_k}$.

$$g = \min_{1 \le k \le M} g_{x_k} \in \bar{A} \quad \text{and} \quad f(z) - \varepsilon \le g \le f(z) + \varepsilon$$

$\Rightarrow \|g - f\|_{C_{\mathbb{R}}(K)} < \varepsilon \Rightarrow A$ is dense in $C_{\mathbb{R}}(K)$. This proves sufficiency.

<u>Necessity</u>: If A vanishes at some $x_0 \in K$, then 1 cannot be approximated uniformly by elements of A. If A does not separate points x, y, then the function f: $f \in C_{\mathbb{R}}(K)$, $f(x) = 1$, $f(y) = 0$ cannot be approximated. Such a function exists by Uryson's lemma ($K_1, K_2 \subset K$ - compact, $K_1 \cap K_2 \neq \emptyset \Rightarrow \exists f \in C_{\mathbb{R}}(K)$. $f|_{K_1} = 1$, $f|_{K_2} = 0$). $\blacksquare$

<u>Example</u>: Let $K = \mathbb{T} = \{|z| = 1\}$, $C(\mathbb{T})$ the complex Banach space of continuous functions. Consider $A = \mathcal{P} = \operatorname{span}\{z^k \mid z \in \mathbb{Z}_+\}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. A is an algebra, A does not vanish ($1 \in A$), A separates points ($z \in A$) (but we have a complex algebra).

<u>Observation</u>: $\int_{\mathbb{T}} \bar{z}\, \overline{p(z)}\, \underbrace{dm(z)}_{\text{Lebesgue measure on } \mathbb{T}, \text{ normalised by } m(\mathbb{T}) = 1} = 0 \quad \forall p \in A.$

Indeed, this follows from the fact that

$$\int_{\mathbb{T}} \bar{z} \cdot \bar{z}^k = 0 \quad \forall k \in \mathbb{Z}_+$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+1)t} dt = \frac{1}{2\pi} \frac{e^{-i(k+1)t}}{-i(k+1)} \Bigg|_0^{2\pi} = 0$$

Conclusion: $\bar{z} \perp A$ in $L^2(m)$. So, A cannot be dense, because otherwise $\bar{z} \perp C(\mathbb{T})$ in $L^2(m) \Rightarrow \bar{z} \perp L^2(m)$. Contradiction.
So, A is not dense and SW-theorem (real version) does not work in the complex space.

<u>Definition:</u> A is a complex Stone-Weierstrass algebra if
1) A does not vanish.
2) A separates points
3) A is closed under conjugation $f \longmapsto \bar{f}$.

In the example above, 3) does not hold.

<u>Theorem</u> [SW, complex]: An algebra A closed under conjugation is dense in $C(K) \Leftrightarrow A$ is a SW complex algebra.

<u>Proof:</u> A is SW complex algebra $\Rightarrow \mathrm{Re} f, \mathrm{Im} f \in A \ \forall f \in A \Rightarrow$
$\Rightarrow \mathrm{Re} A$ is a real SW-algebra $\Rightarrow \mathrm{Re} A$ is dense in $C_{\mathbb{R}}(K) \Rightarrow$
$\Rightarrow \mathrm{Re} A + i \, \mathrm{Re} A$ is dense in $C(K) = C_{\mathbb{R}}(K) + i\, C_{\mathbb{R}}(K)$ and it is contained in A. The other direction is trivial. $\blacksquare$

C^*-algebras: Gelfand-Naimark theorem

November 19, 2025

<u>Definition</u>: A Banach algebra A is called an algebra with involution, if there is an operation $*$ s.t.

1) $(\alpha x+\beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^* \quad \forall \alpha,\beta \in \mathbb{C} \quad \forall x,y \in A$
2) $(xy)^* = y^* x^* \quad \forall x,y \in A$
3) $(x)^{**} = x \quad \forall x \in A$

If moreover,

4) $\|x^*\| = \|x\| \quad \forall x \in A$
5) $\|x^* x\| = \|x^*\| \cdot \|x\| \quad \forall x \in A$

then A is called a C^*-algebra.

<u>Example</u>: Let K be a Hausdorff compact. Then $C(K)$ is a C^*-algebra, $f^* := \bar{f}$.

<u>Proposition</u>: Let $T \in \mathcal{B}(H)$, where H is a Hilbert space. Then $\|T\| \overset{(*)}{=} \sup_{\substack{\|x\|\le 1 \\ \|y\|\le 1}} |\langle Tx, y\rangle|$.

<u>Proof</u>: We will use the fact that $\forall h \in H.\ \|h\| = \sup_{\|x\|\le 1} |\langle h, x\rangle|$.
(indeed, $|\langle h,x\rangle| \le \|h\|\cdot\|x\| \le \|h\|$ by CS inequality, and taking $x = \frac{h}{\|h\|}$, we get $\langle h,x\rangle = \|h\|$.

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\| = \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} |\langle Tx,y\rangle| = \sup_{\substack{\|x\|\le 1 \\ \|y\|\le 1}} |\langle Tx,y\rangle| \qquad \blacksquare$$

<u>Definition</u>: Let $T \in \mathcal{B}(H)$, then $T^* \in \mathcal{B}(H)$ is the operator such that $\langle Tx,y\rangle = \langle x, T^* y\rangle \quad \forall x,y \in H$.

<u>Proposition</u>: $T^* \in \mathcal{B}(H)$ exists for every $T \in \mathcal{B}(H)$, moreover, $\|T^*\| = \|T\|$.

<u>Proof</u>: We will use Riesz theorem, which says that $\forall \varphi \in H^*$

$\exists!\, h \in H$ such that $\phi(x) = \langle x, h\rangle \ \forall x \in H$. Having this theorem, define: $\phi_y(x) := \langle Tx, y\rangle$, $x \in H$, where $y \in H$ is fixed

$\phi_y \in H^*$: $\sup\limits_{\|x\|\le 1} |\phi_y(x)| = \sup\limits_{\|x\|\le 1} |\langle Tx, y\rangle| \le \|T\| \cdot \|y\|$.

$\Rightarrow \exists!\, h \in H$. $\phi_y(x) = \langle x, h\rangle$. Let's define $T^*y := h$.

Since $\phi_{\alpha y_1 + \beta y_2} = \bar{\alpha}\phi_{y_1} + \bar{\beta}\phi_{y_2}$

$$\langle x, T^*(\alpha y_1 + \beta y_2)\rangle = \bar{\alpha}\underbrace{\langle x, T^* y_1\rangle}_{=\langle x, \alpha T^* y_1\rangle} + \bar{\beta}\underbrace{\langle x, T^* y_2\rangle}_{=\langle x, \beta T^* y_2\rangle} = \langle x, \alpha T^* y_1 + \beta T^* y_2\rangle \ \forall x \in H$$

$\Rightarrow T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2$.

We have used the fact that $\langle h, z_1\rangle = \langle h, z_2\rangle \ \forall h \Rightarrow z_1 = z$.
(Proof: $h: z_1 - z_2 \quad \langle z_1 - z_2, z_1 - z_2\rangle = 0$)

To compute the norm of T^*, we note that

$$\|T^*\| = \sup_{\substack{\|y\|\le 1\\ \|x\|\le 1}} |\langle T^* y, x\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle y, Tx\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\overline{\langle Tx, y\rangle}| = \|T\|$$

$\Rightarrow \|T^*\| = \|T\|$, in particular, $T^* \in \mathcal{B}(H)$. ▨

Proposition: $\|T^*T\| = \|T^*\| \cdot \|T\| \quad \forall T \in \mathcal{B}(H)$

Proof:
$$\|T^*T\| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle T^*Tx, y\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} \|\langle Tx, Ty\rangle\|$$
$$\ge \sup_{\|x\|\le 1} \|\langle Tx, Tx\rangle\|$$
$$= \|T\|^2$$

$\Rightarrow \|T\|^2 \le \|T^*T\| \le \|T^*\| \cdot \|T\| = \|T\|^2 \Rightarrow \|T\|^2 = \|T^*\| \cdot \|T\|$

In particular, $\|T^*T\| = \|T^*\| \cdot \|T\|$. ▨

Example: $\mathcal{B}(H)$ is a C^*-algebra. We only need to check that $(\alpha T_1 + \beta_2 T_2)^* = \bar{\alpha} T_1^* + \bar{\beta} T_2^*$, $(T_1T_2)^* = T_2^* T_1^*$, $(T^*)^* = T$.

This follows directly from the definition e.g. $\langle Tx,y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, (T^*)^*x\rangle} = \langle (T^*)^*x, y\rangle$

$\Rightarrow T^{**}$ and T have the same bilinear forms $\Rightarrow$

$\Rightarrow \|T^{**} - T\| = 0$ by $(*)$ for $T^{**} - T$ in place of T.

<u>Definition</u>: $T \in \mathcal{B}(H)$ is called normal if $T^*T = TT^*$.

<u>Example</u>: $A = \overline{\operatorname{span}\{T^k (T^*)^j \mid k, j \geq 0\}}$ is a commutative C^*-algebra for every normal operator $T \in \mathcal{B}(H)$.

The only nontrivial thing here is that $S \in A \Rightarrow S^* \in A$. This follows from the fact that $S_n \longrightarrow S$ in $\mathcal{B}(H) \Rightarrow S_n^* \longrightarrow S^*$ in $\mathcal{B}(H)$ $(\|S_n^* - S^*\| = \|S_n - S\|)$.

November 20, 2025

<u>Goal</u>: A is a commutative C^*-algebra $\Rightarrow A \cong C(K)$ for some Hausdorff compact K.
In fact, we will see that $K = A^*_{mult}$ with some topology, that we will now define.

<u>Definition</u>: Let X be a Banach space, X^* the dual space to X. Then $\sigma(X^*, X)$, the weak*-topology, is defined on X^* as follows: for $\phi \in X^*$, $\varepsilon > 0$, $F \in X$ - a finite subset, we define.

$V_{F,\varepsilon}(\phi) := \{\psi \in X^* \mid |\phi(x) - \psi(x)| < \varepsilon \;\; \forall x \in F\}$ (neighbourhood of ϕ, corresponding to F, ε)

Open subsets in $\sigma(X^*, X)$ are precisely those subsets S that have the following property: $\forall \phi \in S. \exists F, \varepsilon. V_{F,\varepsilon}(\phi) \subset S$.

<u>Remarks</u>: (1) If X is separable, then this topology is metrizable on each bounded subset of X, and closed subsets could be defined as follows: S is closed $\Leftrightarrow \forall \phi_n \in S. \phi_n(x) \longrightarrow \phi(x)$ $\forall x \in X$ we have $\phi \in S$.

(2) In the general situation $\sigma(X^*, X^*)$ is Hausdorff.
Indeed, take $\phi_1, \phi_2 \in X^*$, $\phi_1 \neq \phi_2 \Rightarrow \exists\, x \in X.\ \phi_1(x) \neq \phi_2(x) \Rightarrow$
$\Rightarrow V_{F_1,\varepsilon}(\phi_1) \cap V_{F_2,\varepsilon}(\phi_2) = \emptyset$ for $F_1 = \{x\} = F_2$, $\varepsilon = \frac{|\phi_1(x) - \phi_2(x)|}{2}$

(3) The importance of this topology is explained by the following theorem.

→ Proof: Functional Analysis

Theorem [Banach-Alaoglu]: Any bounded $\underbrace{\text{closed}}_{\text{in norm (in the initial topology)}}$ subset of X^* is compact in the $\sigma(X^*, X)$-topology for every Banach space X.

Lemma: Let A be a commutative Banach algebra. Then A^*_{mult} with the induced topology from $\sigma(X^*, X)$ is Hausdorff compact.

Proof: Since $A^*_{mult} \subset \mathcal{B}_{X^*}[0,1]$, by B-A theorem, we only need to check that A^*_{mult} is closed. Assume that $\phi_n \in A^*_{mult}$, $\phi_n \longrightarrow \phi$ in $X^* \Rightarrow \phi \in X^*$ and $\phi(xy) = \lim_{n\to\infty} \phi_n(xy) = \phi(x) \cdot \phi(y)$ $\forall x, y \in X$. ∎

Definition: Let A be a commutative Banach algebra. and for every $x \in A$ define $\hat{x} : \phi \longrightarrow \phi(x)$, $\phi \in A^*_{mult}$. The mapping $x \longmapsto \hat{x}$ is called the Gelfand transform.

Lemma: For every $x \in A$ we have $\hat{x} \in C(A^*_{mult})$.

Proof: Take $x \in A$, $U \subset \mathbb{C}$-open subset. We need to show that $\hat{x}^{-1}(U)$ is open in $\sigma(X^*, X)$. It is enough to prove that $\hat{x}^{-1}(B(z_0, \varepsilon))$ is open $\forall z_0 \in U \cap \hat{x}(A^*_{mult})$ $\forall \varepsilon > 0$.
Let $z_0 \in \hat{x}(A^*_{mult})$, i.e., $\exists \psi.\ z_0 = \hat{x}(\psi) = \psi(x)$.
Then $\hat{x}^{-1}(B(z_0, \varepsilon)) = \{\phi \in A^*_{mult} \mid |\hat{x}(\phi) - z| < \varepsilon\} = \underbrace{V_{F,\varepsilon}(\psi) \cap A^*_{mult}}_{\text{open in induced topology}}$,
where $F = \{x\}$. ∎

Lemma: Let A be a C^*-algebra and $x \in A$ normal $(x^*x = xx^*)$. Then $\|x\| = r(x)$.

Proof: At first assume $x^* = x$, then

$$\|x^{2^n}\| = \|x^{2^{n-1}} \cdot x^{2^{n-1}}\| = \|(x^{2^{n-1}})^* x^{2^{n-1}}\| = \|x^{2^{n-1}}\|^2 = [\text{iterate}] = \|x\|^{2^n}$$

$$\Rightarrow r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \lim_{n\to\infty} \|x\| = \|x\|.$$

Now the general case: $x^*x = xx^*$. Consider $y = x^*x$, and note that $\|y\| = r(y)$, because $y^* = y$. We have $\|x\|^2 = \|y\|$, and

$$r(y) \overset{x \text{ normal}}{=} \lim \|(x^*)^n x^n\|^{\frac{1}{n}} = r(x)^2$$

$\llcorner$ C^* property and the definition of $r(x)$

$\Rightarrow r(x) = \|x\|.$ ▨

Lemma: Let A be a Banach algebra, $x, y \in A$, $xy = yx$. Then $e^x \cdot e^y = e^{x+y}$, where $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in A$.

Proof:
$$e^x \cdot e^y = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{j=0}^{\infty} \frac{y^j}{j!}\right) = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{k+j=n} x^k \cdot y^j \frac{1}{k!} \cdot \frac{1}{j!} \cdot (k+j)!\right)}_{xy=yx \,\to\, (x+y)^n} \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e^{x+y}$$ ▨

Theorem [Gelfand–Naimark]: Let A be a commutative C^*-algebra. Then the Gelfand transform is the isomorphism of C^*-algebras A and $C(A^*_{mult})$, i.e.

1) $\widehat{\alpha x + \beta y} = \alpha \hat{x} + \beta \hat{y}$
2) $\widehat{xy} = \hat{x}\hat{y}$
3) $\widehat{x^*} = \overline{\hat{x}}$
4) $\|\hat{x}\|_{C(A^*_{mult})} = \|x\|$
5) $x \mapsto \hat{x}$ is a bijection between A and A^*_{mult}.

Proof: 1) $\widehat{\alpha x + \beta y}(\phi) = \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y) = \alpha\hat{x} + \beta\hat{y}$

2) $\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}\hat{y}$

3) Take $x \in A$, and observe that $x = \mathrm{Re}\,x + i\,\mathrm{Im}\,x$.

$$\mathrm{Re}\,x = \frac{x + x^*}{2}, \quad \mathrm{Re}\,x = (\mathrm{Re}\,x)^*$$

$$\mathrm{Im}\,x = \frac{x - x^*}{2}, \quad \mathrm{Im}\,x = (\mathrm{Im}\,x)^*$$

Moreover, $\widehat{(\mathrm{Re}\,x + i\,\mathrm{Im}\,x)^*} = \widehat{\mathrm{Re}\,x} - i\,\widehat{\mathrm{Im}\,x}$

$\Rightarrow$ it suffices to prove that $\hat{x} = \overline{\hat{x}}$ for every $x = x^*$.

$$\hat{x} = \overline{\hat{x}} \iff \phi(x) = \overline{\phi(x)} \iff |e^{i\phi(x)}| = 1 \iff |\phi(e^{ix})| = 1$$

$$\underset{\text{identity}}{1 = |e_A|} = |\phi(e_A)| \underset{\substack{\text{lemma, } ix - ix = 0 \\ e^0 = e_A}}{=} |\phi(e^{ix}) \cdot \phi(e^{-ix})| \leq \|e^{-ix}\| \cdot \|e^{ix}\| \underset{C^*\text{-property } \& \ (e^{ix})^* = e^{-ix}}{=} \|(e^{ix})^* e^{ix}\| = \|e_A\| = 1$$

$\Rightarrow$ all inequalities are in fact equalities, $|\phi(e^{ix})| \cdot |\phi(e^{-ix})| = 1$.

4) $\|x\| \overset{A \text{ commutative, so } x^*x = xx^*}{=} r(A) = \sup\{|\lambda|, \lambda \in \sigma(x)\}$

description of spectrum in commutative Banach algebra

$$= \sup\{|\lambda|, \lambda \in \{\phi(x), \phi \in A^*_{mult}\}\}$$

$$= \sup\{|\phi(x)|, \phi \in A^*_{mult}\}$$

$$= \sup_{\phi \in A^*_{mult}} |\hat{x}(\phi)| = \|\hat{x}\|_{C(A^*_{mult})}$$

5) $\hat{A}$ is a closed subalgebra in $C(A^*_{mult})$ because it is an isometric image of a closed algebra ($= A$).

It remains to check that $\hat{A}$ does not vanish and separates points (SW-theorem then implies $\hat{A} = \mathrm{clos}\,\hat{A} = C(A^*_{mult})$).

i) $\hat{A}$ does not vanish $\iff \forall \phi \in A^*_{mult}\ \exists x \in A.\ \hat{x}(\phi) \neq 0 \iff \phi \neq 0$. This holds, because $\phi(e_A) = 1$.

ii) $\hat{A}$ separates points $\iff \forall \phi \neq \psi.\ \exists x \in A.\ \hat{x}(\phi) \neq \hat{x}(\psi) \iff \phi \neq \psi$. ∎

Application: functional calculus for C^*-algebras

Definition: Let A be a commutative C^*-algebra, $x \in A$, $\sigma(x)$ the spectrum of x, $f \in C(\sigma(x))$. Then we define:

$$f(x) := (f(\hat{x}))^{\vee},$$

where "$\vee$" is the inverse Gelfand transform.

Remark: $f(x)$ is defined correctly, in other words, $\operatorname{Ran}\hat{x} \subset \operatorname{Dom} f$.
Indeed, $\operatorname{Ran}\hat{x} = \{\hat{x}(\phi),\ \phi \in A^*_{mult}\} = \{\phi(x),\ \phi \in A^*_{mult}\} = \sigma(x)$.

Remark: If f is a polynomial, $f = c_0 + c_1 z + \dots + c_n z^n$, then $f(x) = c_0 + c_1 x + \dots + c_n x^n$ for every $x \in A$.
Indeed, it suffices to check this for $f_n = z^n$, $z \in \mathbb{C}$:
$f_n(x) = (z^n(\hat{x}))^{\vee} = (\widehat{x^n})^{\vee} = x^n$.

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Remark: For $f(z) = \bar{z}$ we have $f(x) = x^*$ because
$$\bar{z}(\hat{z}(\cdot))^{\vee} = (\overline{\hat{x}(\cdot)})^{\vee} = (\widehat{x^*})^{\vee} = x^*.$$

Remark: If $f_1, f_2 \in \mathcal{C}(\sigma(x))$, then $(f_1 \cdot f_2)(x) = \underbrace{f_1(x)}_{\in A} \overset{\text{mult. in } A}{\cdot} \underbrace{f_2(x)}_{\in A}$.

Theorem [Functional calculus theorem]: Assume that A is a commutative C^*-algebra generated by some element $x \in A$ (polynomials of x, x^* are dense in A). Then $A \cong \mathcal{C}(\sigma(x))$ as C^*-algebras. Moreover, the isomorphism between $\mathcal{C}(\sigma(x))$ and A is given by the map $f \mapsto f(x)$, where $f(x) = (f(\hat{x}(\cdot)))^{\vee}$. In particular, we have:

(1) $\|f(x)\| = \max\limits_{z \in \sigma(x)} |f(z)| \quad \forall f \in \mathcal{C}(\sigma(x))$

(2) $\sigma_A(f(x)) = f(\sigma(x)) \quad \forall f \in \mathcal{C}(\sigma(x))$

spectral mapping theorem

Lemma 1: Let K_1, K_2 be Hausdorff compacts, and let $h: K_1 \longrightarrow K_2$ be a continuous bijection. Then h is a homeomorphism.

Proof: We need to check $h^{-1}: K_2 \longrightarrow K_1$ is continuous $\Leftrightarrow$
$(h^{-1})^{-1}(U)$ is open in K_2 for every U-open in K_1 $\Leftrightarrow$
$(h^{-1})^{-1}(C)$ is closed in K_2 for every C-closed in K_1 $\Leftrightarrow$
$h(C)$ is compact for every C-compact in K_1.
Take some open cover $\{V_\alpha\}_{\alpha \in I}$ of $h(C)$, and consider $\{h^{-1}(V_\alpha)\}_{\alpha \in I}$ - open cover of C, so $\exists \alpha_1, \dots, \alpha_N$ s.t. $\{h^{-1}(V_{\alpha_k})\}_{k=1}^{N}$ subcover of C
$\Rightarrow \{V_{\alpha_k}\}_{k=1}^{N}$ is a subcover of $h(C)$. ▨

Lemma 2: Assume that $K_1, K_2, h: K_1 \to K_2$ are as in the previous lemma. Then $\mathcal{C}(K_1) \cong \mathcal{C}(K_2)$ as C^*-algebras, and the isomorphism is given by $\underset{\in \mathcal{C}(K_1)}{f(\cdot)} \longmapsto f(h^{-1}(\cdot))$.

Proof: Exercise $\left(\|f\|_{C(K_1)} = \|f(h^{-1})\|_{\mathcal{C}(K_2)}, \text{ etc.} \right)$

Proof of the theorem: We know that $A \cong \mathcal{C}(A^*_{mult})$ by GN theorem. To prove $A \cong \mathcal{C}(\sigma(x))$, we will check that $\hat{x}: \phi \longmapsto \hat{x}(\phi) = \phi(x)$ is a homeo from A^*_{mult} onto $\sigma(x)$ and then apply the last lemma.

1) $\hat{x}(A^*_{mult}) = \sigma(x)$ description of the spectrum in a commutative B.A.

$\{\hat{x}(\phi),\ f \in A^*_{mult}\} = \{\phi(x) \mid \phi \in A^*_{mult}\}$ surjectivity

2) Let $\hat{x}(\phi_1) = \hat{x}(\phi_2)$ for some $\phi_1, \phi_2 \in A^*_{mult}$ $\Leftrightarrow$

$\Leftrightarrow \phi_1(x) = \phi_2(x) \Rightarrow \begin{array}{l} \phi_1(x) = \phi_2(x) \\ \phi_1(x^*) = \phi_2(x^*) \end{array} \Rightarrow \phi_1(p(x, x^*)) = \phi_2(p(x, x^*))$ for any polynomial $p(z, \bar{z})$

$\Rightarrow \phi_1 = \phi_2$ by continuity

$$\left(\begin{array}{ccc} \phi_k(x^*) & = & \overline{\phi_k(x)} \\ \| & & \| \\ \widehat{x^*}(\phi_k) & \underset{GF}{=} & \overline{\hat{x}(\phi_k)} \end{array} \quad \text{(part of GN theorem, because } f^* = \bar{f} \text{ in } \mathcal{C}(K)) \right)$$

3) We also know that $\hat{a} \in \mathcal{C}(A^*_{mult})$ for every $a \in A$ (again GN).
In particular, for $a = x$ we get continuity of the map $\hat{x} : \phi \mapsto \phi(x)$.
By Lemma 1, we conclude that $\hat{x}$ is a homeomorphism and by Lemma 2, that $A \cong \mathcal{C}(\sigma(x))$. Moreover:

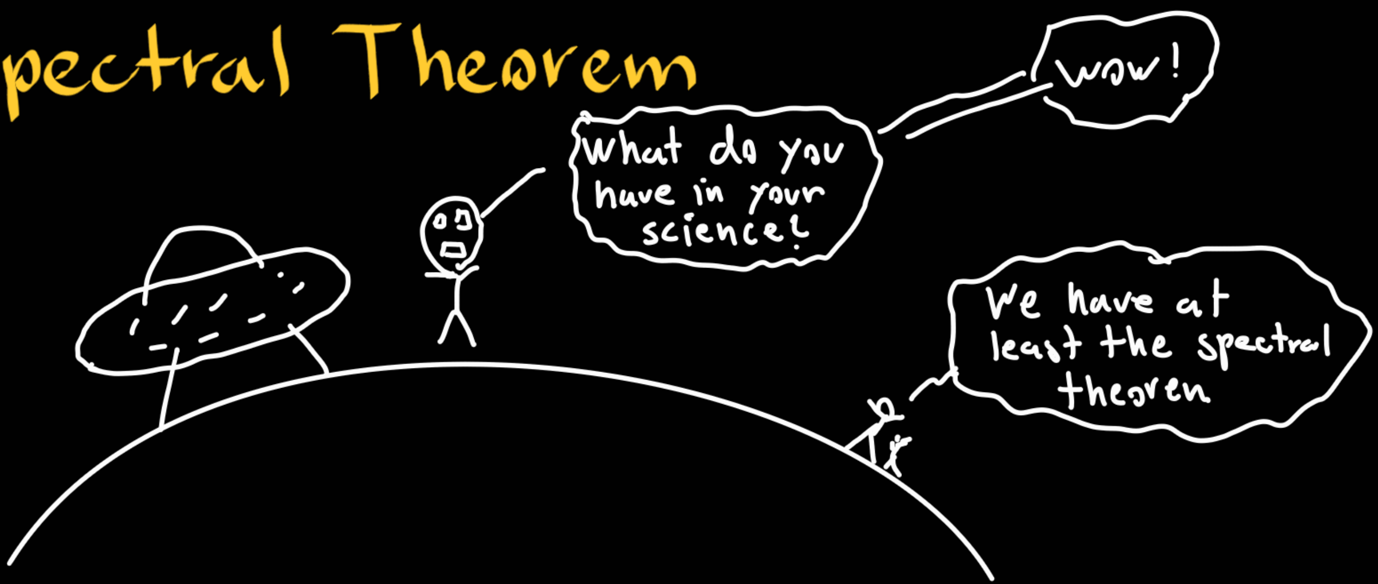
$$A \cong \mathcal{C}(A^*_{mult}), \quad a \mapsto \hat{a} \qquad \mathcal{C}(A^*_{mult}) \cong \mathcal{C}(\sigma(x)), \quad f(\hat{x}(\cdot)) \longleftarrow f \qquad A \cong \mathcal{C}(\sigma(x)), \quad (f(\hat{x}(\cdot)))^\vee \longleftarrow f$$

$$(f(\hat{x}(\cdot)))^\vee \overset{\text{definition}}{=} f(x)$$

Finally: (1) $\|f(x)\|_A \overset{\text{iso}}{=} \|f\|_{\mathcal{C}(\sigma(x))} = \max_{z \in \sigma(x)} |f(z)|$

(2) $\sigma_A(f(x)) = \sigma_{\mathcal{C}(\sigma(x))}(f) = f(\sigma(x))$ ← know from before ∎

Spectral Theorem



Theorem [spectral theorem]: Let H be a separable Hilbert space, $T \in \mathcal{B}(H)$: $T^*T = TT^*$. Then T is unitary equivalent to the direct sum of multiplicative operators: there are Borel measures μ_k, $\operatorname{supp} \mu_k \subseteq \sigma(T)$ and a unitary operator $\mathcal{U}$ such that $\mathcal{U}^{-1} T \mathcal{U} = \bigoplus M_{k,z}$, where $M_{k,z} : f \mapsto zf$ on $L^2(\mu_k)$.

Particular examples:

1) $A = A^*$ $(\langle Ax, y\rangle = \langle x, Ay\rangle \ \forall x, y \in H)$ — Physics: symmetry

2) $\|Ax\| = \|x\|$, $A : H \to H$ is bijection $\left(\begin{matrix} A \text{ is unitary} \\ \text{or } A^*A = AA^* = \mathbb{1}\end{matrix}\right)$ — conservation laws

Definition: Let $\{H_k\}_{k=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$ be a sequence of Hilbert spaces. Then $\bigoplus_{k=1}^N H_k$ is the Hilbert space $\{\{h_k\}_{k=1}^N : \sum_{k=1}^N \|h_k\|^2 < \infty, h_k \in H_k\}$ with the inner product $\langle \{h_k\}, \{g_k\} \rangle_{\bigoplus_{k=1}^N H_k} = \sum_{k=1}^N \langle h_k, g_k \rangle_{H_k}$.

Example: $\ell^2(\mathbb{Z}) = \bigoplus_{k=-\infty}^{\infty} \mathbb{C}$ $\quad (\{a_n\} \subset \mathbb{C}, \sqrt{\sum |a_n|^2} = \|\{a_n\}\|)$

Definition: Let $T_k : H_k \longrightarrow H_k$ be a bounded linear operator for every k. Assume that $\|T_k\| \leq C$ $\forall k = 1, \dots, N$. Then $T = \bigoplus_{k=1}^N T_k$ is an operator on $H = \bigoplus_{k=1}^N H_k$ defined by $T(\{h_k\}_1^N) = \{T_k h_k\}_1^N \in H$.

Remark: $\|T\| = \sup_{1 \leq k \leq N} \|\overset{C}{T_k}\|$, because

$$\|T(\overbrace{\{h_k\}}^{h})\|_H^2 = \sum_1^N \|T_k h_k\|^2 \leq C^2 \sum_1^N \|h_k\| = C^2 \|h\|^2 \Rightarrow \|T\| \leq C$$

$\|T\| \geq C - \varepsilon$ for every ε, because $\|T\| \geq \|T_k\|$ $\forall k$.

(consider h of the form $\{0, 0, \dots, 0, g, 0, \dots, 0\}$
$\uparrow$ k-th position, arbitrary $g \in H_k$ with $\|g\| = 1$

Definition: An operator U is unitary if $U^*U = UU^* = I$.

Proposition: Let $U \in \mathcal{B}(H)$. TFAE:

1) U is unitary
2) U is a bijection from H onto H and $\|U(x)\| = \|x\|$ for every $x \in H$.

Proof: Exercise. $\quad \langle Ux, Ux \rangle = \langle x, x \rangle$, $\langle Ux, Ux \rangle = \langle U^*Ux, x \rangle$ $\leftarrow$ hint

Definition: Let $T \in \mathcal{B}(H)$. Then $E \subset H$ is a reducing subspace of T if $TE \subset E$ and $T^*E \subset E$.

Proposition: E is reducing for $T \Leftrightarrow TE \subset E$, $TE^\perp \subset E^\perp$, where $E^\perp = \{h \in H \mid \langle h, g \rangle = 0 \ \forall g \in E\}$.

Proof: $TE^\perp \subset E^\perp \Leftrightarrow TE^\perp \perp E \Leftrightarrow E^\perp \perp T^*E \Leftrightarrow T^*E \subset E$. ▨

$\langle Tg^\perp, h \rangle = 0 \Leftrightarrow \langle g^\perp, T^*h \rangle = 0$