Operator theory

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PART I

Compact and Fredholm operators

Preliminaries

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<u>Def:</u> (X, ρ) - metric space if X-set, and ρ is a metric:

i) $\rho(x,y) \geq 0 \quad \forall x,y \in X$. $\rho(x,y) = 0 \Leftrightarrow x = y$

ii) $\rho(x,y) = \rho(y,x) \quad \forall x,y \in X$

iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \forall x,y,z \in X$

<u>Def:</u> $\mathcal{U} \subseteq X$ is open if $\forall x \in \mathcal{U}$. $\exists \delta > 0$. s.t. $B(x, \delta) \subset \mathcal{U}$ $\quad (B(x,\delta) = \{y \in X \mid \rho(x,y) < \delta\})$

<u>Def:</u> $K \subset X$ is compact if every open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of K has a finite subcover.

cover: $\{\mathcal{U}_\alpha\}_\alpha$ is a cover of K if $\bigcup_{\alpha \in I} \mathcal{U}_\alpha \supset K$)

<u>Def:</u> A precompact set $A \subset X$ is a set $A \subset X$ s.t. $\overline{A}$ is compact.

closure of A in X

Def: $\{x_j\}_{j\geq 1}$ is Cauchy sequence in X if
$\forall \varepsilon>0.\ \exists N=N(\varepsilon).\ \rho(x_j,x_k)<\varepsilon\ \forall j,k\geq N(\varepsilon).$

Def: X is complete if $\forall$ Cauchy $\{x_j\}_{j\geq 1}\subset X.$
$\exists x\in X.\ \rho(x_j,x)\to 0$ as $j\to\infty$
(Every Cauchy sequence converges.)

Ex. $\left(\mathbb{R}^n,\ \rho_{\mathbb{R}^n}(\{x_i\},\{y_j\}):=\sqrt{\sum|x_i-y_i|}\right)$ – complete metric space

Ex. $(\mathbb{Q},\ \rho_{\mathbb{R}^n})$ – metric space but non-complete

Ex. $[0,1]$ is a compact-subset of $(\mathbb{R},\rho_{\mathbb{R}})$

Th: $K\subseteq\mathbb{R}^n$ is compact $\Leftrightarrow$ closed and bounded

Def: $A\subset(X,\rho)$ is bounded if $\exists x\in X.\ \exists R>0.\ A\subset B(x,R)$

Ex.: $(X,\rho)=\ell^2(\mathbb{Z})=\left(\left\{\{x_j\}_{j\in\mathbb{Z}}\ \middle|\ \sum_{j\in\mathbb{Z}}|x_j|^2<\infty,\right.\right.$ ($x_j\in\mathbb{C}$)

$$\rho(\{x_j\},\{y_j\})=\sqrt{\sum_{j\in\mathbb{Z}}|x_j-y_j|^2}$$

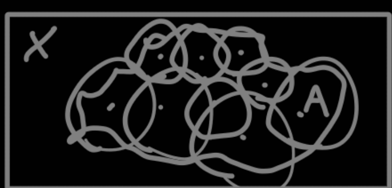
$$B[0,1]=\{y\in\ell^2(\mathbb{Z}):\rho(0,y)\leq 1\}$$

– bounded, closed but not compact HW

Theorem: Let (X,ρ) be a complete metric space, $A\subset X$. The following assertions are equivalent:

i) A is precompact

ii) $\forall\varepsilon>0.\ \exists_N$ a Finite ε-net $\{x_j\}_{j=1}^{N_\varepsilon}$ in A, that is, $\bigcup_{j=1}^{N}B(x_j,\varepsilon)\supset A.$



iii) $\forall \{x_j\}_{j\geq 1} \subset A$ there is a converging subsequence to some element $x \in X$.

<u>Proof</u>: i) $\Rightarrow$ ii) $\{U_x\}_{x\in A} = \{B(x,\varepsilon)\}_{x\in A}$ - open cover of A.

- is an open cover of $\bar{A}$:

$(\forall y \in \bar{A}. \exists x \in A. \rho(x,y) < \varepsilon/2 \Rightarrow y \in B(x,\varepsilon))$

definition of closure

$\Rightarrow \exists \{U_{x_j}\}_{j=1}^{N}$ - finite subcover of $\bar{A}$

$\Rightarrow \{U_{x_j}\}_j^N$ is a ε-net in A

ii) $\Rightarrow$ iii) Observe that $\forall \varepsilon > 0$. any sequence $\{y_i\} \subset A$ has an infinite subsequence that is contained in some $B(x,\varepsilon)$. (we have a finite ε-net)

Assume that $\{y_k\}$ is arbitrary in A.

Cantor diagonal process

$y_1, \underbrace{y_2, y_3, y_4, \dots}$

$\underbrace{y_1}, y_{k_2}, \underbrace{y_{k_3}, \dots}$ where $\{y_{k_j}\} \subset B(x_2, \tfrac{1}{2})$

$\underbrace{y_1, y_{k_2}}, y_{kj_3}, y_{kj_4}$ where $\{y_{kj_s}\} \subset B(x_3, 1/3)$, $\in X$

$\underbrace{y_1, y_{k_2}, y_{kj_3}}, y_{kjs_4}$ ...

Consider $z_1 = y_1$

$z_2 = y_{k_2}$

$z_3 = y_{kj_3}$

$z_4 = y_{kjs_4}$

$\vdots$

Claim: $\{z_j\}$ is a Cauchy sequence.

Indeed $\underset{k<j}{}$ $\rho(z_j, z_k) < \frac{1}{k}$

because $z_j, z_k \in B(x_k, \frac{1}{k})$ $\rho(z_j, z_k) < \underset{\to \infty}{\frac{2}{K}}$ $\Rightarrow$ ok

X is complete $\Rightarrow$ $\{z_j\}$ converges

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iii) $\Rightarrow$ i):

Plan: a) A contains a dense countable subset

a)+b)+c) under assumption 3)

b) If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $\bar{A}$
$\Rightarrow \exists \{U_{\alpha_j}\}_{j \in J}$ - an open coutable subcover of $\bar{A}$

c) $\Rightarrow \{U_{\alpha_j}\}_{j=1}^{r}$ is a cover of $\bar{A}$

a) Observe that $\forall \varepsilon > 0$ there exists at most $N(\varepsilon)$ points $A \supset \{y_j(\varepsilon)\}_{1 \le j \le N(\varepsilon)}$ s.t. $\rho(y_k(\varepsilon), y_j(\varepsilon)) > \varepsilon$ $\forall k \ne j$.
(If this is not true, then $\exists \{y_j(\varepsilon)\}_{j=1}^{\infty}$ such that $\rho(y_j(\varepsilon), y_k(\varepsilon)) > \varepsilon$ and it cannot contain a convergent subsequence by Cauchy criterion.)

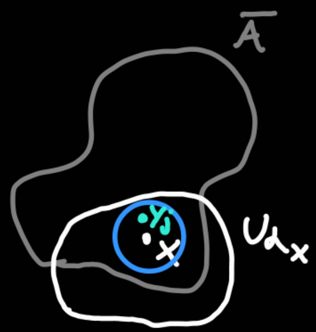
Now $E = \left\{ y_k\{\frac{1}{n}\} \mid 1 \le k \le N(\frac{1}{n}),\ n \ge 1 \right\}$ is a dense countable subset.
(E is dense since $\forall n . \forall x \in A .$ $\min\limits_{1 \le k \le N(\frac{1}{n})} (\rho(x, y_k(\frac{1}{n}))) \le \frac{1}{n}$ by construction)

b) Assume that $\{U_\alpha\}_{\alpha \in I}$ is some open cover of $\bar{A}$.
For every $x \in \bar{A}$ define

$$\varepsilon(x) \overset{(*)}{:=} \frac{\sup\{\varepsilon > 0 \mid B(x, \varepsilon) \subset U_\alpha \text{ for some } \alpha\}}{2} > 0$$

Claim: if $\{y_j\}_{j=1}^{\infty}$ is a countable dense subset in A, then

$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$.



Take $x \in \bar{A}$ $\exists \alpha_x.\ x \in U_{\alpha_x}$, and let $\eta(x) > 0$ such that $B(x, \eta(x)) \subset U_{\alpha_x}$ (U_{α_x} is open $\Rightarrow \eta(\varepsilon)\ \exists$)

Find y_j such that $\rho(x, y_j) < \frac{\eta(x)}{10}$.

Then, since $\eta(x) \leq 2\varepsilon(x)$, $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow \rho(x, y_j) < \varepsilon(y_j)$

$\varepsilon(y_j) \geq \frac{\eta(x)}{10}$ - Because $B(y_j, \frac{\eta(x)}{5}) \subset U_{\alpha_x}$ by triangle inequality and $\varepsilon(y_j)$ satisfies ⊛.

Since $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, then $\{U_{\alpha_{y_j}}\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, where $U_{\alpha_{y_j}}$ is the set U_α from the definition of $\varepsilon(y_j)$ (that is, $U_{\alpha(y_j)} \supseteq B(y_j, \varepsilon(y_j))$).

Since we have ⊛⊛, $\bigcup_{j=1}^{\infty} U_{y_{\alpha_i}} \supset \bar{A}$.

c) <u>Claim</u>: $\exists N.\ \{U_{\alpha_{y_j}}\}_{j=1}^{N}$ is a cover of $\bar{A}$.

Suppose this is not the case $\Rightarrow \forall j \geq 1.\ \exists x_j \in A_j \setminus \bigcup_{k=1}^{j} U_{y_j}$.
Consider $\{x_j\}_{j=1}^{\infty}$, and assume that the sequence $\{x_{j_k}\}$ - convergent to some $x \in X$. Note that $x \in \bar{A}$ ($x_j \in \bar{A}$).
$\Rightarrow \exists j_*.\ x \in U_{\alpha_{j_*}} \Rightarrow \exists \delta > 0.$ s.t. $B(x, \delta) \subset U_{\alpha_{j_*}}$, but $x_{j_k} \notin U_{\alpha_{j_*}}$ for large k by construction.
(in particular, $x_{j_k} \notin B(x, \delta)$, hence $\rho(x, x_{j_k}) > \delta$, but this contradicts the fact that $x_{j_k} \longrightarrow x$).
We have shown that $\{x_j\} \subset \bar{A}$ cannot have a convergent subsequence.
Then if $\tilde{x}_j \in A.\ \rho(\tilde{x}_j, x_j) < \frac{1}{j}$, then $\{\tilde{x}_j$ also has no convergent subsequence. So, we assumed there is no

finite subcover $\{U_{\alpha_{y_j}}\}$ and found a sequence $\{\tilde{x}_j\} \subset A$ that has no converging subsequence, a contradiction with 3). Therefore 3) $\Rightarrow$ 1). ▨

<u>Examples of compact sets and their properties:</u>

1) $K \subset (X, \rho)$ is compact $\Rightarrow$ K is bounded

Indeed, if K is not bounded, then $\{B(x, n)\}_{n \geq 1}$ is an open cover without a finite subcover.

2) $K \subset (X, \rho)$ is compact, then it is closed
($\Leftrightarrow \{x_j\} \subset K$ such that $x_j \to x$ in (X, ρ) we also have $x \in K$)

Let's check that $X \setminus K$ is open. Take $y \in X \setminus K$, take $x \in K$, let $\delta(x) > 0$. $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$
$\{B(x, \delta(x))\}_{x \in K}$ is an open cover, let $\{B(x_j, \delta(x_j))\}_{j=1}^{N}$ be a finite subcover, then $\delta := \min_{1 \leq j \leq K} \delta(x_j)$, $B(y, \delta) \cap K = \emptyset$
$\Rightarrow X \setminus K$ is open.

<u>Another proof</u>: Suppose $\{y_j\} \subset K$ s.t. $y_j \to y$, $y \notin K$.

$$U_j = \{x \in X \mid \rho(x, y) > \frac{1}{j}\}$$

$\{U_j\}_{j=1}^{\infty}$ open cover, $\{U_{j_k}\}_{k=1}^{N}$ finite subcover

$\varepsilon := \min_{1 \leq k \leq N} \left(\frac{1}{j_k}\right)$, $\rho(x, y) > \varepsilon \quad \forall x \in K$ contradiction ▨

linear space = vector space

3) Let X be a finite-dimensional complete linear normed space. Then $E \subset X$ is compact $\Leftrightarrow$ E is closed and bounded.

$(\Rightarrow)$: Examples 1+2.

$(\Leftarrow)$: $X = \{ \sum_1^N a_k e_k \mid c_k \in \mathbb{C} \}$, $N = \dim X$, $\{e_k\}_1^N$ - basis

$$\varphi\left(\sum a_k e_k\right) = \max_{1 \le k \le N} |c_k| \quad \text{- norm on } X$$

Since all norms on finite dimensional vec. spaces are equivalent.

$\exists\, A, B > 0.\quad A\|x\| \le \varphi(x) \le B\|x\| \quad \forall x \in X.$

in particular the set $\left\{ \{c_k(x)\}_1^N, \; \begin{matrix} x \in E \\ x = \sum c_k(x) e_k \end{matrix} \right\}$

is bounded (= bdd) in $\mathbb{C}^n$ For every bounded $E \subset X$.

$$\sup_{x \in E} \varphi(x) \le B \cdot \sup_{x \in E} \|x\| < \infty$$

$\Rightarrow$ for any sequence $\left\{ \{c_k(x_j)\}_{k=1}^N \right\}_{j=1}^\infty$ one can extract a converging subsequence in $\mathbb{C}^n$, i.e.

$$c_k(x_{j_n}) \longrightarrow c_k \qquad n \to \infty.$$

But then $\sum_1^N c_k(x_{j_n}) e_k \longrightarrow \sum c_k e_k$ in X

$\Rightarrow$ bounded subsets are precompact in X

$\Rightarrow$ bdd + closed sets are compact

4) $\ell^2(\mathbb{Z}) = \left\{ \{c_k\}_{k \in \mathbb{Z}} \;\middle|\; \sum |c_k|^2 < \infty \right\}$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^\infty |c_k|^2}$$

$$B[0,1] = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \;\middle|\; \|\{c_k\}\| \le 1 \right\}$$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite $\frac{1}{2}$-net in $B[0,1]$, beca $\rho(e_k, e_j) > 1/2$ for $e_k = (0,\dots,0,\underset{\text{k-th place}}{\hat{1}},0,\dots,0)$.

Definition: X is a Banach space if it is a linear normed space such that X is complete with respect to this norm.

Example: Let (K, ρ) be a metric compact space.

$$C(K) := \{ f: K \to \mathbb{C} \mid \text{cont. in the metric } \rho \}$$
$$(\Leftrightarrow f(x_j) \to f(x) \quad \forall x_j \to x \quad \text{in } (K,\rho))$$

$$\|f\|_{C(K)} = \|f\| := \max_{x\in K} |f(x)| \quad \text{- norm in } C(K)$$

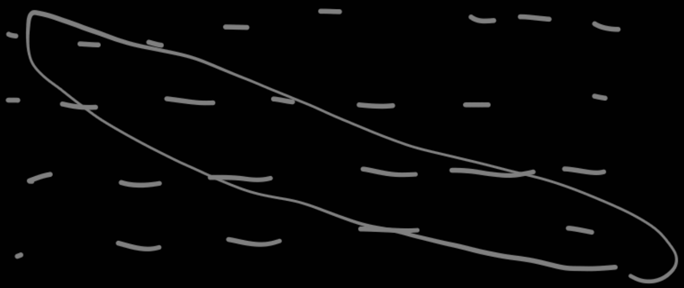
Theorem: [Arzela-Ascoli]: Assume that K is a complete compact metric space. $E \subset C(K)$ is precompact $\Leftrightarrow$

$\Leftrightarrow$ 1) E is bounded in C

2) Functions in E are equicontinuous, that is, $\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0.\ |f(x) - f(y)| < \varepsilon \quad \forall x, y \in K.\ \rho(x,y) < \delta_\varepsilon \quad \forall f \in E$

We will need 1)+2) $\Rightarrow$ precompactness.

Proof: Find a dense sequence $\{x_j\}$ in K.
(such sequence exists because K is compact)
Then take $\{f_n\}$ arbitrary sequence in E.
We want to find a converging subsequence of $\{f_n\}$
(then E - precompact)
For this find a subsequence $\{f_{n_k}\}$ such that
$$f_{n_k}(x_j) \longrightarrow f(x_j) \quad \text{for every } j$$
(cantor diagonalization process + uniform boundedness)



look at the proof from the 1st lecture

<u>Claim</u>: f_{n_n} is Cauchy sequence in $C(K)$.

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<u>Aim</u>: $\|f_{n_s} - f_{n_m}\|_{C(K)} \longrightarrow 0$ as $s, m \to \infty$

For simplicity let $g_s = f_{n_s}$ $s \geq 1$.

Idea:

$$|g_s(x) - g_m(x)| \leq \underbrace{|g_s(x) - g_s(x_j)|}_{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x}} + \underbrace{|g_s(x_j) - g_m(x_j)|}_{\substack{\text{take } s,m \text{ large} \\ \text{enough:} \\ \leq \varepsilon/3}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x \\ \text{(equicontinuity)}}}$$

To make the idea work we need to check that in this construction we can deal only with finite number of points x_j, $j = 1, \dots, N(\varepsilon)$.

For this it suffices to find $N(\delta_\varepsilon)$ such that $\rho(x, x_j) < \delta_\varepsilon$ for every $x \in K$ and x_j, $j = 1 \dots N_\varepsilon$.
($\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is δ_ε-net).

So, it remains to show that if $\{x_j\}_{j=1}^{\infty}$ is dense then $\forall \delta_\varepsilon > 0 . \exists N(\delta_\varepsilon) . \{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net.
To this end, let $\{y_k\}_1^N$ is a $\delta_\varepsilon/2$-net in K (K is compact)
Let $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ be the part of $\{x_j\}$ such that

$$\operatorname{dist}\left(\{x_j\}_{j=1}^{N(\delta_\varepsilon)}, y_k\right) \leq \delta_\varepsilon/2 \quad \forall 1 \leq k \leq N.$$

$\Rightarrow$ then $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net by triangle inequality. ▨
($\|g_s - g_m\|_{C(K)} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ for s, m large enough)

Example: $K=[0,1]$, $E_A = \left\{ f \in C[0,1],\ f(0)=0,\ f \text{ is Lipshitz with constant at most } A \right\}$, $A>0$

E_A is compact

i) E_A is bounded in $C[0,1]$:

$$|f(x)| \leq |f(x)-f(0)| \leq A|x| \leq A$$

$$E_A \subset B(0,A)$$

ii) $|f(x)-f(y)| \leq A|x-y| \leq A\delta = \varepsilon$ if $\varepsilon>0$, $\delta := \frac{\varepsilon}{A}$,
$x,y \in [0,1]$: $|x-y| \leq \delta$

$i + ii + AA$ theorem $\Rightarrow E_A$ is precompact

iii) E_A is closed

if $f_n \rightrightarrows f$ in $C(K)$ then $f_n(0) \to f(0) \Rightarrow f(0)=0$

$$|f_n(x)-f_n(y)| \leq A|x-y|$$
$$\downarrow$$
$$|f(x)-f(y)| \Rightarrow f \text{ is } Lip(A)$$

Example: Let $E = \left\{ \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x} \text{, where } c_k \in \mathbb{C} : |c_k| \leq \frac{1}{k^2+1} \right\}$

Then E is compact as well in $C[0,1]$.

i) bbs: $f \in E$. $\|f\| \leq \sum_{k\in\mathbb{Z}} \frac{1}{k^2+1}$

Details: exercise

ii) equicontinuity $f = \sum_{|k|\leq N} + \sum_{|k|>N}$ small if N large

Lipshitz with some constant A_N ~ does not depent on f

Compact operators: basic properties

<u>Definition:</u> Let X, Y be Banach spaces, $T: X \to Y$ a linear map. T is called bounded if $T(B(0,1))$ is a bounded in Y set in Y. T is called compact if $T(B(0,1))$ is a precompact set in Y. ($B(0,1) = \{\|x\|_X < 1\}$) bounded linear operator

<u>Some observations:</u>

1) If $S \subset X$ is bdd then $T(S)$ is bdd for any bounded operator
precompact compact

2) T is compact $\Rightarrow$ T is bounded
(precompact sets are bounded)

3) with the norm $\|T\| = \sup\limits_{x \in B(0,1)} \|Tx\|_Y$, the set of bdd linear operators becomes a linear normed space, to be denoted by $B(X,Y)$ or $B(X)$ if $X = Y$.

4) A linear map between Banach spaces X, Y is continuous if and only if it is bounded.
Hint: $\|Tx - Ty\| \leq \|T\| \cdot \|x - y\|$, so bounded operators are Lipshitz.

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<u>Definition:</u> Banach algebra is an associative algebra which is a linear space with a norm $\|\cdot\|$ such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$ for any elements in this algebra.

<u>Proposition:</u> Let X be a Banach space, then $B(X,X)$ is a Banach algebra.

Proof: $\alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{B}(X,X)$ $\quad\forall T_1 T_2 \in \mathcal{B}(X,X)$ (proved)

$T_1 \cdot T_2 \in \mathcal{B}(X,X)$, since $\forall x \in X.\ \|T_1 T_2 x\| \le \|T_1\| \cdot \|T_2 x\|$

$\Rightarrow \underbrace{\|T_1\|}_{} \cdot \|T_2\| \le \|T_1\| \cdot \|T_2\|$ $\quad$ since $T_1 \in \mathcal{B}(X,X)$ we have $\|T_1 y\| \le \|T_1\|\|y\|\ \forall y \in X$

$\sup\limits_{\substack{y\in X\\ \|x\|\le 1}} \|T_1 T_2 x\|$

We see that $T_1 T_2 \in \mathcal{B}(X,X)$ and $\|T_1 T_2\| \le \|T_1\|\|T_2\|$.

Now let us prove that $\mathcal{B}(X,X)$ is Banach.

Let us show that $\sum\limits_{n=1}^{\infty} B_k$ converges in $B(X,X)$ if $\sum \|B_k\| \overset{(*)}{<} \infty$.

$T_n = \sum\limits_{k=1}^{N} B_k$, $x \in X$, $\|T_N x - T_M x\| = \|\sum\limits_{M+1}^{N} B_k x\| \le \sum\limits_{M+1}^{N} \|B_k\| \cdot \|x\| \xrightarrow[M,N\to\infty]{} 0$, $\quad N > M$

because of $(*)$.

$\Rightarrow \{T_N x\}_N$ Cauchy in X, but X-banach $\Rightarrow \exists Tx = \lim\limits_{X\to\infty} T_N x$

Moreover, $\|Tx\| = \lim \|T_N x\| \le \lim\limits_{N\to\infty} \sum\limits_{1}^{N} \|B_k\|\,\|x\| \le \left(\sum\limits_{1}^{\infty} \|B_k\|\right) \cdot \|x\|$.

$\Rightarrow T \in \mathcal{B}(X,X)$, $\|T\| \le \sum\limits_{1}^{\infty} \|B_k\|$

$\sup\limits_{\substack{x\in X\\ \|x\|<1}} \|Tx - T_N x\| = \lim\limits_{M\to\infty} \|T_N x - T_M x\| \le \sup\limits_{\|x\|<1} \lim\limits_{N\to\infty} \sum\limits_{M+1}^{N} \|B_k\| \cdot \|x\| = \sum\limits_{M+1}^{\infty} \|B_k\| \xrightarrow[(*)]{M\to\infty} 0$ ▨

Proposition: The set $S_\infty(X)$ $\left(= S_\infty(X,X)\right)$ (index ∞ will be explained later) of all compact operators on X is a two-sided ideal in $\mathcal{B}(X) = \mathcal{B}(X,X)$: $\forall T_1 \in S_\infty(X). \forall T_2 \in \mathcal{B}(X)$. $T_1 T_2 \in S_\infty(X)$ and $T_2 T_1 \in S_\infty(X)$.

Proof: Take $\{x_n\}_1^\infty$ s.t. $\|x_n\| \le 1$, and let us check that there is a subsequence $\{x_{n_k}\}$: $T_1 T_2 x_{n_k}$ converges.

Note that $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$. T_1 takes $\mathcal{B}_X(0,\|T_2\|)$ into a precompact subset of $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$ - convergent subsequence

Now let's consider $\{T_2 T_1 x_n\}$. Note that $\{T_1 x_{n_k}\}$ - convergent subsequence ($T_1 \in S_\infty(X)$). Then $\{T_2 T_1 x_{n_k}\}$ converges, since T_2 is continuous. ▨

Proposition: $S_\infty(X,Y)$ is a closed subset in $\mathcal{B}(X,Y)$, i.e. $T_n(X,Y)$, $T_n \longrightarrow T$ in $\mathcal{B}(X,Y) \Rightarrow T \in S_\infty(X,Y)$?

Proof: Let's find a finite ε-net in $T(B_X[0,1])$.
Take finite $\varepsilon/3$-net for $T_n B_X(0,1)$ for $n : \|T-T_n\| \leq \varepsilon/2$; denote it by $\{y_k\}_{k=1}^N$, then

$$\|Tx - Tx_k\| \leq \underbrace{\|Tx - T_n x\|}_{A} + \underbrace{\|T_n x - T_n x_k\|}_{B} + \underbrace{\|T_n x_k - T x_k\|}_{C} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \leq \varepsilon.$$

for any $1 \leq k \leq N$
so choose k : $B \leq \varepsilon/2$
and note $A \leq \|T-T_n\| \leq \varepsilon/3$
for every $x \in B_X(0,1)$
$C \leq \varepsilon/3$

Corollary: If T is a limit of finite-rank operators in $\mathcal{B}(X,Y)$, then $T \in S_\infty(X,Y)$.

Proof: Since finite-rank operators are in $S_\infty(X,Y)$, we have $T \in S_\infty(X,Y)$ by the previous proposition. ◪

Remark: At a general Banach space $\exists T \in S_\infty(X,Y)$ such that $\nexists (T_n)_n$: $\operatorname{rank} T_n < \infty$ and $\|T-T_n\| \longrightarrow 0$.

Definition: Let X be a Banach space. $\{e_k\}_{k=1}^\infty$ is a *Schauder basis* if $\forall x \in X.\ \exists! \{c_k(x)\}_{k=1}^\infty$ such that $x = \sum_{k=1}^\infty c_k(x) e_k$, where the series converges in X.

Theorem: Let X be a Banach space with Schauder basis, then $T \in S_\infty(X) \Leftrightarrow \exists T_n.\ \operatorname{rank} T_n \leq n$ and $\|T-T_n\| \to 0$.
(here $\operatorname{rank} S = \dim S(X)$ $\forall S \in \mathcal{B}(X)$)

Proof: ($\Leftarrow$): we already know

$(\Leftarrow)$: Let $T \in S_\infty(X)$, and let $P_n : x \longmapsto \sum_{k=1}^{\infty} c_k(x) e_k$.

<u>P is linear</u>: $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X$. $P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$?

$$\left.\begin{array}{l} \text{If } x = \sum c_k(x) e_k \\ y = \sum c_k(y) e_k \end{array}\right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha c_k(x) + \beta c_k(y)) e_k$$

$$\alpha x + \beta y = \sum_{k=1}^{\infty} c_k(\alpha x + \beta y) e_k$$

by uniqueness   by def. of Schauder basis

$$c_k(\alpha x + \beta y) = \alpha c_k(x) + \beta c_k(y) \quad \forall k$$

Then $P_n(\alpha x + \beta y) = \sum_{n=1}^{n} c_k(\alpha x + \beta y) e_k = \sum_{k=1}^{n} \alpha c_k(x) e_k + \sum_{n=1}^{k} \beta c_k(y) e_k$

$= \alpha P_n(x) + \beta P_n(y) \Rightarrow P_n$ linear

Note that $T_n := P_n T$ are such that $\operatorname{rank}(T_n) \leq n$ because $\dim P_n T(X) \leq \dim P_n(X) \leq n$.

It remains to show that $T_n \longrightarrow T$ in $\mathcal{B}(X)$. Since T is compact, $\forall \varepsilon > 0.\ \exists \{x_k\}_{k=1}^N$ such that $\|x_k\| \leq 1\ \forall k$ and $\{Tx_k\}_{k=1}^N$ is a ε-net in $T(B_X(0,1))$. Now take $x \in B_X(0,1)$ and write $\|Tx - T_n x\| \leq \|Tx - Tx_k\| + \|Tx_k - T_n x\| + \|T_n x_k - T_n x\|$

$$\leq \underbrace{\|Tx - Tx_k\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_k - P_n T x_k\|}_{\substack{\leq \varepsilon \text{ if } n \\ \text{large enough} \\ \text{for any fixed } k}} + \underbrace{\|P_n T x_k - P_n T x\|}_{\leq \|P_n\| \cdot \|Tx_k - Tx\|}$$

$$\leq \varepsilon + \varepsilon + \underbrace{\sup \|P_n\|}_{< \infty} \varepsilon$$

$\sup \|P_n\| < \infty$ by Banach-Schteinhaus theorem theorem on uniform point-wise convergence.

$\|T - T_n\| \leq \varepsilon(2 + \sup \|P_n\|)$ for n large enough $\blacksquare$

Theorem [Banach–Schteinhaus]: Assume that $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ where X is a Banach space, such that

$$\sup_n \|T_n x\| \le C(x) < \infty$$

local information $\rightsquigarrow$ uniform estimate

for every $x \in X$. Then $\sup_n \|T_n\| < \infty$. In particular, one can take C in place of $C(x)$.

Remark: In our situation $\sup_{1\le n<\infty} \|P_n\| < C(x) < \infty$ because $P_n x \longrightarrow x$ in X.

Banach adjoint operators

October 15, 2025

Definition: Let X be a Banach space, then $X^* = \mathcal{B}(X, \mathbb{C})$ is called the dual space to X. The elements of X^* are called functionals.

$$\mathcal{B}(X,\mathbb{C}) = \left\{ \phi : X \to \mathbb{C} \;\middle|\; \begin{matrix}\phi \text{ linear} \\ \|\phi\| < \infty\end{matrix} \right\}$$

Examples: (can ignore, if one does not know measure theory)

i) $L^p(\mu) = \left\{ f : S \longrightarrow \mathbb{C} \;\middle|\; \begin{matrix} f \text{ is measurable with} \\ \text{respect to } \sigma\text{-algebra} \\ \text{of } \mu \end{matrix},\ \int_S |f|^p \, d\mu < \infty \right\} \Big/ \begin{matrix} f = g \\ \text{if } f(x) = g(x) \\ \text{for } \mu\text{-a.e. } x \in S \end{matrix}$, $\quad 1 \le p < \infty$

$$\|f\|_{L^p(\mu)} = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

ii) $\ell^p(\mathbb{Z}) = \left(\left\{ \{x_k\}_{k\in\mathbb{Z}} \;\middle|\; \sum_{k\in\mathbb{Z}} |x_k|^p < \infty \right\},\ \|x\|_{\ell^p(\mathbb{Z})} = \left(\sum |x|^p \right)^{1/p} \right)$, $\quad 1 \le p < \infty$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

i) $g \in L^q(\mu) \leftrightsquigarrow \Phi_g : f \longmapsto \int_S fg \, d\mu$, $\Phi_g : L^p(\mu) \to \mathbb{C}$

ii) $\{y_k\}_{k=1}^{\infty}$ in $\ell^q(\mathbb{Z}) \leftrightsquigarrow \Phi_{\{y_k\}} : \{x_k\} \longmapsto \sum_{k \in \mathbb{Z}} x_k y_k$

$\Phi_{\{y_k\}} : \ell^p(\mathbb{Z}) \to \mathbb{C}$

Remark: i) is non-trivial measure theory

More examples:

iii) $c_0(\mathbb{Z}) = \{ \{x_k\}_{k \in \mathbb{Z}} \mid x_k \to 0 \text{ as } |k| \to \infty \}$

$c_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$ same identification

(Hausdorff is actually sufficient ~ hard)

iv) Let K be a compact metric space, and $X = C(K)$.
Then $X^* = \mathcal{M}(K)$.

set of cont. maps $(K, \delta) \to \mathbb{C}$

$$\mathcal{M}(K) = \left\{ \begin{matrix}\text{the set of Borel}\\ \text{measures on } K\\ \text{(complex valued)}\end{matrix} \;\middle|\; |\mu|(K) = \sup_{\substack{K = \cup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n \in \mathbb{Z}} |\mu(E_n)| < \infty \right\}$$

$\|\mu\| = |\mu|(K)$

Riesz-Markov representation theorem

Here $\mu \in \mathcal{M} \overset{(*)}{\leftrightsquigarrow} \Phi_\mu : f \longrightarrow \int_K f \, d\mu$

We can also define ℓ^p, L^p for $p = \infty$:

- $\ell^\infty(\mathbb{Z}) := \{ \{x_k\} \subset \mathbb{C} : \sup_{k \in \mathbb{Z}} |x_k| < \infty \}$
- $L^\infty(\mu) := \{ f : \text{---}, \text{---} : \operatorname{ess\,sup} |f| < \infty \}$

Remark: If $1 < p < \infty$ then $(L^p)^* = L^q$, $(L^q)^* = L^p$

But for $p=1$ $(L^1)^* = L^\infty$, $(L^\infty)^* \neq L^1$
$\ell^1(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})$, but $(\ell^\infty(\mathbb{Z}))^* \neq \ell^1(\mathbb{Z})$

<u>Definition</u>: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ is defined by

$$T^* : \underset{\in Y^*}{\Psi} \longmapsto \Big(\underset{\in X^*}{(T^*\Psi)} : x \mapsto \langle Tx, \underset{\Psi(Tx)}{\underbrace{T^*\Psi}} \rangle \Big),$$

where $\langle x, \phi \rangle = \phi(x)$ for $x \in X$, $\phi \in X^*$.

<u>Remark</u>: $\langle Tx, \Psi \rangle = \langle x, T^*\Psi \rangle$ $\rightarrow$ this formula is equivalent to the definition of T^*

<u>Remark</u>: Operation that sends x, ϕ into $\phi(x) = \langle x, \phi \rangle$ for $x \in X$, $\phi \in X^*$ is called a pairing of Banach spaces X, X^*.

<u>Example</u>: For $\varphi \in C[0,1]$, μ on $[0,1]$ then the pairing is $\langle \varphi, \mu \rangle = \int_0^1 \varphi \, d\mu$, see (✱).

<u>Theorem</u>: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then the map $T^* : Y^* \longrightarrow X^*$ defined by $\langle x, T^*\Psi \rangle := \langle Tx, \Psi \rangle$, $x \in X$, is an element of $\mathcal{B}(Y^*, X^*)$.
$(\Leftrightarrow \ (T^*\Psi)(x) = \Psi(Tx))$

<u>Lemma 1 [Hahn-Banach theorem]</u>: Let X be a Banach space, $E \subset X$ - subspace in X, $\phi_0 : E \to \mathbb{C}$ is linear and bdd ($\Leftrightarrow \phi_0 \in E^*$). Then $\exists \, \phi \in X^*$ such that $\phi|_E = \phi_0$ and $\|\phi\| = \|\phi_0\|$.

Lemma 2 ["Sufficient amount of functionals"]:

Let $x \in X$, then $\|x\| = \sup_{\|\phi\| \le 1} |\phi(x)|$.

Proof: $|\phi(x)| \le \|\phi\| \cdot \|x\| \le \|x\|$, so $\|x\| \ge \sup_{\|\phi\| \le 1} |\phi(x)|$

To prove "$\le$", define $E = \operatorname{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$,
$\phi_0 : y \longrightarrow c_y$ if $y = c_y x \in E$.
Assume that $\|x\| = 1$, then $\|\phi_0\|_{E^*} = \sup_{\|y\| \le 1} |c_y| =$ (from (**),
$|c_y| = \|y\|$ if $\|x\| = 1$) $= \sup_{\|y\| \le 1} \|y\| = 1$.

Hahn-Banach theorem $\Rightarrow \exists\, \tilde{\phi}_0 \in X^*$: $\|\tilde{\phi}_0\| = 1$, $\tilde{\phi}|_E = \phi_0$.
In particular $\sup_{\|\phi\| \le 1} |\phi(x)| \ge |\tilde{\phi}| = |\phi_0(x)| = 1 = \|x\|$.

We have proved "$\le$" in the case where $\|x\| = 1$.
The general case follows from consideration of $\frac{x}{\|x\|}$ in place of x.

$\boxed{\text{October 21, 2025}}$

We are proving that $T \in \mathcal{B}(X, Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.

Let $T \in \mathcal{B}(X, Y)$, consider

$$\|T^*\| = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} \|T^*\psi\|_{X^*} = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} \sup_{\substack{x \in X \\ \|x\| \le 1}} |(T^*\psi)(x)|$$

$$= \sup \sup_{-\,''\,-} |\langle x, T^*\psi \rangle|$$

$$= \sup \sup |\langle Tx, \psi \rangle|$$

$$= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} \sup_{\substack{x \in X \\ \|x\| \le 1}} |\psi(Tx)|$$

$$= \sup_{\substack{x \in X \\ \|x\| \le 1}} \sup_{\substack{\psi \in Y^* \\ \|\psi\| \le 1}} |\psi(Tx)|$$

$\overset{\text{Lemma "sufficient amount of functionals"}}{=} \|Tx\|$

$$= \sup_{\|x\|\le 1} \|Tx\| = \|T\| < \infty$$

The claim follows. ▨

Corollary: $T \in \mathcal{B}(X,Y)$ is invertible $(\exists T^{-1} \in \mathcal{B}(Y,X))$ iff $T^* \in \mathcal{B}(Y^*, X^*)$ is invertible $(\exists\, (T^*)^{-1} \in \mathcal{B}(X^*, Y^*))$.

We prove just $\Rightarrow$.

Proof: Assume that T is invertible $\Leftrightarrow$ $T^{-1}T = I_X$
$TT^{-1} = I_Y$

Let's take adjoint operators and see:

$$\left.\begin{array}{l} (T^{-1}T)^* = (I_X)^* \\ (TT^{-1})^* = (I_Y)^* \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} T^*(T^{-1})^* = I_X^* \\ (T^{-1})^* T^* = I_Y^* \end{array}\right.$$

Exercise: $(AB)^* = B^* A^*$

It remains to check that $I_X^* = I_{X^*}$, $I_Y^* = I_{Y^*}$. Then, by the previous theorem, $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$, hence T^* is invertible and its bounded inverse is $(T^*)^{-1} = (T^{-1})^*$.

Let's check that $I_X^* = I_{X^*}$. Take $\Phi \in X^*$, $x \in X$.

$$(I_X^* \Phi)(x) = \langle x, I_X^* \Phi \rangle = \langle I_X x, \Phi \rangle = \langle x, \Phi \rangle = \Phi x$$

$$(I_{X^*} \Phi(x)) \underset{\text{def}}{=} (\Phi)(x) = \Phi(x).$$

Similarly, $I_Y^* = I_{Y^*}$. ▨

The „pairing notation" is often not used in literature, but it is very usefull to not make mistakes.

Theorem [Schauder]: We have $T \in S_\infty(X,Y) \Leftrightarrow T^*_\infty(Y^*, X^*)$.

Proof: We will prove just "$\Rightarrow$".

Consider $K = \overline{TB_X(0,1)}$ – a compact set. Let $C(K)$ be the Banach space of continuous functions on K with

$$\|f\|_{C(K)} = \max_{s \in K} |f(s)|, \quad f: K \to \mathbb{C}$$

Let $E := \{\psi \in Y^* \mid \|\psi\|_{Y^*} \leq 1, \ \psi \text{ is considered as a function on } K\}$

$K \subset Y$, K metric space with respect to the metric $\rho(y_1,y_2) = \|y_2 - y_1\|_Y$

So, $E \subset C(K)$ and we claim that E is precompact.

1) uniform boundedness:

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s\in K} |\psi(s)| = \max_{s \in \overline{TB_X(0,1)}} |\psi(s)| \overset{\psi \text{ cont.}}{=} \sup_{x \in B_X(0,1)} |\psi(Tx)|$$

$$\leq \|\psi\| \sup_{\|x\| \leq 1} \|Tx\| \leq \|\psi\| \cdot \|T\| \leq \underbrace{\|T\| < \infty}_{\text{does not dependent on } \psi}$$

2) Equicontinuity: take $s_1, s_2 \in K$, let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \leq \|\psi\| \cdot \|s_1 - s_2\| \leq \|s_1 - s_2\|,$$

so maps from E are Lipschitz with constant 1, hence equicontinuous.

$\Rightarrow$ By Arzela-Askoli theorem, E is precompact.

We are now ready to prove $T^* \in S_\infty(Y^*, X^*)$. For this, we need to check that if $\{\psi_n\}$ is a sequence in $B_{Y^*}(0,1)$, then $\exists \{\psi_{n_k}\}$ such that $T^* \psi_{n_k}$ converges in X^*. So, take $\{\psi_n\} \subset B_{Y^*}(0,1)$ and consider it as elements $E \subset C(K)$.

Let $\{\psi_{n_k}\}$ be such that $\psi_{n_k} \xrightarrow{(*)} \psi$ in $C(K)$. (E is precompact!)

Let's prove that $\{T^*\psi_{n_k}\}$ is Cauchy in X^*, then the theorem will follow.

Take $x \in X$, $\|x\| \leq 1$, and consider

$$\|(T^*\psi_{n_k})(x) - (T^*\psi_{n_j})(x)\| = \|\langle x, T^*\psi_{n_k}\rangle - \langle x, T^*\psi_{n_j}\rangle\|$$
$$= \|\langle Tx, \psi_{n_k}\rangle - \langle Tx, \psi_{n_j}\rangle\|$$
$$= \|\psi_{n_k}(Tx) - \psi_{n_j}(Tx)\|$$
$$\leq \sup_{s \in K} |\psi_{n_k}(s) - \psi_{n_j}(s)|$$
$$= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|_{C(K)}}_{\varepsilon_{kj} \text{ - does not depend on } x} \longrightarrow 0 \quad \text{by } (*).$$

$\Rightarrow \|T^*\psi_{n_k} - T^*\psi_{n_j}\| \leq \varepsilon_{k,j} \longrightarrow 0.$ ▨

Fredholm alternative

Example: Consider the equation $f(t) - \int_0^1 e^{t-s} f(s)\,ds \overset{(**)}{=} g(t)$ in $L^2[0,1]$.

Question: For which $g \in L^2[0,1]$ do we have a solution $f \in L^2[0,1]$?

Observation: g has to satisfy $\int_0^1 e^{-t} g(t)\,dt \overset{(***)}{=} 0$

$\int_0^1 c\,dt = c$

Indeed, $\int_0^1 e^{-t} g(t)\,dt = \int_0^1 e^{-t} f(t)\,dt - \int_0^1 \cancel{e^{-t}} \left(\int_0^1 \cancel{e^{t}}{}^{-s} f(s)\,ds\right) dt = 0$

It is not clear so far if there are other restrictions.

Theorem [Fredholm alternative]: Let X be a Banach space, $T = I - K$ for $K \in \mathcal{S}^\infty(X,X)$. Then
$$\operatorname{Ran} T = \{x \in X \mid \langle x, \Phi\rangle = 0 \ \forall \Phi \in \ker T^*\}.$$

In other words, either:

(1) $\ker T^* = \{0\}$ and the equation $Tf = g$ has solution $\forall g \in X$.

or

(2) $\ker T^* \neq \{0\}$ and the equation $Tf = g$ has solutions only for $g: \langle g, \Phi\rangle = 0 \ \forall \phi \in \ker T^*$.

Let's complete the consideration of the example: we need to check that $K: f \longrightarrow \int_0^1 e^{t-s} f(s)ds$ is compact (exercise) and find $\operatorname{Ker} T^*$.

$$\phi \in \operatorname{Ker} T^* \Leftrightarrow T^* \varphi = 0$$

Adjoint operator T^* is defined by

$$\langle Tf, g\rangle = \langle f, T^* g\rangle \qquad f, g \in L^2[0,1]$$

$$(L^2[0,1]^* = L^2[0,1])$$

$$\Leftrightarrow \quad \langle f - \int_0^1 e^{t-s} f, g\rangle = \int_0^1 f(t) g(t) dt - \int_0^1\int_0^1 e^{t-s} f(s) ds\, g(t) dt$$

$$= \int_0^1 f(t) g(t) dt - \int_0^1 f(s) \left(\int_0^1 e^{t-s} g(t) dt\right) ds$$

$$= \langle f, g - \int_0^1 e^{t-s} g(t) dt \rangle_{L^2[0,1]}$$

$$(T^* g): s \longmapsto g(s) - \int_0^1 e^{t-s} g(t) dt, \quad s \in [0,1].$$

$$T^* g = 0 \Leftrightarrow g(s) = \int_0^1 e^{t-s} g(t) dt \quad \text{a.e. on } [0,1]$$

$$\Leftrightarrow \quad e^s g(s) = \underbrace{\int_0^1 e^t g(t) dt}_{\text{constant}} \quad \text{for almost every } s \in [0,1]$$

$\Rightarrow$ So, $\ker T^* = \{ c \cdot e^{-s}, c \in \mathbb{C}\}$, $\dim(\operatorname{Ker} T^*) = 1$.
By Fredholm theorem, equation (**) is solvable $\Leftrightarrow$ $\forall c \in \mathbb{C}.\ \langle g, c \cdot e^{-s}\rangle = 0 \Leftrightarrow \int_0^1 g(s) e^{-s} ds = 0$, which is (***).

Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let X be a Banach space, $E \subsetneq X$ - a linear closed subspace, $\varepsilon > 0$. Then $\exists x_0 \in X$ such that $\|x_0\| = 1$, $\operatorname{dist}(x_0, E) \geq 1 - \varepsilon$.

Proof: Since $E \neq X$, then $\exists\, \tilde{x}_0 \in X \setminus E$. Since E is closed, we have $\operatorname{dist}(\tilde{x}_0, E) = \delta > 0$ for some $\delta > 0$. Now consider $\tilde{y}_0 \in E$ such that $\delta \leq \|\tilde{x}_0 - \tilde{y}_0\| \leq (1+\eta)\delta$ for some $\eta \in (0,1)$.

Now let $\quad x_\eta := \dfrac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}, \quad \|x_\eta\| = 1.$

$$\operatorname{dist}(x_\eta, E) = \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \operatorname{dist}(\tilde{x}_0 - \tilde{y}_0, E)$$

$$= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \operatorname{dist}(\tilde{x}_0, E)$$

$$= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta}$$

Choosing η so that $\frac{1}{1+\eta} = 1 - \varepsilon$, we are done.

October 22, 2025

Lemma: Let X be a Banach space. Then $I: x \mapsto x$ is compact on $X \Leftrightarrow \dim X < \infty$.

Proof: $\dim X < \infty \Rightarrow I \in S_\infty(X)$ – we already know
$I \in S_\infty(X) \Rightarrow \dim X < \infty$:
Suppose $\dim X = +\infty$, find a sequence $\{e_n\}: \|e_n\| = 1$ $\forall n \in \mathbb{N}$

$e_1 \in X$ – arbitrary
e_2: $\operatorname{dist}(e_2, \operatorname{span}\{e_1\}) \geq 1/2$
e_3: $\operatorname{dist}(e_3, \operatorname{span}\{e_1, e_2\}) \geq 1/2$
e_4: etc

existence of $\{e_n\}$ follows from previous lemma, because $\operatorname{span}\{e_1, \dots, e_k\} \neq X$ $\forall k \in \mathbb{N}$

Then $\{e_n\} \subset B_X[0,1] = I(B_X[0,1])$ but there is no convergent subsequence, because $\|e_k - e_j\| \geq 1/2$ $\forall k, j$. ∎

Lemma: Let X be a Banach space, $K \in S_\infty(X)$, $T = I - K$.
Then: 1) $\dim(\operatorname{Ker} T) < \infty$.
2) $\operatorname{Ran} T$ is closed in X. [closed range Lemma]

Proof: 1): We have $I|_{\operatorname{Ker}T} = \underbrace{(I-K)|_{\operatorname{Ker}T}}_{0} + \underbrace{K|_{\operatorname{Ker}T}}_{\in S_\infty(\operatorname{Ker}T, S)}$

$$I|_{\operatorname{Ker}T} \in S_\infty(\operatorname{Ker}T, X) \Rightarrow I \in S^\infty(\operatorname{Ker}T) \Rightarrow \dim(\operatorname{Ker}T) < \infty$$

2): The statement is equivalent to the fact that if $\{x_n\} \subset X$ s.t. $Tx_n \longrightarrow y$ in X then $\exists x \in X$. $Tx = y$.

2.a) Let $\{x_n\}$: $\|x_n\| \le C$ $\forall n$.
Then $(I-K)(x_n) \longrightarrow y$, $(I-K)(x_{n_k}) \xrightarrow{(*)} y$
For every subsequence x_{n_k}
Let's choose x_{n_k}: Kx_{n_k} converges to $z \in X$
(use $K \in S_\infty(X)$)

Then $x_{n_k} \longrightarrow y + z$ by $(*)$, take $x = y + z$:

$$T(y+z) = \lim_{k\to\infty} Tx_{n_k} = y, \text{ so } Tx = y. \quad \checkmark$$

2.b) $\operatorname{dist}(x_n, \operatorname{Ker}T) \le C$ $\forall n \in \mathbb{Z}$
Take $\tilde{x}_n := x_n - w_n$, where $w_n \in \operatorname{Ker}T$: $\|\tilde{x}_n\| \le 2C$.
We have $\lim_{n\to\infty} T\tilde{x}_n = \lim_{n\to\infty} Tx_n$ by step 2a) $\exists \tilde{x}$: $T\tilde{x} = y$ ✓

2.c) $\operatorname{dist}(x_n, \operatorname{Ker}T) \longrightarrow +\infty$. Let us show that this situation does not occur. Suppose the converse:
Consider $\tilde{x}_n = x_n - \underset{\operatorname{Ker}T}{w_n}$: $\underbrace{\operatorname{dist}(x_n, \operatorname{Ker}T) \le \|\tilde{x}_n\| \le 2\operatorname{dist}(x_n, \operatorname{ker}T)}_{(**)}$

For $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ we have $Tz \longrightarrow 0$.

$$T z_n = T\frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T\frac{x_n}{\|\tilde{x}_n\|} = \frac{T x_n}{\|\tilde{x}_n\|}$$

$T x_n \to y \Rightarrow T x_n$ is bdd in X; $\|\tilde{x}_n\| \to +\infty$

$$\Rightarrow \|T z_n\| \leq \frac{2\|y\|}{\|\tilde{x}_n\|} \longrightarrow 0 \quad \text{For } n \text{ large enough}$$

At the same time, $T z_n = z_n - K z_n$

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $\{K z_{n_k}\}$ converges to some $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{T z_{n_k}}_{\to 0} + \underbrace{K z_{n_k}}_{\to z} \longrightarrow z$$

We have $Tz = 0$ $\left(= \lim T z_{n_k} = \lim T z_n = 0\right)$

$\Leftrightarrow z \in \operatorname{Ker} T$, $0 = \operatorname{dist}(z, \operatorname{Ker} T) =$

$$= \lim_{k\to\infty} \operatorname{dist}(z_{n_k}, \operatorname{Ker} T)$$

$$= \lim_{k\to\infty} \operatorname{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \operatorname{Ker} T\right)$$

$$= \lim \frac{\operatorname{dist}(\tilde{x}_{n_k}, \operatorname{Ker} T)}{\|\tilde{x}_{n_k}\|}$$

$$= \lim \frac{\operatorname{dist}(x_{n_k}, \operatorname{Ker} T)}{\|\tilde{x}_{n_k}\|}$$

$$\overset{(**)}{\geq} \frac{1}{2} \rightsquigarrow \text{contradiction}$$

▨

Lemma: Let X be a Banach space, $T \in \mathcal{B}(X)$: $\operatorname{Ker}(T) = \{0\}$ and $T^{k+1}X = T^k X$ for some $k \geq 0$. Then $\operatorname{Ran} T = X$.

Proof: We need to prove that $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$.
We know that: $T^{k+1}a = T^k \tilde{a}$ for every a and some $\tilde{a}$ depending on a.

$\operatorname{Ker} T^k \neq \{0\} \Rightarrow \operatorname{Ker} T \neq \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}.$$

▨

Theorem [Fredholm]: Let X be a Banach space, $K \in S_\infty(X)$, $T = 1 - K$. Then TFAE:

1) T is invertible in $\mathcal{B}(X)$ 2) $\operatorname{Ker} T = \{0\}$ 3) $\operatorname{Ran} T = X$

1') T^* is invertible in $\mathcal{B}(X)$ 2') $\operatorname{Ker} T^* = \{0\}$ 3') $\operatorname{Ran} T^* = X^*$

Proof: We will prove $2 \Rightarrow 3 \Rightarrow 2' \Rightarrow 3' \Rightarrow 2$, $1 \Rightarrow 1' \Rightarrow 2'$, $2 \& 3 \Rightarrow 1$

(2) $\Rightarrow$ (3): If $T^k X = T^{k+1} X$ for some k, we are done by the lemma. Define $X_k := T^k X$, $k \geq 0$, and note that $X_0 \supset X_1 \supset X_2 \supset X_3 \ldots$ Assume that all inclusions are strict, i.e. $X_k \supsetneq X_{k+1}$ $\forall k$. The subspaces X_k are closed by the closed range lemma (by induction). By the almost orthogonality lemma: $\exists \{y_k\}_0^\infty$ s.t.

i) $y_k \in X_k$ $\forall k$ ii) $\|y_k\| = 1$ $\forall k$ iii) $\operatorname{dist}(y_k, X_{k+1}) \geq 1/2$

Since K is compact $\{Ky_k\}$ contains a convergent subsequence $\{Ky_{k_j}\}_{j=1}^\infty$. On the other hand, if $j < m$

$$Ky_{k_j} - Ky_{k_m} = (Ky_{k_j} - y_{k_j}) - (Ky_{k_m} - y_{k_m}) + y_{k_j} - y_{k_m}$$

$$= y_{k_j} - \underbrace{y_{k_m} - (Ty_{k_j} - Ty_{k_m})}_{\in X_{k_j+1}}$$

Since $y_{k_m} \in X_{k_m}$, $j < m$, $k_j < k_m$, $k_j + 1 \leq k_m$, $X_{k_m} \subset X_{k_j+1}$
$\Rightarrow y_{k_m} \in X_{k_j+1}$; $Ty_{k_m} \in T X_{k_j+1} \subset X_{k_j+1}$; $Ty_{k_j} \in T(X_{k_j}) = X_{k_j+1}$
$\Rightarrow Ky_{k_j} - Ky_{k_m} = y_{k_j} + R$, $R \in X_{k_j+1}$. Since $\|y_{k_j} + R\| \geq 1/2$ by (iii), we get a contradiction ($\{Ky_{k_j}\}$ is not Cauchy).

(3) $\Rightarrow$ (2'): Take $\phi \in \operatorname{Ker} T^*$. We have

$T^*\phi = 0 \iff \langle x, T^*\phi \rangle = 0$ $\forall x \in X \iff \langle Tx, \phi \rangle = 0$ $\forall x \in X \iff$
$\iff \langle y, \phi \rangle = 0$ $\forall y \in X$ (by 3) $\iff \phi = 0$

(2') $\Rightarrow$ (3'): (the same as $2 \Rightarrow 3$ using Schauder's theorem)

$(3') \Rightarrow (2)$: Assume that $\operatorname{Ran} T^* = X^*$ and take $x \in \operatorname{Ker} T$.
We have $Tx = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \ \forall \phi \in X^* \Leftrightarrow \langle x, T^*\phi \rangle \ \forall \phi \in X^*$
$\Leftrightarrow \langle x, \psi \rangle = 0 \ \forall \psi \in X^*$ (by 3') $\Leftrightarrow x = 0$ lemma on sufficient amount of functionals

Conclusion: $(2) \Leftrightarrow (3) \Leftrightarrow (2') \Leftrightarrow (3')$

$(1) \Rightarrow (1')$: We already know for arbitrary $T \in \mathcal{B}(X)$.

$(1') \Rightarrow (2')$: Invertible operators are injective.

$(2') \Leftrightarrow (2), (3)$

$(2 \,\&\, 3) \Rightarrow (1)$: This holds for every $T \in \mathcal{B}(X)$ by the following fundamental theorem from general F.A.:

Theorem [linear mapping theorem]: Let X, Y be a Banach space, $T : X \to Y$ - linear bijection. Then $T \in \mathcal{B}(X,Y) \Leftrightarrow T^{-1} \in \mathcal{B}(Y, X)$.

Check injectivity: $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0$ (by 2) $\Leftrightarrow x_1 = x_2$
surjectivity: $\operatorname{Ran} T = X$ (by 3) $\blacksquare$

Remark: If $T = 1 - K$, $K \in S_\infty(X)$, and $\operatorname{Ker} T = \{0\}$ then the equation $Tx = y$ has a unique solution for every $y \in X$.

Remark: $\operatorname{Ker} T = \{0\} \Leftrightarrow \operatorname{Ker} T^* = \{0\}$, so we have proved half of Fredholm Alternative.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then
$\overline{TX} = \{ x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^* \}$.

Remark: This implies the other half of Fredholm alternative since $\overline{TX} = TX$ for $T = 1 - K$, $K \in S_\infty(X)$.

Lemma [seperation lemma]: Let Y be a Banach space, $Y_0 \subsetneq Y$ - a closed subspace, then $\exists \phi \in Y^* : \phi|_{Y_0} = 0$, $\phi(y) \neq 0$ for some $y \in Y \setminus Y_0$.

Proof: Take $y \in Y \setminus Y_0$, define $\phi_0 : \operatorname{span}\{y, Y_0\} \longrightarrow \mathbb{C}$ by $\phi_0(cy + y_0) \longmapsto c$, for $c \in \mathbb{C}$, $y_0 \in Y_0$.

1) $\{cy, y_0 \mid c \in \mathbb{C}, y_0 \in Y_0\} = \operatorname{span}\{y, Y_0\}$ - clear ✓

2) $\underbrace{cy + y_0}_{e} = \underbrace{\tilde{c}y + \tilde{y}_0}_{\tilde{e}} \Leftrightarrow (c - \tilde{c})y = y_0 - \tilde{y}_0 \in Y_0 \Leftrightarrow c = \tilde{c} \Rightarrow \phi(e) = \phi(\tilde{e})$

$\Rightarrow$ correctness ok

3) ϕ_0 is linear - clear

4) $\phi_0|_{Y_0} = 0$ $\quad (c = 0 \text{ on } Y)$

5) $|\phi_0(cy + y_0)| \overset{?}{\leq} A\|cy + y_0\| \quad \forall c, y_0$

$$|\phi_0(cy + y_0)| = |c| = (\operatorname{dist}(y, \underbrace{Y_0}_{\text{closed}}))^{-1} \cdot |c| \cdot \operatorname{dist}(y, Y_0)$$

$$= \operatorname{dist}(y, Y_0)^{-1} \cdot \operatorname{dist}(|c|y, Y_0) = \operatorname{dist}(y, Y_0)^{-1} \cdot \operatorname{dist}(cy, Y_0)$$

$|c|y = \alpha \cdot e \cdot y_0$, $|\alpha| = 1$

$\operatorname{dist}(|c|y, Y_0) = \operatorname{dist}(\underbrace{\alpha|c|}_{c} y_0, \underbrace{y_0 Y}_{Y})$

$$\leq \operatorname{dist}(y, Y_0)^{-1} \|cy + y_0\|, \qquad \text{so } A = \operatorname{dist}(y, Y_0)^{-1} \text{ works}$$

6) $\phi_0(y) = 1 \neq 0$

7) Use the Hahn-Banach theorem and extend ϕ_0 to the whole Y. $\square$

We actually proved that the lemma holds $\forall y \in Y \setminus Y_0$.

We can now prove the theorem from above.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{TX} = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T\}$.

Proof: We have $TX \subset E$, $E = \{x \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*\}$, because $\langle Tx, \phi \rangle = \langle x, T^* \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*$.
Then $\overline{TX} \subset \overline{E} = E$ since E is closed.
($x_n \to x$ and $\langle x_n, \phi \rangle = 0$ for some $\phi \in X^*$ then $\langle x, \phi \rangle = \lim \langle x_n, \phi \rangle = 0$)

We now need to check $\overline{TX} \supset X$. If not, the inclusion $\overline{TX} \subset E$ is proper and by the seperation lemma $\exists \phi : \phi|_{\overline{TX}} \overset{(*)}{=} 0$ but $\phi(e) \neq 0$ for some $e \in E$.
$(*) \Rightarrow \phi|_{TX} = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \ \forall x \in X \Leftrightarrow \langle x, T^* \phi \rangle = 0 \ \forall x \in X$
$\Leftrightarrow \phi \in \operatorname{Ker} T^* \Rightarrow \phi(e) = 0 \ \forall e \in E$ by definition of E, contradiction. ▨

Classical Form of Fredholm alternative for integral equations

Theorem: Let (S, μ) be a space with measure μ, and let $K(x,y) : S \times S \longrightarrow \mathbb{C}$:

$$\iint_{S \times S} |K(x,y)|^2 \, d\mu(x) \, d\mu(y) < \infty.$$

Then either equation $f(y) + \int_S K(x,y) f(x) \, d\mu(x) \overset{(**)}{=} 0$ has only the trivial solution $f = 0$ and the equation

$$f(y) + \int_S K(x,y) f(x) \, d\mu(x) = g(y)$$

is solvable for every $g \in L^2(S, \mu)$ or the equation $(**)$ has a non-trivial solution in $L^2(S, \mu)$.

Remark: Uniqueness implies existance.

Proof: Let's define $T = I - K$, $(Kf)(y) = \int_S K(x,y) f(x) \, d\mu(x)$.
Consider T as an operator on $L^2(S, \mu)$.

Then $(**) \Leftrightarrow \operatorname{Ker}T = \{0\} \Leftrightarrow TL^2(S,\mu) = L^2(S,\mu)$ by Fredholm alternative, modulo the fact that $K \in S_\infty(L^2(S,\mu))$. $\boxtimes$

↑ This we pospone until Hilbert spaces theory.

Remarks: Further reading:

1) $\dim \operatorname{Ker}T = \dim \operatorname{Ker}T^*$ if $T = 1 - K$, $K \in S_\infty$, it coincides with the dimension of the space of solutions

2) There is a version of Fredholm theory for general operators $T \in \mathcal{B}(X)$: $\dim(\operatorname{Ker}T) < \infty$, $\dim(X/\operatorname{Ran}T) < \infty$.

Spectrum of compact operators

October 29, 2025

Definition: Let $T \in \mathcal{B}(X)$, X Banach space.

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$$

is called the spectrum of T.

Definition: A number $\lambda \in \mathbb{C}$ is called an eigenvalue of T if $\exists e \in X \setminus \{0\}.\ Te = \lambda e$.

Definition: $\sigma_p(T) := \{\lambda \text{ eigenvalue of } T\}$ ... point spectrum of T.

Remark: We have $\sigma_p(T) \subset \sigma(T)$ for every $T \in \mathcal{B}(X)$.

Proof: $\lambda \in \sigma_p(T) \Rightarrow \lambda I - T$ is not injective because $\operatorname{Ker}(\lambda I - T) \neq \{0\}$.

Remark: In general, we might have $\sigma_p(T) \neq \sigma(T)$ and even $\sigma_p(T) = \emptyset$.

Theorem: Let X be a Banach space, $\dim X = +\infty$, and let $K \in S_\infty(X)$. Then $0 \in \sigma(K)$, $\sigma(K) \setminus \{0\} \subset \sigma_p(T)$, moreover, each eigenvalue has a finite multiplicity, and $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$.

Proof: $0 \in \sigma(K)$ since $0 \notin \sigma(K)$, then $\exists K^{-1} \in \mathcal{B}(X)$: $I = \underset{S_\infty(X)}{\underset{\uparrow}{K}}\underset{\mathcal{B}(X)}{\underset{\uparrow}{K^{-1}}} = K^{-1}K$, but then $I \in S_\infty(X) \Rightarrow \dim X < \infty$, contradiction.

Now let's prove that $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$.
Take $\lambda \in \sigma(K)$ and assume that $\lambda \in \sigma_p(K)$. Then

$$\operatorname{Ker}(\lambda I - K) = \{0\} \Leftrightarrow \operatorname{Ker}(I - \tfrac{1}{\lambda}K) = \{0\}$$

$\Rightarrow I - \frac{1}{\lambda}K$ is invertible by Fredholm theorem – contradiction.

Now let's prove that $\dim E_\lambda < \infty$.

$$E_\lambda := \{e \in X \mid Ke = \lambda e\} \quad \left(\begin{array}{l}\text{definition for:}\\ \lambda \text{ has finite multiplicity}\end{array}\right)$$

If this is not the case, there is a sequence $\{e_n\}_{n=1}^\infty$ such that $\operatorname{dist}(e_n, \operatorname{span}\{e_1, \dots, e_{n-1}\}) \geq 1/2$, $\|e_n\| = 1$.
Consider $\{Ke_n\} = \{\lambda e_n\}$: we cannot choose a convergent subsequence from this sequence – contradiction with $K \in S_\infty(X)$.

To prove $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$, assume the converse and let $e_n \in X$: $Ke_n = \lambda_n e_n$, $\|e_n\| = 1$, $\lambda_n \in \sigma_p(K)$, $\lambda_n \neq \lambda_k$ for $k \neq n$.
Define $E_n := \operatorname{span}\{e_1, \dots, e_n\}$ $\forall n$.
Observation: $E_{n+1} \supsetneq E_n$ $\forall n$.
Clearly $E_{n+1} \supset E_n$. If $E_{n+1} = E_n$ for some n, there exists the first such n. Then

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^{n} \lambda_{n+1} \alpha_k e_k$$

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^{n} \alpha_k \lambda_k e_k$$

$$\Rightarrow 0 = \sum_{k=1}^{n} \alpha_k (\lambda_{n+1} - \lambda_k) e_k \Rightarrow \alpha_{n+1} - \lambda_n = 0$$

$$\Rightarrow \lambda_{n+1} = \lambda_n \ \ldots \text{ contradiction}$$

$\Rightarrow$ The observation is true.

Let's choose $y_{n+1} \in E_{n+1}$ such that $\|y_{n+1}\| = 1$, $\operatorname{dist}(y_{n+1}, E_n) \geq \frac{1}{2}$
It remains to prove that $\{Ky_n\}$ does not have a convergent subsequence.

$$y_{n+1} = \alpha_{n+1} e_{n+1} + R_n, \quad \text{where } R_n \in E_n$$

$$Ky_{n+1} - Ky_{m+1} = \lambda_{n+1}\alpha_{n+1}e_{n+1} + \tilde{R}_n - \lambda_{m+1}\alpha_{m+1}e_{m+1} - \tilde{R}_m$$

Assume that $n \geq m+1$, then

$$\underset{E_n}{\underset{\Uparrow}{\tilde{R}_n}} - \underset{E_{m+1} \subset E_n}{\underset{\Uparrow}{\underbrace{\lambda_{m+1}\alpha_{m+1}e_{m+1}}}} + \underset{E_m \subset E_n}{\underset{\Uparrow}{\tilde{R}_m}} \quad \in E_n$$

$$\begin{aligned}
\Rightarrow \|Ky_{n+1} - Ky_{m+1}\| &\geq \operatorname{dist}(\lambda_{n+1}\alpha_{n+1}e_{n+1}, E_n) \\
&= |\lambda_{n+1}| \operatorname{dist}(\alpha_{n+1}e_{n+1}, E_n) \\
&= |\lambda_{n+1}| \operatorname{dist}(y_{n+1}, E_n) \\
&\geq r \cdot \tfrac{1}{2} > 0
\end{aligned}$$

∎

Scheme of the solution of Dirichlet problem in $\mathbb{R}^n, n \geq 3$, by means of Fredholm theory

Dirichlet problem: Find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ $\left(\begin{array}{l} \Omega \text{ is a domain in } \mathbb{R}^n \\ \partial\Omega \in C^2 \end{array} \right)$
such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} \overset{(*)}{=} f \end{cases} \quad \text{where } f \in C(\partial\Omega)$$

Physics interpretation: heat

Scheme for solution: we search for a solution of the form

$$u(x) = \int_{\partial\Omega} K(x,y)\varphi(y)\,dy, \quad K(x,y) = c_n \frac{(x-y,\ n_y)}{\|x-y\|^n_{\mathbb{R}^n}}$$

n_y is the ouward unit normal, $c_n \in \mathbb{R}$

$\Delta u = 0$ for every $\varphi \in C(\partial\Omega)$, we only need to find good φ (such that $(*)$ will hold)

$$u_\varphi(x) = -\varphi(x) + \int_{\partial\Omega} K(x,y)\varphi(y)\,dy \quad \text{if } x \in \partial\Omega$$

$$u_\varphi = (-I + K)\varphi, \quad K\varphi = \int K(x,y)\varphi(y)\,dy$$

To check that $\exists \varphi$: $u_\varphi = f$ on $\partial\Omega$ we just check that $K \in S_\infty((C(\partial\Omega))$ and $\operatorname{Ker}(-I+K) = \{0\}$

$\Rightarrow$ we are done by Fredholm alternative.

This is hard to prove (course on PDEs)

PART II Banach Algebras

November 4, 2025

Definition: A is a Banach algebra if A is a Banach space with the operation of multiplication such that:

1) $(xy)z = x(yz)$; $x, y, z \in A$
2) $(x+y)z = xz + yz$
   $x(y+z) = xy + xz$
3) $(\alpha x)y = x(\alpha y)$; $\alpha \in \mathbb{C}$
4) $\exists e \in A.\ xe = ex = x \quad \forall x \in A.$
5) $\|xy\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A.$

Examples: 1) X Banach space $\Rightarrow \mathcal{B}(X)$ is a Banach algebra with unity $e = I$ and norm $\|T\| = \sup\limits_{\|x\| \leq 1} \|Tx\|$

2) Calkin algebra: $\mathcal{B}(X)/S_\infty(X)$, $e = I + S_\infty(X)$

$$\|T\|_{\mathcal{B}(X)/S_\infty(X)} = \inf_{K \in S_\infty(X)} \|T - K\| = \operatorname{dist}(T, S_\infty(X))$$

3) K - compact Hausdorff space, $C(K)$ is a Banach algebra,
$e = 1$, $\|f\|_{C(K)} = \max\limits_{x \in K} |f(x)|$
$f: K \to \mathbb{C}$

4) $W(\mathbb{T})$ - Wiener algebra on $\mathbb{T} := \{|z| = 1\}$.

$$W(\mathbb{T}) = \left\{ f = \sum_{k \in \mathbb{Z}} c_k z^k \;\middle|\; c_k \in \mathbb{C},\ \sum |c_k| < \infty \right\}$$

$\|f\|_{W(\mathbb{T})} = \sum\limits_{k \in \mathbb{Z}} |c_k|$, $e = 1$, multiplication is the usual multiplication of functions

$\|f \cdot g\|_{W(\mathbb{T})} \leq \|f\|_{W(\mathbb{T})} \cdot \|g\|_{W(\mathbb{T})}$
exercise

5) $L^\infty(\mathbb{R})$, $\|f\| = \operatorname{ess\,sup}_{\mathbb{R}} |f|$

6) $H^\infty(\mathbb{D})$ - set of bounded analytic functions on $\mathbb{D} := \{|z|<1\}$

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{|z|<1} |f(z)|$$

Remark: 1) & 2) are noncomuttative Banach algebras, others are commutative.

Remark: If A is a Banach space with multiplication and properties 1), 2), 3), 5), then we can always add the identity to convert A to a Banach algebra as follows:

$\mathcal{A} = A \times \mathbb{C}$, $\quad (x, \alpha) + (y, \beta) = (x+y, \alpha+\beta)$

$$\gamma(x, \alpha) = (\gamma x, \gamma \alpha)$$

$$(x, \alpha)\cdot(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

$$\left(\text{"}(x+\alpha e)(y+\beta e)\text{"} = xy + \alpha y + \beta x + \alpha\beta e\right) \qquad e = (0,1)$$

$$\|(x,\alpha)\| = \|x\| + |\alpha| \quad \leftarrow \text{! corrected on Nov. 5th}$$

$\Rightarrow \mathcal{A}$ is a Banach algebra with identity and $(A, 0) \subset \mathcal{A}$.

Example: $L^1(\mathbb{R})$, multiplication $f * g = \int_{\mathbb{R}} f(y) g(x-y)\, dy$

$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f|\, dx$ – this is a Banach algebra without identity, and the above construction is equivalent to consideration of $\mathcal{A} = \mathbb{C}\delta_0 + L^1(\mathbb{R})$

$$\left(\underset{\text{measure}}{\delta_0(S)} = \begin{cases} 1; & 0 \in S \\ 0; & 0 \notin S \end{cases}, \quad \delta_0 * f = f * \delta_0 = \int f(y)\, \delta_0(x-y) = f(x) \right)$$

measures $\rightsquigarrow$ no dy

$$\langle \varphi, \delta_0 \rangle = \varphi(0) = \int_{\mathbb{R}} \varphi\, d\delta_0, \quad \varphi \in C(\mathbb{R})$$

Definition: A Banach algebra, $x \in A$. We say $x \in A$ is invertible if $\exists x^{-1} \in A.\ xx^{-1} = x^{-1}x = e$.

If x is invertible, then x^{-1} is unique.

Definition: $G(A) := \{x \in A \mid x \text{ is invertible}\}$.
we use G, because $G(A)$ is a group

Definition: $x \in A$. $\delta(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\}$.

Example: If A is the set of $n \times n$ matrices with complex coefficients, then for $T \in A$, $\delta(T)$ is the set of eigenvalues (the usual spectrum of the matrix).

Example: $A = \mathbb{C}(K)$, $\sigma(f) = ?$
$\lambda \in \mathbb{C}$: $\lambda - f$ is not invertible in $C(K)$.
Since $g \in G(C(K)) \Leftrightarrow \frac{1}{g} \in C(K)$, we have $\lambda - f \in G(C(K)) \Leftrightarrow \frac{1}{\lambda - f} \in C(K)$
$\Leftrightarrow \lambda \notin f(K)$

$\Rightarrow \sigma(f) = f(K)$ ← compact non-empty subset of $\mathbb{C}$

Basic properties

Proposition 1: Let A be a Banach algebra. Then $\|e\| \geq 1$.

Proof: $\|e\| = \|e \cdot e\| \leq \|e\| \cdot \|e\| \Rightarrow \cancel{\|e\| = 0}$ or $\|e\| \geq 1$
$\Updownarrow$
$e = 0$ $\times$

Definition: A Banach algebra is called *unital* if $\|e\| = 1$.

Proposition 2: If A is an arbitrary Banach algebra, then the algebra $A \times \mathbb{C}$ is unital.

Proof: $\|(0,1)\| = \|0\| + |1| = 1$

! **Note**: This was wrong previously and corrected on November 5th.

⚠ From now on, all Banach algebras are unital.

Proposition 3: If $\begin{matrix} x_n \longrightarrow x \\ y_n \longrightarrow y \end{matrix}$ in A, then $x_n \cdot y_n \longrightarrow xy$.

Proof: $\|x_n \cdot y_n - xy\| \leq \underbrace{\|x_n - x\|}_{\to 0} \underbrace{\|y_n\|}_{\text{bdd}} + \|x\| \underbrace{\|y_n - y\|}_{\to 0} \longrightarrow 0$ ▨

Proposition 4: Let $a \in A$, $\|a\| < 1$, then $e - a \in G(A)$.

Proof: Define $(e-a)^{-1} = e + a + a^2 + \cdots = \sum_{k=0}^{\infty} a^k$

this series converges because A_n is a Banach space and $\sum \|a^k\| \leq \sum \|a\|^k < \infty$

Let us check that $(e-a)^{-1}(e-a) = e$

$$\Leftrightarrow \lim_{n \to \infty} \underbrace{\left(\sum_{k=0}^{n} a^k\right)(e-a)}_{e - a^{n+1}} = e$$

$$e - \underbrace{a^{n+1}}_{\to 0} \longrightarrow e, \quad \text{since } \|a^{n+1}\| \leq \|a\|^{n+1} \longrightarrow 0$$

Similarly, $(e-a)(e-a)^{-1} = e$.

Proposition 5: $G(A)$ is open in A.

Proof: Let $a \in G(A)$, $b \in A$, then $a - b = \underset{\substack{\uparrow \\ G(A)}}{a}\underbrace{(e - a^{-1}b)}_{\in G(A) \text{ if } \|a^{-1}b\| < 1}$

$\|a^{-1}b\| < 1$ holds for all b : $\|b\| \leq \frac{1}{\|a^{-1}\|}$

$\Rightarrow B\left(a, \frac{1}{\|a^{-1}\|}\right) \subset G(A) \Rightarrow G(A)$ is open.

Proposition 6: Let $x \in G(A)$, $x_n \longrightarrow x$ in A. Then $x_n \in G(A)$ for n large enough, and $x_n^{-1} \longrightarrow x^{-1}$ as $n \longrightarrow \infty$.

Proof: Write $x_n = x + z_n$, we have $x_n \in G(A)$ for n large enough by Proposition 5.

$$(x + z_n)^{-1} - x^{-1} = (x(e + x^{-1}z_n))^{-1} - x^{-1} = \underbrace{(e + x^{-1}z_n)^{-1}}_{y_n} \cdot x^{-1} - x^{-1} \underset{\text{Prop. 3}}{\longrightarrow} ex^{-1} - x^{-1} = 0$$

$y_n \longrightarrow e$, see Prop. 4 : $\|y_n - e\| \leq \sum_{k=1}^{\infty} \|x^{-1} \cdot z_n\|^k \longrightarrow 0$ $\quad \|z_n\| \longrightarrow 0$ ▨

Theorem: Let A be a Banach algebra, $a \in A$. Then $\sigma(A)$ is a nonempty compact subset of $\mathbb{C}$.

Proof: $\sigma(a) = \mathbb{C} \setminus \rho(a)$, $\rho(a) := \{\lambda \mid \lambda e - a \text{ is invertible}\}$
resolvent set

$\rho(a)$ is open by Prop. 5 $\Rightarrow$ $\sigma(a)$ is closed.
Let's check that $\sigma(a)$ is bounded: $\sigma(a) \subset \{\lambda \mid |\lambda| \le \|a\|\}$
$\Leftrightarrow \rho(a) \supset \{\lambda \mid |\lambda| \geq \|a\|\}$.
Take $\lambda : |\lambda| \geq \|a\|$, then $\lambda e - a = \lambda e \underbrace{\left(e - \frac{a}{\lambda}\right)}_{\text{invertible by Prop. 4}}$

It remains to check that $\sigma(a) \neq \emptyset$.
Assume that $\sigma(a) = \emptyset$ and consider the function

$$f_\phi(\lambda) = \phi\left((\lambda e - a)^{-1}\right) \text{ for some } \phi \in A^*; \quad \lambda \in \mathbb{C}$$

Let's check that f_ϕ is analytic. Take $\lambda_0 \in \mathbb{C}$, consider

$$\lim_{\lambda \to \lambda_0} \frac{f_\phi(\lambda) - f_\phi(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} \phi\left(\frac{(\lambda e - a)^{-1} - (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0}\right)$$

$$[x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}] \quad = \lim_{\lambda \to \lambda_0} \phi\left(\frac{(\lambda e - a)^{-1}\left((\lambda_0 e - \cancel{a}) - (\lambda e - \cancel{a})\right)(\lambda_0 e - a)^{-1}}{\lambda - \lambda_0}\right)$$

$$= \lim_{\lambda \to \lambda_0} \phi\left(\underbrace{(\lambda e - a)^{-1}}_{\to (\lambda_0 e - a)^{-1} \text{ Prop. 6}} (\lambda_0 e - a)^{-1}\right)$$

ϕ continuous, continuity of multiplication

$$= -\phi\left((\lambda_0 e - a)^{-2}\right)$$

$\Rightarrow f_\phi$ is analytic (f_ϕ is $\mathrm{Hol}(\mathbb{C})$) holomorphic

Take $\lambda : |\lambda| \geq 2\|a\|$, $|f_\phi(\lambda)| = \|\phi\| \cdot \|(\lambda e - a)^{-1}\| = \|\phi\| \overset{(*)}{|\lambda|^{-1}} \underbrace{\|(e - \tfrac{a}{\lambda})^{-1}\|}_{\text{uniformly bounded for } \lambda : |\lambda| \geq 2\|a\|}$

f_ϕ is bdd on $\mathbb{C}$ by the max. principle
$\Rightarrow f_\phi = c_\phi \in \mathbb{C}$, but $c_\phi = 0$ by $(*)$
$\Rightarrow \phi(\lambda e - a) = 0 \ \forall \lambda \in \mathbb{C}. \ \forall \phi \in A^* \Rightarrow \lambda e - a = 0 \ \forall \lambda \in \mathbb{C}$
$\rightsquigarrow$ contradiction ▨

November 5, 2025

Definition: Let A_1, A_2 be Banach algebras.
We say that A_1 is *isomorphic* to A_2 ($A_1 \cong A_2$) if there exists a map $j: A_1 \longrightarrow A_2$:

$$j(\alpha a + \beta b) = \alpha j(a) + \beta j(b) \quad (*)$$
$$j(ab) = j(a) \cdot j(b) \quad (**)$$
$$\|j(a)\|_{A_2} = \|a\|_{A_1}$$
$$j \text{ is bijective}$$

Theorem [Banach-Mazur]: Let A be a Banach algebra s.t. $G(A) = A \setminus \{0\}$. Then $A \cong \mathbb{C}$.

Proof: For every $a \in A$ we have $\sigma(a) \neq \emptyset$. So there is a $\lambda(a) \in \mathbb{C}$: $\lambda(a)e - a$ is not invertible $\Leftrightarrow$ $\lambda(a)e - a = 0 \Leftrightarrow a = \lambda(a)e$.
In particular, such $\lambda(a)$ is unique.

Define $j: A \longrightarrow \mathbb{C}$, $a \longmapsto \lambda(a)$. We have:

$$\left.\begin{aligned} a+b &= \lambda(a+b)e \\ a+b &= \lambda(a)e + \lambda(b)e \end{aligned}\right\} \Rightarrow \lambda(a+b) = \lambda(a) + \lambda(b)$$

Similarly, $\lambda(\alpha a) = \alpha\lambda(a)$, $\lambda(ab) = \lambda(a)\cdot\lambda(b) \Rightarrow (*), (**)$ ✓

$\|j(a)\|_{\mathbb{C}} = \|a\|_A \Leftrightarrow |\lambda(a)| = \|a\|_A \Leftrightarrow \|\lambda(a)e\|_A = \|a\|_A$ ✓

$$\left.\begin{aligned} & j(a) = j(b) \Leftrightarrow \lambda(a)e = \lambda(b)e \Leftrightarrow a = b \\ & j(e) = 1 \Rightarrow j(A) \supset j(\mathbb{C}\cdot e) = \mathbb{C} \end{aligned}\right\} j \text{ is a bijection}$$

Definition: Let A be a Banach algebra, then $r(a) := \sup\{|\lambda|, \lambda \in \sigma(a)\}$ is called the *spectral radius of a*.


$\sigma(a)$

Theorem: Let A be a Banach algebra. Then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

In particular, the limit above exists.

Proof: Step 1: $r(a) \le \|a\|$, because
$$\lambda e - a = \underbrace{\lambda\left(e - \frac{a}{\lambda}\right)}_{\text{invertible element in } A \text{ (Prop. 4)}} \qquad \|b\| < 1 \text{ for } \lambda : |\lambda| > \|a\|$$
where $b = \frac{a}{\lambda}$.

$\Rightarrow \lambda e - a \in G(A) \Rightarrow \lambda \in \rho(a) \Rightarrow \sigma(a) \subset \mathcal{B}[0, \|a\|]$. ✓

Step 2: $r(a) \le \|a^n\|^{1/n}$ for every $n \ge 2$.
Take $\lambda \in \mathbb{C}$, and consider
$$\lambda^n e - a^n = (\lambda e - a) \cdot p_\lambda(a), \quad p_\lambda \text{ is a polynomial}$$
If $\lambda \in \sigma(a)$, then $\lambda^n e - a^n = z_1 \cdot z_2$, where $z_1, z_2 \in A$, $z_1 \notin G(A)$, $z_1 z_2 = z_2 z_1$. If $z = \lambda^n e - a^n$ is invertible, $\exists z^{-1} \in A$.
$$z^{-1} z_1 z_2 = z_1 (z_2 z^{-1}) = e$$
$$z^{-1} z_1 z_2 = (z^{-1} z_2) z_1$$
$\Rightarrow z_1 \in G(A)$ ... contradiction

$\Rightarrow \lambda^n \in \sigma(a^n) \underset{\text{Step 1}}{\Rightarrow} |\lambda^n| \le \|a^n\| \Rightarrow |\lambda| \le \|a^n\|^{1/n}$. ✓

Step 3: $\inf_{n \ge 1} \|a^n\|^{1/n} \le \underline{\lim} \|a^n\|^{1/n} \le \overline{\lim} \|a^n\|^{1/n} \le r(a) \le \inf \|a^n\|^{1/n}$
$\hookrightarrow$ This implies the claim.

Notation note: $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$.

All that remains is $\overline{\lim} \|a^n\|^{1/n} \le r(a)$. Take $\phi \in A^*$, $\|\phi\| \le 1$,
$f_\phi(\lambda) := \phi((\lambda e - a)^{-1})$ for $\lambda \in \rho(a)$ (this is an analytic function on $\rho(a)$)

$\varepsilon > 0$, $k \in \mathbb{Z}$:
$$\frac{1}{2\pi i} \oint_{|\lambda| = r(a) + \varepsilon} \lambda^k f_\phi(\lambda)\, d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^k f_\phi(\lambda)\, d\lambda =$$
$$= \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^k \phi\left(\lambda \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n}\right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^{k+1} \sum_{n=0}^{\infty} \frac{\phi(a^n)}{\lambda^n}\, d\lambda$$
$$= \phi(a^{k+2}) \qquad \text{(Cauchy formula from complex analysis)}$$

$$\Rightarrow \phi(a^{k+2}) \le \left| \frac{1}{2\pi i} \oint_{|\lambda| = r(a) + \varepsilon} \lambda^k f_\phi(\lambda)\, d\lambda \right|$$

$$\leq \max_{|\lambda|=r(a)+\varepsilon} \left(|\lambda^k| \cdot |f_\phi(\lambda)| \right) \frac{1}{2\pi} (2\pi(r(a)+\varepsilon))$$

$$= (r(a)+\varepsilon)^{k+1} \underbrace{\|\phi\|}_{\leq 1} \cdot \underbrace{\sup_{|\lambda|=r(a)+\varepsilon} \|(\lambda e - a)^{-1}\|}$$

constant depending only on ε, because $|\lambda| = r(a)+\varepsilon$ is a compact set and $\lambda \mapsto \|(\lambda e - a)^{-1}\|$ is continuous

$$\sup_{\|\phi\|\leq 1} |\phi(a^{k+2})| = \|a^{k+2}\|$$

$$\text{I}\wedge$$

$$C_\varepsilon \cdot (r(a)+\varepsilon)^k$$

$\Rightarrow \overline{\lim} \|a^k\|^{1/k} \leq r(a)+\varepsilon$ for every $\varepsilon > 0$. ▨

November 11, 2025

<u>Example</u>: $V: f \longmapsto \int_0^x f(s)\,ds$, $V \in \mathcal{B}(L^2[0,1])$.
Let's prove that $r(V) = 0 \Leftrightarrow \sigma(\{V\}) = \{0\}$.

<u>Proof 1</u>: $r(V) = \lim_{r\to\infty} \|V^n\|^{1/n}$, so we need a formula for V^n.
By induction: $$(V^n f)(x) \overset{(*)}{=} \int_0^x f(s) \frac{(x-s)^{n-1}}{(n-1)!}\,ds.$$

$n=1$: ✓

Assume $(*)$ for some n and compute

$$(V^{n+1}f)(x) = \int_0^x \left(\int_0^t f(s) \frac{(t-s)^{n-1}}{(n-1)!}\,ds \right) dt$$

$$= \int_0^x \Bigg(f(s) \underbrace{\int_s^t \frac{(t-s)^{n-1}}{(n-1)!}\,dt}_{\frac{(t-s)^n}{n!}\Big|_s^x = \frac{(x-s)^n}{n!}} \Bigg) ds$$

$$= \int_0^x f(s) \frac{(x-s)^n}{n!}\,ds.$$

$$\|V^n f\| \leq \max_{x\in(0,1)} \max_{s\in(0,x)} \frac{(x-s)^{n-1}}{(n-1)!} \left(\int_0^1 \Big[\underbrace{\int_0^x |f(s)|\,ds}_{1\cdot|f(s)|} \Big]^2 dx \right)^{1/2}$$

$$\overset{CS}{\leq} \frac{1}{(n-1)!}\left(\int_0^1 \underbrace{\sqrt{x}}_{\leq 1} \underbrace{\int_0^x |f|^2 ds}_{\leq \|f\|^2} dx\right)^{1/2}$$

$$\leq \frac{\|f\|_{L^2[0,1]}}{(n-1)!}$$

$$\Rightarrow \|V^n\| \leq \frac{1}{(n-1)!}, \quad \lim \|V^n\|^{1/n} \leq \overline{\lim} \left(\frac{1}{(n-1)!}\right)^{1/n}$$

Stirling formula

$$= \overline{\lim} \frac{1}{\left(\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}\right)^{1/n} (1+o(1))^{1/n}}$$

$$= \overline{\lim_{n\to\infty}} \frac{e^{\frac{n-1}{n}}}{(2\pi)^{1/2n}} \cdot \frac{1}{(n-1)^{3/2 \cdot \frac{n-1}{n}}}$$

$\searrow e$ $\quad \to 0$

$$= 0$$

$\square$

<u>Proof 2</u>: $Vf = \int_0^1 K(x,y) f(y) dy, \quad K(x,y) = \chi_{\mathbb{R}_+}(x-y)$

$$\int_0^1\int_0^1 |K(x,y)|^2 dx dy < \int_0^1\int_0^1 dx dy = 1 < \infty$$

$$\Rightarrow V \in S_\infty(L^2[0,1]) \Rightarrow \sigma(V) = \left\{ \lambda \text{ is an eigenvalue of } V \right\} \cup \{0\}$$

$\uparrow$ theorem about the spectrum of compact operators

$$Vf = \lambda f \Leftrightarrow \underbrace{\int_0^x f(s) ds}_{\in C[0,1]} \overset{(**)}{=} \lambda f(x) \quad \Rightarrow f \in C[0,1]$$

$$\Rightarrow \int_0^x f(s) ds \in C^1[0,1] \quad \Rightarrow \dots \Rightarrow f \in C^\infty[0,1].$$

Differentiating (**), we get $\lambda f' = f$ on $[0,1]$.
Substituting 0 into (**), we get $\lambda f(0) = \lambda$.

So, if $\lambda \neq 0$, then $\begin{cases} \lambda f' = f \\ f(0) = 0 \end{cases} \Leftrightarrow \begin{cases} f = c \cdot e^{\lambda x} \\ 0 = c \cdot e^0 \end{cases} \Leftrightarrow f = 0$

$\Rightarrow$ any $\lambda \neq 0$ is not an eigenvalue $\Rightarrow \sigma(V) \cap (\mathbb{C} \setminus \{0\}) = \varnothing$
Since $\sigma(V) \neq \varnothing$, we get $\sigma(V) = \{0\}$.

Commutative Banach algebras

<u>Definition</u>: Let A be a commutative Banach algebra and $J \subset A$. J is called a proper ideal in A if J is a linear subspace such that $a \cdot J \subset J \ \forall a \in A$, and $J \neq \{0\}$, $J \neq A$.

<u>Definition</u>: J is a maximal ideal if J is a proper ideal and there is no proper ideal J' such that $J' \supsetneq J$.

<u>Proposition</u>: Every proper ideal is contained in some maximal ideal. Take some proper ideal J, $J \neq A$, and consider all proper ideals J': $J' \supseteq J$. This set is partially ordered by inclusion, and for every chain $\{J'_\alpha\}_{\alpha \in I}$ of ideals ordered by inclusion, the set $\bigcup_{\alpha \in I} J'_\alpha = J'$ is again a proper ideal.

- linearity: $p \cdot x + q y \in J$ for every $p, q \in \mathbb{C}$ and $x, y \in J'$, because $\exists \alpha_x, \alpha_y$. $x \in J_{\alpha_x}$, $y \in J_{\alpha_y}$ $\Rightarrow$ $x, y \in J_{\alpha_x}$ or $x, y \in J_{\alpha_y}$, then $px + qy$ are in the same J_{α_x} or J_{α_y} ✓
- ideal property $aJ' = \bigcup a \cdot J'_\alpha \subset \bigcup J'_\alpha \ \forall a \in A$.
- properness: $J' \neq A$ (If $J' = A$, then $e \in J'$, then $e \in J'_\alpha \Rightarrow \overset{A}{\overset{\shortparallel}{eA}} \subseteq J'_\alpha \subsetneq A$.)

By Zorn's lemma, the set of all proper J': $J' \supset J$ has a maximal element. ▨

<u>Proposition</u>: If M is a maximal ideal in A, then M is closed.

<u>Proof</u>: Let's prove that $\overline{M}$ is a proper ideal.

- $\overline{M}$ is linear ✓
- $\overline{M} A \subseteq \overline{M}$: true by continuity of multiplication ✓
- $\overline{M} \neq A$ (If $\overline{M} = A$, then $e \in \overline{M} \Rightarrow \exists x \in M .\ \mathrm{dist}(x, e) < 1 \Rightarrow x \in G(A)$
  $\Rightarrow e = x \cdot x^{-1} \in xA \subset M \Rightarrow M = A$, contradiction.) ▨

Example: $A = C(K)$, $M_{x_0} = \{f \in A.\ f(x_0) = 0\}$. $\neq\{x_0\}$

Then M_{x_0} is a maximal ideal for every $x_0 \in K$.

- M_{x_0} is linear ✓
- $M_{x_0} \cdot A \subseteq M_{x_0}$ ✓
- $M_{x_0} \neq A$, because $1 \notin M_{x_0}$ ✓
- M_{x_0} is maximal: If $\exists J$-proper: $J \supsetneq M_{x_0}$ then $\exists f \in J.\ f(x_0) \neq 0$.

But then $\forall g \in A$ we have $g \overset{(***)}{=} c \cdot f + h$ for $c \in \mathbb{C}$ and $h \in M$, where c is such that $(g - cf)(x_0) = 0$, i.e. $c := \frac{g(x_0)}{f(x_0)}$.

So $A \subset \mathbb{C} \cdot f + M \subset J$, contradiction.

Observation: $M_{x_0} = \operatorname{Ker}\phi_{x_0}$, $\phi_{x_0} : f \longrightarrow f(x_0)$

ϕ_{x_0} is a multiplicative functional: $\phi_{x_0}(fg) = \phi_{x_0}(f) \cdot \phi_{x_0}(g)$

Definition: Let $\phi \in A^*$. We say ϕ is a multiplicative functional if $\phi(fg) = \phi(f) \cdot \phi(g)$ $\forall f, g \in A$, and $\phi \neq 0$.

Theorem: Let A be a commutative Banach algebra. TFAE

1) M is a maximal ideal in A.
2) $M = \operatorname{Ker}\phi$ for some mutiplicative functional $\phi \in A^*$.

Proof: 2) $\Rightarrow$ 1) obvious: i) $\operatorname{Ker}\varphi$ is linear

ii) $x \in \operatorname{Ker}\varphi, a \in A$, $\varphi(xa) = \varphi(x)\varphi(a) = 0 \Rightarrow xa \in \operatorname{Ker}\varphi$

iii) $\operatorname{Ker}\varphi \neq A$ because $\varphi \neq 0$

iv) $\operatorname{Ker}\varphi$ is maximal, because $\operatorname{Ker}\phi + \mathbb{C}a = A$ $\forall a : \phi(a) \neq 0$ (see $***$)

1) $\Rightarrow$ 2) Note that A/M is a Banach algebra (M is closed) in which every nonzero element is invertible. If $[a] \in A/M$, $\neq [0]$ then $a \cdot A$ is an ideal in A containing M, but not equal to $M \Rightarrow aA + M = A$

$aA + M \ni e \Rightarrow ab + M \ni e$ for some $b \in A \Rightarrow [a][b] = [e]$.

By Banach-Mazur theorem, there is an isomorphism j of Banach algebras A/M and $\mathbb{C}$. Let $\phi(x) := j([x])$.

i) Φ is linear, because j is linear.
ii) Φ is multiplicative, because j is multiplicative.
iii) $\operatorname{Ker}\Phi = \{x \mid j([x]) = 0\} \Leftrightarrow [x] = 0 \Leftrightarrow x \in M$. $\square$

November 12, 2025

Notation: $\mathcal{M}(A)$ - set of all maximal ideals in A
A^*_{mult} - set of multiplicative functionals on A

Last time: $\mathcal{J}: \phi \longmapsto \operatorname{Ker}\phi$ maps A^*_{mult} onto $\mathcal{M}(A)$.

Proposition: $\mathcal{J}$ is a bijection.

Proof: We need to check that $\mathcal{J}(\phi_1) = \mathcal{J}(\phi_2) \Rightarrow \phi_1 = \phi_2$.
For this, note that $\phi(e) = 1 \ \forall \phi \in A^*_{mult}$, because $\begin{cases} \phi(e) = \phi(e)\,\phi(e) \\ \phi(e) \neq 0 \end{cases}$
for every $\phi \in A^*_{mult}$. So, if $\mathcal{J}(\phi_1) = \mathcal{J}(\phi_2)$, we have
$0 = \phi_1(y - \phi_1(y)e) \Rightarrow \phi_2(y - \phi_1(y)e) = 0 \quad \forall y \in A$
$\Rightarrow \phi_2(y) = \phi_1(y)\,\phi_2(e) = \phi_1(y)$, so $\phi_1 = \phi_2$.

Theorem: Let A be a commutative Banach algebra, and $a \in A$.
TFAE:
1) $a \in A \setminus G(A)$
2) $a \in M$ for some $M \in \mathcal{M}(A)$
3) $\exists \phi \in A^*_{mult}: \phi(a) = 0$

Proof: 1) $\Rightarrow$ 2): $J = aA$ - a proper ideal in A $(e \notin J)$
$\Rightarrow \exists M \in \mathcal{M}(A).\ M \supset J$
2) $\Rightarrow$ 3): Take $\phi: \operatorname{Ker}\phi = M \Rightarrow \phi(a) = 0$.
3) $\Rightarrow$ 1): If $\exists b \in A.\ ab = e \Rightarrow \phi(a)\cdot\phi(b) = \phi(e) = 1$, but $\phi(a) = 0$. $\square$

Corollary: $\sigma(a) = \{\phi(a) \mid \phi \in A^*_{mult}\}$

Proof: $\sigma(a) = \{\lambda \mid a - \lambda e \in A \setminus G(A)\} = \{\lambda \mid \exists \phi \in A^*_{mult}.\ \phi(a - \lambda e) = 0\}$
$= \{\lambda \mid \lambda \in \phi(a) \text{ for some } \phi \in A^*_{mult}\}$. ▨

Remark: $\forall \phi \in A^*_{mult}$, $\|\phi\| = 1$, because $\phi(e) = 1$ and $|\phi(a^k)|$ is uniformly bounded for every $a \in B_A(0,1)$.

Applications

Theorem [Wiener]: Let $f = \sum_{k \in \mathbb{Z}} c_k z^k$, and $\sum_{k \in \mathbb{Z}} |c_k| < \infty$. Assume that $f(z) \neq 0$ for every $z \in \mathbb{T} = \{|z| = 1\}$. Then $\frac{1}{f} = \sum_{k \in \mathbb{Z}} b_k z^k$ where $\sum_{k \in \mathbb{Z}} |b_k| < \infty$.

Proof: 1. $W^1(\mathbb{T}) = \{\sum c_k z^k \mid \sum |c_k| < \infty\}$ is a Banach algebra:

Indeed, $W(\mathbb{T})$ is a Banach space with respect to the norm $\|\sum c_k z^k\| = \sum |c_k|$, and

$$\|(\sum c_k z^k)(\sum b_k z^k)\| = \sum_{n \in \mathbb{Z}} \left| \left(\sum_{k \in \mathbb{Z}} c_k b_{n-k} \right) \right|$$
$$\leq \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |c_k| |b_{n-k}|$$
$$= \sum_{k \in \mathbb{Z}} |c_k| \sum_{j \in \mathbb{Z}} |b_j|$$
$$= \|\sum c_k z^k\| \cdot \|\sum b_k z^k\|$$

λ.

2. Identification of $W^1(\mathbb{T})^*_{mult}$:

Let $\phi \in W^1(\mathbb{T})^*_{mult}$, $\lambda = \phi(z)$, then $\phi(\frac{1}{z}) \cdot \phi(z) = 1$, $\phi(\frac{1}{z}) = \frac{1}{\lambda}$
$|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot \|\frac{1}{z}\| = 1$
$|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot |\frac{1}{z}| = 1 \Rightarrow |\lambda| = 1$

$\phi(\sum_{-N}^{N} c_k z^k) = \sum_{-N}^{N} c_k \lambda^k$, and hence $\phi(f) = f(\lambda)$ $\forall f \in W^1(\mathbb{T})$, because $\{\sum_{-N}^{N} c_k z^k\}$ is dense in $W^1(\mathbb{T})$ and ϕ is continuous.

3. Application of invertibility criterion:

$f \in W^1(\mathbb{T})$ is invertible $\Leftrightarrow \nexists\, \phi \in W^1(\mathbb{T})^*_{mult}.\ \phi(f)=0 \Leftrightarrow f(z)=0\ \forall z \in \mathbb{T}$

This is the case in our case.

$\Rightarrow fg=1,\ g \in W^1(\mathbb{T}) \Rightarrow g=\frac{1}{f},\ g=\sum b_k z^k,\ \sum |b_k| < \infty.$ ▨

Bezout equation: Let $\{f_k\}_1^N \subset A(\overline{\mathbb{D}})$. We are interested if there exists $\{g_k\}_{k=1}^N \subset A(\overline{\mathbb{D}}): \sum_1^N f_k g_k = 1.$

Necessary condition: $\nexists\, z_0 \in \overline{\mathbb{D}}.\ f_k(z_0)=0$ for every $1 \le k \le N$.

Theorem: Necessary condition is also sufficient.

Proof: 1. $A(\mathbb{D})$ is a Banach algebra with respect to the norm $\|f\| = \max_{z \in \mathbb{D}} |f(z)|$ ✓

2. Identification of $A(\overline{\mathbb{D}})^*_{mult}$:

$\phi \in A(\overline{\mathbb{D}})^*_{mult} \Leftrightarrow \phi(f)=f(\lambda)$ for some $\lambda \in \overline{\mathbb{D}}$ [exercise]

3. $J = \{\sum_1^N f_k g_k \mid g_k \in A(\overline{\mathbb{D}})\}$ is a proper ideal in $A(\overline{\mathbb{D}})$
$\Leftrightarrow J \subset M,\ M \in \mathcal{M}(A(\overline{\mathbb{D}})) \Leftrightarrow \sum_1^N (f_k g_k)(z_0) = 0\ \ \forall g_k \in A(\overline{\mathbb{D}})$ for some $z_0 \in \overline{\mathbb{D}}$

$\Leftrightarrow f_k(z_0)=0\ \ \forall 1 \le k \le N$

At the same time: J is proper $\Leftrightarrow e \notin J$. ▨

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