PART III. Applications

Classical moment problems

December 10, 2025

Hamburger's moment problem:
Given a sequence of numbers $\{h_k\}_{k=0}^{\infty}$, decide if there exists a measure μ on $\mathbb{R}$ s.t. $h_k = \int_{\mathbb{R}} x^k d\mu$, $k \geq 0$.
($\Leftrightarrow$ which sequences of reals are moment sequences of measures?)

Trigonometric moment problem:
Given a sequence $\{t_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$, is there a $\mu \geq 0$ s.t. $t^k = \int_{\mathbb{T}} z^k d\mu$?

Obvious restrictions:

- Hamburger case: $\sum_{k,j=0}^{N} h_{k+j} a_k \bar{a}_j \overset{(*)}{\geq} 0 \quad \forall \{a_k\}_{k=0}^{N} \subset \mathbb{C}, \ N \geq 0$

$$0 \leq \int_{\mathbb{R}} \Big|\sum_{k=0}^{N} a_k x^k\Big|^2 d\mu = \sum_{1}^{N} a_k \bar{a}_j \int_{\mathbb{R}} x^{k+j} d\mu \overset{N}{=} \sum_{k,j=0} a_k \bar{a}_j h_{k+j}$$

- Trigonometric case: $\sum_{k,j=-N}^{N} t_{k-j} a_k \bar{a}_j \overset{(**)}{\geq} 0 \quad \forall \{a_k\}_{-N}^{N} \subset \mathbb{C}, N \geq 0$

$$0 \leq \int_{\mathbb{T}} \Big|\sum_{-N}^{N} a_k z^k\Big|^2 d\mu = \sum_{-N}^{N} a_k \bar{a}_j \int_{\mathbb{R}} z^k \bar{z}^j d\mu \overset{N}{=} \sum_{-N}^{N} a_k \bar{a}_j t_{k-j}$$

Theorem [Hamburger]: The assumption $(*)$ is sufficient for the solvability of the Hamburger case.

Theorem: The assumption $(**)$ is sufficient for the solvability of the Trigonometric moment problem. Moreover, we have
$\sum_{-N}^{N} t_{k-j} a_k \bar{a}_j > 0 \ \forall \{a_k\}_{k=0}^{N} \Leftrightarrow \{t_k\}$ is the moment sequence of a measure μ such that $\# \operatorname{supp} \mu = +\infty$. $(***)$

Our goal is to prove $(***)$.

Proof: $H_0 = \left(\operatorname{span}\{z^k\}_{k\in\mathbb{Z}}, \left\langle \sum_{-N}^{N} a_k z^k, \sum_{-N}^{N} \tilde{a}_k z^k \right\rangle := \sum t_{k-j} a_k \overline{\tilde{a}_j} \right)$

$\hookrightarrow$ pre Hilbert space, because it is linear, and $\langle \cdot , \cdot \rangle$ is the inner product on H_0, but H_0 is not complete w.r.t. $\| \sum a_z z^k \| = \sqrt{\left\langle \sum_{-N}^{N} a_z z^k, \sum_{-N}^{N} a_z z^k \right\rangle}$

General Functional analysis implies that $\exists H$ - Hilbert space such that $H_0 \subset H$ as a dense linear subset.

$T: \sum_{k=0}^{N} a_k z^k \longmapsto \sum_{k=0}^{N} a_k z^{k+1}$ - densely defined operator on H:

$$\left\| T\left(\sum_{-N}^{N} a_k z^k\right)\right\|^2 = \left\| \sum_{-N}^{N} a_k z^{k+1} \right\|^2 = \sum_{-N+1}^{N+1} t_{k-j} a_{k-1} \overline{a_{j-1}} = \sum_{-N+1}^{N+1} t_{(k-1)-(j-1)} a_{k-1} \overline{a_{j-1}} =$$

$$\left\| \sum_{-N+1}^{N+1} a_{k-1} z^k \right\|^2 \qquad = \sum_{-N}^{N} t_{k-j} a_k \overline{a_j} = \left\| \sum_{-N}^{N} a_k z^k \right\|^2$$

$\Rightarrow T$ is an isometry initially defined on H_0.

Let's extend it to the whole space H. $\Rightarrow T$ is isometry on H, $T(H) \supset H_0$ - dense in H, since $T(H)$ is closed, we have $T(H) = H$.

$\Rightarrow T$ is unitary. Moreover, there is $h = 1$ s.t. $\operatorname{span}\{T^k T^{*j} h\} = (H_0)$ is dense in H.

By the spectral theorem, there is a measure μ s.t. $\operatorname{supp}\mu = \mathbb{T} \subset \mathbb{T}$: $T \cong M_z$ on $L^2(\mu)$.

$$\langle T^k h, h \rangle_H = \langle M_z^k 1, 1 \rangle_{L^2(\mu)} \ \forall k \geq 0 \text{ for } h = 1 \text{ in } H.$$

$$\langle T^k 1, 1 \rangle_H = \langle z^k, 1 \rangle_H = \sum_{0}^{k} t_{i-j} \delta_k(i) \delta_0(j) = t_k$$

$$\langle M_z^k 1, 1 \rangle_{L^2(\mu)} = \langle z^k, 1 \rangle_{L^2(\mu)} = \int_{\mathbb{T}} z^k \, d\mu$$

$\Rightarrow t_k = \int_{\mathbb{T}} z^k \, d\mu$, $k \geq 0$

$\underline{t_{-k} = \overline{t_k}} = \overline{\int_{\mathbb{T}} z^k \, d\mu} = \int_{\mathbb{T}} z^{-k} \, d\mu$, $k \geq 0$

$\Rightarrow t_k$ is the moment sequence

$t_k = \langle z^k, 1 \rangle$

$t_{-k} = \langle z^{-k}, 1 \rangle = \langle T^k z^{-k}, T^k 1 \rangle = \langle 1, z^k \rangle = \overline{\langle z^k, 1 \rangle} = \overline{t_k} \Rightarrow \overline{t_{-k}} = t_k$

It remains to show that the measure μ is such that $\#\operatorname{supp}\mu = \infty$.

$\Leftrightarrow \int \left| \sum_{-N}^{N} a_k z^k \right|^2 d\mu > 0$ (true by assumption). ∎

December 11, 2025