PART III. Applications

Classical moment problems

December 10, 2025

Hamburger's moment problem:
Given a sequence of numbers $\{h_k\}_{k=0}^{\infty}$, decide if there exists a measure μ on $\mathbb{R}$ s.t. $h_k = \int_{\mathbb{R}} x^k d\mu$, $k \geq 0$.
($\Leftrightarrow$ which sequences of reals are moment sequences of measures?)

Trigonometric moment problem:
Given a sequence $\{t_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$, is there a $\mu \geq 0$ s.t. $t^k = \int_{\mathbb{T}} z^k d\mu$?

Obvious restrictions:

- Hamburger case: $\sum_{k,j=0}^{N} h_{k+j} a_k \bar{a}_j \overset{(*)}{\geq} 0 \quad \forall \{a_k\}_{k=0}^{N} \subset \mathbb{C},\ N \geq 0$

$$0 \leq \int_{\mathbb{R}} \Big|\sum_{k=0}^{N} a_k x^k\Big|^2 d\mu = \sum_{1}^{N} a_k \bar{a}_j \int_{\mathbb{R}} x^{k+j} d\mu^N = \sum_{k,j=0} a_k \bar{a}_j h_{k+j}$$

- Trigonometric case: $\sum_{k,j=-N}^{N} t_{k-j}\, a_k \bar{a}_j \overset{(**)}{\geq} 0 \quad \forall \{a_k\}_{-N}^{N} \subset \mathbb{C},\ N \geq 0$

$$0 \leq \int_{\mathbb{T}} \Big|\sum_{-N}^{N} a_k z^k\Big|^2 d\mu = \sum_{-N}^{N} a_k \bar{a}_j \int_{\mathbb{R}} z^k \bar{z}^j d\mu^N = \sum_{-N}^{N} a_k \bar{a}_j t_{k-j}$$

Theorem [Hamburger]: The assumption $(*)$ is sufficient for the solvability of the Hamburger case.

Theorem: The assumption $(**)$ is sufficient for the solvability of the Trigonometric moment problem. Moreover, we have
$\sum_{-N}^{N} t_{k-j} a_k \bar{a}_j > 0 \ \forall \{a_k\}_{k=0}^{N} \Leftrightarrow \{t_k\}$ is the moment sequence of a measure μ such that $\#\operatorname{supp}\mu = +\infty$. $(***)$

Our goal is to prove $(***)$.

Proof: $H_0 = \left(\mathrm{span}\{z^k\}_{k\in\mathbb{Z}},\ \langle \sum_{-N}^{N} a_k z^k, \sum_{-N}^{N} \tilde{a}_k z^k \rangle := \sum t_{k-j} a_k \overline{\tilde{a}_j} \right)$

$\hookrightarrow$ pre Hilbert space, because it is linear, and $\langle\cdot,\cdot\rangle$ is the inner product on H_0, but H_0 is not complete w.r.t. $\|\sum a_z z^k\| = \sqrt{\langle \sum_{-N}^{N} a_z z^k, \sum_{-N}^{N} a_z z^k\rangle}$

General Functional analysis implies that $\exists H$ - Hilbert space such that $H_0 \subset H$ as a dense linear subset.

$T: \sum_{k=0}^{N} a_k z^k \longmapsto \sum_{k=0}^{N} a_k z^{k+1}$ - densely defined operator on H:

$$\|T(\sum_{-N}^{N} a_k z^k)\|^2 = \|\underbrace{\sum_{-N}^{N} a_k z^{k+1}}_{=\sum_{-N+1}^{N+1} a_{k-1} z^k}\|^2 = \sum_{-N+1}^{N+1} t_{k-j} a_{k-1}\overline{a_{j-1}} = \sum_{-N+1}^{N+1} t_{(k-1)-(j-1)} a_{k-1}\overline{a_{j-1}} =$$

$$= \sum_{-N}^{N} t_{k-j} a_k \overline{a_j} = \|\sum_{-N}^{N} a_k z^k\|^2$$

$\Rightarrow T$ is an isometry initially defined on H_0.

Let's extend it to the whole space H. $\Rightarrow T$ is isometry on H, $T(H) \supset H_0$ - dense in H, since $T(H)$ is closed, we have $T(H) = H$.

$\Rightarrow T$ is unitary. Moreover, there is $h=1$ s.t. $\underbrace{\mathrm{span}\{T^k T^{*j} h\}}_{=(H_0)}$ is dense in H.

By the spectral theorem, there is a measure μ s.t. $\mathrm{supp}\,\mu = \mathbb{T} \subset \mathbb{T}$: $T \cong M_z$ on $L^2(\mu)$.

$$\langle T^k h, h\rangle_H = \langle M_z^k 1, 1\rangle_{L^2(\mu)} \ \forall k \geq 0 \text{ for } h=1 \text{ in } H.$$

$$\|$$

$$\langle T^k 1, 1\rangle_H = \langle z^k, 1\rangle_H = \sum_{0}^{k} t_{i-j}\,\delta_k(i)\,\delta_0(j) = t_k$$

$$\langle M_z^k 1, 1\rangle_{L^2(\mu)} = \langle z^k, 1\rangle_{L^2(\mu)} = \int_{\mathbb{T}} z^k \, d\mu$$

$\Rightarrow t_k = \int_{\mathbb{T}} z^k d\mu, \quad k \geq 0$

$\underline{t_{-k} = \overline{t_k}} = \overline{\int_{\mathbb{T}} z^k d\mu} = \int_{\mathbb{T}} z^{-k} d\mu, \ k \geq 0$

$\Rightarrow t_k$ is the moment sequence

$\begin{cases} t_k = \langle z^k, 1\rangle \\ t_{-k} = \langle z^{-k}, 1\rangle = \langle T^k z^{-k}, T^k 1\rangle = \langle 1, z^k\rangle = \overline{\langle z^k, 1\rangle} = \overline{t_k} \Rightarrow \overline{t_{-k}} = t_k \end{cases}$

It remains to show that the measure μ is such that $\#\mathrm{supp}\,\mu = \infty$.

$\Leftrightarrow \int |\sum_{-N}^{N} a_k z^k|^2 d\mu > 0$ (true by assumption). $\blacksquare$

Characters on compact Abelian groups

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<u>Definition</u>: G is a topological group if G is a group with topology whose operation is continuous in the product topology $G \times G$, and the operation of taking the inverses is also continuous.

<u>Definition</u>: G is a compact group if G is a topological group such that G with its topology is a compact Hausdorff space.

<u>Definition</u>: A map $\gamma : G \longrightarrow \mathbb{T}$ is a character if γ is a group homomorphism and $\gamma \in \mathcal{C}(G, \mathbb{T})$.

<u>Remark</u>: We will deal with the abelian (commutative) case, and we will denote the group operation by "+", the inverse element to $x \in G$ by $-x$, and the identity of the group by 0.

<u>Definition</u>: $\hat{G} = \{\gamma \text{ character of } G\}$ is called the dual group to G.

<u>Remark</u>: In our notation, for every $\gamma \in \hat{G}$ we have

$$\left.\begin{array}{l} \gamma(x+y) = \gamma(x) \cdot \gamma(y) \quad \forall x,y \in G \\ |\gamma(x)| = 1 \quad \forall x \in G \\ \gamma \in \mathcal{C}(G, \mathbb{T}) \end{array}\right] \text{ equivalent to } \gamma \in \hat{G}$$

<u>Remark</u>: $\gamma_0 : x \longmapsto 1$ is always in $\hat{G}$

<u>Definition</u>: Let G be a locally compact topological group. Then μ is the Haar measure on G if $\mu(\mathcal{U}+x) = \mu(x+\mathcal{U}) = \mu(\mathcal{U})$ for every Borel set $\mathcal{U}$, $\mu \neq 0$, μ is regular ($\Rightarrow$ finite on compact subsets).

Theorem [Weyl]: Every locally compact topological group has a Haar measure μ, which is unique up to multiplication by a constant.

<u>Agreement</u>: If G is compact, we normalize μ: $\mu(G)=1$. With this normalization the Haar measure is unique.

<u>Theorem [Peter-Weyl]</u>: If G is a commutative compact group, then characters form an orthonormal basis in $L^2(G,\mu)$, where μ is the Haar measure of G.

<u>Examples of characters</u>:

- $G=\mathbb{R}$ (locally compact), $\hat{G}=\{e^{i\lambda x} \mid \lambda\in\mathbb{R}\}$, $\mathbb{R}\cong\hat{\mathbb{R}}$.
- $G=\mathbb{T}$ (compact), $\hat{\mathbb{T}}=\{z^n \mid n\in\mathbb{Z}\}$, $\hat{\mathbb{T}}\cong\mathbb{Z}$.
- $G=\mathbb{Z}/n\mathbb{Z}$ (compact), $\hat{G}=G_n$.
- $G_n=\{\xi\in\mathbb{T} \mid \xi^n=1\}$, $\hat{G}=\mathbb{Z}/n\mathbb{Z}$.

The decomposition of $f=\sum_{k\in\mathbb{Z}} c_k z^k$ for every $f\in L^2(\mathbb{T})$ is just the Fourier decomposition, the map $f\longmapsto\{c_k\}$ is the discrete Fourier transform. In the continuous case $(G=\mathbb{R})$ $f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)e^{ix\lambda}dx$ is the Fourier decomposition, and the map $f\longmapsto g$ is the Fourier transform $(g(t)=\int_{\mathbb{R}} f(x)e^{-ixt}dx)$.

Characters form an orthonormal system:

$$\int_G \gamma_1(x)\overline{\gamma_2(x)}\,d\mu \overset{\mu\text{ is Haar}}{=} \int_G \gamma_1(x-y)\overline{\gamma_2(x-y)}\,d\mu(x)$$
$$= \gamma_1(-y)\overline{\gamma_2(-y)}\int_G \gamma_1(x)\overline{\gamma_2(x)}\,d\mu$$

$$\Rightarrow \langle\gamma_1(x),\gamma_2(x)\rangle_{L^2(\mu)} = \gamma_1(-y)\overline{\gamma_2(-y)}\langle\gamma_1,\gamma_2\rangle_{L^2(\mu)}$$
$$\Rightarrow \langle\gamma_1(x),\gamma_2(x)\rangle\neq 0 \Leftrightarrow \gamma_1(-y)\overline{\gamma_2(-y)}=1 \quad \forall y\in G$$
$$\Leftrightarrow \gamma_1(-y)=\overline{\overline{\gamma_2(-y)}} \quad \forall y\in G$$
$$\Leftrightarrow \gamma_1=\gamma_2$$

If $\gamma_1=\gamma_2=\gamma$, then $\|\gamma\|^2=\int_G \underbrace{|\gamma|^2}_{=1}d\mu=\mu(G)=1$

So, the problem is completeness of $\{\gamma\}_{\gamma \in \hat{G}}$.

<u>Definition</u>: Let $s \in G$. The shift operator T_s is $T_s: f \longmapsto f(\cdot - s)$.

<u>Lemma 1</u>: T_s is unitary on $L^2(G) = L^2(G,\mu)$ (μ is Haar).

<u>Proof</u>: $T_s L^2(G) = L^2(G)$ and

$$\|T_s f\|^2_{L^2(G)} = \int_G |f(x-s)|^2 d\mu = \int_G |f(x)|^2 d\mu = \|f\|^2_{L^2(G)}$$

$\Rightarrow T_s$ is an isometry. $\blacksquare$

<u>Lemma 2</u>: For every $f \in L^2(G)$, we have $T_s f \longrightarrow f$ as $s \longrightarrow 0$ in G
($\Leftrightarrow \forall \varepsilon > 0 . \exists U_\varepsilon$ - open neighbourhood of $0 . \forall s \in U_\varepsilon . \|T_s f - f\| < \varepsilon$).

To prove this lemma, we will use a version of Cantors theorem for compact groups:

<u>Theorem</u>: If $f \in \mathcal{C}(G)$, G is a compact group, then $\forall \varepsilon > 0 . \exists U_\varepsilon$ a neighborhood of 0 such that $|f(x) - f(y)| < \varepsilon$ if $x - y \in U_\varepsilon$. (Without proof.)

<u>Proof of lemma 2</u>: Take $f \in L^2(G)$ and find $g \in \mathcal{C}(G) : \|f - g\|_{L^2(G)} \leq \frac{\varepsilon}{3}$
(this is possible, because μ is a regular measure).

$$\|T_s f - f\|_{L^2(G)} \leq \underbrace{\|T_s(f-g)\|}_{\leq \varepsilon/3} + \underbrace{\|f-g\|}_{\leq \varepsilon/3} + \underbrace{\|T_s g - g\|}_{\leq ?}$$

$$\|T_s g - g\|_{L^2(G)} \leq \|T_s g - g\|_{L^\infty(G)} = \max_{x \in G} |g(x-s) - g(x)| \leq \varepsilon/3$$

by Cantors theorem if $s \in U_{\varepsilon/3}$ ($U_{\varepsilon/3}$ from Cantors theorem) $\blacksquare$

<u>Definition</u>: Let $f, g \in L^1(G)$, then $(f * g)(y) = \int_G f(x) g(y-x) d\mu(x)$.

<u>Remark</u>: $L^p(G) \subset L^1(G)$, because G is compact, so

$$\left(\int |f| d\mu\right) \le \left(\int 1^q\right)^{1/q}\left(\int |f|^p\right)^{1/p} = \|f\|_{L^p(G)}$$

In particular, we can also define $f*g$ for every $f \in L^{p_1}(G)$, $g \in L^{p_2}(G)$

Lemma 3 [Young inequality]: $1 \le p \le \infty$

$$\|f*g\|_{L^p} \le \|f\|_{L^p(G)} \|g\|_{L^1(G)} \quad \forall f \in L^p(G),\ g \in L^1(G).$$

Proof: We may assume that $\|g\|_{L^1(G)} = 1$. Then

$$\|g*f\|_{L^p}^p = \int_G \Big| \int_G f(y-x) \underbrace{g(x) d\mu(x)}_{\text{probability measure}} \Big|^p d\mu(y) \overset{\text{Jensen}}{\le} \int_G \int_G |f(y-x)|^p |g(x)| d\mu(x)\, d\mu(y)$$

$$\overset{\text{Fubini}}{=} \int_G |g(x)| \underbrace{\int_G |f(y-x)|^p d\mu(y)}_{\int_G |f(y)|^p d\mu} d\mu(x) = \underbrace{\|g\|_{L^1}}_{=1} \cdot \|f\|_{L^p}^p$$

$\Rightarrow \|g*f\|_{L^p} \le \|f\|_{L^p} = \|g\|_{L^1} \cdot \|f\|_{L^p}$.

It remains to note that $g*f = f*g$:

$$f*g = \int f(x) g(y-x)\, d\mu(x) \underset{x = y-\tilde{x}}{=} \int f(y-\tilde{x})\, g(\tilde{x})\, d\mu(\tilde{x})$$ ▨

Lemma 4 [Approximation lemma]: $\forall \varphi \in L^2(G,\mu)$ we have

$$\inf_{\substack{U \ni 0 \\ U = -U \\ U \text{ open}}} \left\| \varphi - \varphi * \frac{\chi_U}{\mu(U)} \right\| = 0.$$

Proof: Take $\varphi \in L^2(G,\mu)$ and $\psi \in \mathcal{C}(G)$: $\|\varphi - \psi\|_{L^2(G)} < \varepsilon$.

$$\left\|\varphi - \varphi * \frac{\chi_U}{\mu(U)}\right\|_{L^2} \le \left\|\psi - \psi * \frac{\chi_U}{\mu(U)}\right\|_{L^2} + \underbrace{\|\varphi - \psi\|_2}_{\le \varepsilon} + \underbrace{\left\|(\varphi-\psi) * \frac{\chi_U}{\mu(U)}\right\|_{L^2}}_{\le \|\varphi - \psi\|_{L^2} \underbrace{\left\|\frac{\chi_U}{\mu(U)}\right\|_{L^1}}_{=1} \le \varepsilon}$$

$$\left\|\psi - \psi * \frac{\chi_U}{\mu(U)}\right\|_{L^2(G)} \le \left\|\psi(y) - \int_G \psi(x) \frac{\chi_U(y-x)}{\mu(U)} d\mu\right\|_{L^2(G)} \le$$

$$\le \sup_{y \in G} \int_G \underbrace{|\psi(y) - \psi(x)|}_{\le \varepsilon} \frac{\chi_U(y-x)}{\mu(x)} d\mu \le \varepsilon$$

if $U = U_\varepsilon$ for the function $f = \psi$ in Cantor's theorem ▨

<u>Goal</u>: G- compact abelian group with Haar measure μ, then $\hat{G}$ is an ONB in $L^2(G,\mu)$.

<u>Lemma</u>: Let G be as above, $f \in L^2(G,\mu)$. Then F.A.E:

i) $f = c\cdot\gamma,\ \ \gamma \in \hat{G}$

ii) $T_s f = \lambda_s f$ in $L^2(G,\mu)$ $\forall s \in G$ $\qquad$ $T_s f = f(\cdot - s),\ s\in G$

<u>Proof</u>: (1) $\Rightarrow$ (2): $T_s(c\gamma) = c\gamma(x-s) = c\gamma(-s)\gamma(x)$, so $\lambda_s := \gamma(-s)$.

<u>(2) $\Rightarrow$ (1)</u>: $T_{s+s'} = T_s T_{s'}$ $\quad s, s' \in G$

$\Rightarrow \lambda_{s+s'} f(x) = \lambda_s \lambda_{s'} f(x)$ for μ-a.e. $x \in G$

$\Rightarrow \lambda_{s+s'} = \lambda_s \cdot \lambda_{s'}$ because $\exists x.\ f(x) \neq 0$ (otherwise $\underset{f=1}{\text{one}}$ can take $c=0$)

$\|T_s f - T_{s'} f\| = |\lambda_s - \lambda_{s'}|\,\|f\|$ $\Rightarrow$ the map $s \mapsto \lambda_s$ is continuous from G to $\mathbb{T}$:

($\Leftrightarrow |\lambda_s - \lambda_{s'}| \longrightarrow 0$ if $s' \longrightarrow s$ in G $\Leftrightarrow \|T_s f - T_{s'} f\|_{L^2(\mu,G)} \longrightarrow 0$ if $s' \longrightarrow s$,

which is true for $f \in \mathcal{C}(G)$ by Cantor's theorem and

$$\|T_s f - T_{s'} f\| \le \|T_s(f - \tilde f)\| + \|T_{s'}(f - \tilde f)\| + \|T_s \tilde f - T_{s'} \tilde f\|$$
$$\le \underbrace{2\|f - \tilde f\|}_{\le \frac{\varepsilon}{4}\ \forall s'} + \underbrace{\|T_s \tilde f - T_{s'} \tilde f\|}_{\le \frac{\varepsilon}{2} \text{ for } s' \text{ close to } s} \ )$$

$\Rightarrow \lambda_s$ is a continuous function from G to $\mathbb{C}$

$$\left.\|f\| \underset{\substack{\uparrow \\ \text{isometricity} \\ \text{of } T_s}}{=} \|T_s f\| \underset{\substack{\uparrow \\ \text{by } T_s f = \lambda_s f}}{=} |\lambda_s|\cdot\|f\|\right\} \Rightarrow |\lambda_s| = 1 \text{ for every } s\in G$$

$\Rightarrow \lambda_s \in \mathcal{C}(G, \mathbb{T})$, $\lambda_{s+s'} = \lambda_s\cdot\lambda_{s'} \Rightarrow \lambda_s \in \hat{G}$.

Let us prove that $f = c\,\overline{\lambda_x}$ for some $c\in\mathbb{C}$ $\quad (\gamma := \overline{\lambda_x})$.

$h(x) := \lambda_x f(x)$. We have $T_s h = \lambda_{x-s} f(x-s) = \lambda_x \lambda_s f(x) = \lambda_x f(x) = h$ on a set $E_s \subset G : \mu(E_s) = \mu(G) = 1$. Unfortunately E_s might depend on s and we cannot say $h(x) = h(0-(-x)) \underset{\uparrow\text{problem}}{=} (T_{-x}h)(0) = h(0)$. So we need to argue differently.

Take $g = \frac{\chi_U}{\mu(U)}$ for some U-open set in G, $U = -U$:

$$(h * g)(y) = \int_G h(x) g(y-x)\,d\mu(x) \underset{\substack{\uparrow \\ \mu \text{ is Haar}}}{=} \int_G h(x+y) g(y-(x+y))\,d\mu(x)$$

$$= \int_G h(x) g(-x)\, d\mu(x) \overset{?}{\underset{U=-U}{=}} \int h(x) g(x)\, d\mu = C_U$$

$\Rightarrow (h*g)(y) = C_U \;\forall y \in G$. In fact, $C_U = C$, C does not depend on U:

$$C_U = \int_G C_U\, d\mu = \int_G\int_G h(x) g(y-x)\, d\mu\, d\mu \overset{\text{Fubini}}{=} \int h(x) \Big(\underbrace{\int_G g(y-x)\, d\mu(y)}_{\int_G g(y)\, d\mu(y) = 1} \Big)\, d\mu(x)$$

$$= \int_G h(x)\, d\mu = C$$

$\Rightarrow h * \frac{\chi_U}{\mu(U)} = C$, C does not depend on U.

From the approximation lemma, $\inf_h \|h - h * \frac{\chi_U}{\mu(U)}\|_{L^1(\mu)} = 0$ we have

$\|h - C\|_{L^1(\mu)} = 0 \Rightarrow h = C$ a.e. on G.

$\Rightarrow \lambda_x \varphi_x = C$ a.e. on G

$\Rightarrow \varphi = C \cdot \gamma$ for $\gamma = \overline{\lambda_x}$. ▨

Lemma: $A: f \longmapsto f * \frac{\chi_U}{\mu(U)}$, $U = -U$, $\mu(U) > 0$ – a compact self-adjoint operator on $L^2(G, \mu)$.

Proof: $Af = \int_G k(x,y) f(x)\, d\mu(x)$ for $k(x,y) = \frac{\chi_U(y-x)}{\mu(U)}$

Note that $k(x,y) = k(y,x) = \overline{k(y,x)} \Rightarrow A = A^*$ $\left(A^* f = \int \overline{k(y,x)} f(x)\, d\mu(x)\right)$.

$$\iint_{G\times G} |k(x,y)|^2\, d\mu(x)\, d\mu(y) < \infty \quad \left(\text{in our case, } \iint_{G\times G} |k(x,y)|^2\, d\mu\, d\mu = \frac{1}{\mu(U)} < \infty\right)$$

$\Rightarrow$ From Hilbert-Schmidt test (to be proved later) $A \in S_\infty(L^2(G,\mu))$. ▨

Lemma: If H is a separable Hilbert space, $A \in S_\infty(H)$, $A = A^*$ $\Rightarrow$ $A = \sum_{\lambda_k \in \sigma(A)} \lambda_k P_{E_k}$, where $E = \{h \in H \mid Ah = \lambda_k h\}$, and the series converges in operator norm.

Proof: This is an exercise from the homework.

Lemma: Let H be a finite-dimensional Hilbert space, $\dim H = N < \infty$. Let $\{U_\alpha\}_{\alpha \in I}$ be a family of unitary operators on H, $U_\alpha U_\beta = U_\beta U_\alpha$ $\forall \alpha, \beta \in I$. $\Rightarrow \exists \{e_n\}_{n=1}^N$ – an ONB in H: $U_\alpha e_n = \lambda_{\alpha,n} e_n$ $\forall \alpha \in I$.

Proof: Induction on N.

- $N=1$ $\quad U_\alpha = c_\alpha \cdot I \quad \forall \alpha$ ✓
- $N-1 \longrightarrow N$, $N \geq 2$

either $U_\alpha = c_\alpha I \quad \forall \alpha$ ✓

or $\exists \alpha.\ E_\alpha = \{h \in H \mid U_\alpha h = \lambda_\alpha h\}$ satisfies $E_\alpha \neq \{0\}$, $E_\alpha \neq H$

$\forall \beta \in I$, $\forall h \in E_\alpha$, we have $U_\alpha(U_\beta h)) = U_\beta(U_\alpha h) = U_\beta(\lambda_\alpha h) = \lambda_\alpha U_\beta h$

$\Rightarrow U_\beta h \in E_\alpha$ by definition of E_α

$\Rightarrow U_\beta E_\alpha \subset E_\alpha \Rightarrow U_\beta E_\alpha = E_\alpha \quad (\dim U_\beta E_\alpha = \dim E_\alpha)$.

Moreover, $U_\beta^* E_\alpha = U_\beta^{-1}(E_\alpha) = E_\alpha$. So, E_α is a reducing subspace for U_β, in particular, $U_\beta = (U_\beta|_{\underbrace{E_\alpha}_{\dim \leq N-1}}) \oplus (U_\beta|_{\underbrace{E_\alpha^\perp}_{\dim \leq N-1}}) \quad \forall \beta \in I$.

$\Rightarrow$ by induction assumption, ok. ▨

Proof of Peter-Weyl: We need to prove that $\hat{G}$ is an ONB in $L^2(G,\mu)$.
We know that $\forall \gamma \neq \tilde{\gamma} \quad (\gamma, \tilde{\gamma})_{L^2(G,\mu)} = 0$, so we need to check that $\forall \varphi \in L^2(G,\mu).\ \varphi \in \operatorname{clos}_{L^2(\mu)}(\operatorname{span} \hat{G})$.

Take an open neighbourhood U of 0 s.t. $U = -U$. Consider the compact self-adjoint operator $A_U : f \longmapsto f * \frac{\chi_U}{\mu(U)}$. We have $A_U = \sum_{\lambda_k \in \sigma(A_U)} \lambda_k P_{E_{\lambda_k}}$

Let us show that $A_U \varphi \in \operatorname{clos}_{L^2(\mu)}(\operatorname{span}\hat{G})$. It is enough to check that $P_{E_{\lambda_k}}\varphi \in \operatorname{span}\hat{G}$. Observe that $T_s A_U = A_U T_s$:

$$A_U T_s f = \int f(x-s) \frac{\chi_U(y-x)}{\mu(U)} = \int f(x-s) \frac{\chi_U(y-s-(x-s))}{\mu(U)} d\mu = \int_G f(\tilde{x}) \frac{\chi_U(y-s-\tilde{x})}{\mu(U)} d\mu = T_s A_U f$$

$\Rightarrow$ If $h : A_U h = \lambda_k h \Rightarrow A_U T_s h = T_s A_U h = T_s(\lambda_k h) = \lambda_k T_s h \Rightarrow T_s h \in E_k$

$\Rightarrow T_s E_{\lambda_k} \subset E_{\lambda_k} \Rightarrow [\dim E_{\lambda_k} < \infty$ because A is compact$] \Rightarrow T_s E_{\lambda_k} = E_{\lambda_k}$.

Now we can use lemma for $\{U_\alpha\}_{\alpha \in I} = \{T_s|_{E_{\lambda_k}}\}_{s \in G}$. There is a ONB $\{e_{\lambda_k,n}\} : T_s e_{\lambda_k,n} = h_s e_{\lambda_k,n} \ \forall n \leq \dim E_{\lambda_k} \ \forall s \in G$.

By lemma, $e_{\lambda_k,n} = c_{\lambda_k,n} \gamma_{\lambda_k,n}$ for some $c_{\lambda_k,n} \in \mathbb{C}$, $\gamma_{\lambda_k,n} \in \hat{G}$.

In particular E_{λ_k} is spanned by $\{\gamma_{\lambda_k,n}\}_{n \leq \dim E_{\lambda_k,n}} \Rightarrow P_{E_{\lambda_k}}\varphi \in E \subset \operatorname{span} G$.

So, $A_U \varphi \in \operatorname{clos}_{L^2(G,\mu)}(\operatorname{span}\hat{G})$. By the approximation lemma,
$\inf_U \|A_U\varphi - \varphi\|_{L^2(G)} = 0$ and $A_U\varphi \in \operatorname{clos}\operatorname{span}\hat{G} \Rightarrow \varphi \in \operatorname{clos}\operatorname{span}\hat{G}$. ▨

Minmax principle

January 6, 2026

Theorem [minmax principle]: Let H be a separable Hilbert space, $A \in S_\infty(H)$, $A = A^*$, $\sigma(A) = \{-\lambda_n^-\}_{n=1}^{N_-} \cup \{\lambda_n^+\}_{n=1}^{N_+}$ where $N_\pm \in \mathbb{N} \cup \{\infty\} \cup \{0\}$, $\lambda^\pm > n$ $\forall n$, and each point $\lambda \in \sigma(A)$ appears in $\{-\lambda_n^-\} \cup \{\lambda_n^+\}$ exactly $k(\lambda)$ times, where $k(\lambda)$ is the multiplicity of $\lambda = \dim\{\varphi \mid A\varphi = \lambda\varphi\}$. Assume, moreover, that $\{\lambda_n^+\}$, $\{\lambda_n^-\}$ are non-increasing. Then

$$\pm\lambda_n^\pm = \min_{\substack{L \subset H \\ \operatorname{codim} L \leq n-1}} \max_{x \in L \setminus \{0\}} \frac{\pm\langle Ax, x\rangle}{\langle x, x\rangle}. \qquad (\operatorname{codim} L = \dim(H \ominus L))$$

<u>Proof:</u> Let's consider the decomposition $A = \bigoplus_{\lambda \in \sigma(A)} \lambda P_{E_\lambda}$ ($E_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$, see exercises).

Let's choose orthonormal sequence $\varphi_1^\pm, \varphi_2^\pm, \dots$ such that

$$A = -\sum_{n=1}^{N_-} \lambda_n^- \langle \cdot, \varphi_n^-\rangle \varphi_n^- + \sum_{n=1}^{N_+} \lambda_n^+ \langle \cdot, \varphi_n^+\rangle \varphi_n^+. \qquad (*)$$

Here we use the fact that in $E \subset H$ - subspace, $\{\psi_1, \dots, \psi_m\}$ - ONB in E, then $P_E = \sum_1^m \langle \cdot, \psi_k\rangle \psi_k$ - orthogonal projector in H to E (Proof: add orthonormal sequence $\psi_1^\perp, \psi_2^\perp, \dots$ so that $\underbrace{\{\psi_k\} \cup \{\psi_k^\perp\}}_{\{e_k\}}$ is ONB in H and consider the action P_E on $h = \sum_{k=1}^\infty c_k e_k$

Let's prove that $\lambda_n^+ = \min_{\substack{L \subset H \\ \operatorname{codim} L \leq n-1}} \max_{x \in L \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$.

(the case $-\lambda_n^- = \dots$ follows from this, because $-\lambda_n^-(A) = \lambda_n^+(-A)$, see $(*)$)
Set $F_n = \operatorname{span}\{\varphi_k^+\}_{k=1}^n$. For every $L \subset H$: $\operatorname{codim} L \leq n-1$ we have $F_n \cap L \neq \{0\}$. $(**)$

$$H = L \oplus \overbrace{(H \ominus L)}^{\dim \leq n-1}$$
$$H = (H \ominus F_n) \oplus \underbrace{F_n}_{\dim n}$$

Indeed $P_{H\ominus L} : F_n \longrightarrow H \ominus L$ has a nonzero kernel $\Rightarrow \exists h \in F_n \setminus \{0\}. P_{H\ominus L} h = 0$ $\Leftrightarrow h \in L \Rightarrow h \in F_n \cap L$, $(**)$ ok.

Take $h \in (F_n \cap L) \setminus \{0\}$ and consider $h = \sum_{k=1}^n d_k \varphi_k^+$, $d_k \in \mathbb{C}$ $(h \in F_n)$.

$$\langle Ah, h\rangle \overset{(***)}{=} \left\langle \sum_{k=1}^{n} \lambda_k^+ \langle h, \varphi_k^+\rangle, h\right\rangle$$

$$= \left\langle \sum_{k=1}^{n} \lambda_k^+ \alpha_k \varphi_k^+, \sum_{k=1}^{n} \alpha_k \varphi_k^+ \right\rangle$$

$$= \sum_{k=1}^{n} \lambda_k^+ |\alpha_k|^2$$

$$\geq \lambda_n^+ \Bigg(\underbrace{\sum_{k=1}^{n} |\alpha_k|^2}_{\hookrightarrow \text{norm}}\Bigg) \qquad \text{(sequence is non-increasing)}$$

$$= \lambda_n^+ \langle h, h\rangle$$

$$\Rightarrow \min_{\substack{L\subset H \\ \operatorname{codim} L \leq n-1}} \max_{h\neq 0} \frac{\langle Ah, h\rangle}{\langle h, h\rangle} \geq \lambda_n^+$$

To prove the converse inequality we take $L := H \ominus F_{n-1}$. For every $h \in H \ominus F_{n-1}$ we have

$$P_{(\ker A)^\perp} h = \sum_{k=n}^{N_+} \alpha_k \varphi_k^+ + \sum_{k=1}^{N_-} \beta_k \varphi_k^-.$$

(here we use the fact that $\{\varphi_k^+\}_1^{N_+} \cup \{\varphi_k^-\}_1^{N_-}$ is an ONB in $H \ominus \ker A$, exercise)

$$\langle Ah, h\rangle = \langle A\tilde{h}, \tilde{h}\rangle = \text{...argument similar to } (***)\text{...}$$

$$= \sum_{k=n}^{N_+} \lambda_k^+ |\alpha_k^2| - \underbrace{\sum_{k=1}^{N_-} \lambda_k^- |\beta_k|^2}\ \leq 0$$

$$\leq \lambda_n^+ \cdot \sum_{k=n}^{N_+} |\alpha_k^2|$$

$$= \lambda_n^+ \langle h, h\rangle, \quad \text{equality holds for } h = \varphi_n^+$$

For this $L = H \ominus F_{n-1}$ we proved $\max_{h\neq 0} \frac{\langle Ah, h\rangle}{\langle h, h\rangle} \leq \lambda_n^+$, ok. ▨

Square root of a nonnegative operator

<u>Definition</u>: $A \in \mathcal{B}(H)$ is nonnegative if $\langle Ax, x\rangle \geq 0 \ \forall x \in H$
positive if $\langle Ax, x\rangle > 0 \ \forall x \in H$

<u>Theorem</u>: For every $A \in \mathcal{B}(H)$ s.t. $A \geq 0$ $\exists!\ \sqrt{A}$ s.t.
1) $\sqrt{A} \in \mathcal{B}(H)$.
2) $\sqrt{A} \geq 0$.
3) $\sqrt{A}\sqrt{A} = A$.

<u>Lemma 1 [polarization identity]</u>: For every $A \in \mathcal{B}(H)$, $\forall x, y \in H$ we have

$$\langle Ax, y\rangle = \left\langle \frac{A(x+y)}{2}, \frac{x+y}{2}\right\rangle - \left\langle A\frac{x-y}{2}, \frac{x-y}{2}\right\rangle + i\left(\left\langle A\frac{x+iy}{2}, \frac{x+iy}{2}\right\rangle - \left\langle A\frac{x-iy}{2}, \frac{x-iy}{2}\right\rangle\right).$$

<u>Proof</u>: $\forall z \in \mathbb{C}$ we have $z = \operatorname{Re} z + i \operatorname{Im} z = \operatorname{Re} z + i \operatorname{Re}(-iz)$

$$\begin{aligned}\langle Ax, y\rangle &= \operatorname{Re}\langle Ax, y\rangle + i\operatorname{Re}(-i\langle Ax, y\rangle) \\ &= \operatorname{Re}\langle Ax, y\rangle + i\operatorname{Re}(\langle Ax, iy\rangle)\end{aligned}$$

$$\operatorname{Re}\langle Ax, y\rangle = \frac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle\right)$$ ∎

<u>Corollary 1</u>: Let $A_1, A_2 \in \mathcal{B}(H)$: $\langle A_1x, x\rangle = \langle A_2x, x\rangle \ \forall x \in H \Rightarrow A_1 = A_2$.

<u>Proof</u>: By Lemma 1, $\langle A_1x, y\rangle = \langle A_2x, y\rangle \ \forall x, y \in H$,

$$\|A_1 - A_2\| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle (A_1 - A_2)x, y\rangle| = 0 \Rightarrow A_1 = A_2$$ ∎

<u>Corollary 2</u>: $A \geq 0 \Rightarrow A = A^*$.

<u>Proof</u>: $\langle Ax, x\rangle \underset{\langle Ax,x\rangle = \overline{\langle Ax,x\rangle}}{=} \langle x, Ax\rangle = \langle A^*x, x\rangle \ \forall x \in H \Rightarrow A = A^*$ by Corollary 1. ∎

Proof of theorem: Take $A \geq 0$, by Corollary 2 we have $A = A^*$. By spectral theorem, $A \cong \bigoplus_k M_{x_k}$, where M_{x_k} is the multiplication operator $f \longmapsto xf$ on $L^2(\mu_k)$ for some $\mu_k \geq 0$, $\operatorname{supp} \mu_k \subset \mathbb{R}$.

$A \geq 0 \Rightarrow M_{x_k} \geq 0$

$\Leftrightarrow \langle xf, f \rangle_{L^2(\mu_k)} \geq 0 \quad \forall f \in L^2(\mu_k)$

$\Leftrightarrow \int_{\mathbb{R}} x |f(x)|^2 \, d\mu_k \geq 0 \quad \forall f \in L^2(\mu_k)$

$\Leftrightarrow \operatorname{supp} \mu_k \subset [0, +\infty)$

$\Rightarrow \sigma(A) = \overline{\bigcup \operatorname{supp} \mu_k} \subset [0, \infty)$

Now consider $f = \sqrt{x} \in C(\sigma(A))$. $\sqrt{A} := f(A)$

Then $\sqrt{A} \in \mathcal{B}(H)$ because $\|\sqrt{A}\| = \|f\|_{C(\sigma(A))}$

$\sqrt{A} \cdot \sqrt{A} = f(A) \cdot f(A) = f^2(A) = x(A) = A$

$\sqrt{A} \geq 0$ because $f(A) = \bigoplus_k M_{\sqrt{x_k}}$ and $\langle \sqrt{x} f, f \rangle_{L^2(\mu)} \overset{?}{\geq} 0 \quad \forall f \in L^2(\mu_k)$

Uniqueness: Suppose there is $\widetilde{\sqrt{A}}$ with the same properties 1)-3).
$K := \sigma(\sqrt{A}) \cup \sigma(\widetilde{\sqrt{A}})$ - compact in $\mathbb{R}_+ = [0, \infty)$.
Find p_n- polynomials: $p_n(x) \rightrightarrows \sqrt{x}$ on $[0, L] \subseteq K$, $L \geq 1$.
Define $q_n(x) := p_n(x^2)$, then $q_n(x) \rightrightarrows x$ on $[0, L^2] \supset K$. We have

$$\|q_n(\sqrt{A}) - \sqrt{A}\| \longrightarrow 0$$
$$\|q_n(\widetilde{\sqrt{A}}) - \widetilde{\sqrt{A}}\| \longrightarrow 0$$

by functional calculus

But $q_n(\sqrt{A}) = p_n((\sqrt{A})^2) = p_n(A) = p_n((\widetilde{\sqrt{A}})^2) = q_n(\widetilde{\sqrt{A}}) \quad \forall n$
$\Rightarrow \sqrt{A} = \widetilde{\sqrt{A}}$. ▨

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