PART II Banach Algebras

November 4, 2025

Definition: A is a Banach algebra if A is a Banach space with the operation of multiplication such that:

1) $(xy)z = x(yz)$; $x, y, z \in A$
2) $(x+y)z = xz + yz$
   $x(y+z) = xy + xz$
3) $(\alpha x)y = x(\alpha y)$; $\alpha \in \mathbb{C}$
4) $\exists e \in A.\ xe = ex = x \quad \forall x \in A.$
5) $\|xy\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A.$

Examples: 1) X Banach space $\Rightarrow \mathcal{B}(X)$ is a Banach algebra with unity $e = I$ and norm $\|T\| = \sup\limits_{\|x\| \leq 1} \|Tx\|$

2) Calkin algebra: $\mathcal{B}(X)/S_\infty(X)$, $e = I + S_\infty(X)$

$$\|T\|_{\mathcal{B}(X)/S_\infty(X)} = \inf_{K \in S_\infty(X)} \|T - K\| = \operatorname{dist}(T, S_\infty(X))$$

3) K - compact Hausdorff space, $C(K)$ is a Banach algebra,
   $e = 1$, $\|F\|_{C(K)} = \max\limits_{x \in K} |f(x)|$
   $F: K \to \mathbb{C}$

4) $W(\mathbb{T})$ - Wiener algebra on $\mathbb{T} := \{|z| = 1\}$.

$$W(\mathbb{T}) = \left\{ F = \sum_{k \in \mathbb{Z}} c_k z^k \;\middle|\; c_k \in \mathbb{C}, \sum |c_k| < \infty \right\}$$

$\|f\|_{W(\mathbb{T})} = \sum\limits_{k \in \mathbb{Z}} |c_k|$, $e = 1$, multiplication is the usual multiplication of functions

$\|f \cdot g\|_{W(\mathbb{T})} \leq \|f\|_{W(\mathbb{T})} \cdot \|g\|_{W(\mathbb{T})}$
exercise

5) $L^\infty(\mathbb{R})$, $\|f\| = \operatorname{ess\,sup}_{\mathbb{R}} |f|$

6) $H^\infty(\mathbb{D})$ - set of bounded analytic functions on $\mathbb{D} := \{|z|<1\}$

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{|z|<1} |f(z)|$$

Remark: 1) & 2) are noncomutttative Banach algebras, others are commutative.

Remark: If A is a Banach space with multiplication and properties 1), 2), 3), 5), then we can always add the identity to convert A to a Banach algebra as follows:

$\mathcal{A} = A \times \mathbb{C}$, $\quad (x,\alpha) + (y,\beta) = (x+y, \alpha+\beta)$

$$\gamma(x,\alpha) = (\gamma x, \gamma\alpha)$$

$$(x,\alpha)\cdot(y,\beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

$$\left(\text{"}(x+\alpha e)(y+\beta e)\text{"} = xy + \alpha y + \beta x + \alpha\beta e\right) \qquad e=(0,1)$$

$$\|(x,\alpha)\| = \|x\| + |\alpha| \quad \leftarrow \text{! corrected on Nov. 5th}$$

$\Rightarrow \mathcal{A}$ is a Banach algebra with identity and $(A,0) \subset \mathcal{A}$.

Example: $L^1(\mathbb{R})$, multiplication $f * g = \int_{\mathbb{R}} f(y) g(x-y)\,dy$

$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f|\,dx$ – this is a Banach algebra without identity, and the above construction is equivalent to consideration of $\mathcal{A} = \mathbb{C}\delta_0 + L^1(\mathbb{R})$

$$\left(\overset{\text{measure}}{\delta_0(S)} = \begin{cases} 1; & 0 \in S \\ 0; & 0 \notin S \end{cases}, \quad \delta_0 * f = f * \delta_0 = \int f(y)\,\delta_0(x-y) = f(x) \right)$$

measures $\rightsquigarrow$ no dy

$$\langle \varphi, \delta_0 \rangle = \varphi(0) = \int_{\mathbb{R}} \varphi\, d\delta_0, \quad \varphi \in C(\mathbb{R})$$

Definition: A Banach algebra, $x \in A$. We say $x \in A$ is invertible if $\exists x^{-1} \in A.\ xx^{-1} = x^{-1}x = e$.

If x is invertible, then x^{-1} is unique.

Definition: $G(A) := \{x \in A \mid x \text{ is invertible}\}$.
↳ we use G, because $G(A)$ is a group

Definition: $x \in A$. $\sigma(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\}$.

Example: If A is the set of $n \times n$ matrices with complex coefficients, then for $T \in A$, $\sigma(T)$ is the set of eigenvalues (the usual spectrum of the matrix).

Example: $A = \mathbb{C}(K)$, $\sigma(f) = ?$
$\lambda \in \mathbb{C}$: $\lambda - f$ is not invertible in $C(K)$.
Since $g \in G(C(K)) \Leftrightarrow \frac{1}{g} \in C(K)$, we have $\lambda - f \in G(C(K)) \Leftrightarrow \frac{1}{\lambda - f} \in C(K)$
$\Leftrightarrow \lambda \notin f(K)$

$\Rightarrow \sigma(f) = f(K)$ ← compact non-empty subset of $\mathbb{C}$

Basic properties

Proposition 1: Let A be a Banach algebra. Then $\|e\| \geq 1$.

Proof: $\|e\| = \|e \cdot e\| \leq \|e\| \cdot \|e\| \Rightarrow \cancel{\|e\| = 0}$ or $\|e\| \geq 1$
$\Updownarrow$
$e = 0$ ✗

Definition: A Banach algebra is called unital if $\|e\| = 1$.

Proposition 2: If A is an arbitrary Banach algebra, then the algebra $A \times \mathbb{C}$ is unital.

Proof: $\|(0,1)\| = \|0\| + |1| = 1$

!**Note:** This was wrong previously and corrected on November 5th.

⚠ From now on, all Banach algebras are unital.

Proposition 3: If $\begin{matrix} x_n \longrightarrow x \\ y_n \longrightarrow y \end{matrix}$ in A, then $x_n \cdot y_n \longrightarrow xy$.

Proof: $\|x_n \cdot y_n - xy\| \leq \underbrace{\|x_n - x\|}_{\to 0} \underbrace{\|y_n\|}_{\text{bdd}} + \|x\| \underbrace{\|y_n - y\|}_{\to 0} \longrightarrow 0$ ▨

Proposition 4: Let $a \in A$, $\|a\| \leq 1$, then $e - a \in G(A)$.

Proof: Define $(e-a)^{-1} = e + a + a^2 + \cdots = \sum_{k=0}^{\infty} a^k$

this series converges because A_n is a Banach space and $\sum \|a^k\| \leq \sum \|a\|^k < \infty$

Let us check that $(e-a)^{-1}(e-a) = e$

$$\Leftrightarrow \lim_{n\to\infty} \underbrace{\left(\sum_{k=0}^{n} a^k\right)(e-a)}_{e - \underbrace{a^{n+1}}_{\to 0}} = e$$

$$e - a^{n+1} \longrightarrow e, \quad \text{since } \|a^{n+1}\| \leq \|a\|^{n+1} \longrightarrow 0$$

Similarly, $(e-a)(e-a)^{-1} = e$.

Proposition 5: $G(A)$ is open in A.

Proof: Let $a \in G(A)$, $b \in A$, then $a - b = \underset{G(A)}{a}\,(\underbrace{e - a^{-1}b}_{\in G(A) \text{ if } \|a^{-1}b\| < 1})$

$\|a^{-1}b\| < 1$ holds for all b : $\|b\| \leq \frac{1}{\|a^{-1}\|}$

$\Rightarrow B\left(a, \frac{1}{\|a^{-1}\|}\right) \subset G(A) \Rightarrow G(A)$ is open.

Proposition 6: Let $x \in G(A)$, $x_n \longrightarrow x$ in A. Then $x_n \in G(A)$ for n large enough, and $x_n^{-1} \longrightarrow x^{-1}$ as $n \longrightarrow \infty$.

Proof: Write $x_n = x + z_n$, we have $x_n \in G(A)$ for n large enough by Proposition 5.

$$(x + z_n)^{-1} - x^{-1} = (x(e + x^{-1}z_n))^{-1} - x^{-1} = \underbrace{(e + x^{-1}z_n)^{-1}}_{y_n} \cdot x^{-1} - x^{-1} \underset{\text{Prop. 3}}{\longrightarrow} e x^{-1} - x^{-1} = 0$$

$y_n \longrightarrow e$, see Prop. 4 : $\|y_n - e\| \leq \sum_{k=1}^{\infty} \|x^{-1} \cdot z_n\|^k \longrightarrow 0$ $\quad \|z_n\| \to 0$ ▨

Theorem: Let A be a Banach algebra, $a \in A$. Then $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.

Proof: $\sigma(a) = \mathbb{C} \setminus \rho(a)$, $\rho(a) := \{\lambda \mid \lambda e - a \text{ is invertible}\}$
resolvent set

$\rho(a)$ is open by Prop. 5 $\Rightarrow$ $\sigma(a)$ is closed.
Let's check that $\sigma(a)$ is bounded: $\sigma(a) \subset \{\lambda \mid |\lambda| \le \|a\|\}$
$\Leftrightarrow \rho(a) \supset \{\lambda \mid |\lambda| \geq \|a\|\}$.
Take λ: $|\lambda| \geq \|a\|$, then $\lambda e - a = \lambda e \underbrace{\left(e - \frac{a}{\lambda}\right)}_{\text{invertible by Prop. 4}}$

It remains to check that $\sigma(a) \neq \emptyset$.
Assume that $\sigma(a) = \emptyset$ and consider the function

$$f_\phi(\lambda) = \phi\left((\lambda e - a)^{-1}\right) \text{ for some } \phi \in A^*; \quad \lambda \in \mathbb{C}$$

Let's check that f_ϕ is analytic. Take $\lambda_0 \in \mathbb{C}$, consider

$$\lim_{\lambda \to \lambda_0} \frac{f_\phi(\lambda) - f_\phi(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} \phi\left(\frac{(\lambda e - a)^{-1} - (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0}\right)$$

$[x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}]$

$$= \lim_{\lambda \to \lambda_0} \phi\left(\frac{(\lambda e - a)^{-1}\left((\lambda_0 e - \cancel{a}) - (\lambda e - \cancel{a})\right)(\lambda_0 e - a)^{-1}}{\lambda - \lambda_0}\right)$$

$$= \lim_{\lambda \to \lambda_0} \phi\left(\underbrace{(\lambda e - a)^{-1}}_{\to (\lambda_0 e - a)^{-1}\ \text{Prop. 6}}(\lambda_0 e - a)^{-1}\right)$$

ϕ continuous, continuity of multiplication

$$= -\phi\left((\lambda_0 e - a)^{-2}\right)$$

$\Rightarrow f_\phi$ is analytic (f_ϕ is $\mathrm{Hol}(\mathbb{C})$) holomorphic

Take λ: $|\lambda| \geq 2\|a\|$, $|f_\phi(\lambda)| = \|\phi\| \cdot \|(\lambda e - a)^{-1}\| = \|\phi\| \overset{(*)}{|\lambda|^{-1}} \underbrace{\left\|\left(e - \frac{a}{\lambda}\right)^{-1}\right\|}_{\text{uniformly bounded for } \lambda: |\lambda| \geq 2\|a\|}$

f_ϕ is bdd on $\mathbb{C}$ by the max. principle
$\Rightarrow f_\phi = c_\phi \in \mathbb{C}$, but $c_\phi = 0$ by $(*)$
$\Rightarrow \phi(\lambda e - a) = 0 \ \forall \lambda \in \mathbb{C}.\ \forall \phi \in A^*$ $\Rightarrow$ $\lambda e - a = 0 \ \forall \lambda \in \mathbb{C}$
$\rightsquigarrow$ contradiction

▨

November 5, 2025

Definition: Let A_1, A_2 be Banach algebras.
We say that A_1 is *isomorphic* to A_2 ($A_1 \cong A_2$) if there exists a map $j: A_1 \longrightarrow A_2$:

$$j(\alpha a + \beta b) = \alpha j(a) + \beta j(b) \quad (*)$$
$$j(ab) = j(a) \cdot j(b) \quad (**)$$
$$\|j(a)\|_{A_2} = \|a\|_{A_1}$$
j is bijective

Theorem [Banach-Mazur]: Let A be a Banach algebra s.t. $G(A) = A \setminus \{0\}$. Then $A \cong \mathbb{C}$.

Proof: For every $a \in A$ we have $\sigma(a) \neq \emptyset$. So there is a $\lambda(a) \in \mathbb{C}$: $\lambda(a)e - a$ is not invertible $\Leftrightarrow \lambda(a)e - a = 0 \Leftrightarrow a = \lambda(a)e$.
In particular, such $\lambda(a)$ is unique.

Define $j: A \longrightarrow \mathbb{C}$, $a \longmapsto \lambda(a)$. We have:

$$\left.\begin{aligned} a+b &= \lambda(a+b)e \\ a+b &= \lambda(a)e + \lambda(b)e \end{aligned}\right\} \Rightarrow \lambda(a+b) = \lambda(a) + \lambda(b)$$

Similarly, $\lambda(\alpha a) = \alpha\lambda(a)$, $\lambda(ab) = \lambda(a) \cdot \lambda(b) \Rightarrow (*), (**)$ ✓

$\|j(a)\|_{\mathbb{C}} = \|a\|_A \Leftrightarrow |\lambda(a)| = \|a\|_A \Leftrightarrow \|\lambda(a)e\|_A = \|a\|_A$ ✓

$$\left.\begin{aligned} &j(a) = j(b) \Leftrightarrow \lambda(a)e = \lambda(b)e \Leftrightarrow a = b \\ &j(e) = 1 \Rightarrow j(A) \supset j(\mathbb{C} \cdot e) = \mathbb{C} \end{aligned}\right\} j \text{ is a bijection}$$

Definition: Let A be a Banach algebra, then $r(a) := \sup\{|\lambda|, \lambda \in \sigma(a)\}$ is called the *spectral radius of a*.

Theorem: Let A be a Banach algebra. Then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

In particular, the limit above exists.

Proof: __Step 1:__ $r(a) \le \|a\|$, because

$$\lambda e - a = \lambda(e - \underbrace{\tfrac{a}{\lambda}}_{b}) \qquad \|b\| < 1 \text{ for } \lambda : |\lambda| > \|a\|$$

$\lambda(e - \frac{a}{\lambda})$: invertible element in A (Prop. 4)

$\Rightarrow \lambda e - a \in G(A) \Rightarrow \lambda \in \rho(a) \Rightarrow \sigma(a) \subset \mathcal{B}[0, \|a\|]$. ✓

__Step 2:__ $r(a) \le \|a^n\|^{1/n}$ for every $n \ge 2$.

Take $\lambda \in \mathbb{C}$, and consider

$$\lambda^n e - a^n = (\lambda e - a) \cdot p_\lambda(a), \quad p_\lambda \text{ is a polynomial}$$

If $\lambda \in \sigma(a)$, then $\lambda^n e - a^n = z_1 \cdot z_2$, where $z_1, z_2 \in A$, $z_1 \notin G(A)$, $z_1 z_2 = z_2 z_1$. If $z = \lambda^n e - a^n$ is invertible, $\exists z^{-1} \in A$.

$$\underset{\displaystyle (z^{-1} z_2) z_1}{\overset{\shortparallel}{z^{-1} z_1 z_2}} = z_1 (z_2 z^{-1}) = e \qquad \Rightarrow z_1 \in G(A) \;\ldots \text{contradiction}$$

$\Rightarrow \lambda^n \in \sigma(a^n) \underset{\text{Step 1}}{\Rightarrow} |\lambda^n| \le \|a^n\| \Rightarrow |\lambda| \le \|a^n\|^{1/n}$. ✓

__Step 3:__ $\inf\limits_{n \ge 1} \|a^n\|^{1/n} \le \underline{\lim} \|a^n\|^{1/n} \le \overline{\lim} \|a^n\|^{1/n} \le r(a) \le \inf \|a^n\|^{1/n}$

$\hookrightarrow$ This implies the claim.

Notation note: $\overline{\lim}$ = limsup, $\underline{\lim}$ = liminf.

All that remains is $\overline{\lim} \|a^n\|^{1/n} \le r(a)$. Take $\phi \in A^*$, $\|\phi\| \le 1$, $f_\phi(\lambda) := \phi((\lambda e - a)^{-1})$ for $\lambda \in \rho(a)$ (this is an analytic function on $\rho(a)$)

$\varepsilon > 0$, $k \in \mathbb{Z}$: $$\frac{1}{2\pi i} \oint_{|\lambda| = r(a) + \varepsilon} \lambda^k f_\phi(\lambda)\, d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^k f_\phi(\lambda)\, d\lambda =$$

$$= \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^k \phi\left(\lambda \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n}\right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = 2\|a\|} \lambda^{k+1} \sum_{n=0}^{\infty} \frac{\phi(a^n)}{\lambda^n}\, d\lambda$$

$$= \phi(a^{k+2})$$ (Cauchy formula from complex analysis)

$$\Rightarrow \phi(a^{k+2}) \le \left| \frac{1}{2\pi i} \oint_{|\lambda| = r(a) + \varepsilon} \lambda^k f_\phi(\lambda)\, d\lambda \right|$$

$$\le \max_{|\lambda|=r(a)+\varepsilon}\left(|\lambda^k|\cdot|f_\phi(\lambda)|\right)\frac{1}{2\pi}\left(2\pi(r(a)+\varepsilon)\right)$$

$$=(r(a)+\varepsilon)^{k+1}\underbrace{\|\phi\|}_{\le 1}\cdot\underbrace{\sup_{|\lambda|=r(a)+\varepsilon}\|(\lambda e-a)^{-1}\|}$$

constant depending only on ε, because $|\lambda|=r(a)+\varepsilon$ is a compact set and $\lambda\mapsto\|(\lambda e-a)^{-1}\|$ is continuous

$$\sup_{\|\phi\|\le 1}|\phi(a^{k+2})|=\|a^{k+2}\|$$

$$\wedge|$$

$$C_\varepsilon\cdot(r(a)+\varepsilon)^k$$

$\Rightarrow \overline{\lim}\,\|a^k\|^{1/k}\le r(a)+\varepsilon$ for every $\varepsilon>0$. $\square$

November 11, 2025

<u>Example</u>: $V: f\longmapsto\int_0^x f(s)ds$, $V\in\mathcal{B}(L^2[0,1])$.
Let's prove that $r(V)=0 \Leftrightarrow \sigma(\{V\})=\{0\}$.

<u>Proof 1</u>: $r(V)=\lim_{r\to\infty}\|V^n\|^{1/n}$, so we need a formula for V^n.
By induction: $$(V^nf)(x)\overset{(*)}{=}\int_0^x f(s)\frac{(x-s)^{n-1}}{(n-1)!}ds.$$

$n=1$: ✓

Assume $(*)$ for some n and compute

$$(V^{n+1}f)(x)=\int_0^x\left(\int_0^t f(s)\frac{(t-s)^{n-1}}{(n-1)!}ds\right)dt$$

$$=\int_0^x\left(f(s)\underbrace{\int_s^t\frac{(t-s)^{n-1}}{(n-1)!}dt}_{\frac{(t-s)^n}{n!}\Big|_s^x=\frac{(x-s)^n}{n!}}\right)ds$$

$$=\int_0^x f(s)\frac{(x-s)^n}{n!}ds.$$

$$\|V^nf\|\le\max_{x\in(0,1)}\max_{s\in(0,x)}\frac{(x-s)^{n-1}}{(n-1)!}\left(\int_0^1\Big[\underbrace{\int_0^x|f(s)|ds}_{=1\cdot|f(s)|}\Big]^2dx\right)^{1/2}$$

$$\overset{CS}{\leq} \frac{1}{(n-1)!}\left(\int_0^1 \underbrace{\sqrt{x}}_{\leq 1} \underbrace{\int_0^x |f|^2 ds}_{\leq \|f\|^2} dx\right)^{1/2}$$

$$\leq \frac{\|f\|_{L^2[0,1]}}{(n-1)!}$$

$\Rightarrow \|V^n\| \leq \frac{1}{(n-1)!}$, $\quad \lim \|V^n\|^{1/n} \leq \overline{\lim} \left(\frac{1}{(n-1)!}\right)^{1/n}$ Stirling formula

$$= \overline{\lim} \frac{1}{\left(\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}\right)^{1/n}(1+o(1))^{1/n}}$$

$$= \overline{\lim_{n\to\infty}} \underbrace{\frac{e^{\frac{n-1}{n}}}{(2\pi)^{1/2n}}}_{\to e} \cdot \underbrace{\frac{1}{(n-1)^{3/2\cdot\frac{n-1}{n}}}}_{\to 0}$$

$$= 0$$

$\square$

<u>Proof 2</u>: $Vf = \int_0 K(x,y) f(y) dy$, $\quad K(x,y) = \chi_{\mathbb{R}_+}(x-y)$

$$\int_0^1\int_0^1 |K(x,y)|^2 dx dy < \int_0^1\int_0^1 dx dy = 1 < \infty$$

$\Rightarrow V \in S_\infty(L^2[0,1]) \Rightarrow \sigma(V) = \left\{ \begin{matrix}\lambda \text{ is an eigenvalue} \\ \text{of } V\end{matrix}\right\} \cup \{0\}$

theorem about the spectrum of compact operators

$Vf = \lambda f \iff \underbrace{\int_0^x f(s) ds}_{\in C[0,1]} \overset{(**)}{=} \lambda f(x) \Rightarrow f \in C[0,1]$

$\Rightarrow \int_0^x f(s) ds \in C^1[0,1] \quad \Rightarrow \dots \Rightarrow f \in C^\infty[0,1]$.

Differentiating (**), we get $\lambda f' = f$ on $[0,1]$.
Substituting 0 into (**), we get $\lambda f(0) = \lambda$.

So, if $\lambda \neq 0$, then $\begin{cases}\lambda f' = f \\ f(0) = 0\end{cases} \iff \begin{cases} f = c \cdot e^{\lambda x} \\ 0 = c \cdot e^0 \end{cases} \iff f = 0$

$\Rightarrow$ any $\lambda \neq 0$ is not an eigenvalue $\Rightarrow \sigma(V) \cap (\mathbb{C} \setminus \{0\}) = \emptyset$
Since $\sigma(V) \neq \emptyset$, we get $\sigma(V) = \{0\}$.

Commutative Banach algebras

<u>Definition</u>: Let A be a commutative Banach algebra and $J \subset A$. J is called a proper ideal in A if J is a linear subspace such that $a \cdot J \subset J$ $\forall a \in A$, and $J \neq \{0\}$, $J \neq A$.

<u>Definition</u>: J is a maximal ideal if J is a proper ideal and there is no proper ideal J' such that $J' \supsetneq J$.

<u>Proposition</u>: Every proper ideal is contained in some maximal ideal. Take some proper ideal J, $J \neq A$, and consider all proper ideals J': $J' \supseteq J$. This set is partially ordered by inclusion, and for every chain $\{J'_\alpha\}_{\alpha \in I}$ of ideals ordered by inclusion, the set $\bigcup_{\alpha \in I} J'_\alpha = J'$ is again a proper ideal.
- linearity: $p \cdot x + q y \in J'$ for every $p, q \in \mathbb{C}$ and $x, y \in J'$, because $\exists \alpha_x, \alpha_y$. $x \in J_{\alpha_x}$, $y \in J_{\alpha_y}$ $\Rightarrow$ $x, y \in J_{\alpha_x}$ or $x, y \in J_{\alpha_y}$, then $px + qy$ are in the same J_{α_x} or J_{α_y} ✓
- ideal property $aJ' = \bigcup a \cdot J'_\alpha \subset \bigcup J'_\alpha$ $\forall a \in A$.
- properness: $J' \neq A$ (If $J' = A$, then $e \in J'$, then $e \in J'_\alpha \Rightarrow \overset{A}{\overset{\shortparallel}{eA}} \subseteq J'_\alpha \subsetneq A$.)

By Zorn's lemma, the set of all proper J': $J' \supset J$ has a maximal element. ∎

<u>Proposition</u>: If M is a maximal ideal in A, then M is closed.

<u>Proof</u>: Let's prove that $\overline{M}$ is a proper ideal.
- $\overline{M}$ is linear ✓
- $\overline{M} A \subseteq \overline{M}$: true by continuity of multiplication ✓
- $\overline{M} \neq A$ (If $\overline{M} = A$, then $e \in \overline{M} \Rightarrow \exists x \in M.\ \mathrm{dist}(x, e) < 1 \Rightarrow x \in G(A)$
  $\Rightarrow e = x \cdot x^{-1} \in xA \subset M \Rightarrow M = A$, contradiction.) ∎

<u>Example</u>: $A = C(\overset{\neq\{x_0\}}{K})$, $M_{x_0} = \{f \in A.\ f(x_0) = 0\}$.
Then M_{x_0} is a maximal ideal for every $x_0 \in K$.

- M_{x_0} is linear ✓
- $M_{x_0} \cdot A \subseteq M_{x_0}$ ✓
- $M_{x_0} \neq A$, because $1 \notin M_{x_0}$ ✓
- M_{x_0} is maximal: If $\exists J$-proper: $J \supsetneq M_{x_0}$ then $\exists f \in J.\ f(x_0) \neq 0$.

But then $\forall g \in A$ we have $g \overset{(***)}{=} c \cdot f + h$ for $c \in \mathbb{C}$ and $h \in M$, where c is such that $(g - cf)(x_0) = 0$, i.e. $c := \frac{g(x_0)}{f(x_0)}$.
So $A \subset \mathbb{C} \cdot f + M \subset J$, contradiction.

<u>Observation</u>: $M_{x_0} = \operatorname{Ker} \phi_{x_0}$, $\phi_{x_0} : f \longrightarrow f(x_0)$
ϕ_{x_0} is a multiplicative functional: $\phi_{x_0}(fg) = \phi_{x_0}(f) \cdot \phi_{x_0}(g)$

<u>Definition</u>: Let $\phi \in A^*$. We say ϕ is a multiplicative functional if $\phi(fg) = \phi(f) \cdot \phi(g)$ $\forall f, g \in A$, and $\phi \neq 0$.

<u>Theorem</u>: Let A be a commutative Banach algebra. TFAE
1) M is a maximal ideal in A.
2) $M = \operatorname{Ker} \phi$ for some mutiplicative functional $\phi \in A^*$.

<u>Proof</u>: 2) $\Rightarrow$ 1) obvious: i) $\operatorname{Ker} \varphi$ is linear
ii) $x \in \operatorname{Ker} \varphi$, $a \in A$, $\varphi(xa) = \varphi(x)\varphi(a) = 0 \Rightarrow xa \in \operatorname{Ker} \varphi$
iii) $\operatorname{Ker} \varphi \neq A$ because $\varphi \neq 0$
iv) $\operatorname{Ker} \varphi$ is maximal, because $\operatorname{Ker} \phi + \mathbb{C} a = A$ $\forall a : \phi(a) \neq 0$ (see $***$)

1) $\Rightarrow$ 2) Note that A/M is a $\overset{\text{(M is closed)}}{\text{Banach}}$ algebra in which every nonzero element is invertible. If $\overset{\neq[0]}{[a]} \in A/M$, then $a \cdot A$ is an ideal in A containing M, but not equal to $M \Rightarrow aA + M = A$
$aA + M \ni e \Rightarrow ab + M \ni e$ for some $b \in A \Rightarrow [a][b] = [e]$.
By Banach-Mazur theorem, there is an isomorphism j of Banach algebras A/M and $\mathbb{C}$. Let $\phi(x) := j([x])$.

i) Φ is linear, because j is linear.
ii) ϕ is multiplicative, because j is multiplicative.
iii) $\operatorname{Ker}\phi = \{x \mid j([x]) = 0\} \Leftrightarrow [x] = 0 \Leftrightarrow x \in M$. ▨

November 12, 2025

Notation: $\mathcal{M}(A)$ - set of all maximal ideals in A
A^*_{mult} - set of multiplicative functionals on A

Last time: $\mathcal{J}: \phi \longmapsto \operatorname{Ker}\phi$ maps A^*_{mult} onto $\mathcal{M}(A)$.

Proposition: $\mathcal{J}$ is a bijection.

Proof: We need to check that $\mathcal{J}(\phi_1) = \mathcal{J}(\phi_2) \Rightarrow \phi_1 = \phi_2$.
For this, note that $\phi(e) = 1 \ \forall \phi \in A^*_{mult}$, because $\begin{cases} \phi(e) = \phi(e)\,\phi(e) \\ \phi(e) \neq 0 \end{cases}$ for every $\phi \in A^*_{mult}$. So, if $\mathcal{J}(\phi_1) = \mathcal{J}(\phi)_2$, we have
$0 = \phi_1(y - \phi_1(y)e) \Rightarrow \phi_2(y - \phi_1(y)e) = 0 \quad \forall y \in A$
$\Rightarrow \phi_2(y) = \phi_1(y)\,\phi_2(e) = \phi_1(y)$, so $\phi_1 = \phi_2$.

Theorem: Let A be a commutative Banach algebra, and $a \in A$. TFAE:
1) $a \in A \setminus G(A)$
2) $a \in M$ for some $M \in \mathcal{M}(A)$
3) $\exists \phi \in A^*_{mult}: \phi(a) = 0$

Proof: 1) $\Rightarrow$ 2): $J = aA$ - a proper ideal in A $(e \notin J)$
$\Rightarrow \exists M \in \mathcal{M}(A).\ M \subset J$
2) $\Rightarrow$ 3): Take $\phi: \operatorname{Ker}\phi = M \Rightarrow \phi(a) = 0$.
3) $\Rightarrow$ 1): If $\exists b \in A.\ ab = e \Rightarrow \phi(a)\cdot\phi(b) = \phi(e) = 1$, but $\phi(a) = 0$. ▨

Corollary: $\sigma(a) = \{\phi(a) \mid \phi \in A^*_{mult}\}$

Proof: $\sigma(a) = \{\lambda \mid a - \lambda e \in A \setminus G(A)\} = \{\lambda \mid \exists \phi \in A^*_{mult}.\ \phi(a-\lambda e) = 0\}$
$= \{\lambda \mid \lambda \in \phi(a) \text{ for some } \phi \in A^*_{mult}\}$. ▨

Remark: $\forall \phi \in A^*_{mult}$, $\|\phi\| = 1$, because $\phi(e) = 1$ and $|\phi(a^k)|$ is uniformly bounded for every $a \in B_A(0,1)$.

Applications

Theorem [Wiener]: Let $f = \sum_{k\in\mathbb{Z}} c_k z^k$, and $\sum_{k\in\mathbb{Z}} |c_k| < \infty$. Assume that $f(z) \neq 0$ for every $z \in \mathbb{T} = \{|z| = 1\}$. Then $\frac{1}{f} = \sum_{k\in\mathbb{Z}} b_k z^k$ where $\sum_{k\in\mathbb{Z}} |b_k| < \infty$.

Proof: 1. $W^1(\mathbb{T}) = \{\sum c_k z^k \mid \sum |c_k| < \infty\}$ is a Banach algebra:

Indeed, $W(\mathbb{T})$ is a Banach space with respect to the norm $\|\sum c_k z^k\| = \sum |c_k|$, and

$$\begin{aligned} \|(\textstyle\sum c_k z^k)(\sum b_k z^k)\| &= \sum_{n\in\mathbb{Z}} \left|\left(\sum_{k\in\mathbb{Z}} c_k b_{n-k}\right)\right| \\ &\leq \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |c_k||b_{n-k}| \\ &= \sum_{k\in\mathbb{Z}} |c_k| \sum_{j\in\mathbb{Z}} |b_j| \\ &= \|\textstyle\sum c_k z^k\| \cdot \|\sum b_k z^k\| \end{aligned}$$

λ.

2. Identification of $W^1(\mathbb{T})^*_{mult}$:

Let $\phi \in W^1(\mathbb{T})^*_{mult}$, $\lambda = \phi(z)$, then $\phi(\frac{1}{z}) \cdot \phi(z) = 1$, $\phi(\frac{1}{z}) = \frac{1}{}$
$|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot \|\frac{1}{z}\| = 1$
$|\lambda| \leq \|\phi\| \cdot \|z\| = 1$, $|\frac{1}{\lambda}| \leq \|\phi\| \cdot |\frac{1}{z}| = 1 \Rightarrow |\lambda| = 1$

$\phi(\sum_{-N}^{N} c_k z^k) = \sum_{-N}^{N} c_k \lambda^k$, and hence $\phi(f) = f(\lambda)$ $\forall f \in W^1(\mathbb{T})$, because $\{\sum_{-N}^{N} c_k z^k\}$ is dense in $W^1(\mathbb{T})$ and ϕ is continuous.

3. Application of invertibility criterion:

$f \in W^1(\mathbb{T})$ is invertible $\Leftrightarrow \nexists\, \phi \in W^1(\mathbb{T})^*_{mult}.\ \phi(f)=0 \Leftrightarrow f(z)=0\ \forall z \in \mathbb{T}$

This is the case in our case.

$\Rightarrow fg=1,\ g \in W^1(\mathbb{T}) \Rightarrow g=\frac{1}{f},\ g=\sum b_k z^k,\ \sum |b_k| < \infty.$ ▨

Bezout equation: Let $\{f_k\}_1^N \subset A(\overline{\mathbb{D}})$. We are interested if there exists $\{g_k\}_{k=1}^N \subset A(\overline{\mathbb{D}}): \sum_1^N f_k g_k = 1.$

Necessary condition: $\nexists\, z_0 \in \overline{\mathbb{D}}.\ f_k(z_0)=0$ for every $1 \le k \le N.$

Theorem: Necessary condition is also sufficient.

Proof: 1. $A(\mathbb{D})$ is a Banach algebra with respect to the norm $\|f\| = \max_{z \in \mathbb{D}} |f(z)|$ ✓

2. Identification of $A(\overline{\mathbb{D}})^*_{mult}$:

$\phi \in A(\overline{\mathbb{D}})^*_{mult} \Leftrightarrow \phi(f) = f(\lambda)$ for some $\lambda \in \overline{\mathbb{D}}$ [exercise]

3. $J = \{ \sum_1^N f_k g_k \mid g_k \in A(\overline{\mathbb{D}}) \}$ is a proper ideal in $A(\overline{\mathbb{D}})$

$\Leftrightarrow J \subset M,\ M \in \mathcal{M}(A(\overline{\mathbb{D}})) \Leftrightarrow \sum_1^N (f_k g_k)(z_0) = 0\quad \forall g_k \in A(\overline{\mathbb{D}})$ for some $z_0 \in \overline{\mathbb{D}}$

$\Leftrightarrow f_k(z_0) = 0\quad \forall 1 \le k \le N$

At the same time: J is proper $\Leftrightarrow e \notin J$. ▨

November 18, 2025

Stone-Weierstrass Theorem

Definition: $C_{\mathbb{R}}(K)$ - the real Banach space of continous functions on a compact Hausdorff space K.

Definition: $A \subset C_{\mathbb{R}}(K)$ is a (real) Stone-Weierstrass algebra, if it is an algebra, and

(1) A does not vanish at any point $x \in K$, that is $\forall x \in K.\, \exists f \in A.\, f(x) \neq 0$.

(2) A separates points in K, that is, $\forall x, y \in K.\, \exists f \in A.\, f(x) \neq f(y)$.

Theorem [Stone-Weierstrass]: Let $A \subset C_{\mathbb{R}}(K)$ be an algebra. Then A is dense in $C_{\mathbb{R}}(K) \Leftrightarrow A$ is a Stone-Weierstrass algebra.

Example: $K = [0,1]$, $A = \mathcal{P}$ ... the set of polynomials. Indeed, $\mathcal{P}$ is an algebra, and

$$\left.\begin{array}{l} \cdot\ 1 \in \mathcal{P} \Rightarrow (1) \text{ is satisfied} \\ \cdot\ x \in \mathcal{P} \Rightarrow (2) \text{ is satisfied} \end{array}\right\} \Rightarrow \begin{array}{l} \text{polynomials are dense in } [0,1] \\ \text{(Classical Weierstrass theorem)} \end{array}$$

Example: $K = [0,1]$, $A = \operatorname{span}\{\sin(2\pi k x), \cos(2\pi k x) \mid k \in \mathbb{Z}\}$

A is not dense in $C[0,1]$, because 0 and 1 are not seperated. On the other hand, A is dense in $C[0,a]$ for every $0 \leq a < 1$.

For the proof we need some lemmas:

Lemma: Let $a > 0$. Then $\exists \{p_k\} \in \mathcal{P}$ such that $\|p_k - |x|\|_{C[-a,a]} \longrightarrow 0$.

Proof: By scaling, we can assume that $a = 1$. Then we need to approximate $|x| = \sqrt{x^2} = \sqrt{1-y}$, $y = 1 - x^2$ by polynomials. Thus, it suffices to approximate $y \mapsto \sqrt{1-y}$ by polynomials uniformly on $[0,1]$.

Taylor: $\sqrt{1-y} = \sum_{k=0}^{\infty} c_k y^k$ for $c_k = (-1)^k \binom{1/2}{k} = (-1)^k \frac{1/2\,(1/2-1) \cdots (1/2-k+1)}{k!}$.

Observation: $c_0 = 1$, $c_k < 0$ for $k \geq 1$. (e.g. $c_1 = -\frac{1}{2}$, $c_2 = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} < 0$, $c_3 = -1 \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} < 0$)

In particular, we have $\sum_{1}^{\infty} |c_k| = \sup_{0<y<1} \left(-\sum_{k=1}^{\infty} c_k y^k\right) = \sup_{0<y<1} (\underset{\overset{\shortparallel}{1}}{c_0} - \sqrt{1-y}) = 1$.

So $\sum_0^\infty |c_k| < \infty \Rightarrow p_k = \sum_{j=0}^{k} c_j y^j$ are such that

$$\|p_k - \sqrt{1-y}\|_{C[0,1]} \leq \sum_{j=k+1}^{\infty} |c_j| \cdot \underbrace{\|y^j\|_{C[0,1]}}_{1} \longrightarrow 0. \quad ▨$$

Lemma: If A is a SW algebra, then $\forall x,y \in K.\ \exists h \in A.\ h(x)=1, h(y)=0.$

Proof: We know that $\exists f, g \in A.\ f(x) \neq 0,\ g(x) \neq g(y).$

1) If $f(y) = 0$, then $h := \frac{f}{f(x)}$.

2) If $f(y) \neq 0$, and $f(y) \neq f(x)$, then $h := \dfrac{\overset{\in A}{f^2} - \overset{\in A}{f(y) f}}{\underbrace{f^2(x) - f(x) f(y)}_{\neq 0}}$

$h = \frac{f - f(y)}{f(x) - f(y)}$ is not okay $\frac{f}{f(x)-f(y)} \in A$, $\frac{f(y)}{f(x)-f(y)}$ might not be in A, if A does not contain 1

3) If $f(y) \neq 0$, and $f(y) \neq f(x)$, then $\begin{pmatrix} f(x) \\ f(y) \end{pmatrix}$, $\begin{pmatrix} g(x) \\ g(y) \end{pmatrix}$ are not collinear in $\mathbb{R}^2 \Rightarrow \exists\, \alpha, \beta \in \mathbb{R}$.

$$\alpha \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} + \beta \begin{pmatrix} g(x) \\ g(y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow h := \alpha f + \beta g \quad ▨$$

Lemma: Let A be a SW algebra in $C_{\mathbb{R}}(K)$. Then $\forall f, g \in A$ we have $\min(f,g) \in \bar{A}$, $\max(f,g) \in \bar{A}$.

Proof: $\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$, $\max(f,g) = \frac{f+g}{2} + \frac{|f+g|}{2}$

So we only need to show that $|h| \in \bar{A}$ for every $h \in A$.
Let's approximate $|x|$ by polynomials $p_k \in \mathcal{P}$ on $[-a,a]$ for $a = \|h\|_{C_{\mathbb{R}}(K)}$. Then $p_k \circ h \in A$, $p_k(h) \rightrightarrows |h|$ on K. $\Rightarrow |h| \in \bar{A}$.

Lemma: Let $x \in K$, $\varepsilon > 0$, $\varphi \in C_{\mathbb{R}}(K)$. Then $\exists g_x \in \bar{A}$ s.t. $g_x(x) = \varphi(x)$ and $g_x(z) \geq \varphi(z) - \varepsilon$ for every $z \in K$.

Proof: For every $y \in K$ define $g_{x,y}$ to be a function in A: $\begin{array}{l} g_{x,y}(x) = \varphi(x) \\ g_{x,y}(y) = \varphi(y) \end{array}$
Such a function is a linear combination of functions

taking values 0 and 1 at points x, y.
Let $\mathcal{U} = \{z \in K \mid g_{x,y}(z) > f(z) - \varepsilon\}$, $y \in K$.
- For each $x \in K$ we have $K = \bigcup_{y \in K} \mathcal{U}_y$, because $y \in \mathcal{U}_y$.
- Each $\mathcal{U}_y$ is open, because $g_{x,y} - f$ is continuous, and $\mathcal{U}_y$ is the preimage of $(-\varepsilon, +\infty)$ - an open set.

$\Rightarrow$ By compactness, $\exists \{y_k\}_{k=1}^N$. $K = \bigcup_{k=1}^N \mathcal{U}_{y_k}$.

Now $g_x := \max_{1 \le k \le N} g_{x,y_k} \in \bar{A}$ works.

<u>Remark</u>: A SW algebra $\Rightarrow$ $\bar{A}$ SW algebra

<u>Proof of SW theorem</u>: Assume that A is a SW-algebra, and take $f \in C_{\mathbb{R}}(K)$, $\varepsilon > 0$. For every $x \in K$ construct g_x: $g_x(x) = f(x)$, $g_x(z) \ge f(z) - \varepsilon$ $\forall z \in K$. Consider $V_x = \{z \in K \mid g_x(z) < f(z) + \varepsilon\} \ni x$.
Thus, V_x is open $\forall x \in K$, and $K = \bigcup_{x \in K} V_x \Rightarrow \exists \{x_k\}_{k=1}^M$. $K = \bigcup_{k=1}^M V_{x_k}$.

$$g = \min_{1 \le k \le M} g_x \in \bar{A} \quad \text{and} \quad f(z) - \varepsilon \le g \le f(z) + \varepsilon$$

$\Rightarrow \|g - f\|_{C_{\mathbb{R}}(K)} < \varepsilon \Rightarrow A$ is dense in $C_{\mathbb{R}}(K)$. This proves sufficiency.

<u>Necessity</u>: If A vanishes at some $x_0 \in K$, then 1 cannot be approximated uniformly by elements of A. If A does not separate points x, y, then the function f: $f \in C_{\mathbb{R}}(K)$, $f(x) = 1$, $f(y) = 0$ cannot be approximated. Such a function exists by Uryson's lemma ($K_1, K_2 \subset K$ · compact, $K_1 \cap K_2 \neq \emptyset \Rightarrow \exists f \in C_{\mathbb{R}}(K)$. $f|_{K_1} = 1$, $f|_{K_2} = 0$). ▨

<u>Example</u>: Let $K = \mathbb{T} = \{|z| = 1\}$, $C(\mathbb{T})$ the complex Banach space of continuous functions. Consider $A = \mathcal{P} = \operatorname{span}\{z^k \mid z \in \mathbb{Z}_+\}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
A is an algebra, A does not vanish ($1 \in A$), A separates points ($z \in A$) (but we have a complex algebra).
<u>Observation</u>: $\displaystyle\int_{\mathbb{T}} \bar{z}\, \overline{p(z)}\, \underbrace{dm(z)}_{\text{Lebesgue measure on } \mathbb{T}, \text{ normalised by } m(\mathbb{T}) = 1} = 0 \quad \forall p \in A.$

Indeed, this follows from the fact that

$$\int_{\mathbb{T}} \bar{z} \cdot \bar{z}^k = 0 \quad \forall k \in \mathbb{Z}_+$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+1)t}\,dt = \frac{1}{2\pi}\,\frac{e^{-i(k+1)t}}{-i(k+1)}\Bigg|_0^{2\pi} = 0$$

Conclusion: $\bar{z} \perp A$ in $L^2(m)$. So, A cannot be dense, because otherwise $\bar{z} \perp C(\mathbb{T})$ in $L^2(m) \Rightarrow \bar{z} \perp L^2(m)$. Contradiction.
So, A is not dense and SW-theorem (real version) does not work in the complex space.

<u>Definition</u>: A is a complex Stone-Weierstrass algebra if
1) A does not vanish.
2) A separates points
3) A is closed under conjugation $f \longmapsto \bar{f}$.

In the example above, 3) does not hold.

<u>Theorem</u> [SW, complex]: An algebra A closed under conjugation is dense in $C(K) \Leftrightarrow A$ is a SW complex algebra.

<u>Proof</u>: A is SW complex algebra $\Rightarrow \operatorname{Re} f, \operatorname{Im} f \in A \;\forall f \in A \Rightarrow$
$\Rightarrow \operatorname{Re} A$ is a real SW-algebra $\Rightarrow \operatorname{Re} A$ is dense in $C_{\mathbb{R}}(K) \Rightarrow$
$\Rightarrow \operatorname{Re} A + i \operatorname{Re} A$ is dense in $C(K) = C_{\mathbb{R}}(K) + i\, C_{\mathbb{R}}(K)$ and it is contained in A. The other direction is trivial. $\square$

C^*-algebras: Gelfand-Naimark theorem

November 19, 2025

Definition: A Banach algebra A is called an algebra with involution, if there is an operation $*$ s.t.

1) $(\alpha x+\beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^* \quad \forall \alpha,\beta\in\mathbb{C} \quad \forall x,y\in A$
2) $(xy)^* = y^* x^* \quad \forall x,y\in A$
3) $(x)^{**} = x \quad \forall x\in A$

If moreover,

4) $\|x^*\| = \|x\| \quad \forall x\in A$
5) $\|x^* x\| = \|x^*\|\cdot\|x\| \quad \forall x\in A$

then A is called a C^*-algebra.

Example: Let K be a Hausdorff compact. Then $C(K)$ is a C^*-algebra, $f^* := \bar{f}$.

Proposition: Let $T\in\mathcal{B}(H)$, where H is a Hilbert space. Then $\|T\| \overset{(*)}{=} \sup\limits_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle Tx, y\rangle|$.

Proof: We will use the fact that $\forall h\in H.\ \|h\| = \sup\limits_{\|x\|\le 1} |\langle h,x\rangle|$.
(indeed, $|\langle h,x\rangle| \le \|h\|\cdot\|x\| \le \|h\|$ by CS inequality, and taking $x = \frac{h}{\|h\|}$, we get $\langle h,x\rangle = \|h\|$.

$$\|T\| = \sup_{\|x\|\le 1}\|Tx\| = \sup_{\|x\|\le 1}\sup_{\|y\|\le 1}|\langle Tx,y\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}}|\langle Tx,y\rangle| \qquad \blacksquare$$

Definition: Let $T\in\mathcal{B}(H)$, then $T^*\in\mathcal{B}(H)$ is the operator such that $\langle Tx,y\rangle = \langle x, T^*y\rangle \quad \forall x,y\in H$.

Proposition: $T^*\in\mathcal{B}(H)$ exists for every $T\in\mathcal{B}(H)$, moreover, $\|T^*\| = \|T\|$.

Proof: We will use Riesz theorem, which says that $\forall \varphi\in H^*$

$\exists!\, h \in H$ such that $\phi(x) = \langle x, h\rangle$ $\forall x \in H$. Having this theorem, define:

$$\phi_y(x) := \langle Tx, y\rangle,\ x \in H, \text{ where } y \in H \text{ is fixed}$$

$\phi_y \in H^*$: $\sup\limits_{\|x\|\le 1} |\phi_y(x)| = \sup\limits_{\|x\|\le 1} |\langle Tx, y\rangle| \le \|T\| \cdot \|y\|.$

$\Rightarrow \exists!\, h \in H.\ \phi_y(x) = \langle x, h\rangle$. Let's define $T^*y := h$.

Since $\phi_{\alpha y_1 + \beta y_2} = \bar{\alpha}\phi_{y_1} + \bar{\beta}\phi_{y_2}$

$$\langle x, T^*(\alpha y_1 + \beta y_2)\rangle = \bar{\alpha}\underbrace{\langle x, T^* y_1\rangle}_{\langle x, \alpha T^* y_1\rangle} + \bar{\beta}\underbrace{\langle x, T^* y_2\rangle}_{\langle x, \beta T^* y_2\rangle} = \langle x, \alpha T^* y_1 + \beta T^* y_2\rangle \quad \forall x \in H$$

$\Rightarrow T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2.$

We have used the fact that $\langle h, z_1\rangle = \langle h, z_2\rangle\ \forall h \Rightarrow z_1 = z$.
(Proof: $h: z_1 - z_2 \quad \langle z_1 - z_2, z_1 - z_2\rangle = 0$)

To compute the norm of T^*, we note that

$$\|T^*\| = \sup_{\substack{\|y\|\le 1\\ \|x\|\le 1}} |\langle T^* y, x\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle y, Tx\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\overline{\langle Tx, y\rangle}| = \|T\|$$

$\Rightarrow \|T^*\| = \|T\|$, in particular, $T^* \in \mathcal{B}(H)$. ▨

Proposition: $\|T^*T\| = \|T^*\| \cdot \|T\| \quad \forall T \in \mathcal{B}(H)$

Proof:
$$\|T^*T\| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle T^*Tx, y\rangle| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} \|\langle Tx, Ty\rangle\|$$
$$\ge \sup_{\|x\|\le 1} \|\langle Tx, Tx\rangle\|$$
$$= \|T\|^2$$

$\Rightarrow \|T\|^2 \le \|T^*T\| \le \|T^*\| \cdot \|T\| = \|T\|^2 \Rightarrow \|T\|^2 = \|T^*\| \cdot \|T\|$

In particular, $\|T^*T\| = \|T^*\| \cdot \|T\|$. ▨

Example: $\mathcal{B}(H)$ is a C^*-algebra. We only need to check that $(\alpha T_1 + \beta_2 T_2)^* = \bar{\alpha} T_1^* + \bar{\beta} T_2^*$, $(T_1T_2)^* = T_2^* T_1^*$, $(T^*)^* = T$.
This follows directly from the definition

e.g. $\langle Tx,y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, (T^*)^*x \rangle} = \langle (T^*)^*x, y\rangle$

$\Rightarrow$ T^{**} and T have the same bilinear forms $\Rightarrow$

$\Rightarrow$ $\|T^{**} - T\| = 0$ by $(*)$ for $T^{**} - T$ in place of T.

<u>Definition</u>: $T \in \mathcal{B}(H)$ is called normal if $T^*T = TT^*$.

<u>Example</u>: $A = \overline{\operatorname{span}\{T^k (T^*)^j \mid k,j \geq 0\}}$ is a commutative C^*-algebra for every normal operator $T \in \mathcal{B}(H)$.

The only nontrivial thing here is that $S \in A \Rightarrow S^* \in A$. This follows from the fact that $S_n \longrightarrow S$ in $\mathcal{B}(H)$ $\Rightarrow$ $S_n^* \longrightarrow S^*$ in $\mathcal{B}(H)$ $(\|S_n^* - S^*\| = \|S_n - S\|)$.

November 20, 2025

<u>Goal</u>: A is a commutative C^*-algebra $\Rightarrow A \cong C(K)$ for some Hausdorff compact K.

In fact, we will see that $K = A^*_{mult}$ with some topology, that we will now define.

<u>Definition</u>: Let X be a Banach space, X^* the dual space to X. Then $\sigma(X^*, X)$, the weak*-topology, is defined on X^* as follows: for $\phi \in X^*$, $\varepsilon > 0$, $F \in X$ - a finite subset, we define.

$V_{F,\varepsilon}(\phi) := \{\psi \in X^* \mid |\phi(x) - \psi(x)| < \varepsilon \ \forall x \in F\}$ (neighbourhood of ϕ, corresponding to F, ε)

Open subsets in $\sigma(X^*, X)$ are precisely those subsets S that have the following property: $\forall \phi \in S.\ \exists F, \varepsilon.\ V_{F,\varepsilon}(\phi) \subset S$.

<u>Remarks</u>: (1) If X is separable, then this topology is metrizable on each bounded subset of X, and closed subsets could be defined as follows: S is closed $\Leftrightarrow$ $\forall \phi_n \in S.\ \phi_n(x) \longrightarrow \phi(x)$ $\forall x \in X$ we have $\phi \in S$.

(2) In the general situation $\sigma(X^*, X^*)$ is Hausdorff.
Indeed, take $\phi_1, \phi_2 \in X^*$, $\phi_1 \neq \phi_2 \Rightarrow \exists\, x \in X.\ \phi_1(x) \neq \phi_2(x) \Rightarrow$
$\Rightarrow V_{F_1,\varepsilon}(\phi_1) \cap V_{F_2,\varepsilon}(\phi_2) = \emptyset$ for $F_1 = \{x\} = F_2$, $\varepsilon = \frac{|\phi_1(x) - \phi_2(x)|}{2}$

(3) The importance of this topology is explained by the following theorem.

↳ Proof: Functional Analysis

Theorem [Banach-Alaoglu]: Any bounded $\underbrace{\text{closed}}_{\text{in norm (in the initial topology)}}$ subset of X^* is compact in the $\sigma(X^*, X)$-topology for every Banach space X.

Lemma: Let A be a commutative Banach algebra. Then A^*_{mult} with the induced topology from $\sigma(X^*, X)$ is Hausdorff compact.

Proof: Since $A^*_{mult} \subset \mathcal{B}_{X^*}[0,1]$, by B-A theorem, we only need to check that A^*_{mult} is closed. Assume that $\phi_n \in A^*_{mult}$, $\phi_n \longrightarrow \phi$ in $X^* \Rightarrow \phi \in X^*$ and $\phi(xy) = \lim_{n\to\infty} \phi_n(xy) = \phi(x)\cdot\phi(y)$ $\forall x,y \in X$. ▨

Definition: Let A be a commutative Banach algebra. and for every $x \in A$ define $\hat{x} : \phi \longrightarrow \phi(x)$, $\phi \in A^*_{mult}$. The mapping $x \longmapsto \hat{x}$ is called the Gelfand transform.

Lemma: For every $x \in A$ we have $\hat{x} \in C(A^*_{mult})$.

Proof: Take $x \in A$, $U \subset \mathbb{C}$-open subset. We need to show that $\hat{x}^{-1}(U)$ is open in $\sigma(X^*, X)$. It is enough to prove that $\hat{x}^{-1}(B(z_0, \varepsilon))$ is open $\forall z_0 \in U \cap \hat{x}(A^*_{mult})$ $\forall \varepsilon > 0$.
Let $z_0 \in \hat{x}(A^*_{mult})$, i.e., $\exists \psi.\ z_0 = \hat{x}(\psi) = \psi(x)$.
Then $\hat{x}^{-1}(B(z_0, \varepsilon)) = \{\phi \in A^*_{mult} \mid |\hat{x}(\phi) - z| < \varepsilon\} = \underbrace{V_{F,\varepsilon}(\psi) \cap A^*_{mult}}_{\text{open in induced topology}}$,
where $F = \{x\}$. ▨

Lemma: Let A be a C^*-algebra and $x \in A$ normal. Then $\|x\| = r(x)$.
$(x^*x = xx^*)$

Proof: At first assume $x^* = x$, then

$$\|x^{2^n}\| = \|x^{2^{n-1}} \cdot x^{2^{n-1}}\| = \|(x^{2^{n-1}})^* x^{2^{n-1}}\| = \|x^{2^{n-1}}\|^2 = [\text{iterate}] = \|x\|^{2^n}$$

$$\Rightarrow r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \lim_{n\to\infty} \|x\| = \|x\|.$$

Now the general case: $x^*x = xx^*$. Consider $y = x^*x$, and note that $\|y\| = r(y)$, because $y^* = y$. We have $\|x\|^2 = \|y\|$, and

$$r(y) \overset{x \text{ normal}}{=} \lim \|(x^*)^n x^n\|^{\frac{1}{n}} = r(x)^2$$

C^* property and the definition of $r(x)$

$\Rightarrow r(x) = \|x\|.$ ▨

Lemma: Let A be a Banach algebra, $x, y \in A$, $xy = yx$. Then $e^x \cdot e^y = e^{x+y}$, where $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in A$.

Proof:

$$e^x \cdot e^y = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{j=0}^{\infty} \frac{y^j}{j!}\right) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k+j=n} x^k \cdot y^j \frac{1}{k!} \cdot \frac{1}{j!} \cdot (k+j)!\right)}_{xy=yx \;\to\; (x+y)^n} \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e^{x+y}$$ ▨

Theorem [Gelfand–Naimark]: Let A be a commutative C^*-algebra. Then the Gelfand transform is the isomorphism of C^*-algebras A and $C(A^*_{mult})$, i.e.

1) $\widehat{\alpha x + \beta y} = \alpha \hat{x} + \beta \hat{y}$
2) $\widehat{xy} = \hat{x}\hat{y}$
3) $\widehat{x^*} = \overline{\hat{x}}$
4) $\|\hat{x}\|_{C(A^*_{mult})} = \|x\|$
5) $x \mapsto \hat{x}$ is a bijection between A and A^*_{mult}.

Proof: 1) $\widehat{\alpha x + \beta y}(\phi) = \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y) = \alpha\hat{x} + \beta\hat{y}$

2) $\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}\hat{y}$

3) Take $x \in A$, and observe that $x = \mathrm{Re}\,x + i\,\mathrm{Im}\,x$.

$$\mathrm{Re}\,x = \frac{x+x^*}{2}, \quad \mathrm{Re}\,x = (\mathrm{Re}\,x)^*$$

$$\mathrm{Im}\,x = \frac{x-x^*}{2}, \quad \mathrm{Im}\,x = (\mathrm{Im}\,x)^*$$

Moreover, $\widehat{(\mathrm{Re}\,x + i\,\mathrm{Im}\,x)^*} = \widehat{\mathrm{Re}\,x} - i\,\widehat{\mathrm{Im}\,x}$

$\Rightarrow$ it suffices to prove that $\hat{x} = \overline{\hat{x}}$ for every $x = x^*$.

$$\hat{x} = \overline{\hat{x}} \Leftrightarrow \phi(x) = \overline{\phi(x)} \Leftrightarrow |e^{i\phi(x)}| = 1 \Leftrightarrow |\phi(e^{ix})| = 1$$

$$1 = |e_A| = |\phi(e_A)| \underset{\uparrow}{=} |\phi(e^{ix}) \cdot \phi(e^{-ix})| \le \|e^{-ix}\| \cdot \|e^{ix}\| \underset{\uparrow}{=} \|(e^{ix})^* e^{ix}\| = \|e_A\| = 1$$

identity — lemma, $ix - ix = 0$, $e^0 = e_A$ — C^*-property & $(e^{ix})^* = e^{-ix}$

$\Rightarrow$ all inequalities are in fact equalities, $|\phi(e^{ix})| \cdot |\phi(e^{-ix})| = 1$.

A commutative, so $x^*x = xx^*$

4) $\|x\| = r(A) = \sup\{|\lambda|, \lambda \in \sigma(x)\}$

description of spectrum in commutative Banach algebra

$$= \sup\{|\lambda|, \lambda \in \{\phi(x), \phi \in A^*_{mult}\}\}$$

$$= \sup\{|\phi(x)|, \phi \in A^*_{mult}\}$$

$$= \sup_{\phi \in A^*_{mult}} |\hat{x}(\phi)| = \|\hat{x}\|_{C(A^*_{mult})}$$

5) $\hat{A}$ is a closed subalgebra in $C(A^*_{mult})$ because it is an isometric image of a closed algebra $(=A)$.

It remains to check that $\hat{A}$ does not vanish and separates points (SW-theorem then implies $A = \mathrm{clos}\,A = C(A^*_{mult})$).

i) $\hat{A}$ does not vanish $\Leftrightarrow \forall \phi \in A^*_{mult}\ \exists x \in A.\ \hat{x}(\phi) \neq 0 \Leftrightarrow \phi \neq 0$. This holds, because $\phi(e_A) = 1$.

ii) $\hat{A}$ separates points $\Leftrightarrow \forall \phi \neq \psi.\ \exists x \in A.\ \hat{x}(\phi) \neq \hat{x}(\psi) \Leftrightarrow \phi \neq \psi$. ▨

Application: functional calculus for C^*-algebras

Definition: Let A be a commutative C^*-algebra, $x \in A$, $\sigma(x)$ the spectrum of x, $f \in C(\sigma(x))$. Then we define:

$$f(x) := (f(\hat{x}))^\vee,$$

where "$\vee$" is the inverse Gelfand transform.

Remark: $f(x)$ is defined correctly, in other words, $\operatorname{Ran} \hat{x} \subset \operatorname{Dom} f$.
Indeed, $\operatorname{Ran} \hat{x} = \{\hat{x}(\phi),\ \phi \in A^*_{mult}\} = \{\phi(x),\ \phi \in A^*_{mult}\} = \sigma(x)$.

Remark: If f is a polynomial, $f = c_0 + c_1 z + \dots + c_n z^n$, then $f(x) = c_0 + c_1 x + \dots + c_n x^n$ for every $x \in A$.
Indeed, it suffices to check this for $f_n = z^n$, $z \in \mathbb{C}$:
$f_n(x) = (z^n(\hat{x}))^\vee = (\widehat{x^n})^\vee = x^n$.

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Remark: For $f(z) = \bar{z}$ we have $f(x) = x^*$ because

$$\bar{z}(\hat{z}(\cdot))^\vee = (\overline{\hat{x}(\cdot)})^\vee = (\widehat{x^*})^\vee = x^*.$$

Remark: If $f_1, f_2 \in C(\sigma(x))$, then $(f_1 \cdot f_2)(x) = \underbrace{f_1(x)}_{\in A} \overset{\text{mult. in } A}{\cdot} \underbrace{f_2(x)}_{\in A}$.

Theorem [Functional calculus theorem]: Assume that A is a commutative C^*-algebra generated by some element $x \in A$ (polynomials of x, x^* are dense in A). Then $A \cong C(\sigma(x))$ as C^*-algebras. Moreover, the isomorphism between $C(\sigma(x))$ and A is given by the map $f \mapsto f(x)$, where $f(x) = (f(\hat{x}(\cdot)))^\vee$. In particular, we have:

(1) $\|f(x)\| = \max\limits_{z \in \sigma(x)} |f(z)| \quad \forall f \in C(\sigma(x))$

(2) $\sigma_A(f(x)) = f(\sigma(x)) \quad \forall f \in C(\sigma(x))$

spectral mapping theorem

Lemma 1: Let K_1, K_2 be Hausdorff compacts, and let $h: K_1 \longrightarrow K_2$ be a continuous bijection. Then h is a homeomorphism.

Proof: We need to check $h^{-1}: K_2 \longrightarrow K_1$ is continuous $\Leftrightarrow$
$(h^{-1})^{-1}(U)$ is open in K_2 for every U-open in K_1 $\Leftrightarrow$
$(h^{-1})^{-1}(C)$ is closed in K_2 for every C-closed in K_1 $\Leftrightarrow$
$h(C)$ is compact for every C-compact in K_1.
Take some open cover $\{V_\alpha\}_{\alpha \in I}$ of $h(C)$, and consider $\{h^{-1}(V_\alpha)\}_{\alpha \in I}$ - open cover of C, so $\exists \alpha_1, \dots, \alpha_N$ s.t. $\{h^{-1}(V_{\alpha_k})\}_{k=1}^N$ subcover of C
$\Rightarrow \{V_{\alpha_k}\}_{k=1}^N$ is a subcover of $h(C)$. ▨

Lemma 2: Assume that $K_1, K_2, h: K_1 \to K_2$ are as in the previous lemma. Then $\mathcal{C}(K_1) \cong \mathcal{C}(K_2)$ as C^*-algebras, and the isomorphism is given by $\underbrace{f(\cdot)}_{\in \mathcal{C}(K_1)} \longmapsto f(h^{-1}(\cdot))$.

Proof: Exercise $(\|f\|_{\mathcal{C}(K_1)} = \|f(h^{-1})\|_{\mathcal{C}(K_2)}$, etc.$)$

Proof of the theorem: We know that $A \cong \mathcal{C}(A^*_{mult})$ by GN theorem. To prove $A \cong \mathcal{C}(\sigma(x))$, we will check that
$\hat{x}: \phi \longmapsto \hat{x}(\phi) = \phi(x)$ is a homeo from A^*_{mult} onto $\sigma(x)$ and then apply the last lemma.

1) $\hat{x}(A^*_{mult}) = \sigma(x)$ — description of the spectrum in a commutative B.A.
$\{\hat{x}(\phi),\ \phi \in A^*_{mult}\} = \{\phi(x) \mid \phi \in A^*_{mult}\}$ ← surjectivity

2) Let $\hat{x}(\phi_1) = \hat{x}(\phi_2)$ for some $\phi_1, \phi_2 \in A^*_{mult}$ $\Leftrightarrow$
$\Leftrightarrow \phi_1(x) = \phi_2(x) \Rightarrow \begin{matrix} \phi_1(x) = \phi_2(x) \\ \phi_1(x^*) = \phi_2(x^*) \end{matrix} \Rightarrow \phi_1(p(x,x^*)) = \phi_2(p(x,x^*))$ for any polynomial $p(z,\bar{z})$
$\Rightarrow \phi_1 = \phi_2$ by continuity

$$\left(\begin{matrix} \phi_k(x^*) & = & \overline{\phi_k(x)} \\ \| & & \| \\ \widehat{x^*}(\phi_k) & \underset{GF}{=} & \overline{\hat{x}(\phi_k)} \end{matrix} \quad \text{(part of GN theorem, because } f^* = \bar{f} \text{ in } \mathcal{C}(K)\text{)} \right)$$

3) We also know that $\hat{a} \in \mathcal{C}(A^*_{mult})$ for every $a \in A$ (again GN).
In particular, for $a = x$ we get continuity of the map $\hat{x}: \phi \mapsto \phi(x)$.
By Lemma 1, we conclude that $\hat{x}$ is a homeomorphism and by Lemma 2, that $A \cong \mathcal{C}(\sigma(x))$. Moreover:

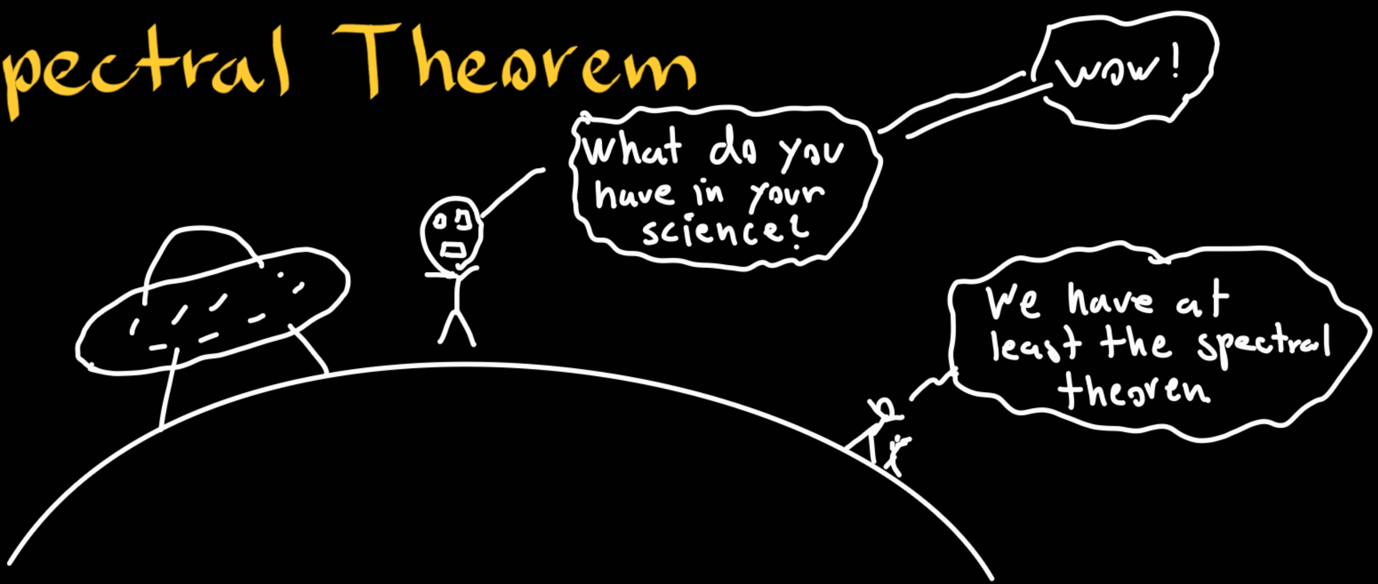
$$A \cong \mathcal{C}(A^*_{mult}), \quad a \mapsto \hat{a} \qquad \mathcal{C}(A^*_{mult}) \cong \mathcal{C}(\sigma(x)), \quad f(\hat{x}(\cdot)) \longleftarrow f \qquad A \cong \mathcal{C}(\sigma(x)), \quad (f(\hat{x}(\cdot)))^\vee \longleftarrow f$$

$$(f(\hat{x}(\cdot)))^\vee \overset{\text{definition}}{=} f(x)$$

Finally: (1) $\|f(x)\|_A \overset{\text{iso}}{=} \|f\|_{\mathcal{C}(\sigma(x))} = \max_{z \in \sigma(x)} |f(z)|$

(2) $\sigma_A(f(x)) = \sigma_{\mathcal{C}(\sigma(x))}(f) = f(\sigma(x))$
know from before

Spectral Theorem



Theorem [spectral theorem]: Let H be a separable Hilbert space, $T \in \mathcal{B}(H)$: $T^*T = TT^*$. Then T is unitary equivalent to the direct sum of multiplicative operators: there are Borel measures μ_k, $\operatorname{supp} \mu_k \subseteq \sigma(T)$ and a unitary operator $\mathcal{U}$ such that $\mathcal{U}^{-1} T \mathcal{U} = \bigoplus M_{k,z}$, where $M_{k,z}: f \mapsto zf$ on $L^2(\mu_k)$.

Particular examples:

1) $A = A^*$ $(\langle Ax, y\rangle = \langle x, Ay\rangle \ \forall x, y \in H)$ Physics: symmetry

2) $\|Ax\| = \|x\|$
$A: H \longrightarrow H$ is bijection $\left(\begin{matrix} A \text{ is unitary} \\ \text{or } A^*A = AA^* = I \end{matrix}\right)$ conservation laws

Definition: Let $\{H_k\}_{k=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$ be a sequence of Hilbert spaces. Then $\bigoplus_{k=1}^N H_k$ is the Hilbert space $\{\{h_k\}_{k=1}^N : \sum_{k=1}^N \|h_k\|^2 < \infty, h_k \in H_k\}$ with the inner product $\langle \{h_k\}, \{g_k\} \rangle_{\bigoplus_{k=1}^N H_k} = \sum_{k=1}^N \langle h_k, g_k \rangle_{H_k}$.

Example: $\ell^2(\mathbb{Z}) = \bigoplus_{k=-\infty}^{\infty} \mathbb{C}$ $\left(\{a_n\} \subset \mathbb{C}, \sqrt{\sum |a_n|^2} = \|\{a_n\}\| \right)$

Definition: Let $T_k : H_k \longrightarrow H_k$ be a bounded linear operator for every k. Assume that $\|T_k\| \leq C$ $\forall k = 1, \dots, N$. Then $T = \bigoplus_{k=1}^N T_k$ is an operator on $H = \bigoplus_{k=1}^N H_k$ defined by $T(\{h_k\}_1^N) = \{T_k h_k\}_1^N \in H$.

Remark: $\|T\| = \sup_{1 \leq k \leq N} \|\overset{C}{T_k}\|$, because

$$\|T(\overbrace{\{h_k\}}^{h})\|_H^2 = \sum_1^N \|T_k h_k\|^2 \leq C^2 \sum_1^N \|h_k\| = C^2 \|h\|^2 \Rightarrow \|T\| \leq C$$

$\|T\| \geq C - \varepsilon$ for every ε, because $\|T\| \geq \|T_k\|$ $\forall k$.

(consider h of the form $\{0, 0, \dots, 0, g, 0, \dots, 0\}$

$\uparrow$ k-th position, arbitrary $g \in H_k$ with $\|g\| = 1$

Definition: An operator U is unitary if $U^* U = U U^* = I$.

Proposition: Let $U \in \mathcal{B}(H)$. TFAE:

1) U is unitary
2) U is a bijection from H onto H and $\|U(x)\| = \|x\|$ for every $x \in H$.

Proof: Exercise. $\begin{array}{c}\langle Ux, Ux \rangle = \langle x, x \rangle \\ \| \\ \langle U^* U x, x \rangle\end{array}$ $\leftarrow$ hint

Definition: Let $T \in \mathcal{B}(H)$. Then $E \subset H$ is a reducing subspace of T if $TE \subset E$ and $T^* E \subset E$.

Proposition: E is reducing for $T \Leftrightarrow TE \subset E$, $TE^\perp \subset E^\perp$, where $E^\perp = \{h \in H \mid \langle h, g \rangle = 0 \quad \forall g \in E\}$.

Proof: $TE^\perp \subset E^\perp \Leftrightarrow TE^\perp \perp E \Leftrightarrow E^\perp \perp T^*E \Leftrightarrow T^*E \subset E$. $\blacksquare$

$\langle Tg^\perp, h \rangle = 0 \Leftrightarrow \langle g^\perp, T^*h \rangle = 0$

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Proposition: Let H be a separable Hilbert space and $T \in \mathcal{B}(H)$ be normal. Then there exist subspaces $\{H_k\} \subset H$:

1) $\bigoplus H_k = H$
2) H_k are reducing for T
3) $T = \bigoplus T_k$, $T_k \in \mathcal{B}(H_k)$ normal
4) $\exists h_k \in H_k$. $\operatorname{clos}\{p(T, T^*)h_k \mid p = p(z, \bar{z}) \text{ is a polynomial}\} = H_k$

Proof: Take a sequence $\{g_k\}_{k=1}^{\infty}$ dense in H (H separable).

$h_1 := g_1 \qquad H_1 := \operatorname{clos}\{p(T, T^*) \mid p \in \mathcal{P}\}$, $\mathcal{P}$ ... polynomials in $z, \bar{z}$

$h_2 := P_{H_1^\perp} g_{k_2}$, where k_2 is the minimal integer k such that

$\perp$ orthogonal projection $\quad P_{H_1^\perp} g_k \neq 0 \quad (\Leftrightarrow g_k \notin H_1)$

$H_2 := \operatorname{clos}\{p(T, T^*) h_2, \ p \in \mathcal{P}\}$.

$h_3 := P_{(\operatorname{span}(H_1, H_2))^\perp} g_{k_3}$, where k_3 is the minimal integer k such that

$P_{(\operatorname{span}(H_1, H_2))^\perp} g_k \neq 0 \quad (g_k \notin \operatorname{span}(H_1, H_2))$

$\vdots$

We get a sequence of vector $\{h_k\}_{k=1}^{N}$ and subspaces $\{H_k\}_{k=1}^{N}$ where $1 \leq N \leq +\infty$ (if there is no integer k_{n+1}, the procedure stops).

Claim: $H_j \perp H_n$ if $j \neq n$.

Let $j > n$. We need to check that

$$\langle p(T, T^*) h_j, q(T, T^*) h_n \rangle = 0 \qquad \forall p, q \in \mathcal{P}$$

$$\langle \underbrace{h_j}_{\in \operatorname{span}(H_1, \dots, H_{j-1})^\perp}, \underbrace{(\bar{p} \cdot q)(T, T^*) h_n}_{\in H_n} \rangle = 0$$

True, because H_n is among $H_1, \dots, H_{j-1}$.

Claim: $\bigoplus H_k = H$

If this is not the case, $\exists h \in H \setminus \{0\}$. $h \perp \bigoplus H_k \Rightarrow h \perp H_k \ \forall k$

$\Rightarrow h \perp g_k$ for every $k \geq 1 \Rightarrow h \perp$ (dense subset in H) $\Rightarrow h = 0$ $\lightning$

<u>Claim</u>: H_k is reducing for T – this is just by construction $Tp(T,T^*) = \tilde{p}(T,T^*)$, $T^* p(T,T^*) = \tilde{\tilde{p}}(T,T^*)$, where $\tilde{p} = z \cdot p$, $\tilde{\tilde{p}} = \bar{z} \cdot p$.

<u>Claim</u>: $T = \oplus T_k$, T_k are normal

$T_k = T|_{H_k}$, $\quad 0 = TT^* - T^*T = \oplus (T_k T_k^* - T_k^* T_k) \Rightarrow T_k T_k^* - T_k^* T_k = 0 \quad \forall k$

<u>Claim</u>: $\text{clos}\{p(T,T^*)h_k, \; p \in \mathcal{P}\} = H_k$

By construction (definition of H_k). $\blacksquare$

Theorem [Riesz-Markov]: Let K be a compact Hausdorff space. Then $C(K)^* = \mathcal{M}(K)$ (set of Borel measures), i.e. for every linear continuous functional $\phi: C(K) \longrightarrow \mathbb{C}$ there exists a unique Borel measure μ on K (complex-valued) such that

$$\phi(f) \overset{(*)}{=} \int_K f \, d\mu, \quad f \in C(K) \text{ and } \|\phi\| = |\mu|(K) = \sup_{\substack{\cup K_j = K \\ K_j \cap K_n = \emptyset}} \sum_{j=1}^{\infty} |\mu(K_j)|.$$

Conversely, every functional ϕ of the form $(*)$ belongs to $C(K)^*$. Moreover, if $\forall f \in C(K).\, f \geq 0 \Rightarrow \phi(f) \geq 0$, then the representing measure is a non-negative measure.

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<u>Proof of Spectral Theorem</u>: By previous consideration, we know that $T = \oplus T_k$ and $H = \oplus H_k$ where $T_k \in \mathcal{B}(H_k)$ are such that $\exists h_k \in H_k$. $\text{span}\{T^k T^{*j} h_k \mid k,j \geq 0\}$ is $\overset{(*)}{\text{dense}}$ in H_k. Let us show that $T_k \cong M_z$ on some $L^2(\mu_k)$ for every k. Define $\Phi(f) := (f(T_k)h_k, h_k)$, $f \in \mathcal{C}(\tilde{\sigma}(T_k))$, where $\tilde{\sigma}(T_k)$ is the spectrum of T^k in the C^*-algebra generated by T_k (in fact, $\sigma(T_k) = \tilde{\sigma}(T_k)$ but we do not know this yet).

$\Phi(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \Phi(f_1) + \alpha_2 \Phi(f_2)$ $\forall \alpha_1, \alpha_2 \in \mathbb{C}.\, \forall f_1, f_2 \in \mathcal{C}(\tilde{\sigma}(T_k))$, because the functional calculus $f \mapsto f(T_k)$ is linear.

$|\Phi(f)| = |(f(T_k)h_k, h_k)| \leq \|f(T_k)\| \cdot \|h_k\|^2 \leq \|f\|_{\mathcal{C}(\tilde{\sigma}(T_k))} \cdot \|h_k\|^2$

property of functional calculus

$\Rightarrow \|\Phi\|_{C(\tilde{\sigma}(T_k))^*} < \infty \Rightarrow$ by Riesz-Markov theorem $\exists \mu_k \in \mathcal{M}(\tilde{\sigma}(T_k))$ such that $\Phi(f) = \int_{\tilde{\sigma}(T_k)} f \, d\mu_k$.

Let us show that the measure μ_k is nonnegative. We need to check that $\forall \varphi \in \mathcal{C}(\tilde\sigma(T_k))$. $\varphi \geq 0 \Rightarrow \Phi(\varphi) \geq 0$. We have

$$\Phi(\varphi) = (\varphi(T_k)h_k, h_k) = ((\sqrt{\varphi}\cdot\sqrt{\varphi})(T_k)h_k, h_k) = (\sqrt{\varphi}(T_k)^*\sqrt{\varphi}(T_k)h_k, h_k)$$
$$= \|\sqrt{\varphi}(T_k)h_k\|^2 \geq 0$$

$\Rightarrow \mu_k \geq 0$

Define $V_k : \varphi(T_k)h_k \longmapsto \varphi$ on a dense subset of H_k formed by $\{\varphi(T_k)h_k \mid \varphi \in \mathcal{C}(\tilde\sigma(T_k))\} = \mathring{H}_k$ (this subset is dense because of (*)) and consider it as an operator from $\mathring{H}_k$ to $L^2(\mu_k)$.

We need to prove V_k is defined correctly and isometric on $\mathring{H}_k$:

$$\|V_k\varphi(T_k)h_k\|^2_{L^2(\mu_k)} = \int_{\tilde\sigma(T_k)} |\varphi|^2 d\mu_k$$

$$\|\varphi(T_k)h_k\|^2_{L^2(\mu_k)} = (\varphi(T_k)h_k, \varphi(T_k)h_k) = (|\varphi|^2(T_k)h_k, h_k) = \Phi(|\varphi|^2) = \int_{\tilde\sigma(T_k)} |\varphi|^2 d\mu_k$$

In particular, $\|V_k\varphi_1(T_k)h_k - V_k\varphi_2(T_k)h_k\|^2 = \int|\varphi_1-\varphi_2|^2 d\mu$, so if $\varphi_1(T_k)h_k = \varphi_2(T_k)h_k$, then $V\varphi_1(T_k)h_k = V\varphi_2(T_k)h_k$ ($\Rightarrow$ correctness of definition).

V_k is a bijection from H_k to $L^2(\mu_k)$ after extension to the whole set H_k by continuity:

1) V_k is an injection, because it is an isometry

2) V_k is a surjection, because the range of V_k is closed (V_k is an isometry) and dense in $L^2(\mu_k)$ (range contains $C(\tilde\sigma(T_k))$ - a dense subset in $L^2(\mu_k)$).

$\Rightarrow V_k$ is a unitary operator. Let's check that $V_kT_kV_k^{-1} = M_z$:

Take $\varphi_1, \varphi_2 \in \mathcal{C}(\tilde\sigma(T_k))$ and consider

$$(T_k\varphi_1(T_k)h_k, \varphi_2(T_k)h_k) = ((\bar\varphi_2\cdot z\cdot\varphi_1)(T_k)h_k, h_k) = \Phi(\bar\varphi_2 z\varphi_1) = \int_{\tilde\sigma(T_k)} \bar\varphi_2\cdot z\,\varphi_1 d\mu_k =$$
$$= (z\varphi_1, \varphi_2)_{L^2(\mu_k)} = (M_z\varphi_1, \varphi_2)_{L^2(\mu_k)} = (M_zV_k(\varphi_1(T_k)h_k), V(\varphi_2(T_k)h_k)) =$$
$$= (V_k^* M_z V_k\varphi_1(T_k)h_k, \varphi_2(T_k)h_k).$$

$T_k = V_k^* M_z V_k \Leftrightarrow V_kT_kV_k^{-1} = M_z$ because $V_k^* = V_k^{-1}$ for unitary operators.

It remains to check that $\operatorname{supp}\mu_k \subset \sigma(T)$. We will prove more: $\overline{\bigcup \operatorname{supp}\mu_k} = \sigma(T)$. For this we need a separate lemma:

Lemma: Let $M_z : f \longmapsto zf$ be the multiplication operator on $L^2(\mu)$ for some compactly supported Borel measure μ. Then $\sigma(M_z) = \tilde\sigma(M_z) = \operatorname{supp}\mu$.

Proof: $\rho(M_z) \supset \tilde{\rho}(M_z) \Rightarrow \sigma(M_z) \subset \tilde{\sigma}(M_z)$ (just because $\mathcal{B}(H) \supset C^*$-algebra generated by M_z). If $\lambda \in \mathbb{C} \setminus \operatorname{supp}\mu \Rightarrow \frac{1}{\lambda - z} \in \mathcal{C}(\operatorname{supp}\mu) \Rightarrow \frac{1}{\lambda - z} \overset{(**)}{=} \lim_{k\to\infty} p_k(z, \bar{z})$ in $\mathcal{C}(\operatorname{supp}\mu_k)$

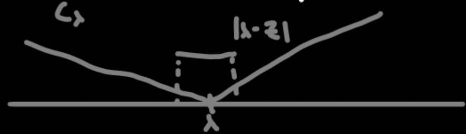
$(\lambda I - M_z)(M_{\frac{1}{\lambda - z}}) = I \Rightarrow$ for $\lambda \in \mathbb{C}\setminus \operatorname{supp}\mu$: $\lambda I - M_z$ is invertible, and, moreover, $(\lambda I - M_z)^{-1} = M_{\frac{1}{\lambda - z}} \in C^*$-algebra generated by M_z, because $\|M_{\frac{1}{z-\lambda}} - p_k(M_z, M_z^*)\| \le \|\frac{1}{\lambda - z} - p_k(z, \bar{z})\|_{\mathcal{C}(\operatorname{supp}\mu_k)} \longrightarrow 0$ by (**).

$\mathbb{C}\setminus \operatorname{supp}\mu \subset \tilde{\rho}(M_z) \Leftrightarrow \tilde{\sigma}(M_z) \subset \operatorname{supp}\mu \Rightarrow \sigma(M_z) \subset \tilde{\sigma}(M_z) \subset \operatorname{supp}\mu$

It remains to check that $\operatorname{supp}\mu \subset \sigma(M_z)$. Take $\lambda \in \operatorname{supp}\mu$ and assume that $\exists (\lambda I - M_z)^{-1} \in \mathcal{B}(L^2(\mu))$. Then

$$\|f\|^2_{L^2(\mu)} = \|(\lambda I - M_z)^{-1}(\lambda I - M_z) f\|^2 \le \underbrace{\|(\lambda I - M_z)^{-1}\|^2}_{= C_\lambda} \cdot \int_{\operatorname{supp}\mu} |\lambda - z| |f|^2 d\mu$$

$$\Rightarrow \int_{\operatorname{supp}\mu} |\lambda - z| |f(z)|^2 d\mu \ge \frac{1}{C_\lambda} \|f\|^2_{L^2(\mu)}$$



Take $f_\varepsilon := \frac{\chi_{B(\lambda,\varepsilon)}}{\underbrace{\sqrt{\mu(B(\lambda,\varepsilon))}}_{\neq 0\text{, because } \in \operatorname{supp}}}$, $\|f_\varepsilon\|_{L^2(\mu)} = 1$

$$\int \underbrace{|\lambda - z|}_{\le \varepsilon} |f_\varepsilon(z)|^2 d\mu \le \varepsilon \int_{B(\lambda,\varepsilon)} |f_\varepsilon|^2 d\mu = \varepsilon \Rightarrow \varepsilon \ge \frac{1}{C_\lambda} \quad \times$$

∎

The end of the proof of the spectral theorem:

We have shown that $T \cong \oplus M_z$ on $\oplus L^2(\mu_k)$, where

$$\operatorname{supp}\mu_k = \tilde{\sigma}(T_k) = \tilde{\sigma}(M_z) \overset{\text{Lemma}}{=} \sigma(M_z) = \sigma(T_k) \subset \sigma(\oplus T_k) \subset \sigma(T)$$

$\Rightarrow \operatorname{supp}\mu_k \subset \sigma(T) \Rightarrow \bigcup_k \operatorname{supp}\mu_k \subset \sigma(T)$

$$\Rightarrow \overline{\bigcup_k \operatorname{supp}\mu_k} \subset \sigma(T) \quad (\sigma(T) \text{ closed})$$

But for every $\lambda \notin \overline{\bigcup_k \operatorname{supp}\mu_k}$ we have $\operatorname{dist}(\lambda, \operatorname{supp}\mu_k) > \varepsilon \ \forall k$.

$\Rightarrow \|(\lambda I - M_z)^{-1}\| \le \frac{1}{\operatorname{dist}(\lambda, \operatorname{supp}\mu_k)} \le \frac{1}{\varepsilon}$

$\Rightarrow \|\oplus(\lambda I - M_z)^{-1}\| \le \frac{1}{\varepsilon}$ and $(\oplus(\lambda I - M_z)^{-1})(\oplus(\lambda I - M_z)) = \oplus I_k = I$

$\Rightarrow \lambda \in \sigma(T) \Rightarrow \sigma(T) = \overline{\bigcup \operatorname{supp}\mu_k}$.

∎

Simplest consequences of the spectral theorem

Proposition: Let $T \in \mathcal{B}(H)$: $T^*T = TT^*$. Then

- T is unitary $\Leftrightarrow \sigma(T) \subset \{|z|=1\}$.
- T is self-adjoint $\Leftrightarrow \sigma(T) \subset \mathbb{R}$.

Proof: $T \cong \oplus M_z$ $\quad$ T is unitary $\Leftrightarrow \begin{cases} T^*T = I \\ TT^* = I \end{cases}$

$\Leftrightarrow \begin{array}{l} \oplus M_z^* M_z = \oplus I_k \\ \oplus M_z M_z^* = \oplus I_k \end{array}$ $\Leftrightarrow M_z M_z^* = M_z^* M_z$ For every block

$\Leftrightarrow$ $M_z : f \longmapsto zf$ is unitary on $L^2(\mu_k)$

$M_{|z|^2} = I$ on $L^2(\mu_k) \Leftrightarrow |z|^2 f = f \;\; \forall f \in L^2(\mu_k) \Leftrightarrow |z| = 1$ on $\operatorname{supp}\mu_k$

Since $\sigma(T) = \overline{\bigcup \operatorname{supp}\mu_k} \Leftrightarrow \sigma(T) \subset \{|z|=1\}$.

T is self adjoint $\Leftrightarrow T - T^* = 0 \Leftrightarrow \oplus M_z - M_z^* = 0 \Leftrightarrow$

$\Leftrightarrow M_{z-\bar{z}} = 0$ on $L^2(\mu) \Leftrightarrow (z-\bar{z})f = 0 \;\; \forall f \in L^2(\mu_k)$

$\Leftrightarrow z - \bar{z} = 0$ on $\operatorname{supp}\mu_k \Leftrightarrow \operatorname{supp}\mu_k \subset \mathbb{R} \;\; \forall k$.

so, $T = T^* \Leftrightarrow \overline{\bigcup \operatorname{supp}\mu_k} \subset \mathbb{R} \Leftrightarrow \sigma(T) \subset \mathbb{R}$. $\blacksquare$

Proposition: Let $T \in \mathcal{B}(H)$: $T^*T = TT^*$, and $\lambda \in \mathbb{C} \setminus \sigma(T)$. Then
$\|(\lambda I - T)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}$.

Proof: $\lambda \notin \sigma(T) \Leftrightarrow \lambda \notin \overline{\bigcup \operatorname{supp}\mu_k}$, $\quad (\lambda I - T)^{-1} = \oplus (\lambda I - M_z)^{-1}$

$\|(\lambda I - T)^{-1}\| = \sup \|(\lambda I - M_z)^{-1}\|_{L^2(\mu_k)} = \sup_k \|\varphi_\lambda(M_z)\|_{\mathcal{B}(L^2(\mu_k))}$, $\quad \varphi_\lambda = \frac{1}{z-\lambda}$

By functional calculus theorem:

$$\|\varphi_\lambda(M_z)\|_{\mathcal{B}(L^2(\mu_k))} = \|\varphi_\lambda\|_{C(\operatorname{supp}\mu_k)}.$$

$$\Rightarrow \|(\lambda I - T)^{-1}\| = \sup_k \sup_{\xi \in \operatorname{supp}\mu_k} \frac{1}{|\lambda - \xi|} = \sup_{\lambda \in \sigma(T)} \frac{1}{|\lambda - \xi|} = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}. \quad \blacksquare$$

Proposition: If $T \in \mathcal{B}(H)$, $TT^* = T^*T$, then $\exists$ a nontrivial invariant subspace of T.

Proof: It suffices to note that either T has a reducing subspace or $T \cong M_z$ on $L^2(\mu)$, $\exists\, V: H \longrightarrow L^2(\mu)$ unitary

$\Rightarrow \dim(L^2(\mu)) = +\infty \Rightarrow \#\operatorname{supp}\mu = +\infty$

$\Rightarrow \exists$ subset E: $\mu(E) > 0$, $\mu(\Delta \setminus E) > 0$

$\Rightarrow L^2(\mu) = \underbrace{L^2(\chi_E \mu)} \oplus \underbrace{L^2(\chi_{\Delta \setminus E}\, \mu)}$ $\blacksquare$

invariant under M_z