Exact solution of Navier-Stokes Equations in simple cases

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Complements of General Physics,
a.y. 2016/2017

NSE represent a system of non-linear PDE to whom vectorial velocity field obey.
In the general case of isotropical, "small variations" (newtonian) fluid, the vectorial form of NSE is:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\, \bar{\mathbf{u}} = \bar{\mathbf{g}} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{3}\nu\nabla(\nabla \cdot \bar{\mathbf{u}})$$

3 equations and 23 unknowns $(\rho, \bar{u}, p, \bar{\sigma}, \bar{g}, E_u, \bar{q}, k(T), T)$: many other equations are necessary (Thermodynamics, laws of conservation,...)
If the fluid is incompressible than:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\, \bar{\mathbf{u}} = \bar{\mathbf{g}} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{\mathbf{u}}$$

The formula hides the relation between stress tensor and shear rate tensor:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu U_{ij}$$

The convention adopted is: first-index $\rightarrow$ force-direction, second-index$\rightarrow$surface-normal

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$$\underbrace{\frac{\partial \bar{\mathbf{u}}}{\partial t}}_{\text{TIME}} + \underbrace{(\bar{\mathbf{u}} \cdot \nabla)\, \bar{\mathbf{u}}}_{\text{ADVECTION}} = \underbrace{\bar{\mathbf{g}}}_{\text{EXT}} - \underbrace{\frac{1}{\rho} \nabla p}_{\text{PRESSURE}} + \underbrace{\nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{3} \nu \nabla (\nabla \cdot \bar{\mathbf{u}})}_{\text{VISCOSITY}}$$

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The single i-component of the velocity $\bar{\mathbf{u}} \equiv (u_x, u_y, u_z)$ in a Cartesian system (x, y, z) (usually labeled by i or j) is:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_x\frac{\partial u_i}{\partial x} + u_y\frac{\partial u_i}{\partial y} + u_z\frac{\partial u_i}{\partial z}\right) = \rho g_i - \frac{\partial p}{\partial i} + \mu\left(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} + \frac{\partial^2 u_i}{\partial z^2}\right)$$

Therefore it's clear that Einstein notation is very convenient!

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial j}\right) = \rho g_i - \frac{\partial p}{\partial i} + \mu\left(\frac{\partial}{\partial j}\frac{\partial u_i}{\partial j}\right)$$

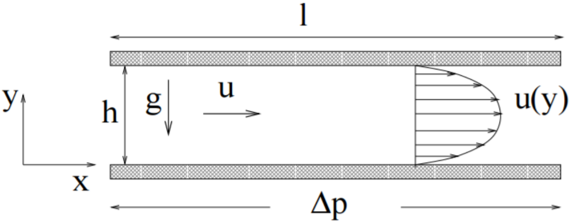
It has not yet been proven that in three dimensions smooth solutions always exist.
An exact solution exists in a degenerate (= simple) case.
Terminology adopted:

- *Boundary Condition*: physically coherent assumption that simplifies the equation, making it degenerate and defining which unknowns remain
- *Initial Condition*: known values of physical quantities that select a solution (set the values of integration constants)

Forgetting dimensional issues and focusing only on mathematical form, many famous equations are contained in the NSE

scalar-NSE	$\frac{\partial u}{\partial t}$	$u \cdot \nabla u$	g	∇p	$\nabla^2 u$
Fick	$\frac{\partial u}{\partial t}$			∇p	
Poisson/Laplace			g		$\nabla^2 u$
Heat ($\approx$ Schrodinger)	$\frac{\partial u}{\partial t}$				$\nabla^2 u$
Conservation/Shock	$\frac{\partial u}{\partial t}$	$u\frac{\partial u}{\partial x}$			
Burgers	$\frac{\partial u}{\partial t}$	$u\frac{\partial u}{\partial x}$			$\frac{\partial^2 u}{\partial x^2}$

Fully developed, no-slip, laminar ($\leftrightarrow$ $Re < 1400$) flow between two parallel moving plates, with pressure difference applied



Boundaries conditions:

1. $\rho = kost \rightarrow \nabla \cdot \bar{\mathbf{u}} = 0$
2. 1-d $\rightarrow$ $u_y = u_z = 0$
3. z-infinite $\rightarrow \frac{\partial}{\partial z} = 0$
4. stationary $\rightarrow \frac{\partial}{\partial t} = 0$
5. gravity $\rightarrow \frac{\partial p}{\partial y} = -\rho g$

Initial conditions:

1. $u_x = U \; if \; y = h$
2. $u_x = 0 \; if \; y = 0$
3. $\frac{\partial p}{\partial x} = kost = \frac{\Delta p}{l}$

Unknowns:

1. $\bar{\mathbf{u}}$
2. σ_{xy}
3. E_{in}
4. T

$\boxed{2} \rightarrow \bar{\mathbf{u}} = (u_x(x,y), 0, 0)$
$\boxed{1} + \boxed{2} \rightarrow \frac{\partial}{\partial x} = 0$
therefore $\rightarrow \bar{\mathbf{u}} = (u_x(y), 0, 0)$

$$\begin{cases} 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} \\ 0 = -\frac{\partial p}{\partial y} - \rho g \\ 0 = -\frac{\partial p}{\partial z} \end{cases}$$

"creeping motion"

from the last two equations:
$p(x,y) = p_0 - \rho g y + f(x)$
(modified pressure) then gravity doesn't in
$\frac{\partial p}{\partial x} = f'(x)$
which is the known value $\boxed{3}$
the first equation becomes $\mu \frac{\partial^2 u_x}{\partial y^2} = \frac{\Delta p}{l}$

Boundaries conditions:

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4. T

The equation obtained $\mu\frac{\partial^2 u_x}{\partial y^2} = \frac{\Delta p}{l}$
can be easily solved giving:
$u_x = \left(\frac{1}{\mu}\frac{\Delta p}{l}\right)\frac{y^2}{2} + C_1 y + C_2$
and using 1 and 2:

$$u_x = \left(\frac{1}{2\mu}\frac{\Delta p}{l}\right)\left(y^2 - yh\right) + U\frac{y}{h} \quad (1)$$

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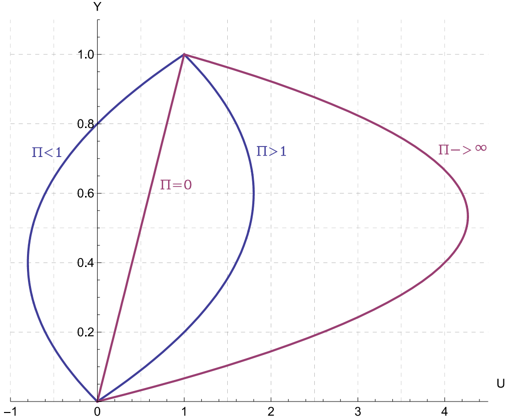
and using 1 and 2:

$$u_x = \left(\frac{1}{2\mu}\frac{\Delta p}{l}\right)\left(y^2 - yh\right) + U\frac{y}{h} \tag{2}$$

it's useful to use a normalized variables $U_x = \frac{u_x}{U}$ and $Y = \frac{y}{h}$ and to put all the constants (with a minus!) in $\boxed{\pi = -\frac{h^2}{2\mu U}\frac{\Delta p}{l}}$ obtaining finally:

$$\boxed{U_x = -\pi(Y^2 - Y) + Y} \tag{3}$$

$$\boxed{U_x = -\pi(Y^2 - Y) + Y} \quad \pi = -\frac{h^2}{2\mu U}\frac{\Delta p}{l}$$



Note that π can change sign due to either pressure or plate velocity. The red coloured graphs represent the degeneration of Coutte flow in the fixed plates pressurized case and in the case without pressure difference

A force acting on a surface (it has not be confused with the total force acting on a CV) can be computed using the formula:

$$\bar{\mathbf{F}} = \int_S \bar{\bar{\sigma}} \cdot \mathbf{d}\bar{\mathbf{S}} = \int_S (-p\delta_{ij} + 2\mu\mathbb{U}_{ij}) \cdot \bar{\mathbf{n}}\,\mathbf{dS} \tag{4}$$

For example,considering a surface of the fluid of area A normal to the y-axis and a shear rate tensor $\mathbb{U}_{ij}$ built using the results obtained above:

$$\bar{\mathbf{F}} = \int_S \begin{pmatrix} -p & \mu\frac{\partial u_x}{\partial y} & 0 \\ \mu\frac{\partial u_x}{\partial y} & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dS = A \begin{pmatrix} \mu\frac{\partial u_x}{\partial y} \\ -p \\ 0 \end{pmatrix} \tag{5}$$

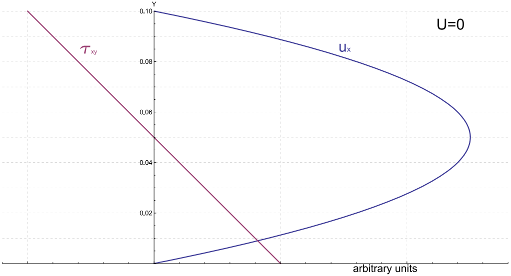
The y-component is the pressure force while the x-component is the well known formula of the shear stress:

$$\boxed{F_{xy} = \tau_{xy} A = A\,\mu\frac{\partial u_x}{\partial y}}$$

$$\tau_{xy} = \mu \frac{\partial u_x}{\partial y}$$

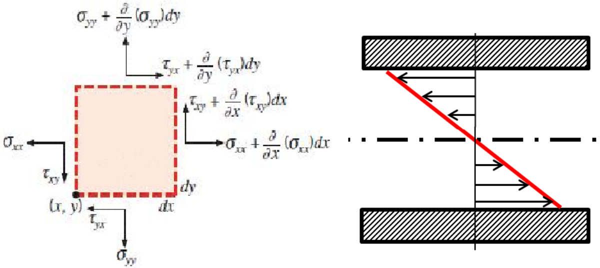
Applying this formula on the u_x expression obtained in (2):

$$\tau_{xy} = \frac{\Delta p}{2l}(2y - h) + \mu U/h \tag{6}$$



For a parabolic velocity profile, the corresponding shear stress is a line (constant if velocity is linear). Furthermore, shear stress changes sign.

Let's consider now the *total* force acting on a CV.
The situation studied is *stationary*: the velocity doesn't change in time



Referring to the sign convention of the stress tensor, the shear stress profile satisfies in every point the equation:

$$\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial p}{\partial x} \tag{7}$$

In order to study the Temperature one more equation is required. Every time a new equation is introduced, it can contain in turn, other unknowns that again, require more equations to close the system.
Let's start writing the equation for the internal energy (the apex ^{m} stay fot unit mass):

$$\rho \frac{dE_i^m}{dt} = \rho \left(\frac{\delta Q^m}{\delta t} + \frac{\delta W^m}{\delta t} \right) = -\nabla \cdot \bar{\mathbf{J}} - p \nabla \cdot \bar{\mathbf{u}} + \Phi \qquad (8)$$

For an incompressible fluid:

$$\boxed{\rho \frac{dE_i}{dt} = -\nabla \cdot \bar{\mathbf{J}} + \Phi} \text{ where } \Phi = 2\mu \mathbb{U}_{ij} \mathbb{U}_{ij} \left[\frac{W}{m^3} \right] \qquad (9)$$

Introducing the well known equations for Internal Energy and Fourier's Law of Heat we finally get:

$$\boxed{\rho c_p \frac{DT}{Dt} = \nabla \cdot (K \nabla T) + \Phi} \qquad (10)$$

DIGRESSION

Before solving the Temperature's equation, the *Power Balance* for the fluid can be checked:

$$\boxed{W_s = W_d}$$

where W_s is the external work made by the pressure forces and $W_d = \int_V \Phi \, dV$ in the dissipation work that increases the internal energy

Considering a CV with dimensions $x \leftrightarrow l$, $y \leftrightarrow h$, $z \leftrightarrow d$ (see slide 8).
In the easiest situation $U = 0$ the (2) becomes

$$u_x = \left(\frac{1}{2\mu}\frac{\Delta p}{l}\right)\left(y^2 - yh\right)$$

$$W_d = \int_{cv} \Phi \, dV = \int_{cv} 2\mu \mathbb{U}_{ij}\mathbb{U}_{ij} \, dV =$$
$$\int_{cv} 2\mu \left(\left(\frac{1}{2}\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{1}{2}\frac{\partial u_x}{\partial y}\right)^2\right) dV = \int_0^h \mu \left(\frac{\partial u_x}{\partial y}\right)^2 dy \, l \, d =$$
$$\frac{\Delta p^2 d}{4\mu l}\int_0^h (2y - h)^2 \, dy = \frac{\Delta p^2 d h^3}{12\mu l}$$

$$W_s = \int_0^h (-\Delta p \, d \, dy)\, u_x(y) = \int_0^h -\Delta p \, h \, d \left(\frac{u_x(y)}{h} dy\right) =$$
$$-\Delta p \, h \, d \, \bar{u}_x = \frac{\Delta p^2 d h^3}{12\mu l}$$

In a stationary condition, the external power is fully converted in internal energy.

END DIGRESSION

The equation (10) can be simplified using the expression of $\bar{\mathbf{u}}$ and assuming a Temperature behaviour similar to the velocity's one, thus: $\frac{\partial T}{\partial x} = \frac{\partial T}{\partial z} = 0$.

In this way we get:

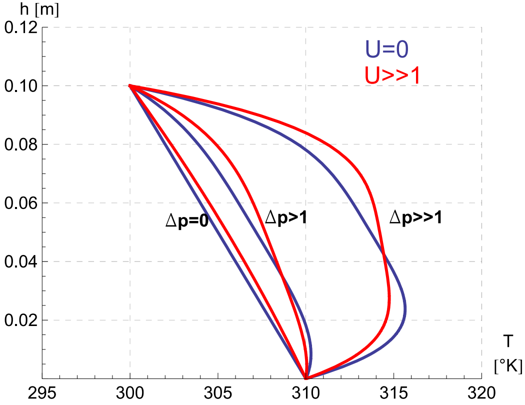
$$0 = \frac{\partial^2 T}{\partial y^2} + \Phi \tag{11}$$

equal to:

$$0 = \frac{\partial^2 T}{\partial y^2} + \frac{1}{\mu K} \left(\frac{\partial u_x}{\partial y} \right)^2 \tag{12}$$

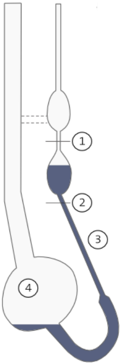
The solution is again straightforward. The initial conditions are the two temperatures of the plates.

```
DSolve[{T''[y] == -u/k * ((G*(2 y - h))/(2*u) + U/h)^2, T[0] == 310, T[h] == 300}, T[y], y]
fun = Table[% /. {k → 0.6 , u → 1*10^-3, G → t, h → 0.1, U → 50}, {t, 0, 100, 50}]
```

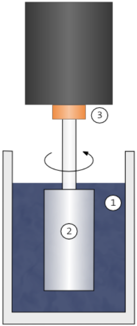


The solution is in general a 4-th order polynomial.When $U = \Delta p = 0$ *the graph of a stationary heat flux through a surface* can be recognized

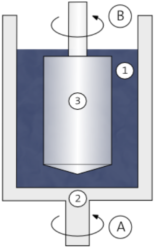
Types of Viscometer



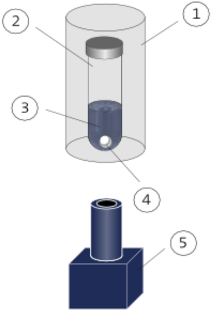
Capillary



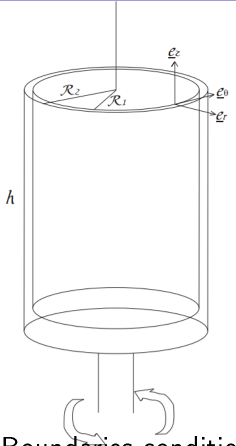
Rotational



Rheometer



EMS



Fully developed, no-slip, laminar Flow between two parallel cylinders (gravity neglected). The fluid is contained in the outer cylinder of radius R_2 that is "open" and is kept at a constant angular velocity Ω by an external engine.
The second cylinder of radius R_1 is hanging inside the first one.
It's connected to a calibrated torsion spring in order to measure the torque M transmitted by the fluid. Cylindrical frame of reference $\rightarrow \bar{\mathbf{u}} = (u_\rho, u_\theta, u_z)$

Boundaries conditions:

1. $\rho = kost \rightarrow \nabla \cdot \bar{\mathbf{u}} = 0$
2. 1-d $\rightarrow u_\rho = u_z = 0$
3. h-high $\rightarrow \frac{\partial}{\partial z} = 0$
4. stationary $\rightarrow \frac{\partial}{\partial t} = 0$

Initial conditions:

1. $u_\theta = \Omega R_2$ if $\rho = R_2$
2. $u_\theta = 0$ if $\rho = R_1$

Unknowns:

1. $\bar{\mathbf{u}}$
2. $M(\eta)$

Condition $\boxed{2}$ implies $\bar{\mathbf{u}} = (0, u_\theta(\rho,\theta), 0)$ which put in $\boxed{1}$ (in polar coordinates: $\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$) set

$$\bar{\mathbf{u}} = (0, u_\theta(\rho), 0$$

In order to write the NSE system is required the expression of the material derivative $\frac{Du}{Dt}$ in cylindrical coordinates. The calculation is quite complicated. The result is:

Boundaries conditions:

1. $\rho = kost \rightarrow \nabla \cdot \bar{\mathbf{u}} = 0$
2. 1-d $\rightarrow$ $u_\rho = u_z = 0$
3. h-high $\rightarrow \frac{\partial}{\partial z} = 0$
4. stationary $\rightarrow \frac{\partial}{\partial t} = 0$

Initial conditions:

1. $u_\theta = \Omega R_2$ if $\rho = R_2$
2. $u_\theta = 0$ if $\rho = R_1$

Unknowns:

1. $\bar{\mathbf{u}}$
2. $M(\eta)$

Condition 2 implies $\bar{\mathbf{u}} = (0, u_\theta(\rho,\theta), 0)$ which put in 1 (in polar coordinates: $\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$) set

$$\bar{\mathbf{u}} = (0, u_\theta(\rho), 0)$$

In order to write the NSE system, the expression of the material derivative $\frac{Du}{Dt}$ in cylindrical coordinates is required. The calculation is quite complicated. The result is:

$$\frac{D\bar{\mathbf{u}}}{Dt} = \begin{pmatrix} \frac{\partial u_\rho}{\partial t} + u_\rho \frac{\partial u_\rho}{\partial \rho} + \frac{u_\theta}{\rho}\frac{\partial u_\rho}{\partial\theta} - \frac{{u_\theta}^2}{\rho} + u_z\frac{\partial u_\rho}{\partial z} \\ \frac{\partial u_\theta}{\partial t} + u_\rho \frac{\partial u_\theta}{\partial \rho} + \frac{u_\theta}{\rho}\frac{\partial u_\theta}{\partial\theta} - \frac{u_\theta u_\rho}{\rho} + u_z\frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial t} + u_\rho \frac{\partial u_z}{\partial \rho} + \frac{u_\theta}{\rho}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z} \end{pmatrix} \tag{13}$$

It's clear to see that in the case considered, all terms of the derivative vanish except one.

This lead to:

Using the cylindrical expression of the derivative, NSE become:

$$\begin{cases} -\frac{{u_\theta}^2}{\rho}\rho = -\frac{\partial p}{\partial \rho} \\ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} - \mu\frac{\partial}{\partial \rho}\left(\frac{1}{r}\frac{\partial}{\partial \rho}(\rho u_\theta)\right) \\ 0 = -\frac{\partial p}{\partial z} \end{cases} \tag{14}$$

from the first and the third equation we get:

$$p(\rho, \theta) = F(\rho) + G(\theta) \tag{15}$$

therefore the second one become:

$$\mu\frac{\partial}{\partial \rho}\left(\frac{1}{r}\frac{\partial}{\partial \rho}(\rho u_\theta)\right) = \frac{\partial G(\theta)}{\partial \theta} \tag{16}$$

The two terms of the equations depend on different variables: they can be equal only if they are constant

$$\rho\mu\frac{\partial}{\partial \rho}\left(\frac{1}{r}\frac{\partial}{\partial \rho}(\rho u_\theta)\right) = \frac{\partial G(\theta)}{\partial \theta} = kost \tag{17}$$

$$\frac{\partial G(\theta)}{\partial \theta} = kost \rightarrow G(\theta) = C_1 + kost\,\theta \tag{18}$$

But, the unique value of *kost* that satisfies the condition $G(2\pi) = G(0)$ is $kost = 0$ there the (17) becomes:

$$\rho\mu \frac{\partial}{\partial \rho}\left(\frac{1}{r}\frac{\partial}{\partial \rho}(\rho u_\theta)\right) = 0 \tag{19}$$

Elementary integrations with the conditions 2 + 1 lead to:

$$\boxed{v_\theta = \frac{R_2^2}{R_2^2 - R_1^2}\frac{\Omega}{\rho}(\rho^2 - R_1^2)} \tag{20}$$

The behaviour is almost-linear in the interval $R1 - R2$ with a slope depending on Ω

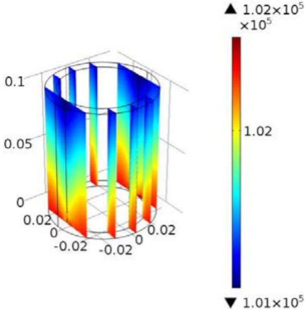


Figure 3. Pressure profile of the fluid at viscometer rotational speed of 55rpm and time 5 secs

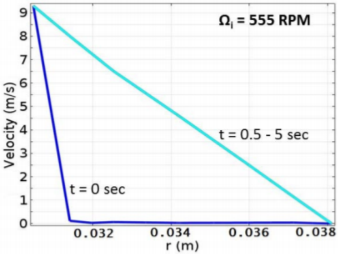


Figure 11. Developing velocity profiles during inner cylinder start up

The final goal is to obtain a formula for the Torque acting on the inner cylinder. It can be done by applying the general formula of the Force acting on a surface of the fluid for the surface S_i of the cylinder.
The rate of shear tensor is built using operators in cylindrical coordinates as above.

$$\bar{\mathbf{F}} = \int_{S_i} \bar{\bar{\sigma}} \cdot \mathbf{d\bar{S}} = \int_{S} \begin{pmatrix} -p & \mu\frac{\partial u_\theta}{\partial y} & 0 \\ \mu\frac{\partial u_\theta}{\partial y} & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dS = \\ = 2\pi R_1 h \begin{pmatrix} -p \\ \mu\frac{\partial u_\theta}{\partial y} \\ 0 \end{pmatrix} \tag{21}$$

The Torque is referred to the the θ-component of the force:

$$F_{\theta\rho} = 4\pi\mu R_1 h\Omega \frac{R_2^2}{R_2^2 - R_1^2} \tag{22}$$

$$\boxed{M = |R_1 \wedge F_{\theta\rho}| = 4\pi\mu h\Omega \frac{R_2^2 R_1^2}{R_2^2 - R_1^2}} \tag{23}$$

The formula lets you measure the dynamic viscosity of a material without references to other viscosity sample but just measuring different physical quantities.

Furthermore, graphing M vs Ω let studying the Newtonian behaviour of the fluid.