# Class Notes on Spacetime

José A. Navarro Ramón

Sunday 19th March, 2023

# Contents

# Chapter 1

# Topological spaces

## 1.1 Introduction

At the coarsest level, spacetime is a set. It just consists of points which are the elements of the set. However, this level is not enough to talk even about the simplest notions that we would like to talk about in Classical Physics, as the notion of continuity of maps.

Why would we want to talk about continuity of maps? Well, in Classical Physics there is the idea of curves in which there are no jumps. Some particle is running somewhere and we do not have the situation that, all of a sudden, there is a jump and the trajectory of the particle continues abruptly in a different place. We do not want that, so we need to require continuity of maps.

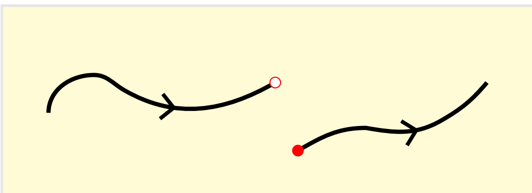

Figure 1.1: A particle is not allowed to jump all of a sudden from one point to another in Classical Physics.

It turns out that a set is not enough structure to be able to talk about curves being continuous or not on that set. You could, of course, imagine all kinds of structure on a set that allows you to talk about continuity. For instance, you could implement a *distance measure* of some kind. But we need to be very minimal and very economic in order to not introduce undue asumptions. So *we are interested in the weakest structure that we can establish on a set* which allows a good definition of continuity on a set. Mathematicians know the weakest such structure, and it is called *a topology*. This is our motivation to study topology in these lectures.

## 1.2 Topology

### 1.2.1 Definition

Let $M$ be a set. A *topology*[1] $\mathcal{O}$ is a subset of the power set of M

$$\mathcal{O} \subseteq \mathcal{P}(M) \tag{1.1}$$

that satisfies three axioms

1. The empty set and $M$ must be always part of the collection

$$\emptyset, M \in \mathcal{O} \tag{1.2}$$

2. For every two elements of the collection, their intersection is always in it

$$\forall\, U, V \in \mathcal{O} \Longrightarrow U \cap V \in \mathcal{O} \tag{1.3}$$

3. Given a finite or infinite number of elements in the collection, their union is always in the collection

$$\forall\, U_\alpha \in \mathcal{O} \Longrightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{O} \tag{1.4}$$

where $\alpha \in A$ is an arbitrary index and $A$ is any set, finite, countable infinite or even uncountable. In particular, the set $A$ may be the whole real line, $\mathbb{R}$.

### 1.2.2 Comments on the defition of topology

- The power set $\mathcal{P}(M)$ of a set $M$ is the set containing all subsets of $M$. So to get a topology of $M$ you start by choosing a certain subset of the power set of $M$, $\mathcal{O} \subseteq \mathcal{P}(M)$.
- The empty set and $M$ must be elements of $\mathcal{P}(M)$.
- The intersection of *any* two sets of $\mathcal{O}$ is also in $\mathcal{O}$. To check for this, you have to check for all $\mathcal{O}$ pairs.
- The last axiom is deceptively similar to the preceding one. But in this case we are not restricted to check for every pair of elements in $\mathcal{O}$, but we can choose any subset of elements of $\mathcal{O}$, whether finite, countable-infinite or even uncountable-infinite elements of the power set. In either case, the union of all these elements must also be an element of $\mathcal{O}$.

### 1.2.3 Examples

**A first try**

Let's have a set $M$ that just consists of three elements, $M \equiv \{\ 1, 2, 3\ \}$. We write down the entire power set of $M$, just for reference. It has $2^3 = 8$ elements

$$\mathcal{P}(M) \equiv \{\ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\ \}$$

[1] $\mathcal{O}$ for *open*.

We now choose an arbitrary subset, $\mathscr{O}$ of this power set

$$\mathscr{O} \equiv \{ \varnothing, \{1\}, \{2\}, \{1,2,3\} \}$$

And we want to find if $\mathscr{O} \subseteq \mathscr{P}(M)$ is a topology of $M$. Our subset choice obey the first two axioms, but it fails on the third:

$$\{1\} \cup \{2\} = \{1,2\} \notin \mathscr{O}$$

**The simple examples of topology**

- The most simple example of topology of a set $M$ is called the *chaotic topology*, and only contains the empty set and the set

  $$\mathscr{O}_{\text{chaotic}} \equiv \{ \varnothing, M \}$$

  It is very easy to prove that this is a topology.

- The other exameple of topology of a set $M$ is the whole power set of $M$. It is called the *discrete topology*

  $$\mathscr{O}_{\text{discrete}} \equiv \mathscr{P}(M)$$

  It is also very easy to prove that this is a topology. The power set contains all subsets of $M$, including $\varnothing$ and $M$. The intersection and union of subsets of the power set is also a subset of $M$.

Pity that these two topologies are utterly useless. It is worth introducing them as they are extreme cases (the topology with the minimum and maximum number of elements).

### 1.2.4 Standard topology

This is a very important example that will reconcile our intuition aboout continuity built from undergraduate analysis courses. The standard topology will be used throughout these lectures.

Contrary to the chaotic and discrete topologies, which can be defined for any set $M$, the standard topology can only be defined in $M = \mathbb{R}^d$, which is the set of all tuples of the form

$$M = \mathbb{R}^d = \underbrace{\mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{\text{d times}} \equiv \left\{ (p_1, \cdots, p_d) \,\middle|\, p_i \in \mathbb{R} \right\}$$

Then, a standard topology is a subset of the power set of $\mathbb{R}^d$

$$\mathscr{O}_{\text{std}} \subseteq \mathscr{P}(\mathbb{R}^d)$$

The standard topology may contain non-countable many elements, so it cannot be defined explicitly.

The definition proceeds in two steps:

**1. Soft ball definition:**

A soft ball is a set $B_r(\mathbf{p})$ where $p \in \mathbb{R}^d$ is a point and $r$ is an element of the positive reals, $r \in \mathbb{R}^+$, is called the soft ball radius. You can think of $\mathbf{p}$ as the center of the ball

$$B_r(\mathbf{p}) = \left\{ (q_1, \cdots, q_d) \,\middle|\, \sum_{i=1}^{d} (q_i - p_i)^2 < r^2 \right\} \tag{1.5}$$

We might argue that we have written down the Euclidian norm. But in this case it is just a formula. To talk about the Euclidian norm we need a vector space structure and a dot product, which is not necessary the case.

**2. Standard topology definition:**

$\mathscr{O}_{\text{std}}$ is a standard topology if for any $U \in \mathscr{O}_{\text{std}}$ and for all $\mathbf{p} \in U$, there exists at least one soft ball $B_r(\mathbf{p})$ which is included in $U$

$$\forall U \in \mathscr{O}_{\text{std}}, \forall \mathbf{p} \in U \implies \exists r \in \mathbb{R}^+ | B_r(\mathbf{p}) \subseteq U \tag{1.6}$$

**Comments**

- A soft ball, $B_r(\mathbf{p})$ in $\mathbb{R}^n$ is the multidimensional equivalent of an open interval in $\mathbb{R}$. This is because the *less than* relational operator in the definition (1.5). Open intervals in $\mathbb{R}$ do not include their endpoints, whereas closed intervals include them. Accordingly, a soft ball consists only of *inner points* and does not include the points in its surface, although we can get as near as we like to it. In figure (1.2) we represent both, a soft ball and a closed ball, in a $\mathbb{R}^2$ plane.

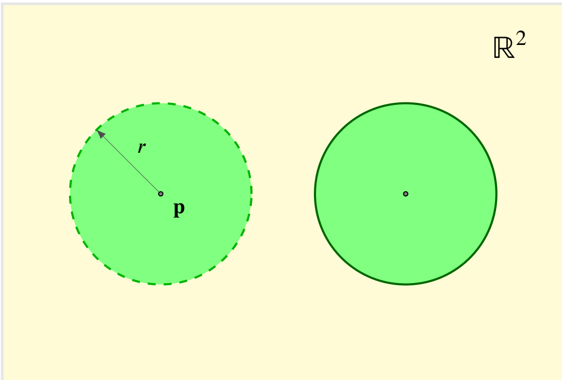


Figure 1.2: To the left there is a soft ball $B_r(\mathbf{p})$, and to the right we find a closed ball.

- It's a simple exercise to prove that this two step definition of the standard topology satisfies the three axioms of a topology. We leave this task to the reader.

- We can think of a standard topology $\theta_{\text{std}}$ in a very simple and graphical way. Let $U$ be any set in the topology, $U \subseteq \theta_{\text{std}}$. It can be proved that this leads to the fact that $U$ has no border, much like a soft ball. For this reason, any set belonging to the standard topology is also called as an *open set*. We will define more properly an open set very soon.

On the left side of picture 1.3 a possible set of a standard topology is shown. We can see its border as a non-continuous line, meaning that all its elements are *inner points*.

According to the definition of standard topology, *for every point* $p$ in any $U \in \theta_{\text{std}}$ we can find at least one soft ball centered in this point that it is included in $U$. On the right side of the same figure an arbitrary set $U$ of this topology and two arbitrary points $p_1$ and $p_2$ are shown. $U$ is an element of $\theta_{\text{std}}$ because we can find a soft ball of small enough radius centered on any point of the set that is completely inside $U$. If the point is extremely near the border, the radius of the soft ball must be also extremely small.

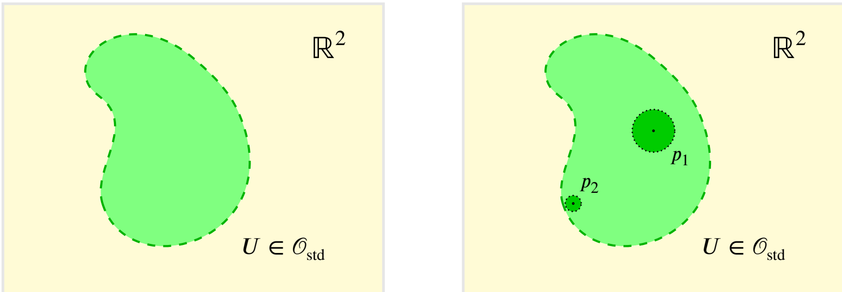


Figure 1.3: On the left side, a possible set of the standard topology is represented. We see that the boundary doesn't belong to the set (roughly speaking). On the right side we suggest that for any point of $U$, we could find a radius small enough such that the soft ball $B_r(\mathbf{p})$ lies entirely in the set $U$.

In contrast, on the left side of figure 1.4 we represent a set $V$ that is not an element of the standard topology. This is because there are some points in its border, marked with a continuous line, that are elements of $V$. To the right of the figure we see that any soft ball —no matter how small— around the points in the border are not completely inside $V$.

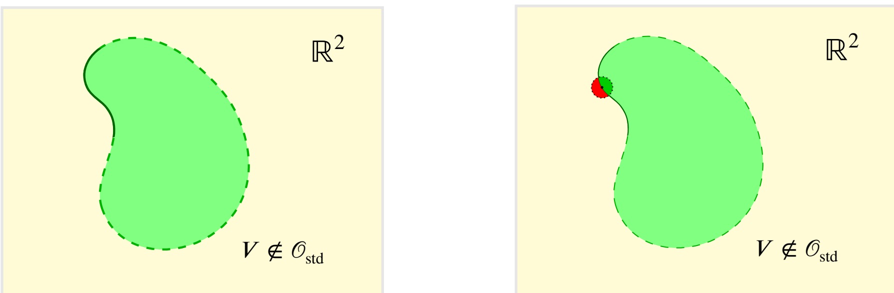


Figure 1.4: On the left side of the picture we outline a set $V$ which does not belong to the standard topology, since some points of its border are elements of $V$ (the ones in the solid line). On the right we show a soft ball centered on one point of this border, revealing that some parts of the soft ball, the ones in red, are not in $V$.

- Although the standard topology is very important, it doesn't mean that it is the only topology. There might be opportunities to equip $M = \mathbb{R}^d$ with a different topology if you have other than the standard purposes in mind. Its name is somewhat deceptive, but there is nothing special about the standard topology, other than you like it because you met it at the beginning of your mathematical studies. But there is no reason for spacetime to carry it. Furthermore, it can only be defined in some sets of the form $\mathbb{R}^d$.

### 1.2.5 Open and closed sets

Later on, we are going to think of $M = \mathbb{R}^d$ as the points of spacetime, and we will ask ourselves what topology should the physical spacetime carry.

Remember that we define topology in order to talk about continuity of maps, so please, do not be deterred if you find that this is a little abstract. It is necessary to make explicit some notions that we may take for granted.

When Einstein wrote down General Relativity, he didn't understand many explicit assumptions, and it took him many years to overcome intuitive notions (some would say obvious notions) that nevertheless didn't apply, making implicit things explicit to really understand the concepts needed for General Relativity.

#### Some terminology

We talk aobut $M$ being a set, meaning a *collection* of *elements*. But for really understand these concepts we need to postulate some axioms, although in this text we are not going that far, and we will be satisfied with the intuitive meaning.

- $M$ is some set that we need to study.
- $\mathcal{O}$ is a topology, which is some set of subsets of $M$. These subsets are also called *open sets*. So a topology $\mathcal{O}$ is a set of open sets. Note that the term *open* is defined only by choosing a certain topology (if we would choose a different topology, the open sets would be different).
- The pair $(M, \mathcal{O})$ is called a *topological space*. It is more than a set. It is a set with more information about it.
- If $U$ is a subset of $M$ that lies in $\mathcal{O}$, $U \in \mathcal{O}$, then we call $U$ an *open set*.
- If $A$ is as subset of $M$, it is called a *closed set* if the complement of $A$ lies on the topology
$$A \subseteq M \text{ is a closed set } \iff M \setminus A \in \mathcal{O}$$
- The *open/closed* terminology is somewhat deceptive: Some would say that *open is the opposite of closed in some way*.

  Well no, not in any way. There are open sets that are closed at the same time (think of the empty set $\varnothing$). There are sets that are open, but not closed (see figure 1.3). There are sets that are closed, but not open (Set on the right of figure 1.2). And there are sets that are not open and not closed (look at the set pictured in figure 1.4).

  But of course, when we choose a certain topology these two terms *open* and *closed* are connected in this way

$$\text{If } U \in \mathcal{O} \iff: \text{ call } U \subseteq M \text{ an } \textit{open set}$$
$$\text{If } M \setminus A \in \mathcal{O} \iff: \text{ call } A \subseteq M \text{ a } \textit{closed set}$$

## 1.3 Maps

Topology yields to the notion of continuous maps.

### 1.3.1 Types of maps

A map $f$ connects every element of a set $M$, called *domain*, to some element of the set $N$, called *target*.

$$f : M \longrightarrow N$$

In figure 1.5 we picture two sets $M = \{ a, b, c, d \}$ and $N = \{ i, j, k, l, m \}$ by the collection of their elements (as points), and consider an arbitrary map $f$ which takes every point in the domain to some point in the target.

If every point in the target set is the image of one element at most, then the map is said to be *injective* or *one to one*. In our example we can see that the target point $i$ is being hit twice, so this map is *non-injective*.

If every point in the target is the image of one or more elements in the domain, the map is called *surjective*. But there is no need that every point is hit, in our example, points $j$ and $l$ are not hit, so the map is *non-surjective*.

A map is called *bijective* if it is injective and surjective.

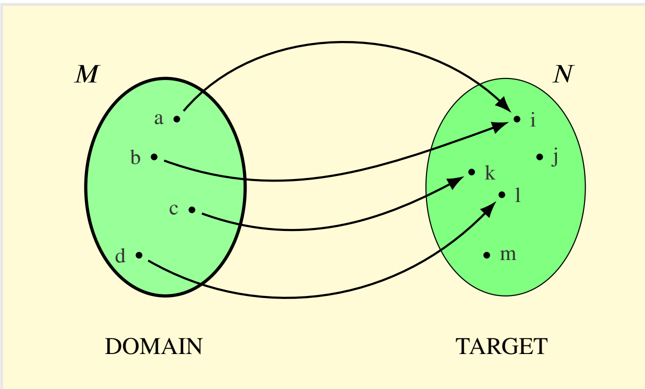


Figure 1.5: An example of map $f : M \to N$.

### 1.3.2 Concept of preimage

Given a certain map $f : M \longrightarrow N$, the *preimage* of some subset of the target $V \subseteq N$ with respect to $f$ is the set of elements of the domain whose image is $V$

$$\text{preim}_f(V) := \{ m \in M \mid f(m) \in V \}$$

Where

$$\begin{array}{rccc} f : & M & \longrightarrow & N \\ & m & \mapsto & f(m) \end{array}$$

In figure 1.6 we can see the preimage with respect to $f$ of the set of the set $V \in N$, $V := \{ i \}$, which is $\text{preim}_f(V) := \{ a, b \}$ in our example.

You may say: The preimage is kind of an inverse $f^{-1}$! Well, careful, because if $f$ is not bijective, then $f^{-1}$ does not exist as a map. Look at figure 1.6. There, $f$ is not

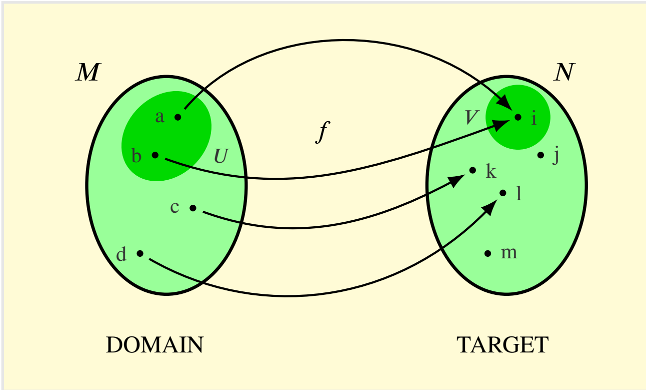


Figure 1.6: The preimage of $V \in N$ with respect of map $f : M \to N$ is the set marked as $U \in M$ in the domain.

bijective, so the element $i$ would have to images, $a$ and $b$, $f^{-1}(i) = \{ a, b \}$, but in a map an element cannot have two images. If this weren't enough, the element $m \in N$ does not have an image $f^{-1}$ in $M$. Only, if $f$ is bijective we can write $\text{preim}_f(V) = f^{-1}(V)$. The point is that although $f^{-1}$ is not a map, there exists a preimage of every set of $N$ in $M$.

### 1.3.3 Continuous maps

A map is only defined by two sets. We need no further structure on a set to define what a map is. But we now ask ourselves if some arbitrary map, like the one in figure 1.5 is continuous. The answer is "it depends".

The answer to the question of whether a map

$$f : M \longrightarrow N$$

is continuous, depends —by definition— on which topologies are chosen on the domain $M$ and on the target space $N$.

So in order to decide continuity of a map, you need to decide a topology for the domain and another topology for the target.

#### Definition

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be two topological spaces. Then, a map $f : M \longrightarrow N$ is continuous —with respect to the topologies $\mathcal{O}_M$ and $\mathcal{O}_N$— if for every set open set $V$ in the target[2] $(V \in \mathcal{O}_N)$ then, the preimage with respect to $f$ of this chosen set is an open set in the domain

$$\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$$

and that's it, no epsilons, no deltas, it's very simple.

If the map was bijective (*one-to-one*), then it is invertible, and the preimage could have been written as the inverse of the map. And if the map was continuous in both directions, then not only the preimages of open sets are open, but also the images of open sets are open because the preimage of the inverse is the image of the map.

[2]We must check for every open set in the target.

In the map of figure 1.5 map, the preimage of element $j$ is the empty set $\varnothing$, and the empty set is always an element of a topology. So if $j$ is an open set in $N$, then it doesn't destroy the notion of continuity of this map (of course, we must also check for every other sets in the topology).

If maps were bijective and continuous in both directions, the topologica structure is preserved and they have a specific name called homeomorphisms and they are structure preserving maps of topology. Mathematicians always clasify structures by identifying in them if certain structure preserving maps in them exist.

I would like to give a mnemonic phrase to remember this definition:

> "A map is continuous if and only if the preimages of all open sets (in the target space) are open sets (in the domain)."

### First example

Let's say that set $M$ and $N$ are the same set —why not?—

$$M = \{ 1, 2 \} \qquad N = \{ 1, 2 \}$$

Now we invent a very simple map $f : M \longrightarrow N$. Just for fun, we map the 1 to the 2 and the 2 to the 1, as represented in figure 1.7

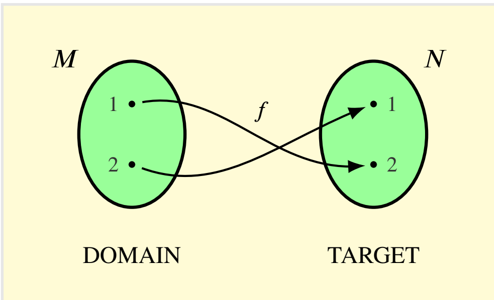


Figure 1.7: A very simple example map $f : M \longrightarrow N$ to discuss map continuity.

We choose some topologies for each map[3]

$$\begin{aligned}&\text{Domain: } \mathcal{O}_M \coloneqq \{ \varnothing, \{ 1 \}, \{ 2 \}, \{ 1, 2 \} \} \\ &\text{Target: } \quad \mathcal{O}_N \coloneqq \{ \varnothing, \{ 1, 2 \} \}\end{aligned}$$

Now we would like to check whether the map $f$ is continuous —not all by itself, but with respect to the chosen topologies—.

We must check that for every open subset of $N$, its preimage with respect to $f$ is an open set in $M$. It's easy in this example:

- The empty set is an open subset in the target $N$, $\varnothing \in \mathcal{O}_N$. Its preimage is the empty set in the domain $M$, which is also an open subset

$$\text{preim}_f(\varnothing) = \varnothing \in \mathcal{O}_M$$

[3]Note that, although the sets are the same, we are free choose a different topology for each one.

- The whole $N = \{ 1, 2 \}$ is the other open set in the target $N$, $\{ 1, 2 \} \in \mathcal{O}_N$, and its preimage is $M = \{ 1, 2 \}$, which is an open set in $M$.

$$\text{preim}_f(\{ 1, 2 \}) = \{ 1, 2 \} \in \mathcal{O}_M$$

Thus we conclude that $f$ is continuous.

### Second example

The map in the first example was bijective, so we can talk about its inverse $f^{-1}$, and we keep the same topologies for the sets

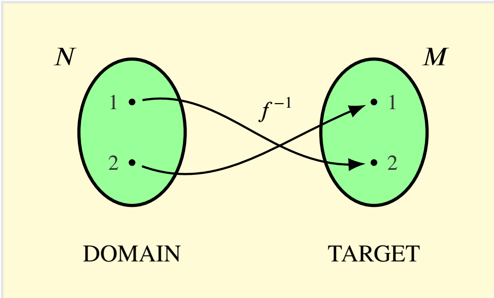


Figure 1.8: The inverse of the map in example one, $f^{-1} : N \longrightarrow B$ to discuss map continuity.

We keep the same topologies as before

$$\begin{aligned} &\text{Domain: } \mathcal{O}_N := \{ \emptyset, \{ 1, 2 \} \} \\ &\text{Target: } \quad \mathcal{O}_M := \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 1, 2 \} \} \end{aligned}$$

As always, we must check if the preimage of all open subsets in the target is an open set in the domain. We start with the open subset $\{ 1 \} \in \mathcal{O}_M$ in the target. Its preimage is the subset set $\{ 2 \}$ in the domain. But this subset is not an open set in the domain, $\{ 2 \} \notin \mathcal{O}_N$. So we conclude that $f^{-1}$ is not continuous[4].

$$\text{preim}_{f^{-1}}(\{ 1 \}) = \{ 2 \} \notin \mathcal{O}_M$$

In particular, if 1 was mapped to 1 and 2 was mapped to 2, then $f$ would be the identity map as $f^{-1}$, but the identity $f$ would be continuous while the identity $f^{-1}$ would not[5].

> "If a map is continuous and invertible, its inverse might not be continuous.
> It all depends on the topologies chosen for the sets."

Although the next thing we are going to talk is not about continuity, let us remember something about open an closed sets. In particular we are going to recall that an open subset may be also closed. Look at subset $\{ 2 \}$ in $M$. This subset is in the topology $\mathcal{O}_M$, so it is an open subset

$$\{ 2 \} \in \mathcal{O}_M \Longrightarrow \{ 2 \} \text{ is an open subset in } M$$

[4]Although $f$ itself was continuous with the same topological spaces, $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$.

[5]As you may guess, it all depends on the topologies associated to the sets. You see how dangerous it is to not explicitly talk about the topologies.

But the entire set without the subset $\{2\}$ is also in the topology, so it is closed at the same time

$$M \setminus \{2\} = \{1\} \in \mathscr{O}_M \Longrightarrow \{2\} \text{ is an closed subset in } M$$

Also, remember that being open or closed has nothing to do with the image of the boundary being there or not. We can define the boundary for every topology, so these notions carry over, but our images of the boundary, as in figure 1.2 not necessarily do.

So we have introduced the notion of continuity for general maps between any two topological spaces. In the beginning we drew this curve in figure 1.1 in a set that makes a jump all of a sudden in order to picture a non-discontinuity. Well the curve could go from the real line into some other set wich may be another real line, but we need to equip these two sets with a topology, which may be the same or different. So in order to start talking about continuity we must establish the two topological spaces first.

In the rest of these lectures, every time we have a topological space involving the real numbers as $\mathbb{R}, \mathbb{R}^2 \cdots \mathbb{R}^d$, and nothing else is said, *we assume they carry the standard topology*: the standard topology on the real line, the standard topology on the real plane, …But in the other spaces we will have, in particular, spacetime will not be $\mathbb{R}^3$, nor $\mathbb{R}^4$, or something like that. Spacetime is a topological space all by itself.

### 1.3.4 Composition of continuous maps

Composition of maps is defined at the set level —we don't need to define a topology on the sets in order to define map composition—. But now we are interested in the composition of continuous maps —so we need to define a topology on the sets in order to talk about continuity—. We want to know whether the composition of two continuous maps is continuous or not.

Let's have a map $f$ which goes from $M$ to $N$ and another map $g$ which goes from $N$ go $P$

$$M \xrightarrow{f} N \xrightarrow{g} P$$

We can define a new map $g \circ f$ ($g$ after $f$) which is a map from $M$ to $P$

$$\begin{aligned} g \circ f \,:\, M &\longrightarrow P \\ m &\mapsto (g \circ f)(m) \end{aligned}$$

Where the composition is defined as $(g \circ f)(m) \coloneqq g(f(m))$

#### Theorem and proof

"If $f$ and $g$ are continuous, then $g \circ f$ is also continuous."

Let $V$ be an open subset in $P$, that is, $V \in \mathscr{O}_P$. The preimage of $V$ by $g \circ f$ is

$$\begin{aligned} \text{preim}_{g \circ f}(V) &\coloneqq \{ m \in M \mid (g \circ f) \in V \} \\ &= \{ m \in M \mid g(f(m)) \in V \} \\ &= \{ m \in M \mid f(m) \in \text{preim}_g(V) \} \\ &= \text{preim}_f(\text{preim}_g(V)) \end{aligned}$$

But $g$ is continuous, so the preimage of $V$ with respect to $g$ is an open subset

$$\text{preim}_g(V) \in \mathscr{O}_N$$

and $f$ is also continuous which tells us that the preimage of an open subset ($\text{preim}_g(v)$) in $N$, with respec to $f$ is also an open subset in $M$

$$\text{preim}_f(\text{preim}_g(V)) \in \mathcal{O}_M$$

So the theorem has been proved, since

$$\text{preim}_{g\circ f}(V) \in \mathcal{O}_M$$

### 1.3.5 Inheriting a topology

In this section we are interested in constructing a new topological space from other or others that we've already got. There are many useful ways to inherit a topology from some given topological space —or spaces—.

For spacetime physics it is important the following: Assume we have the set $M$, on which we are happy enough to already be given a topology $\mathcal{O}_M$. So we already have a topological space. Now we consider a subset $S$ of this topological space $(M, \mathcal{O}_M)$, where $S$ *is any subset of* $M$, not necessarily an open subset

$$\begin{tikzcd}
 & \mathcal{O}_M \arrow[dl, rightsquigarrow] \\
S \subseteq M &
\end{tikzcd}$$

Question: Can we make on $S$ a topology from the topology $\mathcal{O}_M$ on $M$? Yes, we can always invent a topology on $S$, but maybe we want a special topology that in some way inherits from $\mathcal{O}_M$ —we'll see soon enough why this is interesting—.

#### Definition of subset topology

We define a topology $\mathcal{O}|_S \subseteq \mathcal{P}(S)$, called *subset topology* (inherited from the topology on the superset $M$). Every element of $\mathcal{O}|_S$ must be writeable as $U \cap S$, where $U$ is any set in the given topology $\mathcal{O}_M$

$$\mathcal{O}|_S := \{\, U \cap S \mid U \in \mathcal{O}_M \,\}$$

We claim that $\mathcal{O}|_S$ is a topology. To prove that, we need to check the axioms for a topology:

1. *The empty set $\varnothing$ and the whole set$S$ must be part of $\mathcal{O}|_S$:*

   - The empty set is an element of the given topology $\varnothing \in \mathcal{O}_M$, so the intersection of the empty set with $S$ is the empty set, which is also an element of $\mathcal{O}|_S$
   $$\varnothing = \varnothing \cap S \in \mathcal{O}|_S$$
   - The whole set $S$ needs to be part of $\mathcal{O}|_s$. We know that $M$ is an element of $\mathcal{O}_M$, so this is also true
   $$S = M \cap S \in \mathcal{O}|_S$$

2. *The intersection of any two elements of the topology $\mathcal{O}|_S$ must be in the topology:* Let's take two elements of the topology
$$A, B \in \mathcal{O}|_S \Longrightarrow \exists \tilde{A}, \tilde{B} \in \mathcal{O}_M \text{ such that } A = \tilde{A} \cap S, B = \tilde{B} \cap S$$

So

$$A \cap B = (\tilde{A} \cap S) \cap (\tilde{B} \cap S) = (\tilde{A} \cap \tilde{B}) \cap S \in \mathcal{O}|_S$$

because

$$\mathcal{O}_M \text{ is a topology } \Longrightarrow \tilde{A} \cap \tilde{B} \in \mathcal{O}_M$$

3. *Given a finite or infinite number of elements in $\mathcal{O}|_S$, their union is always in $\mathcal{O}|_S$:*

$$\forall U_\alpha \in \mathcal{O}|_S \Longrightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}|_S \tag{1.7}$$

Let's take a number —finite or infinite— of elements of $U_\alpha \in \mathcal{O}|_S$. There must be a number of elements $\tilde{U}_\alpha \in \mathcal{O}_M$ such that $U_\alpha = \tilde{U}_\alpha \cap S$ Now

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha (\tilde{U}_\alpha \cap S) = \left(\bigcup_\alpha \tilde{U}_\alpha\right) \cap S \in \mathcal{O}|_S$$

because $\bigcup_\alpha \tilde{U}_\alpha \in \mathcal{O}_M$.

**Use of this specific way to inherit a topology from a superset**

Very often is very easy for us to judge whether a certain map $f$ from $M$ to $N$, with topologies $\mathcal{O}_M$ and $\mathcal{O}_N$, is continuous

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ \mathcal{O}_M \arrow[u, rightsquigarrow] & \mathcal{O}_N \arrow[u, rightsquigarrow] \end{tikzcd}$$

We'd like to know what happens when we change $M$ with a certain subset of $M$, say ($S \subseteq M$) as the domain, keeping the same target space $N$ and its topology $\mathcal{O}_N$, This new map $f|_S$ is called *restriction of $f$ only to the subset $S$*. Then, if we decide to equip $S$ with an arbitrary topology, we wouldn't know beforehand if it is continuous or not, because it depends on the chosen topology. But if we choose the subset topology $\mathcal{O}|_S$ in the domain, then we are guaranteed that the restriction map $f|_S : S \longrightarrow N$ is continuous.

$$\begin{tikzcd} S \arrow[r, "f|_S"] & N \\ \mathcal{O}|_S \arrow[u, rightsquigarrow] & \mathcal{O}_N \arrow[u, rightsquigarrow] \end{tikzcd}$$

and we'd like to do this very often.

As a simple example, imagine we have a plane (hot plate). Every point of it has a certain temperature. This temperature field is a map from a subset of $\mathbb{R}^2$ into $\mathbb{R}$.

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

We know that this map is continuous. But imagine that some mouse runs across the plate along some trajectory. The mouse is only interested in the temperature on the line (a subset $S$ of the points in the plane). Now we have a restriction of $f$ only to the subset $S$, $f|_S$.

$$f|_S : S \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

If we keep the topology on the target, then the subset topology $\mathcal{O}|_S$ guarantees that the new map is also continuous.