Convergence of collocation methods for delay differential equations

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Outline

- ▶ DDE-Biftool approach to distributed delays and renewal equations
- ▶ Convergence analysis problems for DDEs
- ▶ Convergence of discretization & Newton iteration

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Distributed delays

Linear DDEs: Representation Theorem ensures r.h.s. has form

$$x'(t) = \sum A_j x(t - \tau_j) + \int_0^{\tau_{\max}} G(s) x(t - s) \, ds$$

Nonlinear DDEs:

$$x'(t) = f\left(x(t - \tau_j), \int_{s_1}^{s_2} g(s, x(t - s), p) \, ds, \dots \right) \quad \textbf{?}$$

No interface for general nonlinear functional of $x_t = x(t + (\cdot))$

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DDE-Biftool

uses formulation as differential-algebraic problem:

$$Mx'(t) = f(x(t - \tau_m) \ldots, p)$$

$$0 = \int_0^{\tau_d} g_d(s, x_\ell(t - s), p_i) ds - x_k(t)$$

$$\ldots$$

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$$\ldots$$

nonsquare,
can be singular

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index provided by user

state or parameter (index provided by user)

function provided by user

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DDE-Biftool

uses formulation as differential-algebraic problem:

$$Mx'(t) = f(x(t - \tau_m) \ldots, p)$$

distributed delay, can be used with delay

$$0 = \int_0^{\tau_d} g_d(s, x_\ell(t-s), p_i)ds - x_k(t)$$

nonsquare, can be singular

$\ldots$

state or parameter (index provided by user)

function provided by user

index provided by user

- ▶ permits multiple nested integrals as ℓ and k can overlap,
- ▶ x_k can be multidimensional,
- ▶ several distributed delays possible
- ▶ approximated by N discrete delays $\tau_j = s_j \tau_d$

$$\int \ldots \approx \sum_{j=1}^{N} w_j \tau_d \, g(s_j \tau_d, x_\ell(t - s_j \tau_d), p_i)$$

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DDE-Biftool example renewal equation (RE)

Breda *et al.* 2016

$$x(t) = \frac{\gamma}{2} \int_{\tau_2}^{\tau_1 + \tau_2} x(s)(1 - x(s)) \mathrm{d}s$$

implemented as

$$0 = x(t) - \frac{\gamma}{2} y(t - \tau_2)$$

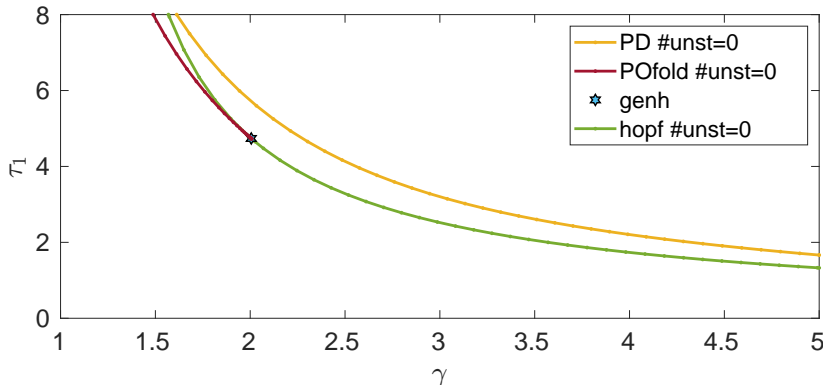$$0 = \int_0^{\tau_1} x(s)(1 - x(s)) \mathrm{d}s - y(t)$$

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DDE-Biftool example renewal equation (RE)

Breda *et al.* 2016

$$0 = x(t) - \frac{\gamma}{2} y(t - \tau_2)$$

$$0 = \int_0^{\tau_1} x(s)(1 - x(s)) \, ds - y(t)$$

$$p = (\gamma, \tau_1, \tau_2), \ u = (x, y)$$

Complex example: size structured Daphnia population model

Diekmann *et al.* 2010, Andò 2020

resource:

$$\dot{r}(t) = f_0(r(t)) - f_c(r(t))p_{\text{eff}}(a_{\max}, t),$$

maturation threshold:
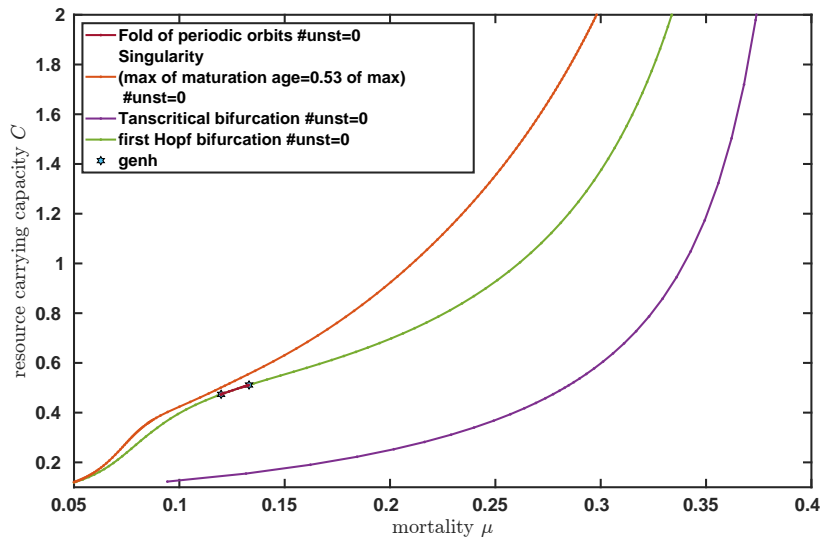
$$0 = f_{\text{thr}}(a_m(t), s(a_m(t), t)) - S_m,$$

birth rate:

$$b(t) = p_{\text{eff}}(a_{\max}, t) - p_{\text{eff}}(a_m(t), t),$$

cohort size:

$$s(a, t) = g_0(a) + \int_0^a g_r(\alpha, f_r(r(t-\alpha))\mathrm{d}\alpha,$$

effective population:

$$p_{\text{eff}}(a, t) = \int_0^a h(\alpha, s(\alpha, t), r(t))b(t-\alpha)\mathrm{d}\alpha.$$
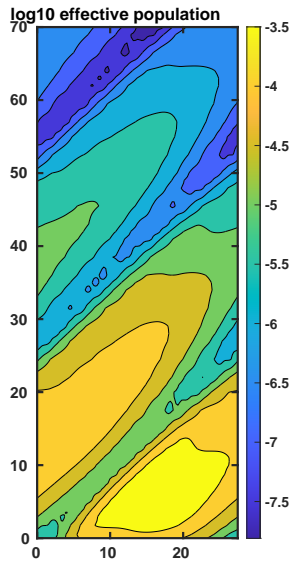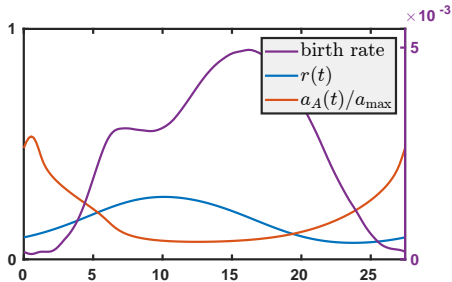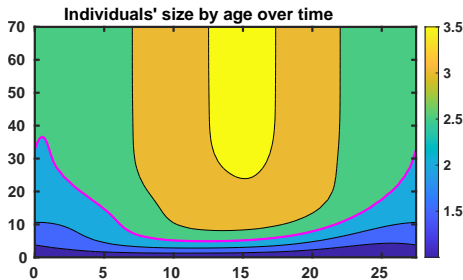
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Complex example: size structured Daphnia population model

Diekmann *et al.* 2010, Andò 2020

Size structured Daphnia population model: shock
Diekmann *et al.* 2010, Andò 2020

DDE-Biftool distributed delays conclusion

- more complex, flexible mechanism for nested integrals $\Rightarrow$ daphnia demo

- uses interface for state-dependent delays to avoid introducing N coupled parameters, $\tau(j, x, p) = s_j \tau_d$,

- discretized renewal equations $\sim$ neutral equations:

$$x(t) = \sum_{j=1}^{N} w_j \tau_d g(s_j \tau_d, x(t - s_j \tau_d), p)$$

$\Rightarrow$essential spectral radius of time-1 map > 1
$\Rightarrow$high-frequency instability
$\Rightarrow$ignore high-frequency eigenvalues of equilibria
$\Rightarrow$ignore Floquet multipliers with highly oscillatory eigenfunctions.

- Renewal equations can be converted to equivalent DDEs

- vectorized g mandatory

Convergence of numerical discretization

DDE-Biftool:

$$\dot{x}(t) = f(x(t - \tau_m), p)$$

time rescaling $\Rightarrow$ $x'(t_k) = Tf(x(t_k - \tau_m/T), p)$ at $L \times n_{\deg}$ times t_k

+continuity & periodicity for piecewise continuous polynomial x with L pieces, degree $n_{\deg}$.

Convergence proof for constant delay:

Engelborghs & Doedel'02: stability for linear DDEs

thought this implies convergence, but

$$F : (x, T) \mapsto Tf(x((\cdot) - \tau_m/T))$$

is not continuously differentiable w.r.t. unknown period T

(term $x'((\cdot) - \tau_m/T)\tau_m/T$ shows up)   (solved by Andò 2020)

$F : C^k \to C^\ell$ is only $C^{k-\ell}$ if $k \geq \ell$

DDEs with state-dependent delays

$$F(x)(t) = f(x_t) \quad x_t(s) := x(t+s), \quad f: C \to \mathbb{R}^n \text{ functional}$$

is cont. diff. only if delays constant.

$$F(x)(t) = x(t+x(t)) \quad \Rightarrow \quad [\partial F(x)y](t) = y(t+x(t)) + x'(t+x(t))y(t)$$

Instead: **mild differentiability** concept (Hartung *et al.*'06)

$$[\partial^k F(x)(y)^k] \quad \text{depends on } x, x', \ldots, x^{(k)}, y', \ldots, y^{(k-1)},$$

(continuously), but **not** $y^{(k)}$.

Result: $(0 = \Phi(x^*), \ 0 = \Phi_L(x_L))$

- ▶ $\|x_L - x^*\|_{0,1} \sim L^{-n_{\deg}}$ if F is $\geq n_{\deg}$ times mild. diff. & $\partial \Phi(x^*)$ is invertible

- ▶ Newton iteration convergence limited by $\|x_L - x^*\|_{0,1}$,

- ⇒ better convergence for higher-accuracy solutions

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Issues:

- ‼ Φ_L only cont. diff. if x_L is cont. diff., but x_L' discontinuous

- ⇒ Jacobian $\partial\Phi_L(\cdot)$ is discontinuous on solution space, violates standard assumptions for convergence of discretization and Newton iteration

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DDEs with state-dependent delays — example error plot

$x'(t) = -x(t - \tau - x(t))$ $+r_{\text{art}}(t)$
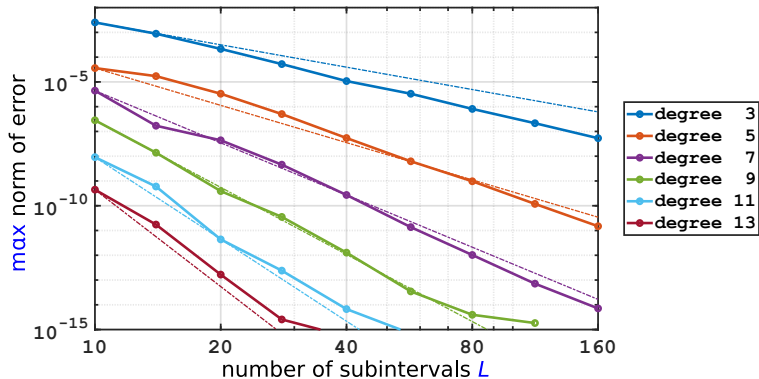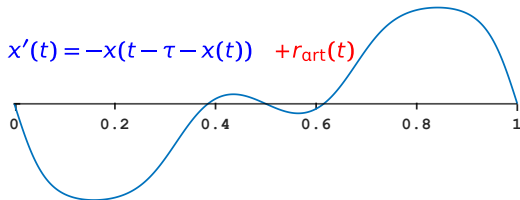
Conclusion

- ▶ bifurcation analysis for DDEs with distributed delays and Renewal Equations feasible

- ▶ linear stability analysis for REs suffers instabilities

- ▶ expectation management for speed

- ▶ convergence proof of numerical method surprisingly recent for constant delays (Andò'20), current preprint for state-dependent delays

- ▶ difficulty: lack of continuous differentiability of r.h.s.

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