

# **New developments for bifurcation analysis of delay equations**

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joint work with

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# Outline

- ▶ DDE-Biftool approach to distributed delays and renewal equations
- ▶ Convergence analysis problems for DDEs
- ▶ Convergence of discretization & Newton iteration

## Distributed delays

Linear DDEs: Representation Theorem ensures r.h.s. has form

$$x'(t) = \sum A_j x(t - \tau_j) + \int_0^{\tau_{\max}} G(s)x(t-s)ds$$

Nonlinear DDEs:

$$x'(t) = f\left(x(t - \tau_j), \int_{s_1}^{s_2} g(s, x(t-s), p)ds, \dots\right) \quad ?$$

No interface for general nonlinear functional of  $x_t = x(t + (\cdot))$

## DDE-Biftool

bifurcation analysis for DDEs,

$$Mx'(t) = f(x(t), x(t - \tau_1), \dots, x(t - \tau_m), p),$$

- ▶ originally developed by Engelborghs, Roose, Luzyanina, Samaey (1999, KU Leuven)
- ▶ equilibria: tracking, stability, bifurcation tracking (KUL)  
periodic orbits: tracking, stability (KUL)  
connecting orbits (KUL)
- ▶ periodic orbits local bifurcation tracking (Orosz⇒JS)
- ▶ linear stability pseudospectral methods (Breda⇒JS)
- ▶ equilibrium normal form analysis (Wage, Bosschaert, Kuznetsov)
- ▶ singular  $M$ , neutral DDEs (Szalai, Barton, Terrien, Hessel, Javaloyes, Gurevich. . . )
- ▶ distributed delays (Humphries⇒JS)

## DDE-Biftool

uses formulation as differential-algebraic problem:

$$Mx'(t) = f(x(t - \tau_m), \dots, p)$$

$$0 = \int_0^{\tau_d} g_d(s, x_\ell(t-s), p_i) ds - x_k(t)$$

...

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distributed delay, can be used with delay

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- ▶ permits multiple nested integrals as  $\ell$  and  $k$  can overlap,
- ▶  $x_k$  can be multidimensional,
- ▶ several distributed delays possible
- ▶ approximated by  $N$  discrete delays  $\tau_j = s_j \tau_d$

$$\int \dots \approx \sum_{j=1}^N w_j \tau_d g(s_j \tau_d, x_\ell(t - s_j \tau_d), p_i)$$

## DDE-Biftool example renewal equation (RE)

Breda *et al.* 2016

$$x(t) = \frac{\gamma}{2} \int_{\tau_2}^{\tau_1 + \tau_2} x(s)(1 - x(s))ds$$

implemented as

$$0 = x(t) - \frac{\gamma}{2} y(t - \tau_2)$$

$$0 = \int_0^{\tau_1} x(s)(1 - x(s))ds - y(t)$$

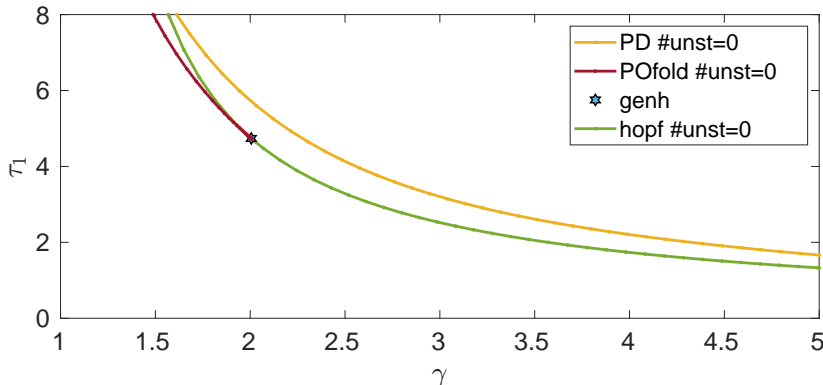
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$$p = (\gamma, \tau_1, \tau_2), \quad u = (x, y)$$



# Complex example: size structured *Daphnia* population model

Diekmann *et al.* 2010, Andò 2020

resource:  $\dot{r}(t) = f_0(r(t)) - f_c(r(t))p_{\text{eff}}(a_{\text{max}}, t),$

maturation threshold:  $0 = f_{\text{thr}}(a_m(t), s(a_m(t), t)) - S_m,$

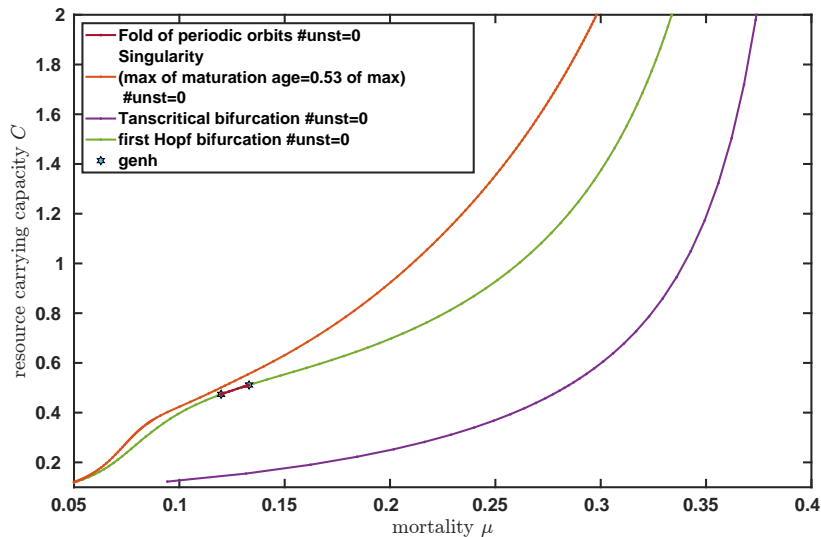
birth rate:  $b(t) = p_{\text{eff}}(a_{\text{max}}, t) - p_{\text{eff}}(a_m(t), t),$

cohort size:  $s(a, t) = g_0(a) + \int_0^a g_r(\alpha, f_r(r(t - \alpha)))d\alpha,$

effective population:  $p_{\text{eff}}(a, t) = \int_0^a h(\alpha, s(\alpha, t), r(t))b(t - \alpha)d\alpha.$

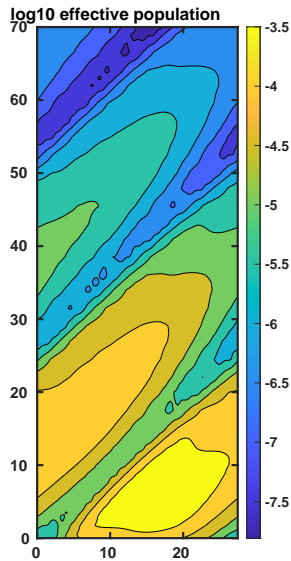
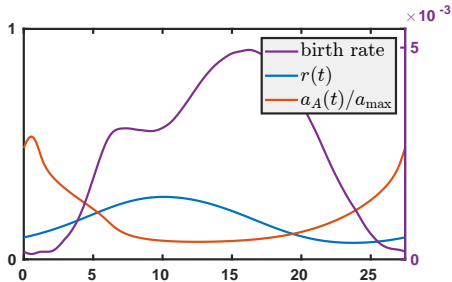
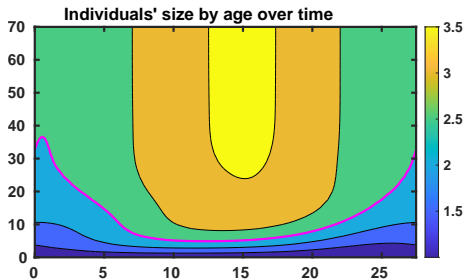
# Complex example: size structured *Daphnia* population model

Diekmann *et al.* 2010, Andò 2020



# Size structured *Daphnia* population model: shock

Diekmann *et al.* 2010, Andò 2020



## DDE-Biftool distributed delays conclusion

- ▶ uses interface for state-dependent delays to avoid introducing  $N$  coupled parameters,  $\tau(j, x, p) = s_j \tau_d$ ,
- ▶ discretized renewal equations  $\sim$  neutral equations:

$$x(t) = \sum_{j=1}^N w_j \tau_d g(s_j \tau_d, x(t - s_j \tau_d), p)$$

⇒ essential spectral radius of time-1 map  $> 1$

⇒ high-frequency instability

⇒ ignore high-frequency eigenvalues of equilibria

⇒ ignore Floquet multipliers with highly oscillatory eigenfunctions.

- ▶ Renewal equations can be converted to equivalent DDEs
- ▶ vectorized  $g$  mandatory

# Convergence of numerical discretization

DDE-Biftool:

$$\dot{x}(t) = f(x(t - \tau_m), p)$$

time rescaling  $\Rightarrow x'(t_k) = Tf(x(t_k - \tau_m/T), p)$  at  $L \times n_{\text{deg}}$  times  $t_k$

+continuity & periodicity for piecewise continuous polynomial  $x$  with  $L$  pieces, degree  $n_{\text{deg}}$ .

## Convergence proof for constant delay:

Engelborghs & Doedel'02: stability for linear DDEs

thought this implies convergence, but

$$F : (x, T, p) \mapsto Tf(x((\cdot) - \tau_m/T), p)$$

is not continuously differentiable w.r.t. unknown period  $T$

(term  $x'((\cdot) - \tau_m/T)\tau_m/T$  shows up) (solved by Andò 2020)

$F : C^k \rightarrow C^\ell$  is only  $C^{k-\ell}$  if  $k \geq \ell$



## DDEs with state-dependent delays

$$F(x)(t) = f(x_t) \quad x_t(s) := x(t+s), \quad f: C \rightarrow \mathbb{R}^n \text{ functional}$$

is cont. diff. only if delays constant.

$$F(x)(t) = x(t+x(t)) \Rightarrow [\partial F(x)y](t) = y(t+x(t)) + x'(t+x(t))y(t)$$

Instead: **mild differentiability** concept (Hartung et al.'06)

$$[\partial^k F(x)(y)^k] \text{ depends on } x, x', \dots, x^{(k)}, y, y', \dots, y^{(k-1)},$$

(continuously), but **not**  $y^{(k)}$ .

**Result:** ( $0 = \Phi(x^*)$ ,  $0 = \Phi_L(x_L)$ )

- ▶  $\|x_L - x^*\|_{0,1} \sim L^{-n_{\text{deg}}}$  if  $F$  is  $\geq n_{\text{deg}}$  times mild. diff. &  $\partial\Phi(x^*)$  is invertible
- ▶ Newton iteration convergence limited by  $\|x_L - x^*\|_{0,1}$ ,  
 $\Rightarrow$  better convergence for higher-accuracy solutions

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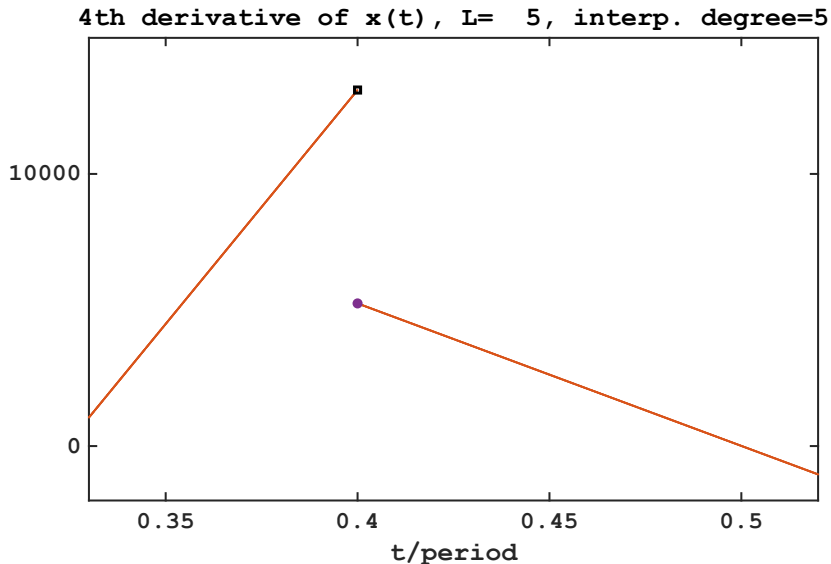
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### Issues:

- !!  $\Phi_L$  only cont. diff. if  $x_L$  is cont. diff., but  $x'_L$  discontinuous  
⇒ Jacobian  $\partial\Phi_L(\cdot)$  is discontinuous on solution space,  
violates standard assumptions for convergence of  
discretization and Newton iteration

## DDEs with state-dependent delays — issues

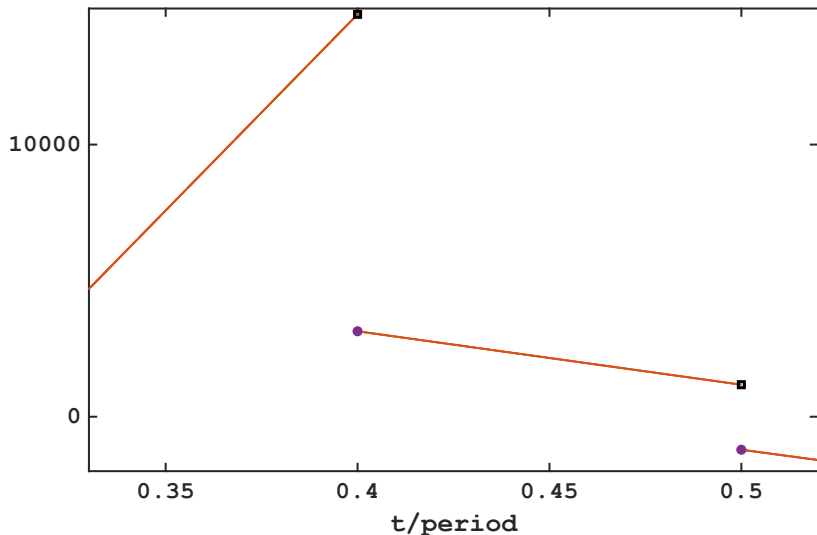
All derivatives discontinuous, but converge to true solution:



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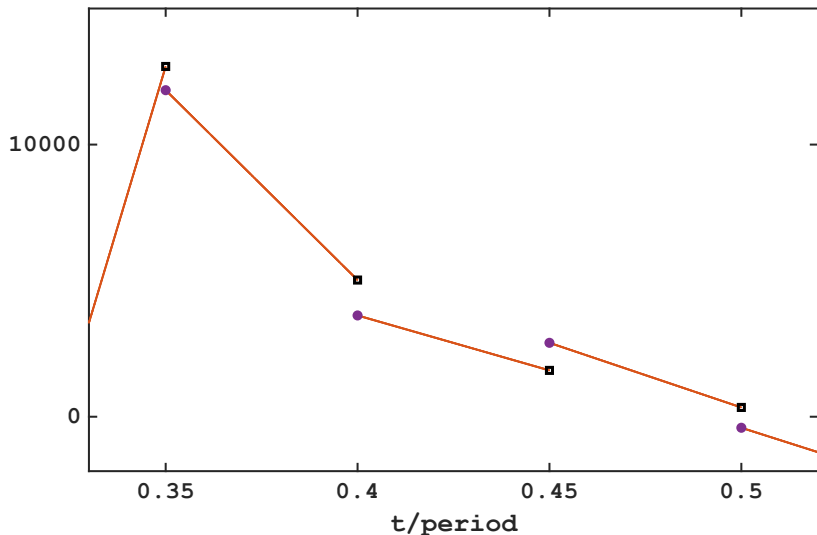
4th derivative of  $x(t)$ ,  $L=10$ , interp. degree=5



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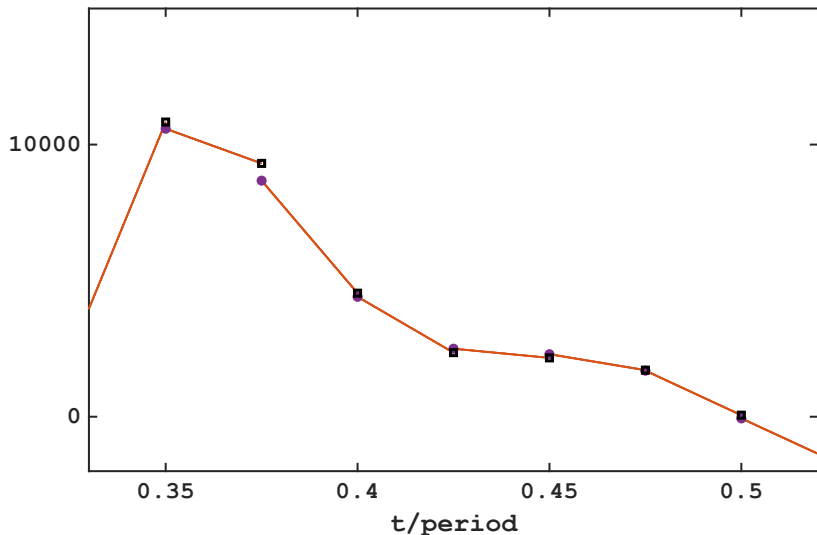
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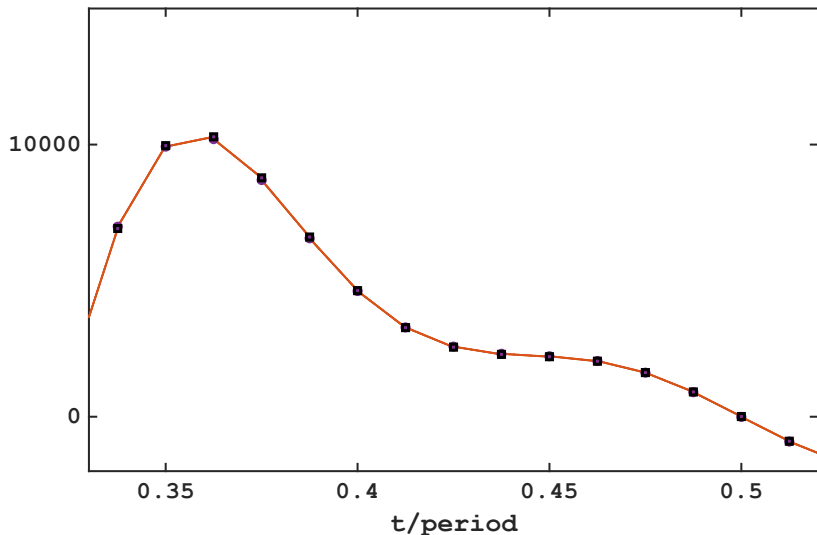
4th derivative of  $x(t)$ ,  $L=40$ , interp. degree=5



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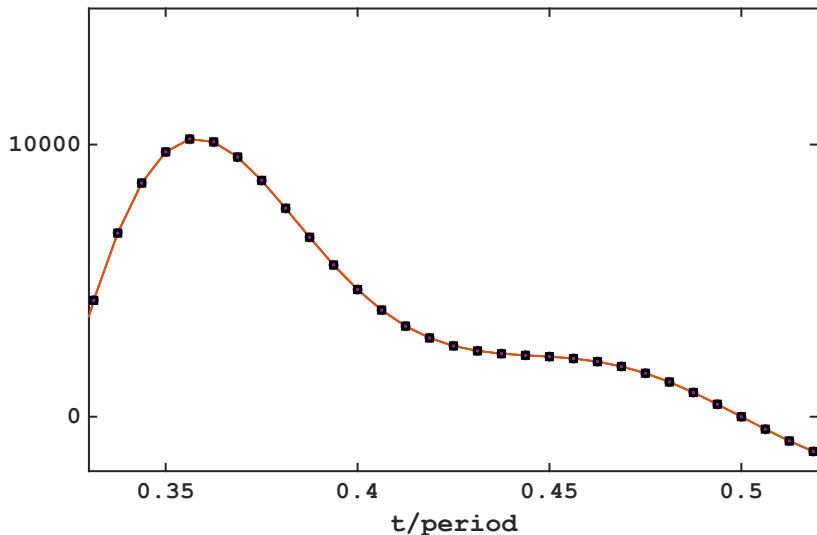
4th derivative of  $x(t)$ ,  $L=80$ , interp. degree=5



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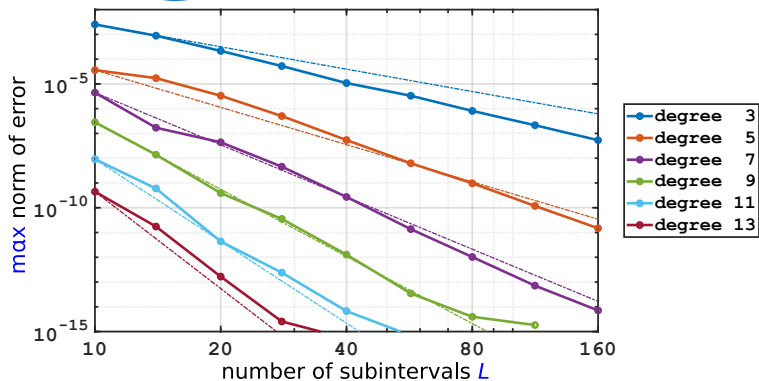
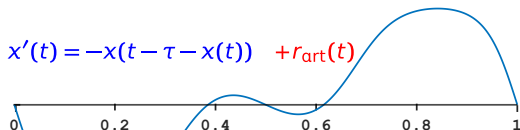
All derivatives discontinuous, but converge to true solution:

4th derivative of  $x(t)$ ,  $L=160$ , interp. degree=5





## DDEs with state-dependent delays — example error plot



## Conclusion

[sourceforge.net/p/ddebiftool/git/ci/master/tree/](https://sourceforge.net/p/ddebiftool/git/ci/master/tree/)

- ▶ bifurcation analysis for DDEs with distributed delays and Renewal Equations feasible
- ▶ linear stability analysis for REs suffers instabilities
- ▶ expectation management for speed
- ▶ convergence proof of numerical method surprisingly recent for constant delays (Andò'20), current preprint for state-dependent delays
- ▶ difficulty: lack of continuous differentiability of r.h.s.