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SYMMETRIC EVENT COLLISIONS IN DYNAMICAL SYSTEMS WITH DELAYED SWITCHES

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Abstract.
We study dynamical systems that switch between two different vector fields depending on a discrete variable. When the delay is sufficiently small one expects that switching between the two vector fields occurs always at certain submanifolds of the state space. When the delay reaches a problem-dependent critical value so-called event collisions occur. We show that at these event collisions the switching manifolds can increase their dimension, giving rise to higher-dimensional dynamics near the periodic orbit than expected. In many practical applications in control engineering the dynamical system has additional symmetry, which adds difficulty in the analysis because event collisions generically occur at several points along the periodic orbit simultaneously.

1. **Introduction.** Dynamical systems that switch between different vector fields depending on a discrete variable (so-called hybrid dynamical systems) are common in applications in control engineering [7, 9]. The switching is typically a control action that aims to stabilize or steer the system. For example, let us consider the problem of stabilizing an unstable equilibrium by feedback control in the presence of a delay in the control loop:

$$\dot{x} = Ax - bg(x(t-\tau)) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix with unstable eigenvalues and $b \in \mathbb{R}^n$ is a suitably chosen control direction. Whenever the delay τ is larger than a problem-dependent critical value τ_c it becomes impossible to stabilize the origin linearly with a linear (or smooth) feedback law $g : \mathbb{R}^n \mapsto \mathbb{R}$, say, $g(x) = k^T x$ (see [14] for a discussion of classical examples such as the inverted pendulum). However, if g is implemented as a switch $g(x) = \operatorname{sign} h(x)$ then one can guarantee the existence of a stable periodic

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orbit for arbitrarily large τ by choosing an appropriate switching law h (at least if A is of saddle type, see [13]).

The motivation behind this paper is the observation that hybrid dynamical systems arising in practical applications often show a surprisingly intricate dynamical behavior if their switch is subject to a delay [3, 4, 8]. The observed behavior often does not match the simple classification of possible codimension-one bifurcations as given in [13]. The reason behind this phenomenon is that practical systems often have special symmetries. For example, piecewise affine systems as arising in control engineering have a full reflection symmetry. The presence of this symmetry causes a violation of the genericity assumptions made in [13]. Most prominently, periodic orbits of systems with delayed switching typically have ‘corners’. A classical event collision corresponds to the case when, varying a parameter, one of the corners of the orbits crosses a switching boundary [13]. In the examples of [3, 4, 8] the symmetry enforces that one of the other corners of the periodic orbit crosses a switching boundary *simultaneously* (at least for symmetric periodic orbits), which violates the assumptions of [13]. This paper studys the simplest but most common case of symmetric hybrid systems, namely systems with full reflection symmetry. This symmetry is present in all examples studied in [3, 4, 8].

Let us consider hybrid dynamical systems of the form

$$\dot{x}(t) = f(x(t), \operatorname{sign} h(x(t-\tau))) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \times \{-1, 1\} \mapsto \mathbb{R}^n$ are smooth and $h : \mathbb{R}^n \mapsto \mathbb{R}$ has a nonzero gradient $h'(x) = \partial_x h(x)$. We use the convention $\operatorname{sign} 0 = 1$.

In control problems, typically, x is the state variable, and $\operatorname{sign} h$ is a discrete control input governed by a switching law. In an open and dense set of times t the state variable x in system (2) follows one of the two flows φ_+ or φ_- where $\varphi_\pm$ is generated by the ordinary differential equation $\dot{x} = f(x, \pm 1)$. That is, $x(t+\delta) = \varphi_+^\delta x(t)$ if $h([t-\tau, t-\tau+\delta]) \geq 0$, and $x(t+\delta) = \varphi_-^\delta x(t)$ if $h([t-\tau, t-\tau+\delta]) < 0$.

A special feature of system (2) is that the switch introduced by $\operatorname{sign} h(x)$ is subject to a delay. That is, the evolution of x depends on the value of x from time τ ago. Systems with delayed switches often admit non-stationary, periodic or more complicated, dynamics [2, 6, 8]. The phase space of system (2) is infinite-dimensional because it is necessary to keep track of the history of x in $[t-\tau, t]$ to determine the forward evolution of x. More precisely, an appropriate initial value for (2) is a history segment $\xi \in C([-\Theta, 0]; \mathbb{R}^n)$ where $\Theta \geq \tau$ (denoting by C the space of continuous functions). Note that the forward evolution $E^t(\xi) : C([-\Theta, 0]; \mathbb{R}^n) \mapsto C([-\Theta, 0]; \mathbb{R}^n)$ defined by (2), mapping the history initial segment ξ to the history segment at time t, is always well defined (using the variation-of-constants formulation of (2)). That is, no *sliding* (a regime where the trajectory follows neither φ_+ nor φ_- but a convex combination of φ_+ and φ_- for a time interval) can occur for (2). This is in contrast to piecewise smooth systems of type (2) with $\tau = 0$. In general $E^t(\xi)$ is not continuous with respect to ξ. However, E^t is continuous for all times t in all points of a periodic orbit $L = (x(\cdot))$ of (2) (say, $x(t) = x(t+p)$ for some $p > \Theta$ and all $t \in \mathbb{R}$) if L has only finitely many intersection points with the switching manifold $\{h = 0\}$, that is, $h(x(t_i)) = 0$ only for finitely many $t_i \in [0, p)$. See [13] for the precise definition of the forward evolution E^t and its continuity properties. What is more, the long time behavior near L can be described locally by a finite-dimensional smooth local return (*Poincaré*) map M if L satisfies the following two genericity conditions [13].

Condition 1. (Generic periodic orbits) The periodic orbit $L = (x(\cdot))$ with period p satisfies for all times $t \in \mathbb{R}$:

1. *(Absence of event collision)* if $h(x(t)) = 0$ then $h(x(t-\tau)) \neq 0$,
2. *(Transversality)* if $h(x(t)) = 0$ then $h'(x(t))\dot{x}(t) \neq 0$.

Condition 1.1 automatically guarantees that $x(\cdot)$ is continuously differentiable in t if $h(x(t)) = 0$, making the time derivative of x in Condition 1.2 well defined. If the orbit L is also *slowly oscillating*, that is, if the time differences between subsequent roots of $h(x(\cdot))$ are all larger than the delay τ, then the local return map M is $(n-1)$-dimensional. In this case the map M can be obtained by recording the first return map (Poincaré map) to a local cross-section (Poincaré section) in $\mathbb{R}^n$, transversal to the graph of $x(\cdot)$ in a point $x(t)$ where $h(x([t-\tau, t])) \neq 0$. A reduction of the description of the dynamics of (2) near L to the smooth finite-dimensional map M links the bifurcation theory of slowly oscillating periodic orbits satisfying Condition 1 to the classical bifurcation theory of smooth finite-dimensional maps [10]. Reference [13] proves that this reduction to (higher-dimensional) smooth maps works also for periodic orbits that are not slowly oscillating as long as Condition 1 is satisfied. Furthermore, [13] classifies what can happen generically near slowly oscillating periodic orbits that violate one of the conditions 1.1 or 1.2. Reference [13] shows that the local return maps are piecewise smooth $(n-1)$-dimensional maps assuming certain secondary genericity conditions. This links the theory of codimension-one discontinuity-induced bifurcations of slowly oscillating periodic orbits to the theory of finite-dimensional piecewise smooth maps [1, 5, 11, 12].

The motivation behind this paper is the observation that many systems arising in applications have special symmetry properties that obstruct the application of the generic theory outlined in [13]. The secondary genericity condition in the study of generic event collisions in [13] is the absence of simultaneous collisions. This means, if $h(x_*(t_*)) = h(x_*(t_* - \tau)) = 0$ for the colliding periodic orbit $L_* = (x_*(\cdot))$ and a time $t_* \in [0, p)$ then $h(x_*(t_i - \tau)) \neq 0$ for all other times $t_i \in [0, p)$ with $h(x_*(t_i)) = 0$.

The symmetry of the periodic orbit often implies that a collision (that is, the violation of Condition 1.1) for one crossing time t_* leads automatically to a simultaneous collision for another crossing time t_i, which violates the secondary conditions assumed in [13]. This has been observed in the example system studied extensively in [3, 4] as well as in many of the examples discussed in [8]. The major source of examples of systems of the form (2) is control engineering where often f and the switching law are affine, that is,

$$\begin{aligned} f(x, \pm 1) &= Ax \pm b \\ h(x) &= h^T x \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$, and $b, h \in \mathbb{R}^n$. The form (3) implies that system (2) has the $\mathbb{Z}_2$ symmetry of full reflection at the origin $x \mapsto -x$, which occurs if the right-hand-side of (2) satisfies

$$\begin{aligned} f(x, \pm 1) &= -f(-x, \mp 1) \\ h(x) &= -h(-x). \end{aligned} \tag{4}$$

The $\mathbb{Z}_2$ reflection symmetry (4) makes symmetric periodic orbits $L = (x(\cdot))$, satisfying $x(t-T) = -x(t)$ for the half-period $T = p/2$ and all t, a robust feature. If $h(x(t)) = h(x(t-\tau)) = 0$ then, due to symmetry, $h(x(t-T)) = h(x(t-T-\tau)) = 0$.

Thus, an event collision of this symmetric periodic orbit for a crossing time t_* automatically induces a simultaneous event collision for the crossing time $t_* - T$, a scenario which is not covered by the classification of [13]. We point out that affine systems of the form (2), (3) can exhibit complex behavior, including chaos, even though all ingredients of the right-hand-side are linear. The switch governing (2) is a strong nonlinearity which is a common cause of complicated dynamics.

2. **Local return maps of symmetric periodic orbits at event collisions.** Let us suppose that system (2), (4) has a symmetric periodic orbit $L_* = (x_*(\cdot))$ of half-period T which, for a critical delay τ_*, experiences the event collision $h(x_*(0)) = h(x_*(-\tau)) = 0$, and, enforced by the reflection symmetry, $h(x_*(T)) = h(x_*(T-\tau)) = 0$. For compactness of presentation let us assume that 0 and T are the only crossing times of L_*, that is, $h(x_*(t)) < 0$ if $t \in (0,T)$ (without loss of generality). Thus, $T = \tau_*$, and x_* switches between the flows φ_+ and φ_- at the crossing times 0 and τ_*. Consequently, (without loss of generality) L_* consists of the two segments

$$x_*([0,\tau_*]) = \varphi_+^{[0,\tau_*]}(x_*(0)), \text{ and}$$
$$x_*([\tau_*, 2\tau_*]) = \varphi_-^{[0,\tau_*]}(-x_*(0)) = -x_*([0,\tau_*]).$$

Moreover, $h(x_*(0)) = h(x_*(\tau_*)) = 0$. The following transversality condition guarantees that the evolution E^t of system (2), (4) is attracted to a n-dimensional locally invariant manifold $\mathcal{C}$.

Condition 2. (Transversal event collision of symmetric orbits) The orbit L_* intersects the switching manifold $\{x : h(x) = 0\}$ transversally at time 0, that is

$$q := h'(x_*(0))f_+ \cdot h'(x_*(0))f_- > 0$$

where $f_+ = f(x_*(0), 1)$ and $f_- = f(x_*(0), -1)$.

Condition 2 means that, even though the orbit $x_*(\cdot)$ is not differentiable in its crossing times 0 and τ_*, it still crosses the switching manifold $\{h(x) = 0\}$ transversally in the sense that the left- and the right-sided time derivatives of $x_*(\cdot)$ both point through the switching manifold and both point in the same direction. Condition 2 is formulated for crossing time 0. The reflection symmetry implies that the same condition automatically holds also for the crossing time τ_*.

Let $\sigma > 0$ be sufficiently small. The point $\xi_* = (t \mapsto x_*(\sigma + t)) \in C([-\Theta, 0]; \mathbb{R}^n)$ is an element of L_* and, by definition and due to symmetry, it is located in the local hypersurface of codimension one (in $C([-\Theta, 0]; \mathbb{R}^n)$)

$$\mathcal{S}_* = \{\xi \in C([-\Theta, 0]; \mathbb{R}^n) : h(\xi(-\tau_*)) + h(x_*(\sigma)) = 0\}.$$

Condition 2 and the fact that the gradient of h is non-zero guarantee that the set $\{x : h(x) + h(x_*(\sigma)) = 0\} \subset \mathbb{R}^n$ is also a well defined hypersurface in the vicinity of $-x_*(0)$ and that the orbit $x_*(\cdot)$ intersects this curve transversally.

As mentioned in Section 1, and explained in detail in [13], $E^t(\xi)$ is continuous with respect to ξ in $\xi = \xi_*$ for all t. This continuity guarantees that E induces a unique map M_* of first return

$$M_* : \mathcal{S}_* \cap U_1(\xi_*) \mapsto \mathcal{S}_* \cap U_2(\xi_*)$$

for a pair of sufficiently small neighborhoods $U_1(\xi_*)$ and $U_2(\xi_*)$. (We will use the notation $U(\cdot)$ for neighborhoods of points in $\mathbb{R}^n$ as well as points in $C([-\Theta, 0]; \mathbb{R}^n)$.)

The following reduction lemma is (due to Condition 2) a straightforward extension of the reduction lemma for generic orbits in [13]. The proof of [13] carries over word by word.

Lemma 1 (Invariant manifold reduction)**.** *Let the length Θ of the history segment be less than $\tau_*+\sigma$. The image of the return map M_* intersected with $U_2(\xi_*)$ is contained in a local n-dimensional manifold $\mathcal{C}$. The manifold $\mathcal{C}$ is parametrizable by points $x\in U(x_*(\sigma))\subset\mathbb{R}^n$. The graph $C:U(x_*(\sigma))\subset\mathbb{R}^n\mapsto U_2(\xi_*)\subset C([-\Theta,0];\mathbb{R}^n)$ of the manifold $\mathcal{C}$ is given by*

$$C(x)(t)=\begin{cases}\varphi_+^{(t+\theta(x))}x & \text{if } t\in[-\theta(x),0]\\ \varphi_-^{(t+\theta(x))}x & \text{if } t\in[-\Theta,-\theta(x)]\end{cases}\tag{5}$$

where $\theta(x)$ is implicitly defined by

$$h(\varphi_-^{(\theta(x)-\tau_*)}(x))+h(x_*(\sigma))=0.\tag{6}$$

Lemma 1 states that all trajectories starting in $U_1(\xi_*)$ after the first period follow φ_+ from $U(x_*(0))$ to $U(-x_*(0))$ then switch exactly once from φ_+ to φ_-, follow φ_- back to $U(x_*(0))$ where they switch exactly once back to φ_+. The small time shift σ has been introduced to guarantee that we have exactly one time shift within the history segment. The coordinate parametrizing the graph C of the manifold is the location x of the switch in the state space $U(x_*(0))\subset\mathbb{R}^n$. The time $\theta(x)-\tau_*<0$ is the traveling time along φ_- from the switching location x back to the curve $\{y\in U(-x_*(0)):h(y)+h(x_*(\sigma))=0\}$. For x close to $x_*(0)$ this traveling time $\theta(x)$ is uniquely defined and depends smoothly on x due to the transversality asserted in Condition 2. The graph of the manifold is Lipschitz continuous but in general not smooth due to the dependence of θ on x and the fact that $f(x,-1)\neq f(x,1)$ for $y\in U(x_*(0)$. For delays τ close to the critical τ_* the reduction to a finite-dimensional manifold persists (τ_* changes to τ in the definition (6) of $\theta(x)$).

The dimension reduction of Lemma 1 implies that the long-time behavior of system (2), (4) for delays τ near τ_* and initial conditions near the periodic orbit $x_*(\cdot)$ can be described completely by a map m on the coordinates of the manifold graph C, which is a map from $U(x_*(0))\subset\mathbb{R}^n$ back to $U(x_*(0))$. The following theorem provides a formula for the piecewise smooth map m.

Theorem 1. *(Dynamics near symmetric collisions) Let τ be sufficiently close to τ_*. The return map for elements of the invariant manifold $C(U(x_*(0)))$ is given as $m=F\circ F$ where $F:U(x_*(0))\mapsto U(x_*(0))$ is defined as*

$$F(x)=-\varphi_+^{\tau+t(x)}x\tag{7}$$

and $t(x)\in(-\tau,\tau)$ is the unique time such that

$$\begin{cases}h\left(\varphi_+^{t(x)}x\right)=0 & \text{if } h(x)>0,\\ h\left(\varphi_-^{t(x)}x\right)=0 & \text{if } h(x)\leq 0.\end{cases}\tag{8}$$

Proof: Due to the reflection symmetry of the orbit $x_*(\cdot)$ the full return map m is the second iterate of the negative of its half-return map, which we call F. Let x be arbitrary in $U(x_*(0))$. The element $C(x)$ of the manifold switches in x from φ_- to φ_+. The image $y=-F(x)$ of x under the half-return map $-F$ is the switching point from φ_+ back to φ_- near $U(-x_*(0))$. Thus, $y=\varphi_+^{\tau+t(x)}x$ where the time $-t(x)$ is the time system (2), (4) needs from the last crossing of the manifold $\{h=0\}$ to x.

Consequently, $t(x)$ determined implicitly by the relation $h(C(x)(t(x))) = 0$, which has the form (8) due to the form (5) of the graph C. □

The expression (8) traveling time $t(x)$ implies that F is continuous in $U(x_*(0))$ and smooth in its two subdomains $D_- := U(x_*(0)) \cap \{x : h(x) \leq 0\}$ and $D_+ := U(x_*(0)) \cap \{x : h(x) > 0\}$ but, in general, its derivative has a discontinuity along the boundary D_0 between D_- and D_+. The existence and uniqueness of the traveling time is a consequence of the transversality Condition 2. The linearizations of both parts of F with respect to x and τ in $x_*(0)$ and τ_* are (appending the parameter τ as a second argument of F):

$$F(x_*(0) + \xi; \tau_* + \delta) - x_*(0) = -A^{\tau_*} \left[\left[I - \frac{f_+ H_*}{g} \right] \xi + \delta f_+ \right] + O(|(\xi, \delta)^2|) \tag{9}$$

where $H_* = h'(x_*(0))$, $A^{\tau_*} = \partial_x \left[\varphi_+^{[0,\tau_*]} x \right]_{x = x_*(0)}$, $f_\pm = f(x_*(0), \pm 1)$, and

$$g = \begin{cases} H_* f_+ & \text{if } x_*(0) + \xi \in D_+, \\ -H_* A^{\tau_*} f_+ & \text{if } x_*(0) + \xi \in D_- \setminus D_0. \end{cases} \tag{10}$$

Condition 2 asserts that the product q of $H_* f_+$ and $H_* f_- = -H_* A^{\tau_*} f_+$ is nonzero. Thus, Condition 2 implies that g is nonzero in both cases. The affine approximation of the boundary D_0 between D_- and D_+ in $x_*(0)$ is given by $\{x_*(0) + \xi : H_* \xi = 0\}$. The map F projects the whole subdomain D_+ onto the $(n-1)$-dimensional local submanifold $\{x \in U(x_*(0)) : h(-\varphi_+^{-\tau} x) = 0\}$, which is the reflected delayed switching manifold. Correspondingly, its linearization (9) projects ξ linearly by $I - f_+ H_* / (H_* f_+)$ before propagating it by $-A^{\tau_*}$. Consequently, an event collision for a symmetric periodic orbit satisfying Condition 2 increases the dimension of the image of the local return map from $n - 1$ in D_+ (where the projection applies) to n in D_-.

Remark. It may occasionally be useful to consider the extensions of the two smooth parts of the map F to the whole domain $U(x_*(0))$. This is possible due to transversality Condition 2:

$$\begin{aligned} F_+(x) &= -\varphi_+^{\tau + t_+(x)} x \text{ where } \quad h\left(\varphi_+^{t_+(x)} x\right) = 0, \text{ and} \\ F_-(x) &= -\varphi_+^{\tau + t_-(x)} x \text{ where } \quad h\left(\varphi_-^{t_-(x)} x\right) = 0. \end{aligned} \tag{11}$$

The return map on the invariant manifold is equal to the map F_+ (a smooth 1D map) in D_+ and equal to the map F_- (a smooth 2D map) in D_-.

3. **Illustrative example.** This section applies the theory developed in Section 2 to a prototype example, an unstable oscillator in controllable normal form subject to a delayed linear switch. This example falls into the class (3) of affine systems. Consider

$$\begin{gathered} \ddot{x}(t) - 2\rho \dot{x}(t) + \left[\omega^2 + \rho^2\right] x(t) = b \operatorname{sign} h[x(t-\tau), \dot{x}(t-\tau)], \\ \text{where} \quad h[\eta, \zeta] = \eta \cos\alpha + \zeta \sin\alpha \end{gathered} \tag{12}$$

By scaling time and x we can adjust the frequency ω of the oscillator to $\omega = 1$ and the amplitude of the control input b to $b = 1$. This reduces the example (12) to a problem with a two-dimensional physical space $\mathbb{R}^2$ of $(x, \dot{x})$ depending on three parameters: the expansion rate $\rho > 0$ (if the oscillator is unstable), the delay $\tau > 0$,

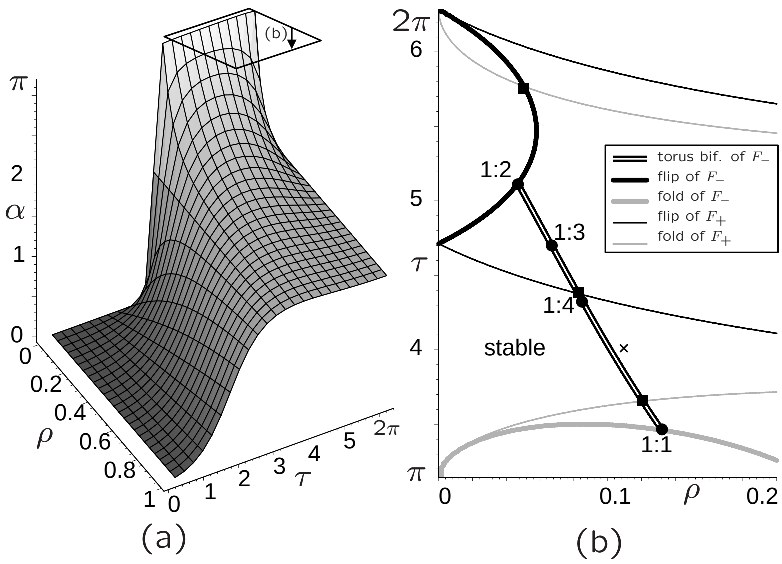

FIGURE 1. Panel (a): codimension-one surface of parameter triples (τ, ρ, α) where a colliding symmetric periodic orbit exists. Panel (b): Projection onto the collision surface along the direction of α, showing the classical bifurcation curves (on the collision surface) of the colliding orbit as a fixed point of each of the smooth maps F_- and F_+. The coincidence of any of these bifurcations with the collision constitutes a codimension-two non-smooth bifurcation of the return map F. Each codimension-two classical bifurcation (for example, the strong resonances) corresponds to a non-smooth codimension-three bifurcation of F.

and the angle α of the unit normal vector of the switching line in $[0, 2\pi]$. The affine flows $\varphi_\pm$ are given by

$$\varphi_\pm^t z = A(t)z \pm v(t)$$

where

$$A(t) = e^{\rho t}\cos t - e^{\rho t}\sin t \begin{bmatrix} \rho & -1 \\ 1+\rho^2 & -\rho \end{bmatrix}, \quad v(t) = \frac{1}{1+\rho^2}\begin{bmatrix} \rho\sin t - \cos t + e^{-\rho t} \\ (1+\rho^2)\sin t \end{bmatrix}$$

and have equilibria at $(\pm 1/(1+\rho^2), 0)^T$. The condition for the existence of a colliding symmetric periodic orbit is

$$z_1\cos\alpha + z_2\sin\alpha = 0, \text{ where } \quad z = -(I + A(\tau)]^{-1} v(\tau). \tag{13}$$

That is, $z = -\varphi_+^\tau z$ is the point where the colliding periodic orbit switches from φ_- to φ_+ and lies exactly on the switching line.

Figure 1(a) shows the surface (a codimension-one object in the 3D parameter space) of parameter triples (τ, ρ, α) where condition (13) for the existence of a colliding periodic orbit is satisfied. For parameters "below" the surface the symmetric periodic orbit is *slowly oscillating*, that is, its half-period is larger than the delay τ and its return map is F_+ (having a one-dimensional image). For parameters "above" the surface the symmetric periodic orbit has a half-period shorter than the delay (that is, it is *rapidly oscillating*). The local return map is F_- and has a two-dimensional image in this parameter region. What happens dynamically near the periodic orbit near or on the surface in Figure 1(a) depends on the properties of the smooth maps F_+ and F_-, as given by (11), at the collision parameter. Figure 1(b) is a view on the collision surface "from the top" in the region of small ρ and $\tau \in [\pi, 2\pi]$; see sketch in Figure 1(a).

For parameters on the collision surface in the region marked as "stable" in Figure 1(b) both smooth maps F_+ and F_- are stable, that is, the linearization of F_- in the symmetric periodic orbit has two non-zero eigenvalues inside the unit circle and the linearization of F_+ has one non-zero eigenvalue that is inside the unit circle. The curves in Figure 1(b) mark the codimension-two bifurcation curves when the collision coincides with a classical bifurcation (a change of linear stability) of F_+ (thin) or F_- (thick). The changes of stability of F_+ occur if the eigenvalue of its linearization is at $+1$ (fold) or -1 (flip). The changes of stability of F_- occur if either one eigenvalue of its linearization is at $+1$ (fold) or -1 (flip), or if both eigenvalues are complex and on the unit circle (torus bifurcation). Accordingly, whenever F_- has a codimension-two degeneracy, its coincidence with the collision is a codimension-three bifurcation (the linear degeneracies of F_- are marked by black circles in Figure 1(b)). We must expect that near the codimension-two curves, secondary bifurcation surfaces of other objects branch off (for example, a collision of non-symmetric periodic orbits at any of the flip bifurcation curves). A coincidence between linear degeneracies of F_+ and F_- is another possibility for a codimension-three bifurcation (black squares in Figure 1(b)). Details of the emerging secondary bifurcations and the expected dynamics near any of the codimension-two curves are not systematically classified yet (in contrast to the classical smooth bifurcation theory [10]).

Figure 2 shows an attractor of the delayed relay system (12) that is born when one crosses the collision surface at the cross ($\tau = 4$, $\rho = 0.111$) in Figure 1(b). "Beneath" the collision surface the return map of (12) is described by the 1D map F_+, which has a stable fixed point. After crossing the surface (increasing α) the attractor is a non-smooth invariant curve densely filled with a chaotic orbit. Figure 2(b) zooms in to shows that the invariant curve is indeed densely filled. (Transients of length 200π have been discarded.) The symmetric periodic orbit is an unstable spiral, that is, initial conditions near the symmetric orbit spiral toward the invariant curve. The dynamics on the invariant curve are given by a discontinuous 1D map.

Remark. The 1:1 resonance appears degenerate in Figure 1(b) in the sense that the torus bifurcation curve (thick black hollow) does not approach the fold bifurcation curve (thick gray) tangentially, as should generically be expected [10], but transversally. This is due to the projection onto the collision surface. Indeed, the locus of fold bifurcations of F_+ in the three-dimensional parameter space (a surface) does not intersect the collision surface of Figure 1(a) transversally but touches it tangentially in the fold curve shown in Figure 1(b).

Furthermore, the point $(\rho, \tau) = (0, 2\pi)$ is highly degenerate because transversality Condition 2 is violated in this point in the symmetric periodic orbit.

4. **Conclusion.** The paper discusses the dynamics near periodic orbits in dynamical systems with delayed switches. It is motivated by the fact that in many practical applications the presence of symmetry prevents the generic and simple bifurcation scenarios as classified in [13] but instead gives rise to intricate and counter-intuitive event collision phenomena [3, 4, 8].

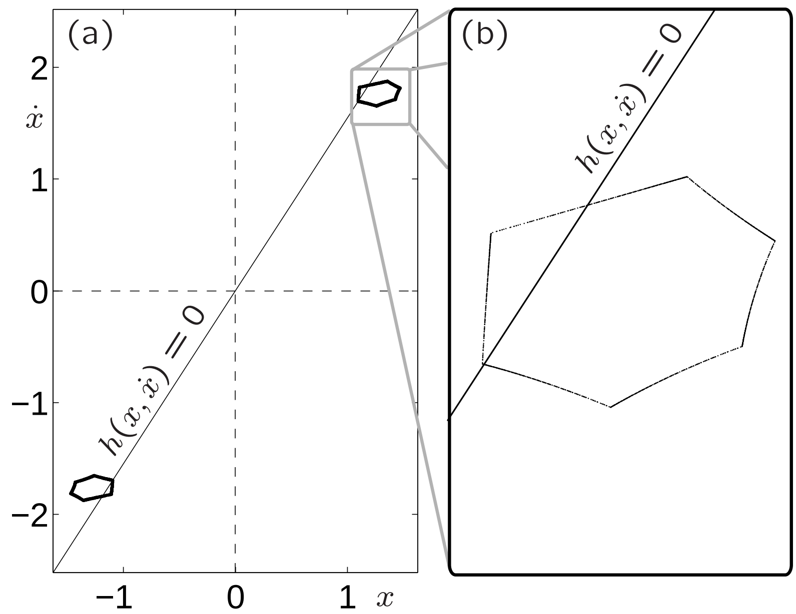

FIGURE 2. Attractor of system (12) near the symmetric periodic orbit for $\tau = 4$, $\rho = 0.111$, $\alpha = 2.5678$ (see cross in Figure 1(b)). Panel (b) zooms into the neighborhood of the invariant curve to show that it is densely filled.

Then we describe the simplest and most common case of an event collision in a symmetric system as it occurs, for example, in a piecewise affine system which is switching with a delay. In this case two corners of a symmetric periodic orbit simultaneously collide with a switching manifold. We proved in Theorem 1 that at this collision the return map near the periodic orbit can be reduced to a finite-dimensional piecewise smooth map, for which we derive an implicit expression. This effectively reduces the analysis of a piecewise smooth delay differential equation to the study of low-dimensional piecewise smooth maps (which have been studied, for example, in [1, 5, 11, 12]). We demonstrated how this reduction can be used to classify possible bifurcations near a collision with a prototype two-dimensional piecewise linear example, which has a symmetric periodic orbit, classifying its changes of stability close to the event collision.

The analysis of the possible dynamics in the unfolding of the event collision is far from complete but the initial theoretical results presented in this paper will lead to classifications of practically relevant behavior for concrete systems.

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