New developments for bifurcation analysis of delay equations

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Outline

- DDE-Biftool approach to distributed delays and renewal equations
- Convergence analysis problems for DDEs
- Convergence of discretization & Newton iteration

Distributed delays

Linear DDEs: Representation Theorem ensures r.h.s. has form

$$x'(t) = \sum A_j x(t-\tau_j) + \int_0^{\tau_{\max}} G(s)x(t-s)\mathrm{d}s$$

Nonlinear DDEs:

$$x'(t) = f\left(x(t-\tau_j), \int_{s_1}^{s_2} g(s, x(t-s), p)\mathrm{d}s, \ldots\right) \quad \textbf{?}$$

No interface for general nonlinear functional of $x_t = x(t+(\cdot))$

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DDE-Biftool

bifurcation analysis for DDEs,

$$Mx'(t) = f(x(t), x(t-\tau_1), \dots, x(t-\tau_m), p),$$

- originally developed by Engelborghs, Roose, Luzyanina, Samaey (1999, KU Leuven)
- equilibria: tracking, stability, bifurcation tracking (KUL)
  periodic orbits: tracking, stability (KUL)
  connecting orbits (KUL)
- periodic orbits local bifurcation tracking (Orosz⇒JS)
- linear stability pseudospectral methods (Breda⇒JS)
- equilibrium normal form analysis (Wage, Bosschaert, Kuznetsov)
- singular M, neutral DDEs
  (Szalai, Barton, Terrien, Hessel, Javaloyes, Gurevich...)
- distributed delays (Humphries⇒JS)

DDE-Biftool

uses formulation as differential-algebraic problem:

$$
\begin{aligned}
Mx'(t) &= f(x(t-\tau_m)\ldots,p) \\
0 &= \int_0^{\tau_{\mathrm{d}}} g_{\mathrm{d}}(s,x_\ell(t-s),p_i)\mathrm{d}s - x_k(t) \\
&\ldots
\end{aligned}
$$

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$$\ldots$$

nonsquare,
can be singular

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index provided by user

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$\ldots$

nonsquare,
can be singular

index provided by user

state or parameter (index provided by user)

function provided by user

DDE-Biftool

uses formulation as differential-algebraic problem:

distributed delay,
can be used with delay

$$Mx'(t) = f(x(t-\tau_m)\dots, p)$$

$$0 = \int_0^{\tau_\mathrm{d}} g_\mathrm{d}(s, x_\ell(t-s), p_i)\mathrm{d}s - x_k(t)$$

$$\dots$$

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- permits multiple nested integrals as ℓ and k can overlap,
- x_k can be multidimensional,
- several distributed delays possible
- approximated by N discrete delays $\tau_j = s_j\tau_\mathrm{d}$

$$\int \dots \approx \sum_{j=1}^{N} w_j\tau_\mathrm{d}\, g(s_j\tau_\mathrm{d}, x_\ell(t-s_j\tau_\mathrm{d}), p_i)$$

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DDE-Biftool example renewal equation (RE)

Breda *et al.* 2016

$$x(t) = \frac{\gamma}{2}\int_{\tau_2}^{\tau_1+\tau_2} x(s)(1-x(s))\mathrm{d}s$$

implemented as

$$0 = x(t) - \frac{\gamma}{2}y(t-\tau_2)$$
$$0 = \int_0^{\tau_1} x(s)(1-x(s))\mathrm{d}s - y(t)$$

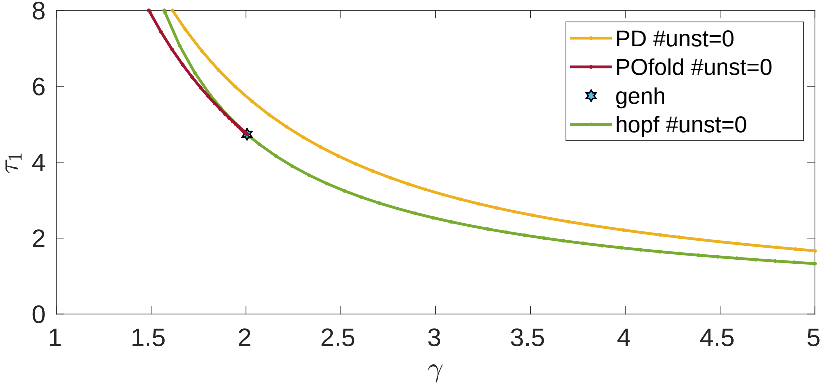
DDE-Biftool example renewal equation (RE)

Breda *et al.* 2016

$$0 = x(t) - \frac{\gamma}{2} y(t - \tau_2)$$

$$0 = \int_0^{\tau_1} x(s)(1 - x(s))\mathrm{d}s - y(t)$$

$$p = (\gamma, \tau_1, \tau_2),\ u = (x, y)$$



Complex example: size structured Daphnia population model

Diekmann *et al.* 2010, Andò 2020

resource: $$\dot{r}(t) = f_0(r(t)) - f_c(r(t))p_{\text{eff}}(a_{\text{max}}, t),$$

maturation threshold: $$0 = f_{\text{thr}}(a_m(t), s(a_m(t), t)) - S_m,$$

birth rate: $$b(t) = p_{\text{eff}}(a_{\text{max}}, t) - p_{\text{eff}}(a_m(t), t),$$

cohort size: $$s(a, t) = g_0(a) + \int_0^a g_r(\alpha, f_r(r(t-\alpha)))\mathrm{d}\alpha,$$

effective population: $$p_{\text{eff}}(a, t) = \int_0^a h(\alpha, s(\alpha, t), r(t))b(t-\alpha)\mathrm{d}\alpha.$$

Complex example: size structured Daphnia population model

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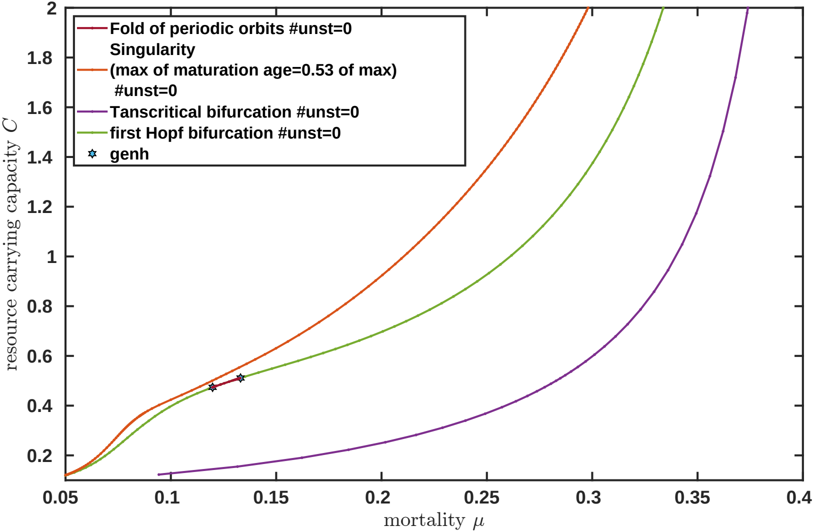
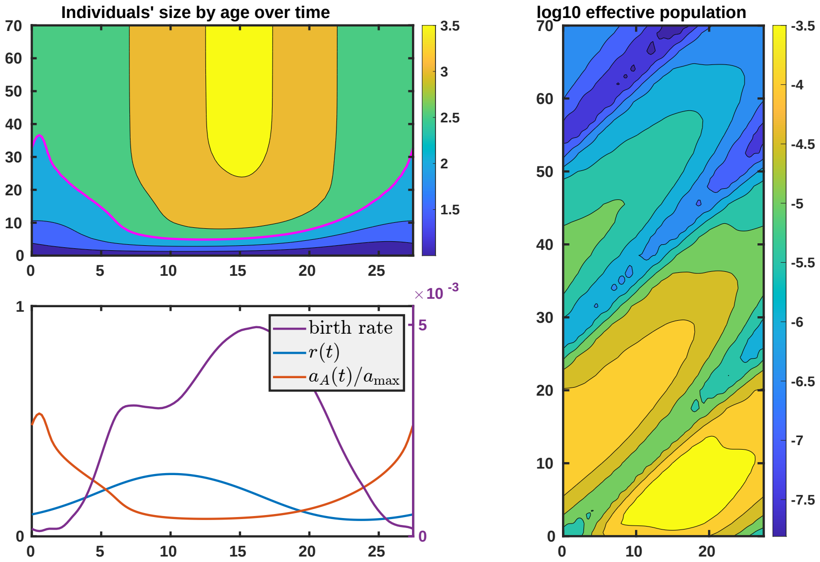


Size structured Daphnia population model: shock

Diekmann *et al.* 2010, Andò 2020



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DDE-Biftool distributed delays conclusion

- uses interface for state-dependent delays to avoid introducing N coupled parameters, $\tau(j,x,p)=s_j\tau_\mathrm{d}$,
- discretized renewal equations $\sim$ neutral equations:

$$x(t)=\sum_{j=1}^{N} w_j\tau_\mathrm{d} g(s_j\tau_\mathrm{d}, x(t-s_j\tau_\mathrm{d}), p)$$

  ⇒essential spectral radius of time-1 map >1
  ⇒high-frequency instability
  ⇒ignore high-frequency eigenvalues of equilibria
  ⇒ignore Floquet multipliers with highly oscillatory eigenfunctions.
- Renewal equations can be converted to equivalent DDEs
- vectorized g mandatory

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Convergence of numerical discretization

DDE-Biftool:

$$\dot{x}(t) = f(x(t-\tau_m), p)$$

time rescaling $\Rightarrow$ $x'(t_k) = Tf(x(t_k - \tau_m/T), p)$ at $L \times n_{\mathrm{deg}}$ times t_k

+continuity & periodicity for piecewise continuous polynomial x with L pieces, degree n_{deg}.

Convergence proof for constant delay:

Engelborghs & Doedel'02: stability for linear DDEs

thought this implies convergence, but

$$F : (x, T, p) \mapsto Tf(x((\cdot) - \tau_m/T), p)$$

is not continuously differentiable w.r.t. unknown period T

(term $x'((\cdot) - \tau_m/T)\tau_m/T$ shows up) (solved by Andò 2020)

$F : C^k \to C^\ell$ is only $C^{k-\ell}$ if $k \geq \ell$

DDEs with state-dependent delays

$$F(x)(t) = f(x_t) \quad x_t(s) := x(t+s), \quad f : C \to \mathbb{R}^n \text{ functional}$$

is cont. diff. only if delays constant.

$$F(x)(t) = x(t+x(t)) \quad \Rightarrow \quad [\partial F(x)y](t) = y(t+x(t)) + x'(t+x(t))y(t)$$

Instead: **mild differentiability** concept (Hartung *et al.*'06)

$$[\partial^k F(x)(y)^k] \quad \text{depends on } x, x', \ldots, x^{(k)}, y', \ldots, y^{(k-1)},$$

(continuously), but **not** $y^{(k)}$.

Result: ($0 = \Phi(x^*)$, $0 = \Phi_L(x_L)$)

- $\|x_L - x^*\|_{0,1} \sim L^{-n_{\mathrm{deg}}}$ if F is $\geq n_{\mathrm{deg}}$ times mild. diff. & $\partial\Phi(x^*)$ is invertible
- Newton iteration convergence limited by $\|x_L - x^*\|_{0,1}$,

$\Rightarrow$ better convergence for higher-accuracy solutions

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- Newton iteration convergence limited by $\|x_L - x^*\|_{0,1}$,

⇒ better convergence for higher-accuracy solutions

Issues:

!! Φ_L only cont. diff. if x_L is cont. diff., but x_L' discontinuous

⇒ Jacobian $\partial\Phi_L(\cdot)$ is discontinuous on solution space,

violates standard assumptions for convergence of discretization and Newton iteration

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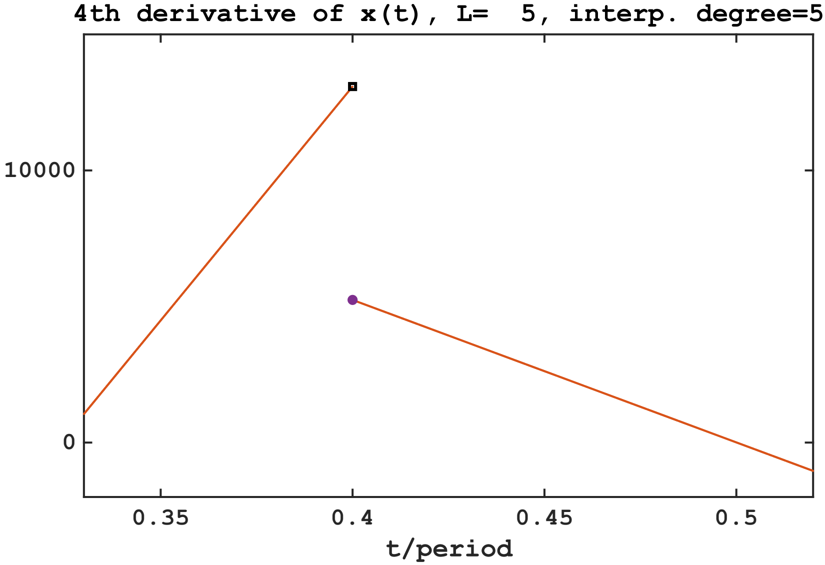
DDEs with state-dependent delays — issues

All derivatives discontinuous, but converge to true solution:



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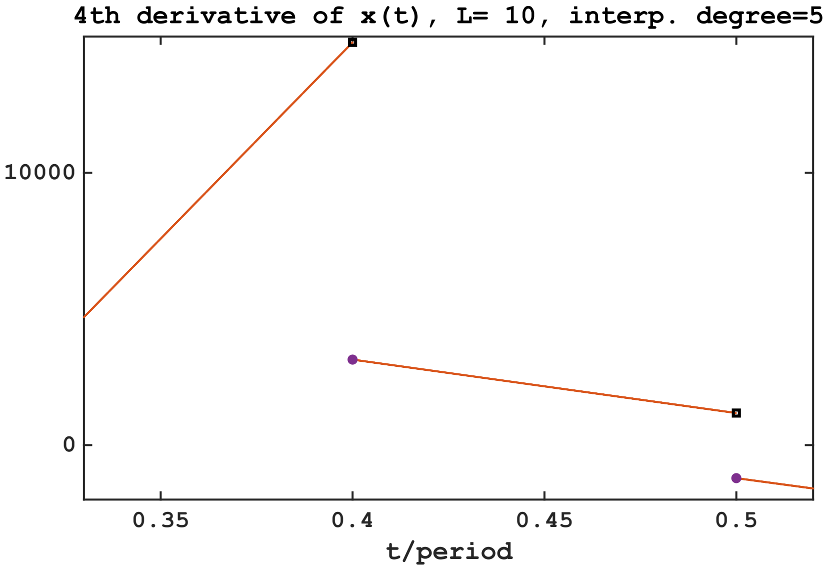

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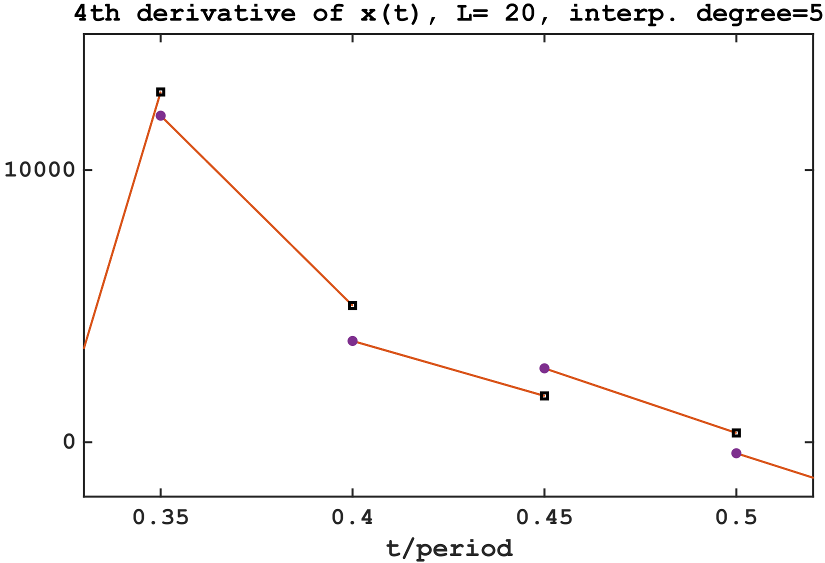

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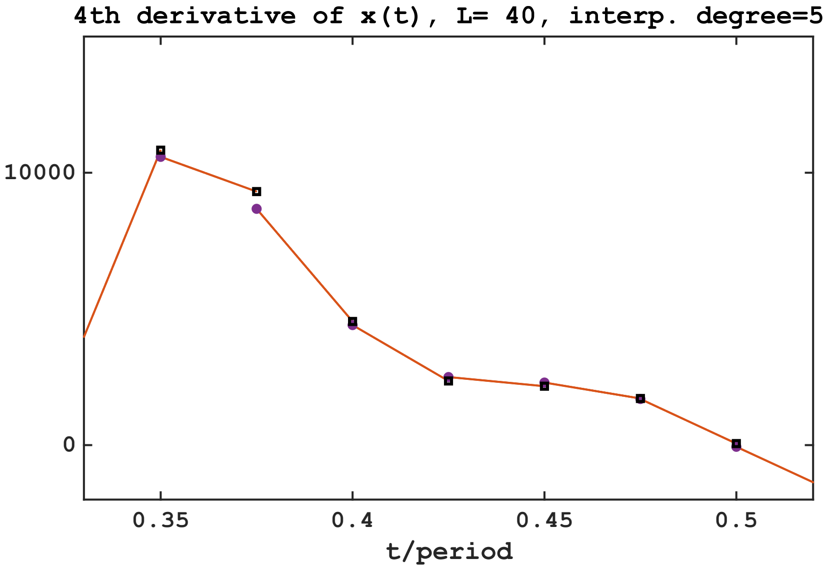

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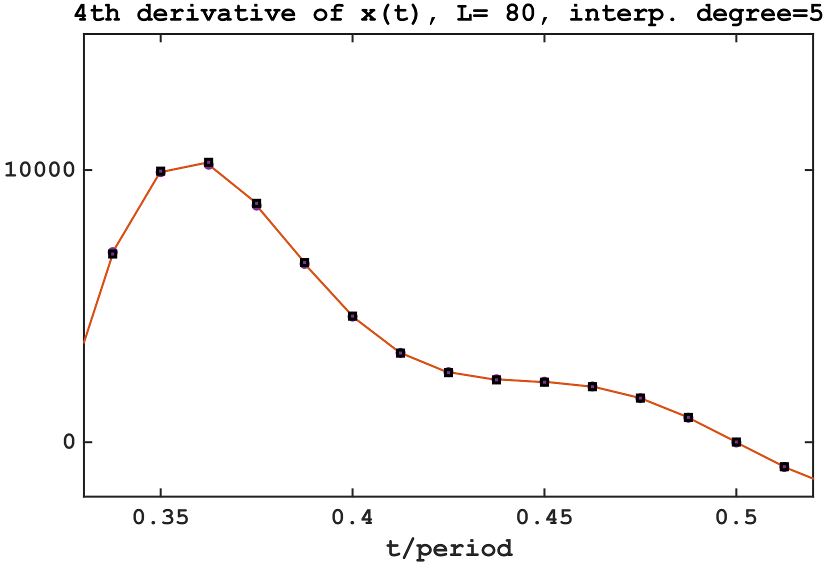

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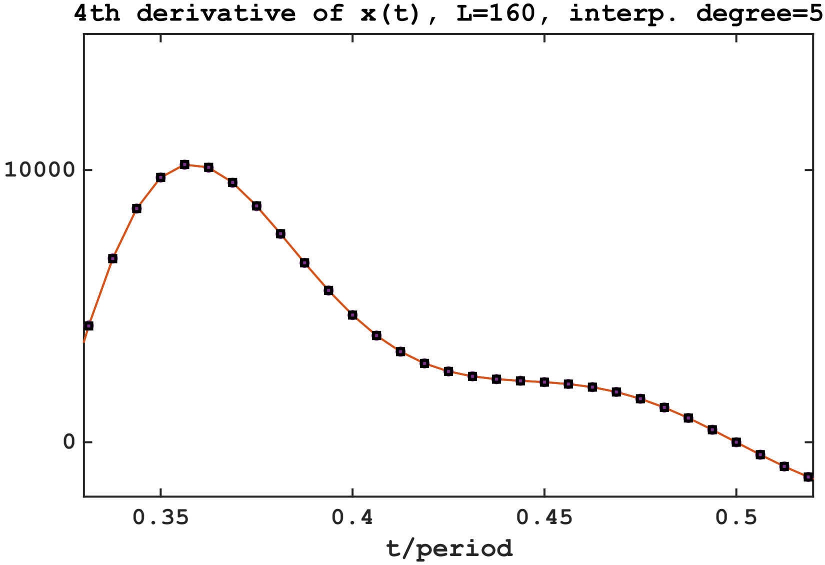
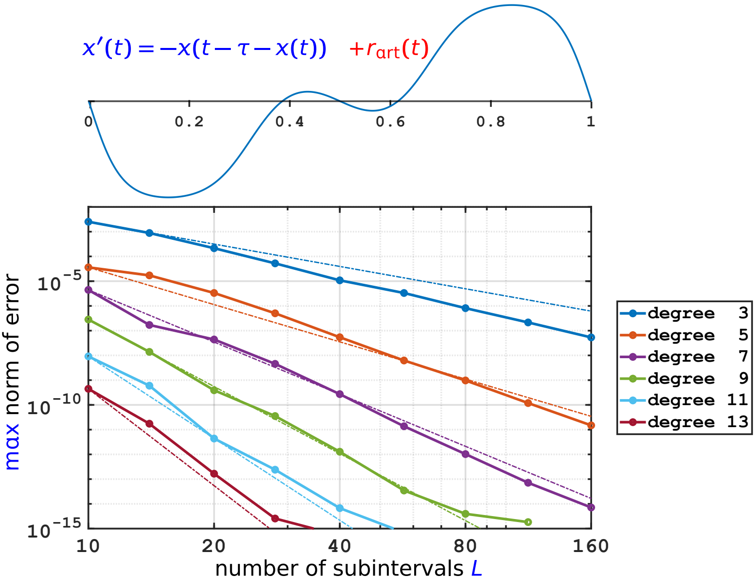


DDEs with state-dependent delays — example error plot



Conclusion

sourceforge.net/p/ddebiftool/git/ci/master/tree/

- bifurcation analysis for DDEs with distributed delays and Renewal Equations feasible
- linear stability analysis for REs suffers instabilities
- expectation management for speed
- convergence proof of numerical method surprisingly recent for constant delays (Andò'20), current preprint for state-dependent delays
- difficulty: lack of continuous differentiability of r.h.s.