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CONTROL-BASED TRACKING OF NONLINEAR OSCILLATIONS

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1 INTRODUCTION

We demonstrate a method for tracking oscillations and their stability boundaries (bifurcations) in nonlinear systems [6]. Our method does not require an underlying model of the dynamical system but instead relies on feedback stabilizability. This gives the approach the potential to transfer the full power of numerical bifurcation analysis techniques [1] from the purely computational domain to real-life experiments.

One important application and test case for this method are so-called hybrid (or real-time substructuring) experiments in civil and mechanical engineering. Hybrid experiments couple mechanical experiments and computer simulations bidirectionally and in real-time. One major aim of these experiments is finding and tracking stability boundaries.

Our method allows one to determine bifurcations of the dynamical system without the need to observe the transient oscillations for a long time to determine their decay or growth. Moreover, in the context of hybrid experiments our method is able to overcome the presence of coupling delays (or, more generally, unknown actuator dynamics), which is a fundamental problem that is currently limiting the use of hybrid testing [5].

We illustrate the basic ideas with a prototype nonlinear hybrid experiment, a real pendulum coupled at its pivot to a computer simulation of a vertically excited mass-spring-damper (MSD) system as sketched in Figure 1. The original combined MSD-pendulum system (a parametrically excited two-degree of freedom oscillator) shows a rich bifurcation structure, which can be explored systematically numerically using the methods implemented in AUTO [1] and explained in [4]. This makes the MSD-pendulum system an ideal test candidate, both, for our method and for hybrid testing of nonlinear dynamical phenomena in general. This abstract focuses on the simplest bifurcation of the system, the period doubling of the hanging-down state.

2 ESSENTIAL INSTABILITIES IN DELAY-COUPLED MECHANICAL SYSTEMS

Let us consider the system in Figure 1 in the configuration where the mass m of the pendulum is larger than the mass M of the mass block in the mass-spring-damper system. In non-dimensionalized form it is governed by the equations

$$(1+p)\ddot{y}+\beta\dot{y}+\alpha y+p[\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta] = a\cos(\Omega t) \quad (1)$$

$$\ddot{\theta}+\zeta\dot{\theta}+(1+\ddot{y})\sin\theta = 0 \quad (2)$$

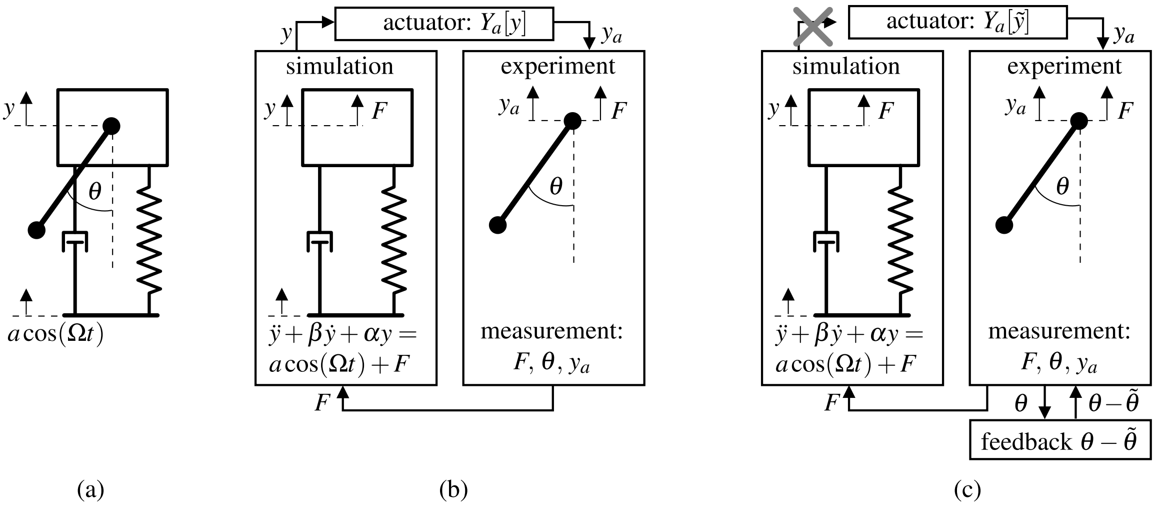

Figure 1. Sketch of decomposition of the overall mechanical system into a computer simulation of a mass-spring-damper system and a real pendulum. Panel (a): original (emulated) system, panel (b): bidirectionally real-time coupled system as studied in [2, 5], panel (c): partially decoupled system with interface matching by Newton iteration.

where $p = m/M$. We use the time unit $\omega^{-1} = \sqrt{l/g}$ where g is the acceleration due to gravity and l is the length of the pendulum. This configuration was found to be technically impossible to run as a hybrid experiment (that is, as a coupled system as shown in Figure 1 (b); see [5]). The analysis in [5] explains this obstruction with the small delay in the coupling between simulation and experiment. Idealizing the dynamics of the actuator in Figure 1 (b) one can assume that the control of the actuator is exactly following the demand signal $y(t)$ only with a fixed pure time delay τ, that is,

$$y_a(t) = Y_a[y](t) = y(t-\tau). \tag{3}$$

This delay introduces a delay in the coupling between (1) and (2), changing the system to

$$\begin{gathered}\ddot{y} + p\ddot{y}_\tau + \beta\dot{y} + \alpha y + p[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta] = a\cos(\Omega t)\\ \ddot{\theta} + \zeta\dot{\theta} + (1+\ddot{y}_\tau)\sin\theta = 0\end{gathered} \tag{4}$$

by inserting the dynamics of the actuator (3). In system (4) the index τ means that this dependent variable is evaluated at time $t-\tau$ instead of t. The appearance of a delayed highest derivative makes (4) a neutral delay differential equation [3]. If the mass ratio p is not less than one then an arbitrarily small delay τ causes an essential instability of delay-coupled system (4). More precisely, in the hanging-down state $\theta = 0$ the delay-coupled system is a compact perturbation of the linear difference equation $y(t) = -py(t-\tau)$, which has infinitely many eigenvalues with real part $\tau^{-1}\log p$. If $p > 1$ these eigenvalues are unstable regardless of the delay τ. The growth rate even gets larger when the delay is decreased, making the problem practically ill-posed for small delays. This instability carries over to the full system (4) linearized at $\theta = 0$, also leading to infinitely many unstable eigenvalues for all delays $\tau > 0$; see [5]. The essential spectrum of the time-τ map of (4) is located outside the unit circle (hence, we refer to this instability as *essential*). This instability is, of course, not present in the emulated system (1), (2).

The physical reason behind this severe instability is the coupling of the two subsystems at a fixed joint (in contrast to a spring) in combination with prescribing displacements and measuring forces at the interface (see Figure 1 (b)). If $p < 1$ or the coupling is at a spring then instabilities can still occur but they involve only a small number of eigenvalues because the delay is small (typically ≈ 10 ms). Classical delay compensation is suitable for these non-essential instabilities [7] but fails for $p > 1$ for any delay $\tau > 0$. The accuracy of the delay compensation can be measured in the experiment by observing the synchronization error $e = y_a - y$, which is the difference between the output of the simulation and the actual motion of the actuator.

A real actuator is not capable of supporting an instability at infinitely many frequencies. Typically, the actuator will be a stiff approximation of the idealization (3), for example, $\ddot{y}_a + \omega_s \dot{y}_a + \omega_s^2[y_a - y(t-\tau)] = F$ for a large $\omega_s > 0$ where F is the force measured at the pivot. This gives rise to a regularization of the ill-posed problem (4) having a large number of strongly unstable eigenvalues for large ω_s and small delays.

3 INTERFACE MATCHING BY NEWTON ITERATION

A consequence of the arguments in Section 2 is that for a mass ratio $p \geq 1$ it is impossible to achieve an approximation of the dynamics of the emulated system in Figure 1 (a) by a bidirectional real-time coupling as in Figure 1 (b). We demonstrate that it is still possible to perform a systematic analysis of the dynamics of the emulated system. In order to achieve this we break the coupling in one direction, match the output at the interface by a Newton iteration and exploit some fundamental statements from bifurcation theory.

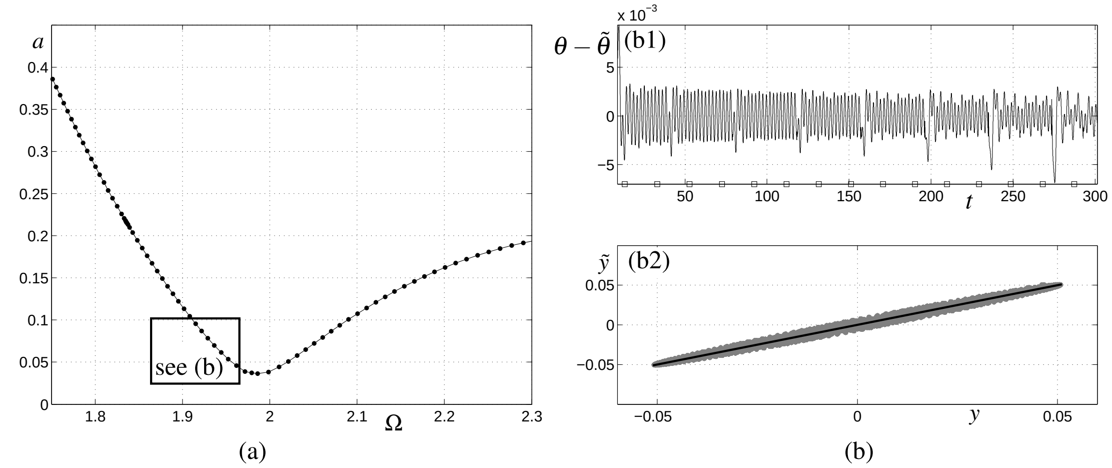

Figure 2. Results of continuation of period doubling. Panel (a): stability curve in two-parameter plane, panel (b1): control effort in feedback loop along a part of the curve, panel (b2): anticipation of y by $\tilde{y}$.

For example, in the original emulated system (1), (2) the hanging-down state $\theta = 0$ loses its dynamical stability in a period doubling bifurcation. Standard bifurcation theory states that near this loss of stability a solution with a small harmonic amplitude of θ and period $4\pi/\Omega$ emerges [4]. Hence, in order to find the boundary of stability of the hanging-down state it is sufficient to track small period-two solutions.

Figure 1 (c) shows how to break the bidirectional coupling. The actuator is fed with a periodic demand $\tilde{y}$. In addition, the pendulum experiment is stabilized by a feedback loop with a periodic demand signal $\tilde{\theta}$. Due to the feedback loop the unidirectionally coupled system will settle (after a transient) into a locally unique periodic state that can be measured and depends on the given demands $\tilde{y}$ and $\tilde{\theta}$ of period $4\pi/\Omega$: the angular displacement $\theta[\tilde{\theta}, \tilde{y}]$, the motion of the pivot $y_a[\tilde{\theta}, \tilde{y}]$ and the output of the simulation $y[\tilde{\theta}, \tilde{y}, \Omega, a]$. The following system of nonlinear equations defines the period doubling bifurcation of the hanging-down state of the original system:

$$0 = y[\tilde{\theta}, \tilde{y}, \Omega, a] - y_a[\tilde{\theta}, \tilde{y}] \qquad \text{synchronization} \tag{5}$$

$$0 = \theta[\tilde{\theta}, \tilde{y}] - \tilde{\theta} \qquad \text{control non-invasive} \tag{6}$$

$$r^2 = \int_0^{4\pi/\Omega} (\tilde{\theta}(t) - \theta_0)^2 \, dt \qquad \text{period doubling} \tag{7}$$

where r is small and θ_0 is the average of θ. The variables of this system are the two parameters a and Ω (the excitation in the simulation) and the two periodic control demands $\tilde{\theta}$ and $\tilde{y}$, which can be expressed by their first two Fourier modes: $\tilde{y}(t) = y_0 + y_1 \exp(it\Omega/2) + y_2 \exp(it\Omega)$ and $\tilde{\theta} = \theta_0 + \theta_1 \exp(it\Omega/2) + \theta_2 \exp(it\Omega)$ where $y_0, \theta_0 \in \mathbb{R}$, y_1, y_2, θ_1, $\theta_2 \in \mathbb{C}$. After a Galerkin projection of (5) and (6) onto the first two Fourier modes one obtains 11 (real-valued) equations for 12 (real-valued) variables. This resulting system defines an implicit curve that can be found by a Newton iteration embedded into a pseudo-arclength continuation. Each evaluation of the right-hand-side of (5) and (6) requires one to run the experiments once until the transients have died. This makes function evaluations expensive compared to purely numerical bifurcation analysis as discussed in [4] and implemented in AUTO [1].

Figure 2 shows the results of a proof-of-concept computer simulation using the idealized actuator model (3) with delay $\tau = 0.01/\omega \approx 0.07$ (10 ms) and the pendulum equation (2) for the pendulum. Figure 2 (a) displays the curve defined by the Galerkin approximation of (5)–(7). Figure 2 (b1) shows a typical time profile during the continuation along the period doubling curve. Note that the transients, occuring whenever the demands and parameters are changed, are typically small because demands and parameters are varied only gradually during the continuation. The squares along the time axis indicate when the system is considered to have settled down, giving one evaluation of the right-hand-sides of (5) and (6). Figure 2 (b2) displays in the synchronization plane that $\tilde{y}$ anticipates the output of the simulation y slightly. Importantly, we achieve synchronization without expressly exploiting the knowledge about the actuator model (3).

4 CONCLUSION AND FURTHER WORK

Proof-of-concept computer simulations, including adverse effects such as coupling delay and measurement inaccuracies, propose that bifurcation analysis should be possible even for hybrid experiments that are genuinely ill-posed. The incorporation of control-based bifurcation analysis into the hybrid experiment itself is currently in preparation. The most pressing problems for the future, apart from experimental validation, are the incorporation of other bifurcations (some have been studied in [6]), of strongly nonlinear phenomena (such as homoclinics), and of non-periodic responses.

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