Graduate School – Udine 15 – 19 Dec 25

Continuation methods for nonlinear problems

- Mutual Introduction
- website: https://github.com/jansieber/udine25-coco-gradschool
- central piece of software (Matlab based) COCO

  https://sourceforge.net/projects/cocotools/

  (copy included into gradschool site)

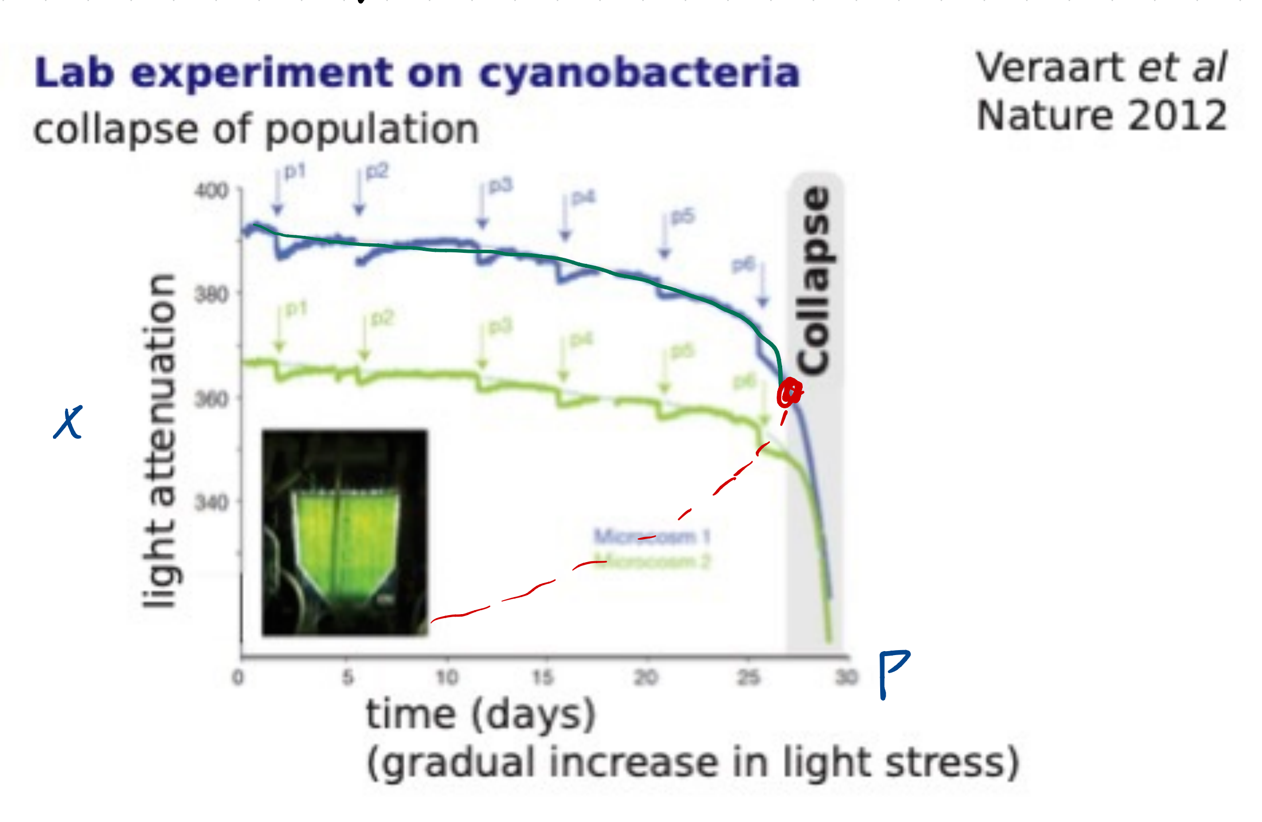
  original authors: Harry Dankowicz, Frank Schilder

- Plan for Monday
  - case study: rate-induced tipping
  - basic bifurcation analysis using COCO

Rate-induced tipping

tipping ~ small change in input, disproportional and irreversible change of output

Prototype ecology of lakes: tipping into



algae are stressed by light (causing increased mortality) but shade each other
=> positive feedback effect

$$\dot{x} = p_0 - p - (x - x_0)^2$$

$$p(t) = p_0 + \varepsilon (t - t_0)$$

+ gradual increase of p through saddle-node bifurcation

Observation by Wieczorek et al '2011 of curious phenomenon (compost bomb)

(*) $\boxed{\dot{x} = (x+L)^2 - a}$, $a>0$ fixed, L depends on time

for all fixed L (*) has stable equilibrium at $-\sqrt{a}-L$

unstable equilibrium at $\sqrt{a}-L$

What happens if L increases rapidly?

- $L(t) = c \cdot t \Rightarrow \dot{x} = (x+ct)^2 - a$, $y(t) = x(t) + ct$

$\Rightarrow \dot{y} = \dot{x} + c = y^2 - a + c$

So if $c>a$: $(\dot{y} = y^2 + c - a)$ no stable equilibrium

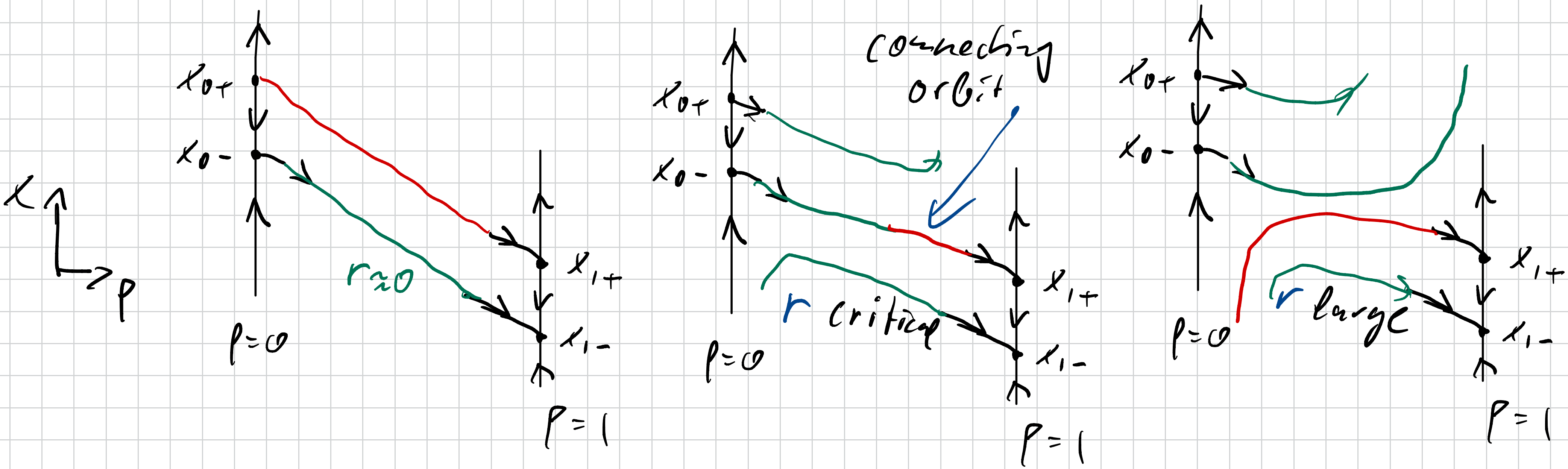
(*) $\boxed{\dot{x} = (x+L(t))^2 - a}$

• (ramp) eg. $L(t) = \dfrac{c}{1+\exp(-2rt)}$

introduce $p(t)$, $\dot{p} = 2rp(1-p) \rightarrow p(t) = \dfrac{1}{1+\exp(-2rt)}$ is sol.

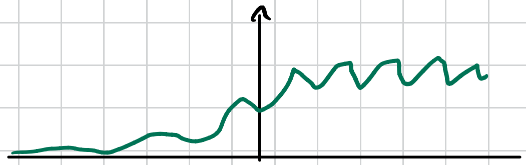
Overall
$$\dot{p} = 2rp(1-p)$$
$$\dot{x} = (x+cp)^2 - a$$
in phase plane equilibria $p=0,\ x_{0,\pm} = \pm\sqrt{a}$

$p=1,\ x_{1,\pm} = \pm\sqrt{a} - c$



$\Rightarrow$ there is critical rate r at which

- ramp and growing oscillations $L(t)$



$$(*)\quad \boxed{\dot{x} = (x+L(t))^2 - a} \qquad L(t) = \frac{c - \beta \sin(\omega t - \varphi)}{1 + \exp(-2rt)}$$

$\Rightarrow$ demo_escape.m $\Rightarrow$ escape depends on phase φ

Use continuation to find precise boundaries and track them in other system parameters: connecting orbit from equilibrium to

Steps:

- Write autonomous ODE for $(*)$

$$(1)\begin{cases} \dot{p} = pr - q\omega - p(p^2+q^2) \\ \dot{q} = p\omega + qr - q(p^2+q^2) \end{cases} \qquad L = c(p^2+q^2) + \beta q \sqrt{p^2+q^2}$$

$$(2)\quad \dot{x} = (x+L)^2 - a$$

(1) drives (2)

Phase portrait of (1)