Basic background / motivation for Bifurcation analysis

Consider ODEs $\dot{x}(t) = f(x(t), p)$ $\quad x(t) \in \mathbb{R}^{n_x}$; $f: \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$

Initial-value problems can be safely solved numerically

$x(0) = x_0$, $\quad \varphi^T(x_0) :=$ solution of ODE at time T when starting from x_0 at time 0,

$x_0 \mapsto \varphi^T(x_0)$ is diffeomorphism of $\mathbb{R}^n$, is approximated by x_N where $x_{k+1} = x_k + h f(x_k, p)$, $\quad \|x_N - \varphi^T(x_0)\| = O(N^{-1})$

$(N \gg 1,\ h = T/N)$

Trouble arises when studying long-time behaviour:

$$\|x_N - \varphi^T(x_0)\| \le \underbrace{\exp(LT)}_{\text{if } T \text{ is large, this is uncontrollably large}} \frac{T}{N}, \quad L \sim \|\partial f\|$$

Observation of the $\varphi^T(x_0)$ converges to invariant sets.

Simple invariant sets can be computed directly:

- **Equilibria:** solve $0 = f(x,p)$ nonlinear system
  $\Rightarrow x_{eq}(p)$
- if $\partial_x f(x_{eq},p) \in \mathbb{R}^{n_x \times n_x}$ has no eigenvalues on imaginary axis
  behaviour of $y = x - x_{eq}(p)$ for small y is governed
  by $\dot{y} = \partial_x f(x_{eq},p)\, y$ (linear ODE)
  all eigenvalues λ have $\mathrm{Re}\,\lambda < 0 \Rightarrow x_{eq}$ stable
  some $\mathrm{Re}\,\lambda > 0 \Rightarrow x_{eq}$ unstable
- Eigenvalues on imaginary axis $\rightarrow$ "**local Bifurcation**"

$$\left.\begin{array}{ll} \lambda = 0 & \Rightarrow \text{Saddle-node} \\ \lambda = \pm i\omega & \Rightarrow \text{Hopf Bifurcation} \end{array}\right\}\ \text{see Kuznetsov Applied Bifurcation theory}$$

Periodic orbits are trajectories $\varphi^{(\cdot)}(x)$ such that $\varphi^T(x) = x$ for some $T>0$, but not all $T>0$

They are found by solving periodic boundary-value problem

$$x'(t) = T f(x(t), p) \quad \Leftarrow \text{ rescaled time}$$
$$0 = x(1) - x(0)$$
$$(\text{e.g.}) \quad 0 = n^T x(0) - c \qquad (\text{Poincaré / return section})$$

where $f(x(0), p)$, n are linearly independent

- periodic orbits branch off near equilibria at Hopf bifurcation
- their stability is determined by eigenvalues of $\partial_x \varphi^T(x)$