

Getting Started with COCO: Basic Bifurcation Analysis

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Contents

1	Introduction	2
2	Starting coco	2
3	Bifurcations of equilibria	3
3.1	Detecting saddle-node points	3
3.2	Detecting branch points	5
3.3	Continuing saddle-node points	6
3.4	Continuing Hopf bifurcations	8
3.5	Performing parameter sweeps	10
4	Bifurcations of periodic orbits	14
4.1	Continuing from Hopf bifurcations	14
4.2	Finding isolated curves and branch points	17
4.3	A web of bifurcations	22
5	Post-processing and analysis	29
5.1	Bifurcation data	29
5.2	Individual solutions	33

6	Function definitions	35
6.1	Vector fields	35
6.2	Partial derivatives	37
6.3	Directional derivatives	39
7	User-initiated enhancements	43
7.1	Monitoring properties of equilibria	43
7.2	Continuing generalized Hopf bifurcations	47
7.3	Monitoring properties of periodic orbits	51
7.4	Tracking and constraining orbit maxima	54
8	Outlook	60

1 Introduction

This text is intended to demonstrate a level of basic usage of the COCO software package that most students (and teachers) of nonlinear dynamics would find useful. It takes inspiration from a set of examples and demos developed by Frank Schilder for the four-day course “Continuation Methods and COCO” given in conjunction with the CRITICS (Critical Transitions in Complex Systems) Winter Workshop in Barnett Hill, UK, in January 2016.

Bifurcation analysis collects techniques for characterizing the dependence of certain classes of solutions of a dynamical system to variations in problem parameters. Common examples include equilibria and periodic orbits, the number and stability of which may vary as parameters vary. Continuation techniques generate continuous families of such solutions in the combined state and parameter space, e.g., curves of periodic orbits or surfaces of equilibria. An advantage to simulation-based approaches is the ability to map out such families independently of the dynamic stability of the equilibria or periodic orbits. Bifurcation diagrams represent families of equilibria and periodic orbits as curves or surfaces in appropriate coordinate systems. Special points, such as bifurcations, are often highlighted in such diagrams.

2 Starting COCO

To get started with COCO, download the most recent release from the online repository. Make a habit of visiting this once a year in the event of minor or major upgrades. If necessary, uncompress the downloaded file and put the content in a preferred location. Navigate in MATLAB to the folder in the COCO installation that contains the file `startup.m` and execute the `startup` command on the command line.

```
>> startup
```

This adds key COCO folders to the top of the MATLAB search path. You can see the current search path by executing the `path` command.

```
>> path
```

You are now ready to use COCO according to the normal MATLAB rules. All examples in this tutorial are included in the COCO release in the `tutorials/Getting Started/examples` folder. Run them as is, reflect on code comments, or copy and modify for your own needs.

3 Bifurcations of equilibria

This section demonstrates the use of COCO for performing continuation along families of equilibria, detecting bifurcations of equilibria, performing continuation along families of bifurcations, and visualizing the results of computation. The calling syntax is deliberately tight and mostly unexplained (but see the corresponding demos in the COCO release).

3.1 Detecting saddle-node points

Suppose that you seek to compute a one-dimensional family of equilibria of the scalar dynamical system $\dot{x} = p - x^2$, known as the *fold normal form*. Such equilibria are roots of the *vector field* $f : (x, p) \mapsto p - x^2$, i.e., points in the (x, p) coordinate system that lie on the curve $p = x^2$. It follows that $p \geq 0$ along this curve and that the curve folds back on itself at the *saddle-node bifurcation* $x = p = 0$. From $\partial_x f(x, p) = -2x$, it follows that equilibria with $x < 0$ are unstable while those with $x > 0$ are locally asymptotically stable.

Use COCO to compute a discrete set of points along the curve of equilibria as follows.

```
>> f = @(x,p) p-x.^2;
>> coco('run1', 'ode', 'isol', 'ep', f, 0.5, 'p', 1, 'p', [-1 1]);
```

STEP	DAMPING		NORMS			COMPUTATION TIMES			
	IT	SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0					7.50e-01	1.50e+00	0.0	0.0	0.0
1	1	3.56e-01	7.50e-01	4.11e-01	1.61e+00	0.0	0.0	0.0	0.0
2	1	1.00e+00	2.68e-01	7.18e-02	1.75e+00	0.0	0.0	0.0	0.0
3	1	1.00e+00	3.47e-02	1.20e-03	1.73e+00	0.0	0.0	0.0	0.0
4	1	1.00e+00	6.02e-04	3.62e-07	1.73e+00	0.0	0.0	0.0	0.0
5	1	1.00e+00	1.81e-07	3.38e-14	1.73e+00	0.0	0.0	0.0	0.0

STEP	TIME	U	LABEL	TYPE	p
0	00:00:00	1.7321e+00	1	EP	1.0000e+00
10	00:00:00	9.9654e-02	2		9.7411e-03
14	00:00:00	1.3380e-06	3	FP	7.8762e-09
14	00:00:00	1.6856e-08	4	SN	7.8780e-09
20	00:00:00	2.1653e-01	5		4.3161e-02
30	00:00:00	1.3657e+00	6		7.4753e-01
31	00:00:00	1.7321e+00	7	EP	1.0000e+00

The screen output shows two stages of computation. First, five iterations of a *Newton method* are used to converge from the (rather poor) initial solution guess $(x, p) = (0.5, 1)$ to an initial equilibrium point for which $p = 1$. In the fifth iteration, a problem residual, denoted by

$\|f\|$ evaluates to 3.38×10^{-14} . In the same iteration, the size of the corresponding Newton correction, denoted by $\|\delta\|$ evaluates to 1.81×10^{-7} . Both are below the default tolerance threshold of 10^{-6} , which is why the first stage terminates at this iteration.

Next, a *pseudo-arclength continuation algorithm* computes 31 additional equilibria, seven of which are captured in the screen output and labeled 1 through 7. The number of steps of continuation is limited from above by a maximum that defaults to 100 in each direction of the curve from the initial point. This limit is not reached here, since the computation starts and ends on the boundary of the *computational domain* $p \in [-1, 1]$.

Of the seven labeled points, the first and last are identified as end points (`EP`), while the third is identified as a fold point (`FP`) along the curve of equilibria and the fourth as a saddle-node bifurcation (`SN`). Although dynamical systems theory predicts that the latter two should coincide, this is only approximately true in this computation due to the use of distinct algorithms to detect the two special points.

You can visualize the results of the computation by graphing an interpolated curve through the computed equilibrium points in a two-dimensional coordinate system with p along the horizontal axis and x along the vertical axis, as shown below (cf. Fig. 1).

```
>> theme = struct('special', {{'SN', 'EP'}});
>> coco_plot_bd(theme, 'run1', 'p', 'x')
```

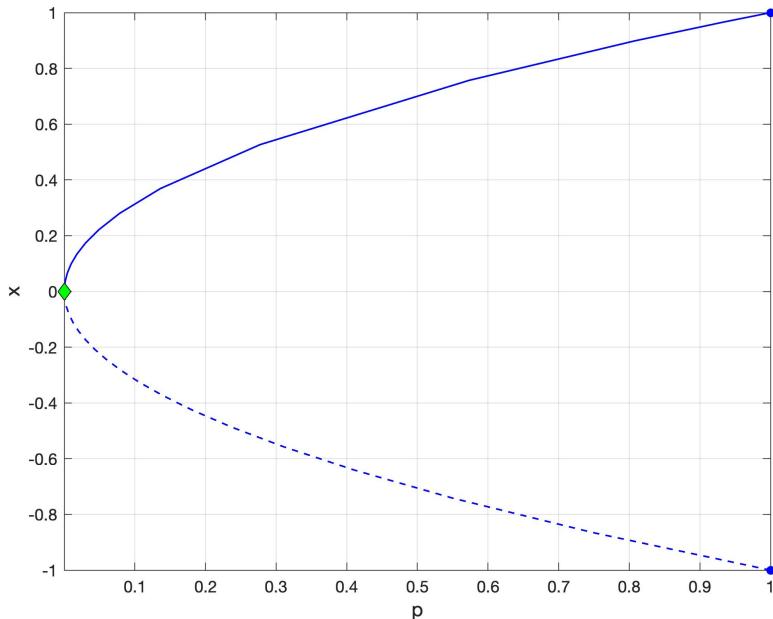


Figure 1: Bifurcation diagram showing branches of asymptotically stable (solid) and unstable (dashed) equilibria for the vector field in Section 3.1. The green diamond marks a saddle-node bifurcation while the small blue disks mark the end points of the computation.

The output distinguishes between portions of the curve representing stable and unstable equilibria and includes markers highlighting the `EP` and `SN` points.

3.2 Detecting branch points

There exists a single curve of equilibria for the fold normal form. For the *transcritical normal form* $\dot{x} = px - x^2$, equilibria correspond to points on the two curves $x = 0$ and $p = x$. These intersect at a *transcritical bifurcation* at $x = p = 0$. From $\partial_x f(x, p) = p - 2x$, it follows that equilibria along $x = 0$ are locally asymptotically stable for $p < 0$ and unstable for $p > 0$, while the reverse holds along $p = x$.

You can use COCO to compute a discrete set of points along the first curve of equilibria.

```
>> f = @(x,p) p.*x-x.^2;
>> coco('run1', 'ode', 'isol', 'ep', f, 0, 'p', -1, 'p', [-1 1]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA    ||d||    ||f||    ||U||   F(x)   DF(x)   SOLVE
        0           0.00e+00  1.41e+00    0.0     0.0     0.0

      STEP      TIME      ||U||  LABEL  TYPE      p
      0  00:00:00  1.4142e+00    1 EP    -1.0000e+00
      5  00:00:00  2.2484e-07    2 SN     1.5899e-07
      5  00:00:00  2.2484e-07    3 BP     1.5899e-07
      8  00:00:00  1.4142e+00    4 EP     1.0000e+00
```

In this case, the first stage of computation terminates immediately, since the initial solution guess $(x, p) = (0, -1)$ lies on the curve $x = 0$. In the second stage, the continuation algorithm computes 8 additional equilibria. Of the equilibria labeled 1 through 4, the first and last are again identified as end points (EP), while the second and third are identified as a saddle-node bifurcation (SN) and branch point (BP), respectively, and approximately coincide with the origin. The characterization of this point as a saddle-node bifurcation is of course contrary to dynamical systems theory, but reflects the default behavior of the detection algorithm.

You can continue from the BP point along the second curve of equilibria as follows.

```
>> BP = coco_bd_labs('run1', 'BP');
>> prob = coco_set(coco_prob, 'cont', 'branch', 'switch');
>> coco('run2', 'ode', 'ep', 'ep', 'run1', BP, 'p', [-1 1]);

      STEP      TIME      ||U||  LABEL  TYPE      p
      0  00:00:00  2.2484e-07    1 EP     1.5899e-07
      7  00:00:00  1.7321e+00    2 EP    -1.0000e+00

      STEP      TIME      ||U||  LABEL  TYPE      p
      0  00:00:00  2.2484e-07    3 EP     1.5899e-07
      1  00:00:00  6.8132e-07    4 BP     4.6035e-07
      1  00:00:00  9.7048e-07    5 SN     6.4659e-07
      7  00:00:00  1.7321e+00    6 EP     1.0000e+00
```

In this case, there is no first stage of computation (since the branch point is a singularity), and the second stage proceeds in two opposite directions along the curve $p = x$. To visualize the two curves of equilibria, use the `coco_plot_bd` command twice (cf. Fig. 2).

```
>> clf
```

```

>> hold on
>> theme1 = struct('special', {{'BP', 'EP'}});
>> coco_plot_bd(theme1, 'run1', 'p', 'x')
>> theme2 = struct('special', {{'EP'}});
>> coco_plot_bd(theme2, 'run2', 'p', 'x')
>> hold off
>> axis tight
>> grid on

```

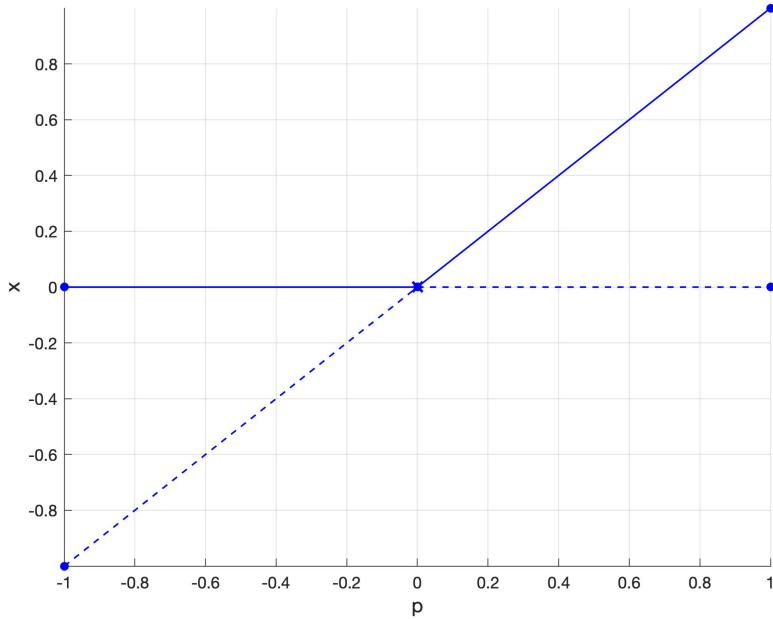


Figure 2: Bifurcation diagram showing branches of asymptotically stable (solid) and unstable (dashed) equilibria for the vector field in Section 3.2. The blue cross marks a branch point while the small blue disks mark the end points of each computation.

3.3 Continuing saddle-node points

As a third example, consider the *pitchfork normal form* $\dot{x} = \kappa + x(\lambda - x^2)$ in the two problem parameters λ and κ . Equilibria lie on the two-dimensional surface $\kappa = x(x^2 - \lambda)$ which includes the curves $s \mapsto (0, s, 0)$, $s \mapsto (s, s^2, 0)$, $s \mapsto (s, 1/2, s(s^2 - 1/2))$, and $s \mapsto (s, 3s^2, -2s^3)$ in the (x, λ, κ) coordinate system. The first two of these intersect at a *pitchfork bifurcation* at $x = \lambda = \kappa = 0$, while the fourth curve consists entirely of saddle-node bifurcations and intersects the third curve when $s^2 = 1/6$. The fourth curve passes through the origin in a fold point known as a *cusp bifurcation*.

To compute the four curves with COCO, use the following sequence of commands.

```

>> f = @(x,p) p(2,:)+x.* (p(1,:)-x.^2);
>> coco('run1', 'ode', 'isol', 'ep', ...
        f, 0, {'la' 'ka'}, [-2; 0], 'la', [-2 2])

```

STEP	DAMPING	NORMS			COMPUTATION TIMES			
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)	DF(x)	SOLVE
0				0.00e+00	2.83e+00	0.0	0.0	0.0

STEP	TIME	$\ U\ $	LABEL	TYPE	la
0	00:00:00	2.8284e+00	1	EP	-2.0000e+00
8	00:00:00	6.4118e-07	2	SN	4.5339e-07
8	00:00:00	6.4118e-07	3	BP	4.5339e-07
10	00:00:00	1.3226e+00	4		9.3520e-01
14	00:00:00	2.8284e+00	5	EP	2.0000e+00


```
>> BP = coco_bd_labs('run1', 'BP');
>> prob = coco_set(coco_prob, 'cont', 'branch', 'switch');
>> coco(prob, 'run2', 'ode', 'ep', 'ep', ...
    'run1', BP, 'la', [-2 2]);
```


STEP	TIME	$\ U\ $	LABEL	TYPE	la
0	00:00:00	6.4118e-07	1	EP	4.5339e-07
1	00:00:00	3.0228e-06	2	SN	-2.3840e-10
1	00:00:00	2.8974e-03	3	FP	-1.8413e-04
10	00:00:00	2.8084e-01	4		6.9276e-02
20	00:00:00	2.0444e+00	5		1.2170e+00
23	00:00:00	3.1623e+00	6	EP	2.0000e+00

STEP	TIME	$\ U\ $	LABEL	TYPE	la
0	00:00:00	6.4118e-07	7	EP	4.5339e-07
1	00:00:00	6.8665e-07	8	BP	4.7487e-07
1	00:00:00	3.4690e-02	9	SN	1.2015e-03
10	00:00:00	6.4635e-01	10		2.7095e-01
18	00:00:01	3.1623e+00	11	EP	2.0000e+00


```
>> coco(prob, 'run3', 'ode', 'isol', 'ep', ...
    f, 0, {'la' 'ka'}, [0.5; 0], 'ka', [-2 2]);
```


STEP	DAMPING	NORMS			COMPUTATION TIMES			
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)	DF(x)	SOLVE
0				0.00e+00	5.00e-01	0.0	0.0	0.0

STEP	TIME	$\ U\ $	LABEL	TYPE	ka
0	00:00:00	5.0000e-01	1	EP	0.0000e+00
6	00:00:00	6.7357e-01	2	FP	-1.3608e-01
6	00:00:00	6.7358e-01	3	SN	-1.3608e-01
10	00:00:00	7.3824e-01	4		-1.2073e-01
20	00:00:00	9.4972e-01	5		1.0362e-01
28	00:00:00	3.1917e+00	6	EP	2.0000e+00

STEP	TIME	$\ U\ $	LABEL	TYPE	ka
0	00:00:00	5.0000e-01	7	EP	0.0000e+00
6	00:00:00	6.7357e-01	8	FP	1.3608e-01
6	00:00:00	6.7358e-01	9	SN	1.3608e-01
10	00:00:00	7.3824e-01	10		1.2073e-01
20	00:00:01	9.4972e-01	11		-1.0362e-01
28	00:00:01	3.1917e+00	12	EP	-2.0000e+00

```

>> SN = coco_bd_labs('run3', 'SN');
>> coco(prob, 'run4', 'ode', 'SN', 'SN', ...
    'run3', SN(1), {'la' 'ka'}, {[ -2 2] [ -2 2]})

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)      DF(x)      SOLVE
      0           5.07e-08   1.31e+00      0.0       0.0       0.0

      STEP      TIME      ||U||      LABEL      TYPE      la      ka
      0  00:00:00  1.3053e+00      1  EP      5.0000e-01  -1.3608e-01
      10 00:00:00  1.0007e+00      2
      14 00:00:00  1.0000e+00      3  FP      9.4219e-09  1.0483e-11
      20 00:00:00  1.0023e+00      4
      30 00:00:00  1.0722e+00      5
      40 00:00:01  3.4694e+00      6  EP      2.0000e+00  1.0887e+00

      STEP      TIME      ||U||      LABEL      TYPE      la      ka
      0  00:00:01  1.3053e+00      7  EP      5.0000e-01  -1.3608e-01
      7  00:00:01  3.4694e+00      8  EP      2.0000e+00  -1.0887e+00

```

The second and fourth curves are here obtained by restarting continuation from a `BP` and `SN` point, respectively, found in the preceding run. The four curves may be visualized simultaneously using the following commands (cf. Fig. 3).

```

>> clf
>> theme1 = struct('special', {{'BP', 'EP'}});
>> theme2 = struct('special', {{'EP'}});
>> theme3 = struct('special', {{'SN', 'EP'}});
>> hold on
>> coco_plot_bd(theme1, 'run1', 'la', 'ka', 'x')
>> coco_plot_bd(theme2, 'run2', 'la', 'ka', 'x')
>> coco_plot_bd(theme3, 'run3', 'la', 'ka', 'x')
>> coco_plot_bd(theme2, 'run4', 'la', 'ka', 'x')
>> hold off
>> axis tight
>> grid on
>> view(-15,25)

```

As with all examples in this tutorial, the visualization is best inspected in a MATLAB figure window.

3.4 Continuing Hopf bifurcations

Next, consider a so-called Brusselator model described by the vector field

$$f(x, p) = \begin{pmatrix} A - x_1(B - x_1 x_2) - x_1 \\ x_1(B - x_1 x_2) \end{pmatrix}$$

in terms of the vector of state variables $x = (x_1, x_2) \in \mathbb{R}^2$ and vector of problem parameters $p = (A, B) \in \mathbb{R}^2$, and encoded in the `brus` function shown below.

```
function f = brus(x, p)
```

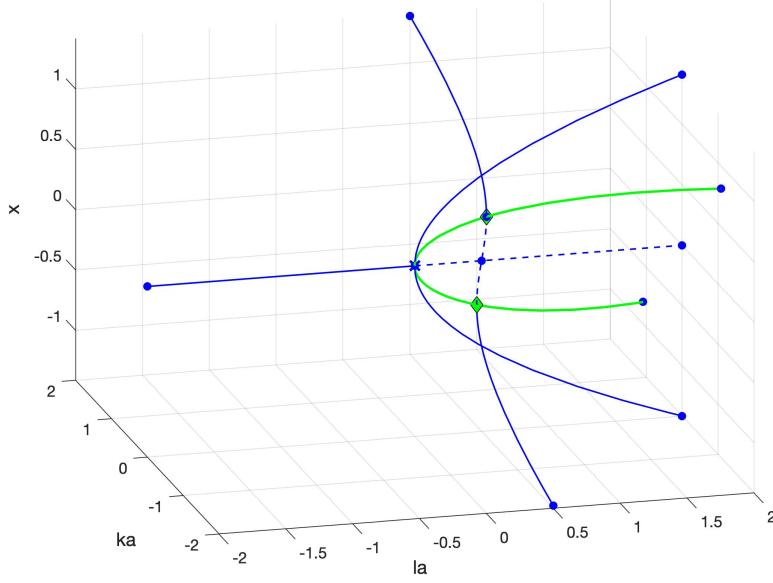


Figure 3: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria and saddle-node bifurcations (solid green) for the vector field in Section 3.3. The blue cross marks a branch point, the green diamonds mark saddle-node bifurcations, and the small blue disks mark the end points of each computation.

```

A = p(1,:);
B = p(2,:);
x1 = x(1,:);
x2 = x(2,:);

BB = x1.* (B - x1.*x2);

f(1,:) = A - BB - x1;
f(2,:) = BB;

end

```

Equilibria are found on the two surfaces $A = x_1 = 0$ and $A = x_1, B = x_1x_2$. The latter includes the curves $s \mapsto (1, s, 1, s)$ and $s \mapsto (s, (1 + s^2)/s, s, 1 + s^2)$ in the (x_1, x_2, A, B) coordinate system. The second of these consists of *Hopf bifurcations*. The two curves may be computed with COCO as follows.

```

>> coco('run1', 'ode', 'isol', 'ep', ...
        @brus, [1; 0], {'A' 'B'}, [1; 0], 'B', [0 3]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)    DF(x)   SOLVE
        0           0.00e+00  1.41e+00

```

STEP	TIME	$\ U\ $	LABEL	TYPE	B
0					

```

0 00:00:00 1.4142e+00      1 EP      0.0000e+00
9 00:00:00 3.7417e+00      2 HB      2.0000e+00
10 00:00:00 4.3853e+00     3          2.3966e+00
13 00:00:00 5.3852e+00     4 EP      3.0000e+00

>> HB = coco_bd_labs('run1', 'HB');
>> coco('run2', 'ode', 'HB', 'HB', ...
    'run1', HB(1), {'A' 'B'}, {[[] [0 3]});

      STEP   DAMPING           NORMS           COMPUTATION TIMES
      IT SIT   GAMMA    ||d||    ||f||    ||U||   F(x)  DF(x)  SOLVE
        0          1.32e-07  4.29e+00   0.0    0.0    0.0

STEP      TIME      ||U||  LABEL  TYPE      A          B
 0 00:00:00 4.2895e+00   1 EP      1.0000e+00 2.0000e+00
10 00:00:00 3.3815e+00  2          5.1142e-01 1.2616e+00
20 00:00:00 4.7582e+00  3          2.4401e-01 1.0595e+00
30 00:00:01 9.5180e+00  4          1.0840e-01 1.0117e+00
40 00:00:01 1.4452e+01  5          7.0111e-02 1.0049e+00
50 00:00:01 1.9421e+01  6          5.1864e-02 1.0027e+00
60 00:00:02 2.4403e+01  7          4.1166e-02 1.0017e+00
70 00:00:02 2.9391e+01  8          3.4131e-02 1.0012e+00
80 00:00:02 3.4383e+01  9          2.9151e-02 1.0008e+00
90 00:00:02 3.9376e+01  10         2.5440e-02 1.0006e+00
100 00:00:03 4.4371e+01 11 EP      2.2568e-02 1.0005e+00

STEP      TIME      ||U||  LABEL  TYPE      A          B
 0 00:00:03 4.2895e+00  12 EP      1.0000e+00 2.0000e+00
 7 00:00:03 6.2849e+00  13 EP      1.4142e+00 3.0000e+00

```

The following sequence of commands visualizes the result (cf. Fig. 4).

```

>> clf
>> hold on
>> theme1 = struct('special', {{'HB', 'EP'}});
>> theme2 = struct('special', {{'EP'}});
>> coco_plot_bd(theme1, 'run1', 'B', 'A', '||x||_2')
>> coco_plot_bd(theme2, 'run2', 'B', 'A', '||x||_2')
>> hold off
>> axis tight
>> grid on
>> view(-65, 40)

```

3.5 Performing parameter sweeps

To construct a family of curves of equilibria for discrete values of a second problem parameter, embed the calls to `coco` and `coco_plot_bd` in a loop. Consider, for example, the vector field

$$f(x, p) = \begin{pmatrix} D(1 - x_1)e^{x_3} - x_1 \\ D(1 - x_1 - \sigma x_2)e^{x_3} - x_2 \\ BD(1 - x_1 + \alpha\sigma x_2)e^{x_3} - (1 + \beta)x_3 \end{pmatrix}$$

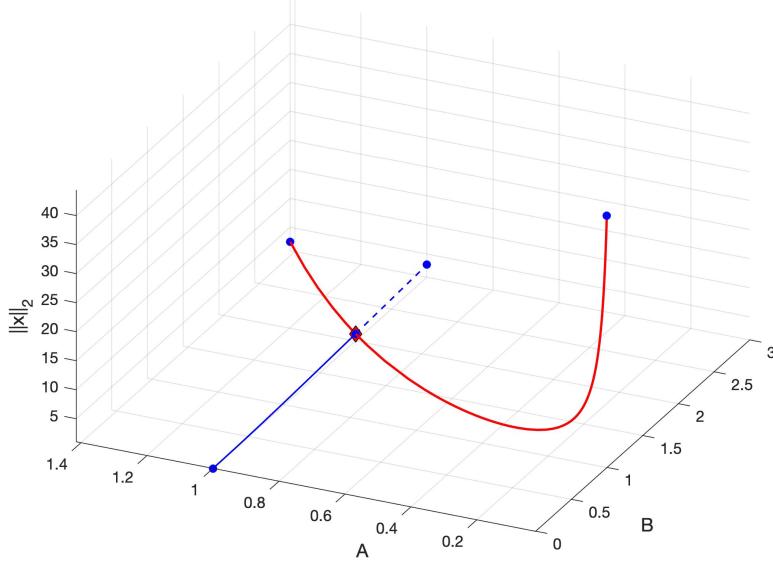


Figure 4: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria and Hopf bifurcations (solid red) for the vector field in Section 3.4. The red diamond marks a Hopf bifurcation, while the small blue disks mark the end points of each computation.

in terms of the vector of state variables $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and vector of problem parameters $p = (\alpha, \sigma, D, B, \beta) \in \mathbb{R}^5$, and encoded in the function `abc` shown below.

```

function f = abc(x,p)

al = p(1,:);
si = p(2,:);
D = p(3,:);
B = p(4,:);
be = p(5,:);

x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);

e3 = exp(x3);

f(1,:) = D.* (1-x1).*e3-x1;
f(2,:) = D.* (1-x1-si.*x2).*e3-x2;
f(3,:) = D.*B.* (1-x1+al.*si.*x2).*e3-x3.* (1+be);

end

```

As in the previous examples, it is possible to derive explicit parameterizations of (five-dimensional) surfaces of equilibria in the $(x_1, x_2, x_3, \alpha, \sigma, D, B, \beta)$ coordinate system and, with greater effort, of four-dimensional surfaces of saddle-node and Hopf bifurcations. As

an alternative, for fixed α , σ , B , and a sequence of values of β , the following sequence of commands graphs the corresponding curves of equilibria in a coordinate system with D along the horizontal axis and $\sqrt{x_1^2 + x_2^2 + x_3^2}$ along the vertical axis (cf. Fig. 5).

```
>> clf
>> hold on
>> axis([0.12 0.22 1 7])
>> grid on
>> Nrun = 0;
>> theme = struct('special', {'SN' 'HB'});
>> for beta = linspace(1.20, 1.42, 23)
    Nrun = Nrun+1;
    runid = sprintf('run_beta_%d', Nrun);
    coco(runid, 'ode', 'isol', 'ep', @abc, [0; 0; 0], ...
        {'al' 'si' 'D' 'B' 'be'}, [1; 0.04; 0.0; 8; beta], ...
        {'D' 'be'}, [0.0 0.25]);
    coco_plot_bd(theme, runid, 'D', '|x||_2');
end
```

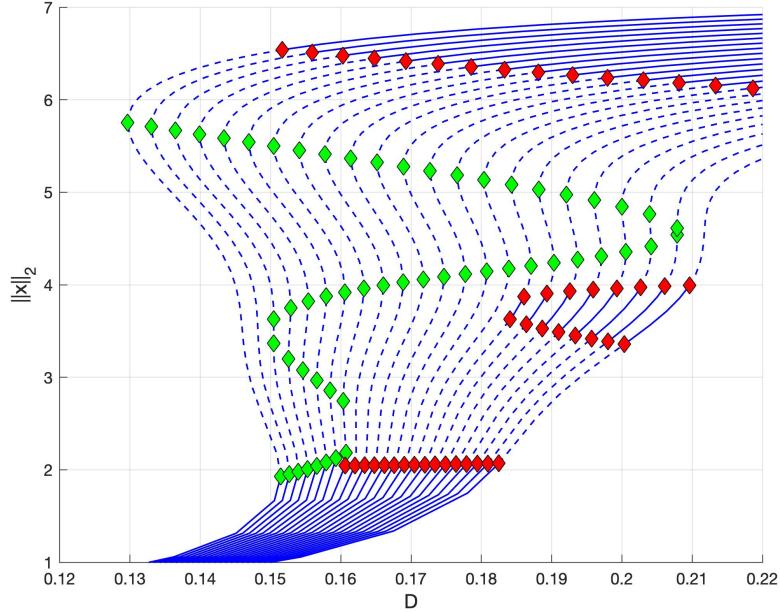


Figure 5: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria for the vector field in Section 3.5. The green and red diamonds mark saddle-node and Hopf bifurcations, respectively.

Although the screen output is omitted, you will see that the computation signals detection of saddle-node bifurcations and Hopf bifurcations. As expected, it is possible to restart continuation from an `SN` or `HB` point in order to compute a family of saddle-node or Hopf bifurcations, respectively. This is accomplished using the commands below.

```
>> runid = 'run_beta_1';
>> SN = coco_bd_labs(runid, 'SN');
```

```

>> coco('run_SN', 'ode', 'SN', 'SN', ...
       runid, SN(1), {'D' 'be'}, [0.1 0.25]);
>> HB = coco_bd_labs(runid, 'HB');
>> prob = coco_set(coco_prob, 'cont', 'PtMX', 150);
>> coco(prob, 'run_HB', 'ode', 'HB', 'HB', ...
       runid, HB(1), {'D' 'be'}, [0.1 0.8]);

```

In the second case, the 'PtMX' setting is changed to increases the maximum number of steps that the continuation algorithm takes along each direction of the curve of equilibria to 150. The resulting curves are then overlaid on top of the diagram generated previously using the following commands (cf. Fig. 6).

```

>> theme_SN = struct('special', {{'FP'}});
>> theme_HB = struct('special', {{'FP' 'BTP'}});
>> coco_plot_bd(theme_SN, 'run_SN', 'D', '| |x| |_2');
>> coco_plot_bd(theme_HB, 'run_HB', 'D', '| |x| |_2');
>> hold off

```

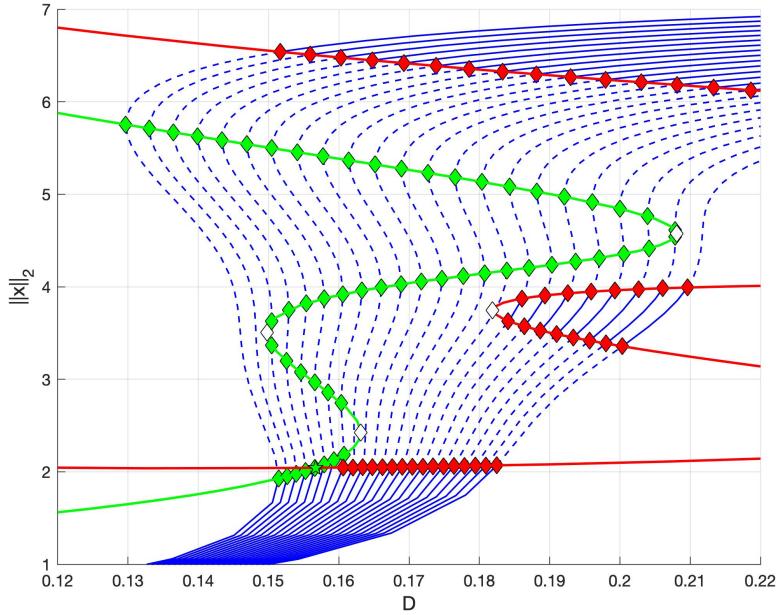


Figure 6: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria, saddle-node bifurcations (solid green), and Hopf bifurcations/neutral saddles (red solid) for the vector field in Section 3.5. The green and red diamonds mark saddle-node and Hopf bifurcations, the white diamonds mark fold points, and the green star at the intersection between the curves of saddle-node bifurcations and Hopf bifurcations/neutral saddles marks a Bogdanov-Takens point.

The visualization includes a marker highlighting a *Bogdanov-Takens bifurcation* denoted by BTP in the corresponding screen output. Although the algorithm continues past this points, the points to the left of this point are not Hopf bifurcations, but *neutral saddles* that satisfy

the algebraic conditions for a Hopf bifurcation, but not the conditions on the eigenvalues of the Jacobian $\partial_x f(x, p)$.

4 Bifurcations of periodic orbits

This section demonstrates the use of COCO for performing continuation along families of periodic orbits, detecting bifurcations of periodic orbits, performing continuation along families of bifurcation points, and visualizing the results of computation. The calling syntax remains tight, but the examples demonstrate a step up in computational complexity relative to the previous section.

4.1 Continuing from Hopf bifurcations

The pattern of analysis and visualization using COCO demonstrated in the previous section also applies to the study of periodic orbits and their bifurcations.

Consider the following unfolding of the Hopf normal form:

$$f(x, p) = \begin{pmatrix} -x_2 + x_1(p_1 + p_2 r^2 - r^4) \\ x_1 + x_2(p_1 + p_2 r^2 - r^4) \end{pmatrix}, r = \sqrt{x_1^2 + x_2^2}$$

in terms of the vector of state variables $x = (x_1, x_2) \in \mathbb{R}^2$ and vector of problem parameters $p = (p_1, p_2) \in \mathbb{R}^2$. Equilibria are found on the two-dimensional surface $x_1 = x_2 = 0$ in the (x_1, x_2, p_1, p_2) coordinate system and are asymptotically stable for $p_1 < 0$ and unstable for $p_1 > 0$. The curve $s \mapsto (0, 0, 0, s)$ consists of Hopf bifurcation points and coincides with the intersection of the surface of equilibria with the two-dimensional family of periodic orbits $x_1(t) = r^* \cos t, x_2(t) = r^* \sin t, p_1 = r^{*4} - p_2 r^{*2}$ for $r^* \geq 0$ with nontrivial Floquet multiplier $\exp(4\pi r^{*2}(p_2 - 2r^{*2}))$. It follows that the Hopf bifurcations are *supercritical* for fixed $p_2 < 0$ and *subcritical* for fixed $p_2 > 0$. In the latter case, the curve $r^* \mapsto (r^* \cos t, r^* \sin t, -r^{*4}, 2r^{*2})$ consists of saddle-node bifurcations of periodic orbits, coincident with a curve of folds along the corresponding curves of periodic orbits for fixed p_2 and under variations in p_1 .

The vector field is encoded in the `hopf` function shown below.

```
function f = hopf(x,p)
    x1 = x(1,:);
    x2 = x(2,:);
    p1 = p(1,:);
    p2 = p(2,:);

    r2 = x1.^2+x2.^2;

    f(1,:) = x1.* (p1+p2.*r2-r2.^2)-x2;
    f(2,:) = x2.* (p1+p2.*r2-r2.^2)+x1;

end
```

As a sometimes optional, but mostly required, alternative to a single call to `coco`, the following demonstrates a separation between problem construction (using the `ode_isol2ep` constructor) and problem analysis for continuation along a curve of equilibria for $p_2 = 1$.

```
>> prob = coco_prob;
>> prob = ode_isol2ep(prob, '', @hopf, [0; 0], {'p1', 'p2'}, [-1; 1]);
>> coco(prob, 'ep_run', [], 'p1', [-1 1]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||    F(x)   DF(x)   SOLVE
        0           0.00e+00  1.73e+00      0.0       0.0       0.0

      STEP      TIME      ||U||  LABEL  TYPE          p1
        0  00:00:00  1.7321e+00      1  EP  -1.0000e+00
        5  00:00:00  1.0000e+00      2  HB  1.5899e-07
        8  00:00:00  1.7321e+00      3  EP  1.0000e+00
```

The following commands use the constructor `ode_HB2po` to continue along a curve of periodic orbits emanating from the `HB` point found in the previous run.

```
>> HB = coco_bd_labs('ep_run', 'HB');
>> prob = coco_prob;
>> prob = ode_HB2po(prob, '', 'ep_run', HB);
>> coco(prob, 'po_run', [], 'p1', [-1 1]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||    F(x)   DF(x)   SOLVE
        0           6.75e-06  8.94e+00      0.0       0.0       0.0
        1      1  1.00e+00  1.96e-06  2.68e-13  8.94e+00      0.0       0.0       0.0
        2      1  1.00e+00  1.14e-12  2.83e-18  8.94e+00      0.0       0.0       0.0

      STEP      TIME      ||U||  LABEL  TYPE          p1
        0  00:00:00  8.9419e+00      1  EP  -9.9933e-07
       10  00:00:00  9.8606e+00      2
       12  00:00:00  1.0251e+01      3  FP  -2.5000e-01
       12  00:00:00  1.0251e+01      4  SN  -2.5000e-01
       20  00:00:00  1.2194e+01      5
       23  00:00:01  1.2762e+01      6  EP  1.0000e+00

      STEP      TIME      ||U||  LABEL  TYPE          p1
        0  00:00:01  8.9419e+00      7  EP  -9.9933e-07
        1  00:00:01  8.9419e+00      8  FP  -1.7508e-10
        2  00:00:01  8.9423e+00      9  BP  -1.7270e-04
       10  00:00:01  9.8546e+00     10
       12  00:00:01  1.0251e+01     11  FP  -2.5000e-01
       12  00:00:01  1.0251e+01     12  SN  -2.5000e-01
       20  00:00:02  1.2188e+01     13
       23  00:00:02  1.2762e+01     14  EP  1.0000e+00
```

The screen output includes three labeled points of type `FP`, two of type `SN`, and one of type `BP`. Of these, the first and third `FP` points denote folds along the curve of periodic orbits, which coincide with (cyclic) saddle-node bifurcations. The second `FP` point and the single `BP` point coincide approximately with the Hopf bifurcation, which is detected as both a branch

point of periodic orbits and a fold along this curve.

The curve of saddle-node bifurcations of periodic orbits that runs through the first SN point found above can be computed using the `ode_po2SN` constructor as shown next.

```
>> SN = coco_bd_labs('po_run', 'SN');
>> prob = coco_prob;
>> prob = ode_po2SN(prob, '', 'po_run', SN(1));
>> coco(prob, 'po_SN_run', [], {'p1', 'p2'}, {[ -1 1] [0.001 3]});
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0				1.21e-05	1.25e+01	0.0	0.0	0.0
1	1	1.00e+00	4.97e-05	7.60e-09	1.25e+01	0.0	0.0	0.0
2	1	1.00e+00	1.70e-08	3.56e-09	1.25e+01	0.0	0.0	0.0

STEP	TIME	U	LABEL	TYPE	p1	p2
0	00:00:00	1.2493e+01	1	EP	-2.5000e-01	1.0000e+00
8	00:00:00	1.3746e+01	2	EP	-1.0000e+00	2.0000e+00

STEP	TIME	U	LABEL	TYPE	p1	p2
0	00:00:00	1.2493e+01	3	EP	-2.5000e-01	1.0000e+00
10	00:00:00	1.1412e+01	4		-6.3961e-04	5.0581e-02
12	00:00:00	1.1357e+01	5	EP	-2.5000e-07	1.0000e-03

The computational domain excludes $p_2 = 0$ to avoid redundant covering of the curve.

Finally, the following commands rely on the `ode_ep2HB` constructor to perform continuation along the curve of Hopf bifurcations with detection of special points at designated values of p_2 .

```
>> prob = coco_prob;
>> prob = ode_ep2HB(prob, '', 'ep_run', HB);
>> prob = coco_add_event(prob, 'UZ', 'p2', setdiff(-2:0.5:3,1));
>> coco(prob, 'ep_HB_run', [], {'p1' 'p2'}, {[ -1 1] [-2.01 3.01]});
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0				2.25e-07	2.65e+00	0.0	0.0	0.0

STEP	TIME	U	LABEL	TYPE	p1	p2
0	00:00:00	2.6458e+00	1	EP	1.5899e-07	1.0000e+00
4	00:00:00	3.0822e+00	2	UZ	1.5399e-07	1.5000e+00
5	00:00:00	3.6056e+00	3	UZ	1.4899e-07	2.0000e+00
6	00:00:00	4.1833e+00	4	UZ	1.4399e-07	2.5000e+00
8	00:00:00	4.7958e+00	5	UZ	1.3899e-07	3.0000e+00
8	00:00:00	4.8083e+00	6	EP	1.3889e-07	3.0100e+00

STEP	TIME	U	LABEL	TYPE	p1	p2
0	00:00:00	2.6458e+00	7	EP	1.5899e-07	1.0000e+00
4	00:00:00	2.3452e+00	8	UZ	1.6399e-07	5.0000e-01
5	00:00:00	2.2361e+00	9	UZ	1.6899e-07	-2.7756e-17
6	00:00:00	2.3452e+00	10	UZ	1.7399e-07	-5.0000e-01
8	00:00:00	2.6458e+00	11	UZ	1.7899e-07	-1.0000e+00
9	00:00:00	3.0822e+00	12	UZ	1.8399e-07	-1.5000e+00
10	00:00:00	3.5341e+00	13		1.8834e-07	-1.9352e+00

11 00:00:00	3.6056e+00	14 UZ	1.8899e-07	-2.0000e+00
11 00:00:00	3.6167e+00	15 EP	1.8909e-07	-2.0100e+00

Each such special point is the basis for a separate continuation run along a curve of periodic orbits using the commands shown below.

```
>> labs = coco_bd_labs('ep_HB_run', 'UZ');
>> for lab=labs
    runid = sprintf('po%d_run', lab);
    prob = coco_prob;
    prob = ode_HB2po(prob, '', 'ep_HB_run', lab);
    coco(prob, runid, [], {'p1' 'p2'}, [-1.01 1]);
end
```

The following sequence of commands visualizes the results of these computations (cf. Fig. 7).

```
>> clf; hold on; grid on; box on
>> axis([-1.5 1 -2 4 0 1.6]); view(3)
>> thm = struct('zlab', '||x||_2');
>> thm.special = {'EP', 'HB'};
>> coco_plot_bd(thm, 'ep_run', 'p1', 'p2', '||x||_2')
>> thm.special = {'EP', 'SN'};
>> coco_plot_bd(thm, 'po_run', 'p1', 'p2', '||x||_{2,MPD}')
>> thm.special = {};
>> coco_plot_bd(thm, 'po_SN_run', 'p1', 'p2', '||x||_{2,MPD}')
>> thm.special = {};
>> coco_plot_bd(thm, 'ep_HB_run', 'p1', 'p2', '||x||_2')
>> thm.special = {'EP'}
>> for lab=labs
    runid = sprintf('po%d_run', lab);
    coco_plot_bd(thm, runid, 'p1', 'p2', '||x||_{2,MPD}')
end
>> hold off
```

Here, the vertical axis represents the Euclidean norm $\|x\|_2$ for the equilibria and a suitably defined scalar magnitude for the periodic orbits.

4.2 Finding isolated curves and branch points

As needed, the default pseudo-arc length continuation algorithm used by COCO relies on finite-difference approximations for derivatives of the vector field with respect to x and p . When exact derivatives are available, these may be provided to COCO in order to eliminate a source of approximation errors, especially during continuation of bifurcations. As an example, the following functions encode the vector field in Section 3.5 and its first derivatives with respect to x and p .

```
function f = abc(x,p)

al = p(1,:);
si = p(2,:);
D = p(3,:);
```

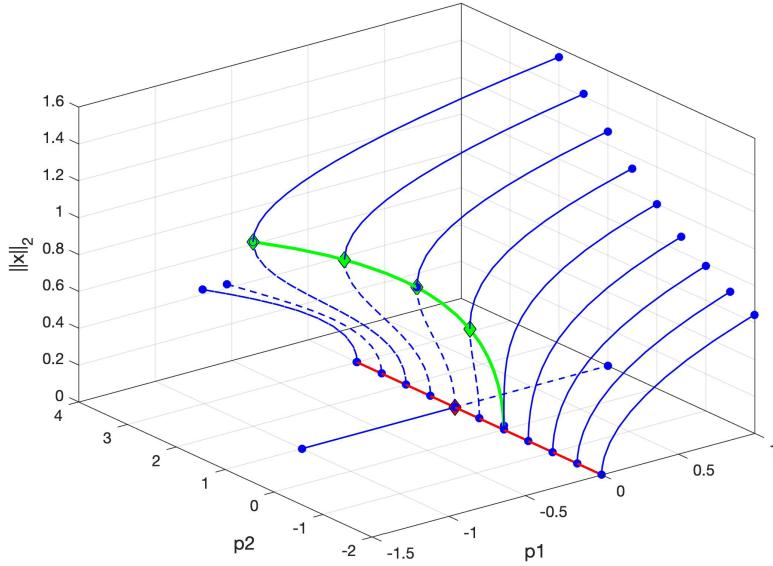


Figure 7: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria and periodic orbits, saddle-node bifurcations of periodic orbits (solid green), and Hopf bifurcations (red solid) for the vector field in Section 4.1. The green and red diamonds mark saddle-node and Hopf bifurcations, and the blue disks mark the end points of each computation.

```

B = p(4,:);
be = p(5,:);

x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);

e3 = exp(x3);

f(1,:) = D.* (1-x1).*e3-x1;
f(2,:) = D.* (1-x1-si.*x2).*e3-x2;
f(3,:) = D.*B.* (1-x1+al.*si.*x2).*e3-x3.* (1+be);

end

```

```

function J = abc_dx(x,p)

al = p(1,:);
si = p(2,:);
D = p(3,:);
B = p(4,:);
be = p(5,:);

x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);

```

```

e3 = exp(x3);

J = zeros(3,3,numel(e3));
J(1,1,:) = -1-D.*e3;
J(1,3,:) = D.*(1-x1).*e3;
J(2,1,:) = -D.*e3;
J(2,2,:) = -1-D.*si.*e3;
J(2,3,:) = D.*(1-x1-si.*x2).*e3;
J(3,1,:) = -D.*B.*e3;
J(3,2,:) = D.*B.*al.*si.*e3;
J(3,3,:) = -1-be+D.*B.*(1-x1+al.*si.*x2).*e3;

end

```

```

function J = abc_dp(x,p)

al = p(1,:);
si = p(2,:);
D = p(3,:);
B = p(4,:);

x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);

e3 = exp(x3);

J = zeros(3,5,numel(e3));
J(1,3,:) = (1-x1).*e3;
J(2,2,:) = -D.*x2.*e3;
J(2,3,:) = (1-x1-si.*x2).*e3;
J(3,1,:) = D.*B.*si.*x2.*e3;
J(3,2,:) = D.*B.*al.*x2.*e3;
J(3,3,:) = B.*(1-x1+al.*si.*x2).*e3;
J(3,4,:) = D.*(1-x1+al.*si.*x2).*e3;
J(3,5,:) = -x3;

end

```

The following then computes branches of equilibria and periodic orbits emanating from Hopf bifurcations for four different values of the β parameter.

```

>> PtMX = [255 90; 260 100; 240 100; 70 100];
>> beta = 1.55:0.01:1.58;
>> for i=1:4
    eprunid = sprintf('run_ep%d', i);
    coco(eprunid, 'ode', 'isol', 'ep', @abc, @abc_dx, @abc_dp, [0; 0; 0], ...
        {'al' 'si' 'D' 'B' 'be'}, [1; 0.04; 0; 8; beta(i)], ...
        'D', [0 0.4]);
    HB = coco_bd_labs(eprunid, 'HB');
    prob = coco_prob;

```

```

porunid = sprintf('run_po1%d', i);
prob = coco_set(prob, 'cont', 'NAdapt', 1, 'PtMX', [0 PtMX(i,1)]);
coco(prob, porunid, 'ode', 'HB', 'po', eprunid, HB(1), ...
'D', [0 0.4]);

prob = coco_prob;
prob = coco_set(prob, 'cont', 'NAdapt', 1, 'PtMX', [0 PtMX(i,2)]);
porunid = sprintf('run_po2%d', i);
coco(prob, porunid, 'ode', 'HB', 'po', eprunid, HB(3), ...
'D', [0 0.4]);
end

```

The 'NAdapt' setting of 1 ensures that the discretization used to approximate the periodic orbits is updated before each step of the continuation algorithm to manage growth of the truncation error. The values for the 'PtMX' setting are chosen (a posteriori) to avoid redundant covering of the curves of periodic orbits. The results of the computation are visualized as follows (cf. Fig. 8).

```

>> clf
>> theme_ep = struct();
>> theme_ep.special = {'HB'};
>> theme_po = struct();
>> theme_po.ls = {{'k-', 'LineWidth', 1}, {'k--, 'LineWidth', 1}};
>> theme_po.special = {'SN'};
>> for i=1:4
    subplot(2,2,i)
    axis([0.15 0.4 1 10])
    grid on
    hold on
    eprunid = sprintf('run_ep%d', i);
    coco_plot_bd(theme_ep, eprunid, 'D', 'MAX(x)', 3);

    porunid = sprintf('run_po1%d', i);
    coco_plot_bd(theme_po, porunid, 'D', 'MAX(x)', 3)

    porunid = sprintf('run_po2%d', i);
    coco_plot_bd(theme_po, porunid, 'D', 'MAX(x)', 3);
    hold off
end

```

The change in organization of the branches of periodic orbits between the bottom two panels suggest the appearance of an isolated closed curve of periodic orbits in the bottom right panel that has pinched off from the main branch at a branch point for some value of β between 1.57 and 1.58. This curve is computed by first continuing from a periodic orbit found in the bottom left panel (label 15 in the corresponding computation) under variations in β , and then holding β fixed and again varying D as shown below.

```

>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 1, 'PtMX', [0 200]);
>> porunid = 'run_po3';
>> coco(prob, porunid, 'ode', 'po', 'po', 'run_po13', 15, ...
'be', [beta(3) beta(4)]);

```

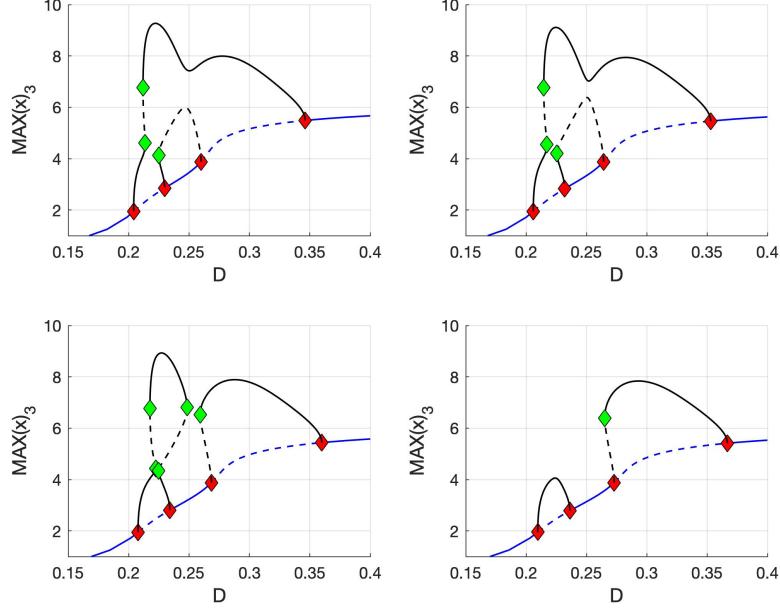


Figure 8: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria, as well as asymptotically stable (black solid) and unstable (black dashed) periodic orbits for the vector field in Section 4.2. The green and red diamonds mark saddle-node and Hopf bifurcations.

```
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 1, 'PtMX', [0 160]);
>> porunid = 'run_po34';
>> coco(prob, porunid, 'ode', 'po', 'po', 'run_po3', 3, ...
    'D', [0 0.4]);
```

The isolated curve is made visible in the bifurcation diagram as follows (cf. Fig. 9).

```
>> hold on
>> coco_plot_bd(theme_po, 'run_po34', 'D', 'MAX(x)', 3);
>> hold off
```

The values of β and D where the isolated curve pinches off from the main branch corresponds to a fold point along the curve of saddle-node bifurcations of periodic orbits found using the following commands.

```
>> porunid = 'run_po11';
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 1, 'PtMX', [0 100], 'h_max', 5);
>> SN = coco_bd_labs(porunid, 'SN');
>> coco(prob, 'po_run_SN', 'ode', 'SN', 'SN', ...
    porunid, SN(1), {'be', 'D'}, {[1.55 1.58], [0 0.4]});
```

STEP IT	SIT	DAMPING GAMMA	NORMS $\ d\ $	NORMS $\ f\ $	NORMS $\ U\ $	COMPUTATION TIMES F(x)	DF(x)	SOLVE
------------	-----	------------------	------------------	------------------	------------------	---------------------------	-------	-------

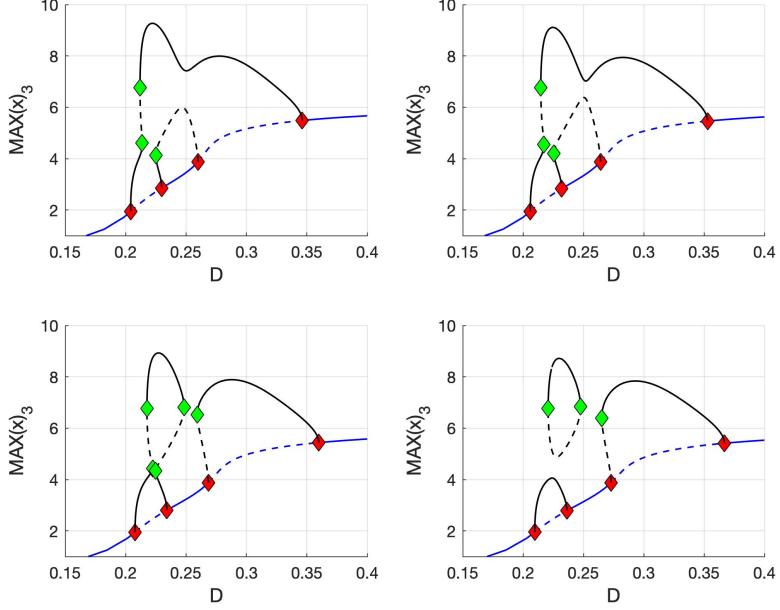


Figure 9: Bifurcation diagram showing branches of asymptotically stable (blue solid) and unstable (blue dashed) equilibria, as well as asymptotically stable (black solid) and unstable (black dashed) periodic orbits for the vector field in Section 4.2. The green and red diamonds mark saddle-node and Hopf bifurcations. The diagram differs from that in Fig. 8 by the inclusion of the isolated curve of periodic orbits in the bottom right panel.

0				3.67e-06	1.68e+02	0.0	0.0	0.0
1	1	1.00e+00	6.90e-04	1.89e-09	1.68e+02	0.0	0.0	0.0
2	1	1.00e+00	1.25e-08	1.83e-13	1.68e+02	0.0	0.0	0.0

STEP	TIME	$\ U\ $	LABEL	TYPE	be	D
0	00:00:00	1.6753e+02	1	EP	1.5500e+00	2.1370e-01
10	00:00:01	1.8824e+02	2		1.5534e+00	2.1481e-01
20	00:00:02	2.3148e+02	3		1.5603e+00	2.1735e-01
30	00:00:03	2.7568e+02	4		1.5676e+00	2.2070e-01
37	00:00:04	3.0056e+02	5	FP	1.5706e+00	2.2365e-01
40	00:00:05	3.0825e+02	6		1.5694e+00	2.2488e-01
50	00:00:06	3.0765e+02	7		1.5574e+00	2.2552e-01
54	00:00:07	3.0433e+02	8	EP	1.5500e+00	2.2496e-01

4.3 A web of bifurcations

Analogously to the detection and continuation of saddle-node bifurcations of periodic orbits, it is straightforward to detect and continue *period-doubling* and *Neimark-Sacker/torus bifurcations* using COCO.

Consider the autonomous dynamical system given by the vector field

$$f(x, p) = \begin{pmatrix} (-(\beta + \nu)x_1 + \beta x_2 - A_3 x_1^3 + B_3(x_2 - x_1)^3)/r \\ \beta x_1 - (\beta + \gamma)x_2 - x_3 - B_3(x_2 - x_1)^3 \\ x_2 \end{pmatrix}$$

in terms of the vector of state variables $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the vector of problem parameters $p = (\nu, \beta, \gamma, r, A_3, B_3) \in \mathbb{R}^6$. In addition to the equilibrium at the origin, there exist two families of equilibria located at

$$(x_1, x_2, x_3) = \left(\pm \frac{\sqrt{-\beta - \nu}}{\sqrt{A_3 + B_3}}, 0, \pm \frac{\sqrt{-\beta - \nu}}{\sqrt{A_3 + B_3}} \frac{A_3 \beta - B_3 \mu}{A_3 + B_3} \right)$$

provided that $\beta + \nu \leq 0$. The three families intersect along $\nu = -\beta$. For the equilibrium at the origin, a pair of conjugate eigenvalues of the Jacobian of the vector field lie on the imaginary axis when

$$\nu = -2 \frac{\beta^2 \gamma + r(r + \beta \gamma)(\beta + \gamma)}{\beta^2 + r(\beta + \gamma)^2 + 2\beta\gamma - \sqrt{(\beta^2 + r(\beta + \gamma)(\beta + \gamma - 2))(\beta^2 + r(\beta + \gamma)(\beta + \gamma + 2))}}$$

provided that the radical is real. Along this curve, the remaining eigenvalue equals

$$\frac{2r(\beta + \gamma)}{\beta^2 + r(\beta + \gamma)^2 + \sqrt{(\beta^2 + r(\beta + \gamma)(\beta + \gamma - 2))(\beta^2 + r(\beta + \gamma)(\beta + \gamma + 2))}}$$

and vanishes when $\beta = -\gamma$, at which point $\nu = -\beta$.

The vector field and its Jacobians are encoded explicitly in the functions `tor`, `tor_dx`, and `tor_dp` shown below.

```
function f = tor(x, p)

x1 = x(1, :);
x2 = x(2, :);
x3 = x(3, :);
nu = p(1, :);
be = p(2, :);
ga = p(3, :);
r = p(4, :);
a3 = p(5, :);
b3 = p(6, :);

f(1, :) = ( -(be+nu).*x1 + be.*x2 - a3.*x1.^3 + b3.* (x2-x1).^3 )./r;
f(2, :) = be.*x1 - (be+ga).*x2 - x3 - b3.* (x2-x1).^3;
f(3, :) = x2;

end
```

```
function J = tor_dx(x, p)
```

```

x1 = x(1,:);
x2 = x(2,:);
nu = p(1,:);
be = p(2,:);
ga = p(3,:);
r = p(4,:);
a3 = p(5,:);
b3 = p(6,:);

J = zeros(3,3,numel(x1));

J(1,1,:) = (- (be+nu) - 3*a3.*x1.^2 - 3*b3.* (x2-x1).^2) ./ r;
J(1,2,:) = (be + 3*b3.* (x2-x1).^2) ./ r;
J(2,1,:) = be + 3*b3.* (x2-x1).^2;
J(2,2,:) = - (be+ga) - 3*b3.* (x2-x1).^2;
J(2,3,:) = -1;
J(3,2,:) = 1;

end

```

```

function J = tor_dp(x,p)

x1 = x(1,:);
x2 = x(2,:);
nu = p(1,:);
be = p(2,:);
r = p(4,:);
a3 = p(5,:);
b3 = p(6,:);

J = zeros(3,6,numel(x1));

J(1,1,:) = -x1./r;
J(1,2,:) = (-x1+x2)./r;
J(1,4,:) = -(- (be+nu).*x1 + be.*x2 - a3.*x1.^3 + b3.* (x2-x1).^3) ./ r.^2;
J(1,5,:) = -x1.^3./r;
J(1,6,:) = (x2-x1).^3./r;
J(2,2,:) = x1-x2;
J(2,3,:) = -x2;
J(2,6,:) = -(x2-x1).^3;

end

```

The following sequence of commands locates a Hopf bifurcation at $\nu \approx -0.589$ along the trivial equilibrium branch under variations in ν , as predicted by theory.

```

>> x0      = [0; 0; 0];
>> pnames = { 'nu', 'be', 'ga', 'r', 'a3', 'b3' };
>> p0      = [-0.65 ; 0.5 ; -0.6 ; 0.6 ; 0.3 ; 0.9];
>> funcs   = {@tor, @tor_dx, @tor_dp};
>> prob = coco_prob;
>> prob = ode_isol2ep(prob, '', funcs{:}, x0, pnames, p0);
>> eprunid = 'ep_run';

```

```

>> coco(prob, eprunid, [], 'nu', [-0.65, -0.55]);

      STEP   DAMPING          NORMS          COMPUTATION TIMES
      IT SIT   GAMMA    ||d||    ||f||    ||U||   F(x)   DF(x)   SOLVE
      0           0.00e+00  1.65e+00    0.0     0.0     0.0

      STEP   TIME   ||U||  LABEL  TYPE          nu
      0  00:00:00  1.6477e+00    1 EP    -6.5000e-01
      1  00:00:00  1.6014e+00    2 HB    -5.8934e-01
      2  00:00:00  1.5732e+00    3 EP    -5.5000e-01

```

Continuation along a family of Hopf bifurcations through this point under simultaneous variations in ν and β is then accomplished by the following sequence of commands.

```

>> HB = coco_bd_labs(eprunid, 'HB');
>> hbrunid = 'ep_hb_run';
>> coco(hbrunid, 'ode', 'HB', 'HB', ...
    eprunid, HB, {'nu', 'be'}, [-0.65, -0.55]);

      STEP   DAMPING          NORMS          COMPUTATION TIMES
      IT SIT   GAMMA    ||d||    ||f||    ||U||   F(x)   DF(x)   SOLVE
      0           8.66e-08  2.23e+00    0.0     0.0     0.0

      STEP   TIME   ||U||  LABEL  TYPE          nu          be
      0  00:00:00  2.2322e+00    1 EP    -5.8934e-01  5.0000e-01
      4  00:00:00  2.3587e+00    2 EP    -6.5000e-01  4.5929e-01

      STEP   TIME   ||U||  LABEL  TYPE          nu          be
      0  00:00:00  2.2322e+00    3 EP    -5.8934e-01  5.0000e-01
      3  00:00:00  2.1755e+00    4 FP    -5.7856e-01  5.3603e-01
      6  00:00:00  2.1308e+00    5 BP    -6.0000e-01  6.0000e-01
      9  00:00:00  2.1338e+00    6 EP    -6.5000e-01  6.6396e-01

```

The screen output shows the existence of a branch point along this family at $\beta = 0.6$ and $\nu = -0.6$, as predicted by the theory.

The following sequence of commands performs continuation along a family of periodic orbits emanating from the initial Hopf bifurcation found above.

```

>> prob = coco_prob;
>> prob = ode_HB2po(prob, '', eprunid, HB);
>> prob = coco_set(prob, 'cont', 'NAdapt', 5, 'PtMX', [100 0]);
>> porunid1 = 'po_run1';
>> coco(prob, porunid1, [], 'nu', [-0.65, -0.55]);

      STEP   DAMPING          NORMS          COMPUTATION TIMES
      IT SIT   GAMMA    ||d||    ||f||    ||U||   F(x)   DF(x)   SOLVE
      0           6.30e-06  1.16e+01    0.0     0.0     0.0
      1   1  1.00e+00  1.27e-05  1.97e-13  1.16e+01    0.0     0.0     0.0
      2   1  1.00e+00  6.52e-11  8.17e-18  1.16e+01    0.0     0.0     0.0

      STEP   TIME   ||U||  LABEL  TYPE          nu
      0  00:00:00  1.1600e+01    1 EP    -5.8934e-01
      1  00:00:00  1.1600e+01    2 FP    -5.8934e-01

```

2	00:00:00	1.1603e+01	3	BP	-5.8925e-01
10	00:00:00	1.2082e+01	4	FP	-5.8414e-01
10	00:00:00	1.2082e+01	5	SN	-5.8414e-01
10	00:00:00	1.2226e+01	6		-5.8450e-01
12	00:00:01	1.2482e+01	7	BP	-5.8669e-01
12	00:00:01	1.2482e+01	8	SN	-5.8670e-01
18	00:00:01	1.5048e+01	9	EP	-6.5000e-01

The 'PtMX' setting is used to restrict continuation to one direction along this family. Continuation along the secondary branch of periodic orbits emanating from the branch point found along the primary branch is accomplished using the following sequence of commands.

```
>> BP = coco_bd_labs(porunid1, 'BP');
>> prob = coco_prob;
>> prob = ode_po2po(prob, '', porunid1, BP(1));
>> prob = coco_set(prob, 'cont', 'NAdapt', 5, 'PtMX', [100 0], ...
    'branch', 'switch');
>> porunid2 = 'po_run2';
>> coco(prob, porunid2, [], 'nu', [-0.65, -0.55]);
```

STEP	TIME	U	LABEL	TYPE	nu
0	00:00:00	1.2482e+01	1	EP	-5.8669e-01
7	00:00:00	1.2633e+01	2	TR	-5.9112e-01
10	00:00:00	1.2888e+01	3		-5.9908e-01
14	00:00:00	1.3342e+01	4	PD	-6.1438e-01
16	00:00:01	1.4385e+01	5	EP	-6.5000e-01

The following sequence of commands perform continuation along a family of saddle-node bifurcations through the point found along the primary branch.

```
>> SN = coco_bd_labs(porunid1, 'SN');
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 5, 'PtMX', [100 13]);
>> porunid3 = 'po_run_SN';
>> coco(prob, porunid3, 'ode', 'SN', 'SN', ...
    porunid1, SN(2), {'nu' 'be'}, [-0.65, -0.55]);
```

STEP	DAMPING	NORMS			COMPUTATION TIMES			
IT	SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0				2.05e-07	1.38e+01	0.0	0.0	0.0

STEP	TIME	U	LABEL	TYPE	nu	be
0	00:00:00	1.3759e+01	1	EP	-5.8414e-01	5.0000e-01
10	00:00:00	1.5243e+01	2		-6.0345e-01	4.2505e-01
18	00:00:01	1.7552e+01	3	EP	-6.5000e-01	3.0597e-01

STEP	TIME	U	LABEL	TYPE	nu	be
0	00:00:01	1.3759e+01	4	EP	-5.8414e-01	5.0000e-01
10	00:00:01	1.3375e+01	5		-5.7956e-01	5.2425e-01
12	00:00:01	1.3234e+01	6	FP	-5.7953e-01	5.2439e-01
13	00:00:02	1.3236e+01	7	BP	-5.7955e-01	5.2430e-01
13	00:00:02	1.3244e+01	8	EP	-5.7964e-01	5.2377e-01

Here, the 'PtMX' setting is chosen to avoid redundant covering of the curve of saddle-node bifurcations.

Similarly, the following sequences of commands perform continuation along families of period-doubling and torus bifurcations through the corresponding points found along the secondary branch.

```

>> PD = coco_bd_labs(porunid2, 'PD');
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 5);
>> porunid4 = 'po_run_PD';
>> coco(prob, porunid4, 'ode', 'PD', 'PD', ...
    porunid2, PD(1), {'nu' 'be'}, [-0.65, -0.55]);
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)	DF(x)	SOLVE
0			2.86e-07	1.56e+01		0.0	0.0	0.0

STEP		TIME	$\ U\ $	LABEL	TYPE	nu	be
0	00:00:00	1.5586e+01		1	EP	-6.1438e-01	5.0000e-01
10	00:00:00	1.7383e+01		2		-6.4880e-01	6.0984e-01
11	00:00:00	1.7441e+01		3	EP	-6.5000e-01	6.1194e-01

STEP		TIME	$\ U\ $	LABEL	TYPE	nu	be
0	00:00:00	1.5586e+01		4	EP	-6.1438e-01	5.0000e-01
3	00:00:00	1.5546e+01		5	FP	-6.1415e-01	4.8983e-01
10	00:00:01	1.6258e+01		6		-6.3888e-01	3.7511e-01
12	00:00:01	1.7498e+01		7	EP	-6.5000e-01	3.4813e-01


```

>> TR = coco_bd_labs(porunid2, 'TR');
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 5, 'PtMX', [32, 29]);
>> porunid5 = 'po_run_TR';
>> coco(prob, porunid5, 'ode', 'TR', 'TR', ...
    porunid2, TR(1), {'nu' 'be'}, [-0.65, -0.55]);
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)	DF(x)	SOLVE
0			1.87e-07	1.49e+01		0.0	0.0	0.0

STEP		TIME	$\ U\ $	LABEL	TYPE	nu	be
0	00:00:00	1.4920e+01		1	EP	-5.9112e-01	5.0000e-01
10	00:00:00	1.5692e+01		2		-5.9437e-01	4.6256e-01
20	00:00:01	1.6041e+01		3		-5.9599e-01	4.5150e-01
30	00:00:01	1.5775e+01		4		-5.9634e-01	4.4933e-01
32	00:00:02	1.5675e+01		5	FP	-5.9636e-01	4.4926e-01
32	00:00:02	1.5672e+01		6	BP	-5.9636e-01	4.4926e-01
32	00:00:02	1.5662e+01		7	EP	-5.9635e-01	4.4927e-01

STEP		TIME	$\ U\ $	LABEL	TYPE	nu	be
0	00:00:02	1.4920e+01		8	EP	-5.9112e-01	5.0000e-01
4	00:00:02	1.4949e+01		9	FP	-5.9085e-01	5.1416e-01
10	00:00:03	1.5093e+01		10		-5.9223e-01	5.4652e-01
20	00:00:03	1.5503e+01		11		-5.9970e-01	5.9847e-01
28	00:00:04	1.5233e+01		12	FP	-6.0000e-01	6.0000e-01

29	00:00:04	1.5233e+01	13	BP	-6.0000e-01	5.9999e-01
29	00:00:04	1.5233e+01	14	EP	-5.9999e-01	5.9995e-01

Finally, we continue along a family of periodic orbits emanating from the period-doubling bifurcation along the secondary branch of periodic orbits, as shown in the following sequence of commands.

```
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 5, 'PtMX', [0 100]);
>> porunid6 = 'po_run_db';
>> coco(prob, porunid6, 'ode', 'PD', 'po', porunid4, 4, ...
    'nu', [-0.65, -0.55]);
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	$F(x)$	$DF(x)$	SOLVE
0				7.06e-02	2.64e+01	0.0	0.0	0.0
1	1	1.00e+00	1.03e-01	2.81e-04	2.64e+01	0.0	0.0	0.0
2	1	1.00e+00	4.32e-03	8.52e-08	2.64e+01	0.0	0.0	0.0
3	1	1.00e+00	8.84e-06	6.04e-13	2.64e+01	0.0	0.0	0.0
4	1	1.00e+00	1.00e-11	8.90e-16	2.64e+01	0.0	0.0	0.0

STEP	TIME	$\ U\ $	LABEL	TYPE	nu
0	00:00:00	2.6416e+01	1	EP	-6.1443e-01
3	00:00:00	2.6447e+01	2	PD	-6.1676e-01
10	00:00:00	2.6865e+01	3		-6.4024e-01
14	00:00:00	2.7095e+01	4	EP	-6.5000e-01

The different curves may now be visualized using the following commands (cf. Fig. 10).

```
>> clf; hold on; grid on
>> thm = struct('zlab', '||x||_2');
>> thm.special = {'EP', 'HB'};
>> coco_plot_bd(thm, eprunid, 'nu', 'be', '||x||_2')
>> thm.special = {'EP', 'BP'};
>> coco_plot_bd(thm, hbrunid, 'nu', 'be', '||x||_2')
>> thm.special = {'EP', 'SN', 'BP'};
>> coco_plot_bd(thm, porunid1, 'nu', 'be', '||x||_{2,MPD}');
>> thm.special = {'EP', 'PD', 'TR'};
>> coco_plot_bd(thm, porunid2, 'nu', 'be', '||x||_{2,MPD}');
>> thm.special = {'EP'};
>> coco_plot_bd(thm, porunid3, 'nu', 'be', '||x||_{2,MPD}');
>> coco_plot_bd(thm, porunid4, 'nu', 'be', '||x||_{2,MPD}');
>> coco_plot_bd(thm, porunid5, 'nu', 'be', '||x||_{2,MPD}');
>> thm.special = {'EP', 'PD'};
>> coco_plot_bd(thm, porunid6, 'nu', 'be', '||x||_{2,MPD}')
>> hold off; view(-192, 44)
```

Inspection of this diagram shows that the curve of saddle-node bifurcations of periodic orbits terminates on the Hopf bifurcation curve for the equilibrium at the origin, while the curve of torus bifurcations terminates at one end on the curve of saddle-node bifurcations and at the other end on the Hopf bifurcation curve for the equilibrium at the origin.

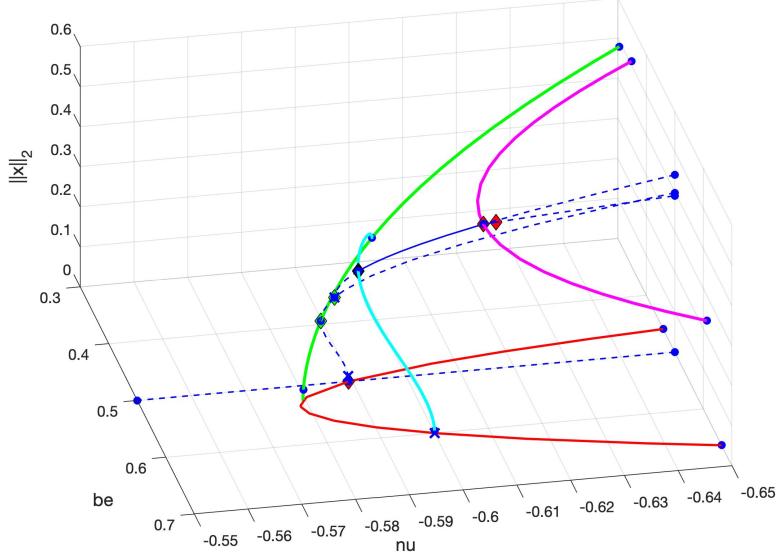


Figure 10: Bifurcation diagram showing branches of unstable (blue dashed) equilibria, Hopf bifurcations of equilibria (red solid) asymptotically stable (blue solid) and unstable (blue dashed) periodic orbits, saddle-node bifurcations of periodic orbits (green solid), period-doubling bifurcations (magenta solid), and torus bifurcations (cyan solid) for the vector field in Section 4.3. The green, blue, and red diamonds along curves of periodic orbits mark saddle-node, torus, and period-doubling bifurcations.

5 Post-processing and analysis

Each successful call to the `coco` function generates data that is stored to disk (in a sub-folder of the `data` folder identified by the name of the run) and that can be explored using `coco_plot_bd` or other post-processing utilities. Detailed information about each labeled point is stored in individual solution files, while a less detailed description of all computed points is stored in a separate file as a MATLAB cell array. This section shows the use of COCO functions for extracting and visualizing stored data.

5.1 Bifurcation data

For workspace access to the output cell array, you can assign the output of the `coco` command to a variable, as shown below.

```
>> f = @(x,p) p-x.^2;
>> bd = coco('run1', 'ode', 'isol', 'ep', f, 0.5, 'p', 1, 'p', [-1 1]);
```

IT	STEP	DAMPING		NORMS			COMPUTATION TIMES		
		SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0					7.50e-01	1.50e+00	0.0	0.0	0.0
1	1	3.56e-01	7.50e-01	4.11e-01	1.61e+00		0.0	0.0	0.0
2	1	1.00e+00	2.68e-01	7.18e-02	1.75e+00		0.0	0.0	0.0
3	1	1.00e+00	3.47e-02	1.20e-03	1.73e+00		0.0	0.0	0.0

```

4   1   1.00e+00   6.02e-04   3.62e-07   1.73e+00     0.0     0.0     0.0
5   1   1.00e+00   1.81e-07   3.38e-14   1.73e+00     0.0     0.0     0.0

STEP      TIME      ||U||      LABEL      TYPE      p
 0  00:00:00  1.7321e+00      1  EP  1.0000e+00
10 00:00:00  9.9654e-02      2  FP  9.7411e-03
14 00:00:00  1.3380e-06      3  FP  7.8762e-09
14 00:00:00  1.6856e-08      4  SN  7.8780e-09
20 00:00:00  2.1653e-01      5          4.3161e-02
30 00:00:00  1.3657e+00      6          7.4753e-01
31 00:00:00  1.7321e+00      7  EP  1.0000e+00

```

Alternative, load this cell array from disk using the `coco_bd_read` utility:

```
>> bd = coco_bd_read('run1');
```

You can explore the content of the `bd` variable in the MATLAB variable editor or on the command line. In this example, `bd` contains 35 rows and 18 columns. The top row contains column headers, while the remaining rows contain data.

The command

```
>> coco_bd_labs('run1', 'SN')
```

```
ans =
```

```
4
```

returns the labels of all points of type `SN` in the run '`run1`'. The same result is returned by replacing '`run1`' with the corresponding `bd` output cell array:

```
>> coco_bd_labs(bd, 'SN')
```

```
ans =
```

```
4
```

To extract all labels, omit the point type or use '`all`':

```
>> coco_bd_labs('run1', 'all')
```

```
ans =
```

```
7      6      5      4      3      2      1
```

This is particularly useful if you want to loop over all labels.

Either of the commands

```
>> coco_bd_lab2idx('run1', 5:7)
```

```
ans =
```

```
12      2      1
```

and

```
>> coco_bd_lab2idx(bd, 5:7)

ans =
    12     2      1
```

returns the row indices in the data portion of the output cell array from the run 'run1' of the points labeled 5 through 7. Either of the commands

```
>> coco_bd_idxs('run1', 'SN')

ans =
    19
```

and

```
>> coco_bd_idxs(bd, 'SN')

ans =
    19
```

returns the row indices in the data portion of the output cell array from the run 'run1' corresponding to points of type SN. To extract the row indices of all labeled points, omit the point type or use 'all':

```
>> coco_bd_idxs('run1', 'all')

ans =
    1     2     12     19     20     24     34
```

The command

```
>> p = coco_bd_col('run1', 'p');
```

assigns to p all the data in the output cell array associated with the 'p' column header. The command

```
>> px = coco_bd_col('run1', {'p', 'x'});
```

assigns to px all the data in the output cell array associated with the 'p' and 'x' column headers. In general, the output format will depend on the type of data stored in a particular column. To extract data from the output cell array associated with particular column headers, but restricted to labeled points or points of a particular type, use the `coco_bd_vals` utility as shown below.

```
>> coco_bd_vals('run1', 'SN', {'p', 'x'})
```

```

ans =
1.0e-07 *
0.0788
-0.1265

```

To extract data associated with all labeled points, leave the second argument empty, as in

```

>> coco_bd_vals(bd, [], {'p' 'x'})
ans =
1.0000    0.7475    0.0432    0.0000    0.0000    0.0097    1.0000
-1.0000   -0.8646   -0.2078   -0.0000    0.0000    0.0987    1.0000

```

The `coco_print_bd` utility extracts a subset of the content of the output cell array associated with labeled points and particular columns, and prints this to screen or to a file. For example, the command

```

>> coco_print_bd('run1', {'p', 'eigs'})

STEP      ||U||  LABEL  TYPE          p        eigs
 31  1.7321e+00    7 EP    1.0000e+00  2.0000e+00
 30  1.3657e+00    6      7.4753e-01  1.7292e+00
 20  2.1653e-01    5      4.3161e-02  4.1550e-01
 14  1.6856e-08    4 SN    7.8780e-09  2.5298e-08
 14  1.3380e-06    3 FP    7.8762e-09  -2.6760e-06
 10  9.9654e-02    2      9.7411e-03  -1.9739e-01
  0  1.7321e+00    1 EP    1.0000e+00  -2.0000e+00

```

includes the values of p and the eigenvalues of the Jacobian of the vector field with respect to the state in the data printed to screen.

The `coco_plot_bd` utility offers opportunities for additional post-processing prior to visualization. Suppose, for example, that you have executed the command

```

>> coco('run1', 'ode', 'isol', 'ep', ...
@brus, [1; 0], {'A' 'B'}, [1; 0], 'B', [0 3]);

      STEP  DAMPING                  NORMS                  COMPUTATION TIMES
      IT  SIT  GAMMA    ||d||    ||f||    ||U||  F(x)  DF(x)  SOLVE
      0           0.00e+00  1.41e+00    0.0    0.0    0.0
      STEP      TIME      ||U||  LABEL  TYPE          B
      0  00:00:00  1.4142e+00    1 EP    0.0000e+00
      9  00:00:00  3.7417e+00    2 HB    2.0000e+00
     10  00:00:00  4.3853e+00    3      2.3966e+00
     13  00:00:00  5.3852e+00    4 EP    3.0000e+00

```

where the encoding of vector field was given in Section 3.4. Then, the commands

```

>> clf
>> hold on
>> theme = struct('special', {{'HB'}});
>> coco_plot_bd(theme,'runl', 'B', @(x) [x; x], 'eigs', @(x) real(x))
>> coco_plot_bd(theme,'runl', 'B', @(x) [x; x], 'eigs', @(x) imag(x))
>> grid on
>> hold off

```

graph the real and imaginary parts of the eigenvalues of the Jacobian of the vector field with respect to the state versus the parameter B along the curve of equilibria (cf. Fig. 11). Alternatively, the commands

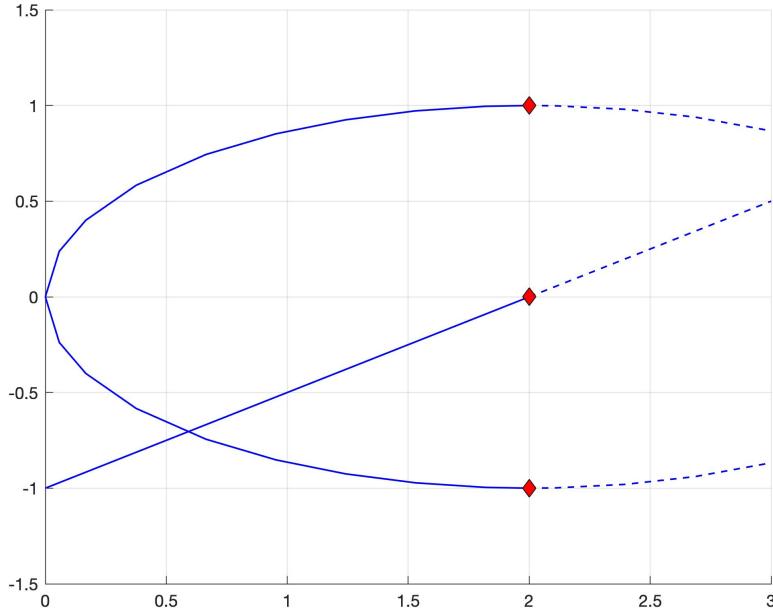


Figure 11: Bifurcation diagram showing real and imaginary parts of the eigenvalues of the Jacobian of the vector field with respect to the state versus the parameter B for the vector field in Section 3.4. The red diamonds mark the Hopf bifurcation.

```

>> clf
>> coco_plot_bd(theme,'runl', 'eigs', @(x) real(x), 'eigs', @(x) imag(x))
>> grid on

```

show the progression of the eigenvalues in the complex plane during continuation (cf. Fig. 12).

5.2 Individual solutions

The commands

```

>> HB = coco_bd_labs('runl', 'HB');
>> sol = ep_read_solution('runl', HB)

```

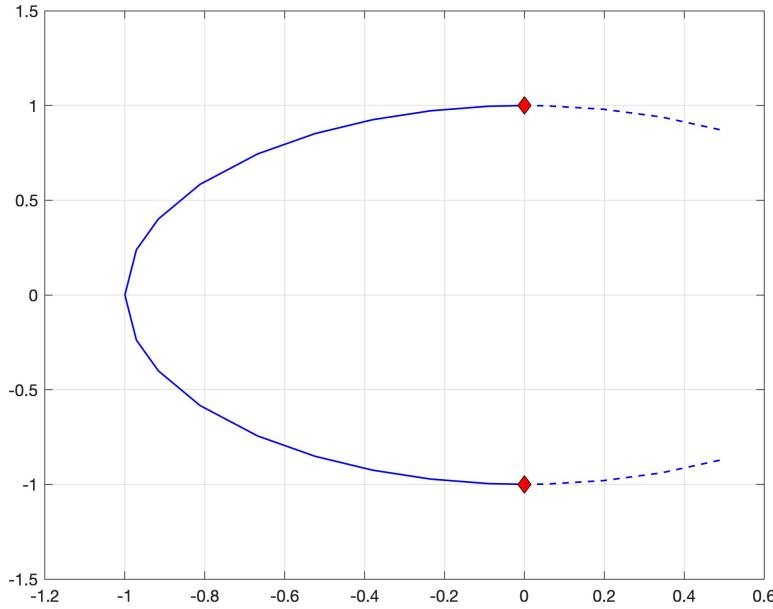


Figure 12: Bifurcation diagram showing the progression in the complex plane of the eigenvalues of the Jacobian of the vector field with respect to the state versus the parameter B for the vector field in Section 3.4. The red diamonds mark the Hopf bifurcation as the pair of eigenvalues crosses the imaginary axis.

```

sol =
  struct with fields:

    format: 'ep.v3'
    branch_type: 'ep'
    pt_type: 'HB'
      u: [4x1 double]
      t: [4x1 double]
    ep_test: [1x1 struct]
      x: [2x1 double]
      p: [2x1 double]
      u0: [4x1 double]
      t0: []
    var: [1x1 struct]
    hb: [1x1 struct]
  
```

assigns detailed information about the Hopf bifurcation found in the 'run1' run above to the variable `sol`. As seen below, `sol.p` and `sol.x` contain the values of p and x at this point.

```

>> sol.x

ans =
  1.0000
  2.0000
  
```

```

>> sol.p

ans =
    1.0000
    2.0000

```

More interestingly, `sol.hb.k` contains the squared angular frequency ω^2 associated with the Hopf bifurcation, while the first column of `sol.var.v` is a unit vector v such that $J^2v + \omega^2v = 0$, where J is the Jacobian of the vector field with respect to the state evaluated at the Hopf bifurcation.

A similar syntax applies to the command `po_read_solution` for accessing data stored to disk during continuation along families of periodic orbits. For example, the commands

```

>> FP = coco_bd_labs('po_run_SN', 'FP');
>> sol = po_read_solution('po_run_SN', FP)
sol =
  struct with fields:

    format: 'po.v2'
  branch_type: 'po.SN'
  pt_type: 'FP'
    u: [375x1 double]
    t: [375x1 double]
    tbp: [101x1 double]
    xbp: [101x3 double]
    T: 6.8465
    p: [5x1 double]
    var: [1x1 struct]
    sn: [1x1 struct]

```

assigns detailed information about the periodic orbit at the fold point found in Section 4.2 to the variable `sol`. In this case, `sol.p`, `sol.T`, `sol.xbp`, and `sol.tbp` contain the value of p , the orbital period, a discretized representation of $x(t)$, and the corresponding temporal mesh, respectively. Individual or families of labeled periodic orbits may be visualized using the `coco_plot_sol` utility, as shown below (cf. Fig. 13).

```

>> clf
>> theme = struct('special', {{'FP'}});
>> coco_plot_sol(theme, 'po_run_SN', '', 'x', 'x', 'x')
>> grid on

```

6 Function definitions

6.1 Vector fields

Consider the following call to `coco` in terms of a run label '`run1`', toolbox family '`ode`', initial point type '`isol`', branch type '`ep`', vector field `f`, initial numerical value for x , parameter name '`p`', initial numerical value for p , active continuation parameter '`p`', and

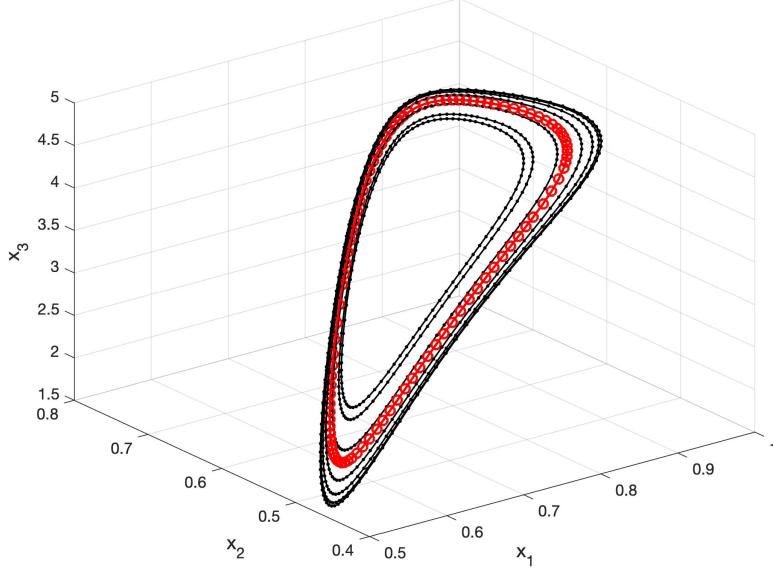


Figure 13: One-dimensional family of periodic orbits at saddle-node bifurcations for the vector field in Section 4.2. The orbit found at a fold point along this family is highlighted in solid red.

computational domain.

```
>> coco('run1', 'ode', 'isol', 'ep', f, 0.5, 'p', 1, 'p', [-1 1]);
```

By default, `f` is here assumed to be a handle to a vectorized encoding of a function of two scalar arguments. For example, if the corresponding vector field is given by $f(x, p) = p - x^2$, then `f` may be given as an anonymous function:

```
>> f = @(x,p) p-x.^2;
```

or as the function handle `@fold` where `fold` is encoded in a function file:

```
function f = fold(x,p)
f = p-x.^2;
end
```

The former is convenient for functions that may be easily expressed in terms of a single computational expression, while the latter is necessary when this is not the case. For a vector field that is defined symbolically, a COCO interface to the symbolic toolbox in MATLAB may be used to generate a corresponding vectorized encoding:

```
>> syms x p
>> F = sco_sym2funcs(p-x^2, {x, p}, {'x', 'p'});
>> f = F('');
```

Finally, rather than a single function handle, the corresponding argument to `coco` can equal

a MATLAB struct with a single field '`f`' whose value equals the function handle, as in

```
>> f = struct('f', F(''));
>> coco('run1', 'ode', 'isol', 'ep', f, 0.5, 'p', 1, 'p', [-1 1]);
```

When vector fields are defined manually (rather than generated symbolically), it is possible, but not encouraged, to build non-vectorized encodings, e.g.,

```
>> f = @(x,p) p-x.^2;
```

In this case, it is necessary to alert `coco` to the non-vectorized encoding by setting the '`vectorized`' setting to false, as shown in the following commands.

```
>> prob = coco_prob;
>> prob = coco_set(prob, 'ode', 'vectorized', false);
>> coco(prob, 'run1', 'ode', 'isol', 'ep', f, 0.5, 'p', 1, 'p', [-1 1]);
```

6.2 Partial derivatives

When explicit partial derivatives are available for a vector field, these may be provided to `coco` and applicable toolbox constructors like `ode_isol2ep` as shown below.

```
>> coco('run1', 'ode', 'isol', 'ep', f, dfdx, dfdp, 0.5, 'p', 1, 'p', [-1 1]);
```

By default, `f`, `dfdx`, and `dfdp` are assumed to equal handles to vectorized encodings of $f(x, p)$, $\partial_x f(x, p)$, and $\partial_p f(x, p)$. For the case of the fold normal form, these may be defined as anonymous functions

```
>> f = @(x,p) p-x.^2; dfdx = @(x,p) -2*x; dfdp = @(x,p) ones(1,numel(p));
```

or as handles `@fold`, `@fold_dx`, and `@fold_dp`, where `fold`, `fold_dx`, and `fold_dp` are encoded in several function files:

```
function f = fold(x,p)
f = p-x.^2;
end
```

```
function J = fold_dx(x,p)
J = -2*x;
end
```

and

```
function J = fold_dp(x,p)
J = ones(1,numel(p));
end
```

The alternative syntax in terms of a MATLAB struct generalizes to this case:

```
>> f = struct('f', @fold, 'dfdx', @fold_dx, 'dfdp', @fold_dp);
>> coco('run1', 'ode', 'isol', 'ep', f, 0.5, 'p', 1, 'p', [-1 1]);
```

Automated generation of such vectorized encodings from a symbolic definition of the vector field is again available using the SYMCOCO interface to the symbolic toolbox in MATLAB:

```
>> syms x p
>> F = sco_sym2funcs(p-x^2, {x, p}, {'x', 'p'});
>> f = struct('f', F(''), 'dfdx', F('x'), 'dfdp', F('p'));
```

A similar syntax applies to functions evaluating to higher-order derivatives, although these may get increasingly cumbersome to encode in order to return the required higher-dimensional arrays for vector-valued functions of vector-valued arguments. Given

$$f(x, p) = \begin{pmatrix} 2p_1 z^2 - 2p_5 x_1^2 - p_3 x_1 x_2 \\ p_2 z - p_6 x_2 - p_3 x_1 x_2 \\ p_4 z - p_4 p_7 x_3 \end{pmatrix}, z = 1 - x_1 - x_2 - x_3$$

the function files

```
function f = bykov(x, p)

x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);
p1 = p(1,:);
p2 = p(2,:);
p3 = p(3,:);
p4 = p(4,:);
p5 = p(5,:);
p6 = p(6,:);
p7 = p(7,:);

z = 1-x1-x2-x3;
f(1,:) = 2*p1.*z.^2-2*p5.*x1.^2-p3.*x1.*x2;
f(2,:) = p2.*z-p6.*x2-p3.*x1.*x2;
f(3,:) = p4.*z-p4.*p7.*x3;

end
```

and

```
function J = bykov_dxdx(x, p)

x1 = x(1,:);
p1 = p(1,:);
p3 = p(3,:);
p5 = p(5,:);

J = zeros(3,3,3,numel(x1));
J(1,1,1,:) = 4*p1-4*p5;
J(1,1,2,:) = 4*p1-p3;
```

```

J(1,1,3,:) = 4*p1;
J(2,1,2,:) = -p3;
J(1,2,1,:) = 4*p1-p3;
J(1,2,2,:) = 4*p1;
J(1,2,3,:) = 4*p1;
J(2,2,1,:) = -p3;
J(1,3,1,:) = 4*p1;
J(1,3,2,:) = 4*p1;
J(1,3,3,:) = 4*p1;

end

```

implement vectorized encodings of $f(x, p)$ and the three-dimensional array of second partial derivatives $\partial_{xx}f(x, p)$. The corresponding automated construction is given by the commands

```

>> syms x1 x2 x3 p1 p2 p3 p4 p5 p6 p7 z
>> z = 1-x1-x2-x3;
>> F = sco_sym2funcs(... 
    [2*p1*z^2-2*p5*x1^2-p3*x1*x2; p2*z-p6*x2-p3*x1*x2; p4*z-p7*p4*x3], ...
    {[x1; x2; x3], [p1; p2; p3; p4; p5; p6; p7]}, {'x', 'p'});
>> f = F(')');
>> dfdx = F({'x', 'x'});

```

as confirmed by the following test:

```

>> x = rand(3,4); p = rand(7,4);
>> size(f(x,p))
ans =
    3      4
>> size(dfdx(x,p))
ans =
    3      3      3      4
>> max(abs(f(x,p)-bykov(x,p)), [], 'all')
ans =
    2.2204e-16
>> max(abs(dfdx(x,p)-bykov_dx(x,p)), [], 'all')
ans =
    4.4409e-16

```

6.3 Directional derivatives

The encoding of vector fields and their derivatives generalizes to directional derivatives, e.g.,

$$(x, p, v) \mapsto D_x f(x, p)[v] := D_h f(x + hv, p)|_{h=0}$$

and

$$(x, p, v, w) \mapsto D_{xx} f(x, p)[v, w] := D_{h_1} D_{h_2} f(x + h_1 v + h_2 w, p)|_{h_1=h_2=0}$$

in terms of the two- and three-tensors $D_x f(x, p)$ and $D_{xx} f(x, p)$. In the case of the vector field encoded with the symbolic function generator in the previous section, function handles

to vectorized encodings of these directional derivatives are assigned to the variables `Dfdx` and `Dfdxdx` in the commands shown below.

```
>> Dfdx = F('x*v');
>> Dfdxdx = F({'x*v', 'x*v'});
```

As an example, we consider the formula¹

$$(x, p) \mapsto \frac{1}{2\omega} \Re \left(w^{*,\top} \cdot \left(D_{xx}f(x, p) [v^*, (2i\omega I - \partial_x f(x, p))^{-1} \cdot D_{xx}f(x, p)[v, v]] \right. \right. \\ \left. \left. - 2D_{xx}f(x, p) [v, (\partial_x f(x, p))^{-1} \cdot D_{xx}f(x, p)[v, v^*]] + D_{xxx}f(x, p)[v, v, v^*] \right) \right)$$

for the *first Lyapunov coefficient* evaluated at a Hopf bifurcation (x, p) in terms of the solutions ω , v , and w to the eigenvalue problems

$$\partial_x f(x, p) \cdot v - i\omega v = 0, \quad w^{*,\top} \cdot \partial_x f(x, p) - i\omega w^{*,\top} = 0$$

such that $v^{*,\top} \cdot v = w^{*,\top} \cdot v = 1$. The sign of the Lyapunov coefficient determines whether the periodic orbits locally along the branch emanating from the bifurcation are asymptotically stable (negative) or unstable (positive).

It follows from the first eigenvalue problem that, given x , p , ω , and $\Re(v)$, we obtain

$$\Im(v) = -\partial_x f(x, p) \cdot \Re(v)/\omega, \quad w^{*,\top} = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x f(x, p) - i\omega I & v \\ v^{*,\top} & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} I \\ 0 \end{pmatrix}$$

The function `lyapunov` below encodes the first Lyapunov coefficient in terms of the arguments `x` (representing x), `p` (representing p), `v` (representing $\Re(v)$), and `k` (representing ω^2).

```
function y = lyapunov(data, x, p, v, k)

n = numel(x);
om = sqrt(k);

A = data.dfdxhan(x,p);
va = v-1i*A*v/om;
va = va/norm(va);
vb = conj(va);
w = ([A-1i*om*eye(n) va; va' 0]\[eye(n); zeros(1,n)])*[zeros(n,1); 1];

B = @([dx1, dx2) data.Dfdxdxhan(x, p, dx1, dx2);
C = @([dx1, dx2, dx3) data.Dfdxdxdxhan(x, p, dx1, dx2, dx3);
y = real(w'* (B(vb, (2*1i*om*eye(n)-A)\B(va,va))...
-2*B(va,A\B(va,vb))+C(va,va,vb)))/2/om;

end
```

¹See Eq. (5.39) in Kuznetsov, Y.A., *Elements of Applied Bifurcation Theory*, Springer New York, NY, 2004, <https://doi.org/10.1007/978-1-4757-3978-7>

The `data` argument is here assumed to be a MATLAB struct with fields `dfdxhan`, `Dfdxdxhan`, and `Dfdxdxdxhan` with values given by the function handles `F('x')`, `F({'x*v', 'x*v'})`, and `F({'x*v', 'x*v', 'x*v'})`.

Consider for example, the following computations.

```
>> syms x1 x2 A B BB
>> BB = x1*(B-x1*x2);
>> F = sco_sym2funcs([A-BB-x1; BB], {[x1, x2], [A,B]}, {'x', 'p'}, ...
    'maxorder', 3);
>> f = struct('f', F()), 'dfdx', F('x'), 'dfdp', F('p'));
>> coco('ep_run', 'ode', 'isol', 'ep', ...
    f, [1; 0], {'A' 'B'}, [1; 0], 'B', [0 3]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)  DF(x)  SOLVE
        0           0.00e+00  1.41e+00      0.0       0.0       0.0

STEP      TIME      ||U||  LABEL  TYPE      B
  0  00:00:00  1.4142e+00    1  EP  0.0000e+00
  9  00:00:00  3.7417e+00    2  HB  2.0000e+00
 10  00:00:00  4.3853e+00    3
 13  00:00:00  5.3852e+00    4  EP  3.0000e+00

>> HB = coco_bd_labs('ep_run', 'HB');
>> sol = ep_read_solution('ep_run', HB);
>> data = struct('dfdxhan', F('x'), 'Dfdxdxhan', F({'x*v', 'x*v'}), ...
    'Dfdxdxdxhan', F({'x*v', 'x*v', 'x*v'}));
>> lyapunov(data, sol.x, sol.p, sol.var.v(:,1), sol.hb.k)

ans =
 -0.5000
```

from which we conclude that the Hopf bifurcation is supercritical. We verify this by first continuing along the corresponding branch of periodic orbits:

```
>> prob = coco_prob;
>> prob = coco_set(prob, 'cont', 'NAdapt', 1);
>> coco(prob, 'po_run', 'ode', 'HB', 'po', 'ep_run', HB, 'B', [0 3])

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)  DF(x)  SOLVE
        0
        1  1.00e+00  2.30e-06  9.65e-14  1.84e+01      0.0       0.0       0.0
        2  1  1.00e+00  3.90e-11  4.71e-15  1.84e+01      0.0       0.0       0.0

STEP      TIME      ||U||  LABEL  TYPE      B
  0  00:00:00  1.8384e+01    1  EP  2.0000e+00
  1  00:00:00  1.8384e+01    2  FP  2.0000e+00
 10  00:00:00  2.2634e+01    3
 20  00:00:01  2.9658e+01    4
 30  00:00:02  3.6243e+01    5
 40  00:00:03  4.3551e+01    6
 50  00:00:03  5.3662e+01    7  EP  3.0000e+00
```

STEP	TIME	$\ U\ $	LABEL	TYPE	B
0	00:00:03	1.8384e+01	8	EP	2.0000e+00
10	00:00:04	2.2683e+01	9		2.1198e+00
20	00:00:05	3.2100e+01	10		2.3670e+00
30	00:00:05	3.6249e+01	11		2.6009e+00
40	00:00:06	4.5893e+01	12		2.8126e+00
50	00:00:07	5.3626e+01	13	EP	3.0000e+00

and then graphing the branches of equilibria and periodic orbits using the following commands (cf. Fig. 14).

```
>> figure(1)
>> clf
>> hold on
>> thm = struct();
>> thm.special = {'EP'};
>> coco_plot_bd(thm, 'po_run', 'B', 'MAX(x)', 1)
>> thm.special = {'EP', 'HB'};
>> thm.ylab = 'max(x_1)';
>> coco_plot_bd(thm, 'ep_run', 'B', 'x')
>> grid on
>> hold off
>> axis([0 3 0 inf])
```

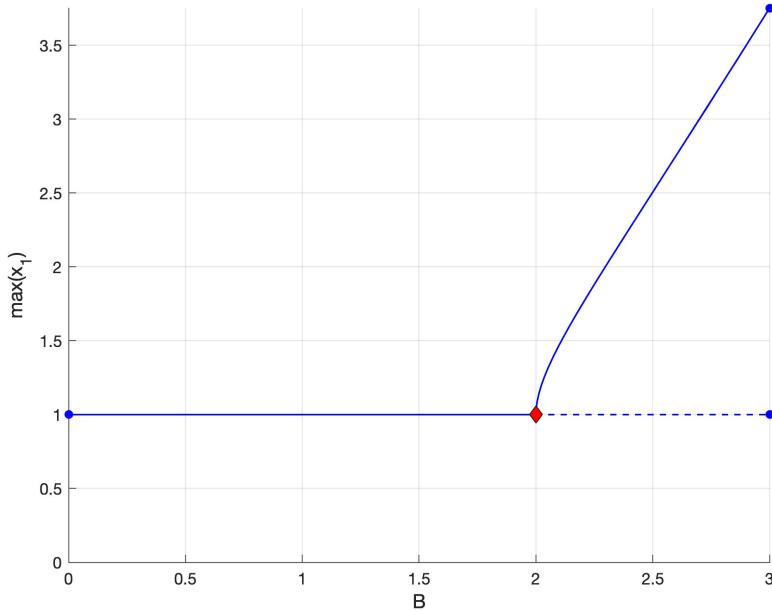


Figure 14: One-dimensional family of equilibria and one-dimensional family of asymptotically stable periodic orbits emanating from a supercritical Hopf bifurcation (red diamond) for the vector field in Section 6.3.

As anticipated, the periodic orbits emanating from the Hopf bifurcation are asymptotically stable.

7 User-initiated enhancements

The scope of bifurcation analysis using COCO may be expanded through user-initiated enhancements to the default behavior. Such enhancements hint at the full power of the COCO package that reaches far beyond a predefined set of problems and analytic functionalities.

7.1 Monitoring properties of equilibria

You can augment data saved to the output cell array during continuation along a family of equilibria by instructing COCO to perform additional computations on each point (x, p) along such a family using the `ep_add_bddat` constructor.

Suppose, for example, that you have entered the following commands:

```
>> syms x y s Q1 Q2 Q3 Q4 Q5 Q6 K
>> z = 1-x-y-s;
>> xp = 2*Q1*z^2-2*Q5*x^2-Q3*x*y;
>> yp = Q2*z-Q6*y-Q3*x*y;
>> sp = Q4*z-K*Q4*s;
>> F = sco_sym2funcs([xp; yp; sp], {[x; y; s], ...
    [Q1; Q2; Q3; Q4; Q5; Q6; K]}, ...
    {'x', 'p'}, 'filename', 'sys_bykov', 'maxorder', 3);
>> x0 = [0.24; 0.04; 0.51];
>> pnames = {'p1' 'p2' 'p3' 'p4' 'p5' 'p6' 'p7'};
>> p0 = [2.5; 0.5; 10; 0.0675; 1; 0.1; 0.4];
>> funcs = {F(''), F('x'), F('p'), F({'x', 'x'}), F({'x', 'p'}), F({'p', 'p'})};
>> prob = coco_prob;
>> prob = ode_isol2ep(prob, '', funcs{:}, x0, pnames, p0);
>> eprunid = 'ep_run';

```

Here, you used the `ode_isol2ep` constructor to initialize the search for equilibrium points of the vector field

$$f(x, p) = \begin{pmatrix} 2p_1 z^2 - 2p_5 x_1^2 - p_3 x_1 x_2 \\ p_2 z - p_6 x_2 - p_3 x_1 x_2 \\ p_4 z - p_4 p_7 x_3 \end{pmatrix}, z = 1 - x_1 - x_2 - x_3$$

The default behavior of COCO produces the output shown below.

```
>> coco(prob, eprunid, [], 'p2', [0.4 3]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT  SIT      GAMMA      ||d||      ||f||      ||U||      F(x)  DF(x)  SOLVE
      0           1.00e+00  5.46e-03  1.06e-02  1.04e+01  0.0    0.0    0.0
      1           1.00e+00  6.56e-05  1.13e-04  1.04e+01  0.0    0.0    0.0
      2           1.00e+00  3.84e-09  5.26e-09  1.04e+01  0.0    0.0    0.0
      3           1.00e+00                3.12e-17  1.04e+01  0.0    0.0    0.0

      STEP      TIME      ||U||  LABEL  TYPE          p2
      0  00:00:00  1.0404e+01      1  EP      5.0000e-01
      2  00:00:00  1.0396e+01      2  EP      4.0000e-01
```

STEP	TIME	$\ U\ $	LABEL	TYPE	p2
0	00:00:00	1.0404e+01	3	EP	5.0000e-01
10	00:00:00	1.0483e+01	4		1.0395e+00
11	00:00:00	1.0483e+01	5	HB	1.0410e+00
17	00:00:01	1.0485e+01	6	FP	1.0522e+00
17	00:00:01	1.0485e+01	7	SN	1.0522e+00
20	00:00:01	1.0485e+01	8		1.0447e+00
21	00:00:01	1.0485e+01	9	SN	1.0420e+00
21	00:00:01	1.0485e+01	10	FP	1.0421e+00
23	00:00:01	1.0489e+01	11	HB	1.0516e+00
30	00:00:02	1.0506e+01	12		1.1183e+00
40	00:00:02	1.0831e+01	13		2.1406e+00
43	00:00:02	1.1235e+01	14	EP	3.0000e+00

The call to `ep_add_bddat` below modifies the default behavior to include a column in the output cell array and in the screen output with the smallest singular value of the Jacobian $\partial_x f(x, p)$ at each equilibrium point.

```
>> prob = ep_add_bddat(prob, '', 'svds', ...
    @(d,x,p) min(svds(feval(F('x'),x,p))));
```

```
>> coco(prob, eprunid, [], 'p2', [0.4 3]);
```

STEP	DAMPING	NORMS				COMPUTATION TIMES		
		IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)
0						1.06e-02	1.04e+01	0.0
1	1	1.00e+00	5.46e-03	5.46e-03		1.13e-04	1.04e+01	0.0
2	1	1.00e+00	6.56e-05	6.56e-05		5.26e-09	1.04e+01	0.0
3	1	1.00e+00	3.84e-09	3.84e-09		3.12e-17	1.04e+01	0.0

STEP	TIME	$\ U\ $	LABEL	TYPE	p2	svds
0	00:00:00	1.0404e+01	1	EP	5.0000e-01	4.4953e-02
2	00:00:00	1.0396e+01	2	EP	4.0000e-01	4.5111e-02

STEP	TIME	$\ U\ $	LABEL	TYPE	p2	svds
0	00:00:00	1.0404e+01	3	EP	5.0000e-01	4.4953e-02
10	00:00:00	1.0483e+01	4		1.0395e+00	1.1742e-02
11	00:00:00	1.0483e+01	5	HB	1.0410e+00	1.0802e-02
17	00:00:01	1.0485e+01	6	FP	1.0522e+00	3.9708e-08
17	00:00:01	1.0485e+01	7	SN	1.0522e+00	6.0449e-09
20	00:00:01	1.0485e+01	8		1.0447e+00	2.3490e-03
21	00:00:01	1.0485e+01	9	SN	1.0420e+00	6.9403e-07
21	00:00:01	1.0485e+01	10	FP	1.0421e+00	8.7757e-06
23	00:00:01	1.0489e+01	11	HB	1.0516e+00	1.2221e-02
30	00:00:02	1.0506e+01	12		1.1183e+00	4.6977e-02
40	00:00:02	1.0831e+01	13		2.1406e+00	2.2317e-02
43	00:00:02	1.1235e+01	14	EP	3.0000e+00	2.1167e-02

As expected, this value is approximately 0 at a saddle-node bifurcation.

Either of the Hopf bifurcations found in the previous run may be used to restart continuation along a family of Hopf bifurcations, as exemplified below.

```
>> labs = coco_bd_labs(eprunid, 'HB');
```

```
>> prob = coco_prob;
```

```

>> prob = ode_HB2HB(prob, '', eprunid, labs(1));
>> prob = coco_set(prob, 'cont', 'PtMX', [90 0]);
>> hbrunid = 'HB-curve';
>> coco(prob, hbrunid, [], {'p2' 'p7'}, {[0.4 3], [0 8]});
```

STEP	DAMPING		NORMS			COMPUTATION TIMES			
	IT	SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0					9.23e-08	1.05e+01	0.0	0.0	0.0
STEP	TIME			U	LABEL	TYPE	p2	p7	
0	00:00:00			1.0538e+01	1	EP	1.0410e+00	4.0000e-01	
10	00:00:00			1.0480e+01	2		7.1662e-01	1.6805e-01	
11	00:00:00			1.0480e+01	3	FP	7.1639e-01	1.6698e-01	
20	00:00:01			1.0486e+01	4		7.6656e-01	1.7425e-01	
30	00:00:01			1.0550e+01	5		1.0661e+00	4.2629e-01	
36	00:00:02			1.0605e+01	6	BTP	1.1612e+00	7.2234e-01	
40	00:00:02			1.0880e+01	7		1.2603e+00	1.7835e+00	
43	00:00:02			1.1183e+01	8	BP	1.2952e+00	2.5338e+00	
50	00:00:03			1.3021e+01	9		1.4705e+00	5.3034e+00	
60	00:00:03			1.3468e+01	10		1.5984e+00	5.8039e+00	
70	00:00:04			1.3332e+01	11		1.6524e+00	5.6316e+00	
76	00:00:04			1.2079e+01	12	FP	1.7317e+00	3.9579e+00	
80	00:00:04			1.1277e+01	13		1.6891e+00	2.5468e+00	
85	00:00:05			1.0695e+01	14	BP	1.4186e+00	9.7401e-01	
85	00:00:05			1.0694e+01	15	BTP	1.4176e+00	9.7140e-01	
90	00:00:05			1.0534e+01	16	EP	1.0229e+00	3.8396e-01	

By default, COCO solves for the square of the limiting angular frequency of oscillation along the family of periodic orbits emanating from such a bifurcation point, but does not store this data in the output cell array. The following sequence of commands modifies this behavior.

```

>> prob = ep_HB_add_bddat(prob, '', 'freq', @(d,x,p,v,k) sqrt(max(k,0)));
>> coco(prob, hbrunid, [], {'p2' 'p7'}, {[0.4 3], [0 8]});
```

STEP	DAMPING		NORMS			COMPUTATION TIMES			
	IT	SIT	GAMMA	d	f	U	F(x)	DF(x)	SOLVE
0						9.23e-08	1.05e+01	0.0	0.0
STEP	...		LABEL	TYPE			p2	p7	freq
0	...		1	EP			1.0410e+00	4.0000e-01	5.9106e-02
10	...		2				7.1662e-01	1.6805e-01	5.6765e-02
11	...		3	FP			7.1639e-01	1.6698e-01	5.6779e-02
20	...		4				7.6656e-01	1.7425e-01	5.7636e-02
30	...		5				1.0661e+00	4.2629e-01	4.9090e-02
36	...		6	BTP			1.1612e+00	7.2234e-01	0.0000e+00
40	...		7				1.2603e+00	1.7835e+00	0.0000e+00
43	...		8	BP			1.2952e+00	2.5338e+00	0.0000e+00
50	...		9				1.4705e+00	5.3034e+00	0.0000e+00
60	...		10				1.5984e+00	5.8039e+00	0.0000e+00
70	...		11				1.6524e+00	5.6316e+00	0.0000e+00
76	...		12	FP			1.7317e+00	3.9579e+00	0.0000e+00
80	...		13				1.6891e+00	2.5468e+00	0.0000e+00
85	...		14	BP			1.4186e+00	9.7401e-01	0.0000e+00
85	...		15	BTP			1.4176e+00	9.7140e-01	2.8516e-05

90	...	16 EP	1.0229e+00	3.8396e-01	5.9254e-02
----	-----	-------	------------	------------	------------

Here, the frequency is reported as 0 at neutral saddle points, which satisfy the algebraic conditions for a Hopf bifurcation but not the condition that there exist a pair of imaginary, complex conjugate eigenvalues of $\partial_x f(x, p)$.

As illustrated below, a further modification appends information about the first Lyapunov coefficient to the output cell array and the screen output.

```
>> data = struct('dfdxhan', F('x'), 'Dfdxdxhan', F({'x*v','x*v'}), ...
    'Dfdxdxdxhan', F({'x*v','x*v','x*v'}), 'nanflag', 1);
>> prob = ep_HB_add_bddat(prob, '', 'L1', @lyapunov, 'data', data);
>> prob = coco_set(prob, 'cont', 'PtMX', [90 0]);
>> coco(prob, hbrunid, [], {'p2' 'p7'}, {[0.4 3], [0 8]});
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)	DF(x)	SOLVE
0			9.23e-08	1.05e+01		0.0	0.0	0.0
STEP	...	LABEL	TYPE	p2	p7	freq	L1	
0	...	1	EP	1.0410e+00	4.0000e-01	5.9106e-02	7.3283e+01	
10	...	2		7.1662e-01	1.6805e-01	5.6765e-02	-1.2709e+02	
11	...	3	FP	7.1639e-01	1.6698e-01	5.6779e-02	-1.2680e+02	
20	...	4		7.6656e-01	1.7425e-01	5.7636e-02	-8.6324e+01	
30	...	5		1.0661e+00	4.2629e-01	4.9090e-02	2.5779e+02	
36	...	6	BTP	1.1612e+00	7.2234e-01	0.0000e+00	NaN	
40	...	7		1.2603e+00	1.7835e+00	0.0000e+00	NaN	
43	...	8	BP	1.2952e+00	2.5338e+00	0.0000e+00	NaN	
50	...	9		1.4705e+00	5.3034e+00	0.0000e+00	NaN	
60	...	10		1.5984e+00	5.8039e+00	0.0000e+00	NaN	
70	...	11		1.6524e+00	5.6316e+00	0.0000e+00	NaN	
76	...	12	FP	1.7317e+00	3.9579e+00	0.0000e+00	NaN	
80	...	13		1.6891e+00	2.5468e+00	0.0000e+00	NaN	
85	...	14	BP	1.4186e+00	9.7401e-01	0.0000e+00	NaN	
85	...	15	BTP	1.4176e+00	9.7140e-01	2.8516e-05	-9.3424e+13	
90	...	16	EP	1.0229e+00	3.8396e-01	5.9254e-02	6.0838e+01	

where `lyapunov` is a slight modification of the function shown in the previous section that returns NaN in the event of a neutral saddle, as shown below.

```
function y = lyapunov(data, x, p, v, k)

n = numel(x);
om = sqrt(k);
if om<1e-6 && data.nanflag % if neutral saddle
    y = NaN;
    return
end

A = data.dfdxhan(x,p);
va = v-1i*A*v/om;
va = va/norm(va);
vb = conj(va);
w = ([A-1i*om*eye(n) va; va' 0]\[eye(n); zeros(1,n)])'...
```

```

*[zeros(n,1); 1];

B = @(dx1, dx2) data.Dfdxdxhan(x, p, dx1, dx2);
C = @(dx1, dx2, dx3) data.Dfdxdxdxhan(x, p, dx1, dx2, dx3);
y = real(w'*(B(vb, (2*pi*i*om*eye(n)-A)\B(va,va))...
-2*B(va,A\B(va,vb))+C(va,va,vb))/2/om;

end

```

We visualize the results of this analysis, including a curve of saddle-node bifurcations, using the following commands.

```

>> figure(1)
>> clf
>> hold on
>> thm = struct();
>> thm.special = {'HB', 'SN'};
>> coco_plot_bd(thm, eprunid, 'p2', 'x')
>> coco_plot_bd(snrnid, 'p2', 'x')
>> thm = struct();
>> thm.special = {'BTP'};
>> thm.ustab = 'L1';
>> thm.ustabfun = @(x) 1+(~isnan(x) & x>0)+2*isnan(x);
>> thm.usept = {};
>> thm.lsspec = {{'r-', 'LineWidth', 1.5}, {'r-.', 'LineWidth', 1.5}};
>> thm.xlab = 'p_2';
>> coco_plot_bd(thm, hbrnid, 'p2', 'x')
>> hold off
>> grid on
>> axis([0.5 2 0 0.16])

```

Here, the sign of the Lyapunov coefficient is used to select the line style for the curve of Hopf bifurcations.

7.2 Continuing generalized Hopf bifurcations

Computations added using `ep_add_bddat`, `ep_SN_add_bddat`, and `ep_HB_add_bddat`, execute after convergence to an equilibrium, saddle-node bifurcation, or Hopf bifurcation, respectively. They allow you to perform *a posteriori* evaluation of properties of each solution point but cannot be used to automatically detect special points associated with particular numerical thresholds. To detect such special points during continuation of Hopf bifurcations, embed the monitor functions within the continuation problem using the `ep_HB_add_func` constructor.

As an example, zero crossings of the first Lyapunov coefficient along a curve of Hopf bifurcations may be detected by embedding the `lyapunov` function as a '`regular`' monitor function, as in the following example code.

```

>> prob = coco_prob;
>> prob = ode_HB2HB(prob, '', eprunid, labs(2));
>> data = struct('dfdxhan', F('x'), 'Dfdxdxhan', F({'x*v', 'x*v'}), ...

```

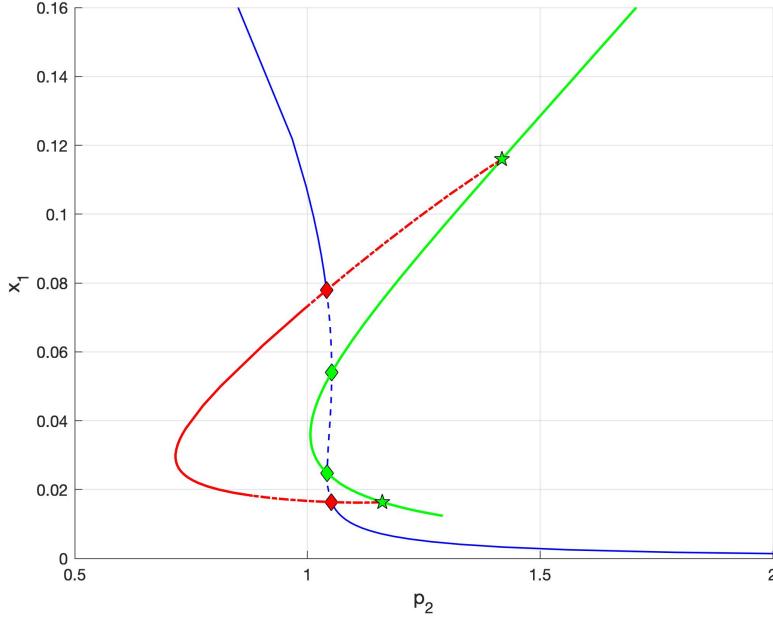


Figure 15: Families of equilibria (blue), saddle-node bifurcations of equilibria (green), and Hopf bifurcations of equilibria (red) for the vector field in Section 7.1. The Hopf bifurcation curve terminates at each end on the saddle-node bifurcation curve at a Bogdanov-Takens points (green star) and consists of both subcritical (dash-dotted) and supercritical (solid) bifurcations.

```
'Dfdxdxdxhan', F({'x*v','x*v','x*v'}), 'nanflag', 1);
>> prob = ep_HB_add_func(prob, '', 'lyap', @lyapunov, data, ...
    'regular', 'L1');
>> prob = coco_add_event(prob, 'GH', 'L1', 0);
>> prob = coco_set(prob, 'cont', 'PtMX', [85 0]);
>> coco(prob, 'HB-curve', [], {'p2' 'p7', 'L1'}, {[0.4 3], [0 8]});
```

STEP	DAMPING		NORMS			COMPUTATION TIMES			
	IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	$F(x)$	$DF(x)$	SOLVE
0				1.86e-07	1.05e+01		0.0	0.0	0.0

STEP	...	LABEL	TYPE	p2	p7	L1
0	...	1	EP	1.0516e+00	4.0000e-01	2.1091e+02
6	...	2	GH	8.9132e-01	2.3249e-01	-1.2993e-05
10	...	3		7.2575e-01	1.6461e-01	-1.1714e+02
14	...	4	FP	7.1639e-01	1.6698e-01	-1.2680e+02
20	...	5		7.4327e-01	1.8907e-01	-1.1155e+02
26	...	6	GH	9.2427e-01	3.0589e-01	6.3144e-06
30	...	7		1.2430e+00	6.3207e-01	3.4470e+02
34	...	8	BTP	1.4176e+00	9.7140e-01	NaN
38	...	9	BP	1.6380e+00	1.9804e+00	NaN
40	...	10		1.7036e+00	2.7909e+00	NaN
44	...	11	FP	1.7317e+00	3.9581e+00	NaN
50	...	12		1.6128e+00	5.7852e+00	NaN
60	...	13		1.5635e+00	5.7815e+00	NaN

70	...	14	1.4025e+00	4.5589e+00	NaN
80	...	15	1.2068e+00	1.0355e+00	NaN
81	...	16	BP	1.1645e+00	7.3900e-01
81	...	17	BTP	1.1612e+00	7.2234e-01
85	...	18	EP	1.0300e+00	3.6606e-01

The zero crossings, here denoted by the GH point type, coincide with *generalized Hopf bifurcations* at the transition between super- and subcritical Hopf bifurcations. The following sequence of commands produces the bifurcation diagram shown in Fig. 16.

```

>> figure(2)
>> clf
>> hold on
>> thm = struct();
>> thm.special = {'HB', 'SN'};
>> coco_plot_bd(thm, eprunid, 'p2', 'x')
>> coco_plot_bd('SN-curve', 'p2', 'x')
>> thm = struct();
>> thm.special = {'BTP', 'GH'};
>> thm.GH = {'kp', 'MarkerFaceColor', 'k', 'MarkerSize', 10};
>> thm.ustab = 'L1';
>> thm.ustabfun = @(x) 1+(~isnan(x) & x>0)+2*isnan(x);
>> thm.usept = {'BTP', 'GH'};
>> thm.lsspec = {{'r-', 'LineWidth', 1.5}, {'r-.', 'LineWidth', 1.5}};
>> thm.xlab = 'p_2';
>> coco_plot_bd(thm, 'HB-curve', 'p2', 'x')
>> hold off
>> grid on
>> axis([0.5 2 0 0.16])

```

You can continue along a curve of generalized Hopf bifurcations by embedding the lyapunov function as an 'inactive' monitor function and fixing its value to 0, as shown below. Since a generalized Hopf bifurcation is a co-dimension-two point, three parameters are allowed to vary along this curve.

```

>> labs = coco_bd_labs('HB-curve', 'GH');
>> prob = coco_prob;
>> prob = ode_HB2HB(prob, '', 'HB-curve', labs(1));
>> data = struct('dfdxhan', F('x'), 'Dfdxdxhan', F({'x*v', 'x*v'}), ...
    'Dfdxdxdxhan', F({'x*v', 'x*v', 'x*v'}), 'nanflag', 1);
>> prob = ep_HB_add_func(prob, '', 'lyap', @lyapunov, data, ...
    'inactive', 'L1');
>> prob = coco_set_parival(prob, 'L1', 0);
>> prob = coco_set(prob, 'cont', 'PtMX', 50);
>> coco(prob, 'GH-curve', [], {'p2' 'p7' 'p1'}, {[0.4 3], [0 8]});

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||   F(x)   DF(x)   SOLVE
        0           3.52e-07  1.08e+01     0.0     0.0     0.0

      STEP      ...      LABEL      TYPE      p2      p7      p1
      0       ...           1      EP      9.2427e-01  3.0587e-01  2.5001e+00
      10      ...          2           8.9383e-01  3.2573e-01  2.3080e+00

```

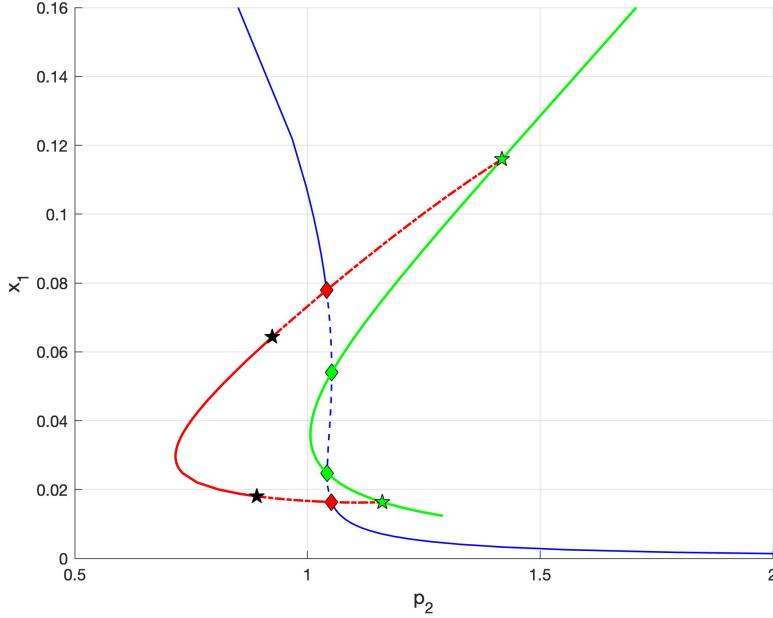


Figure 16: Families of equilibria (blue), saddle-node bifurcations of equilibria (green), and Hopf bifurcations of equilibria (red) for the vector field in Section 7.1. The Hopf bifurcation curve terminates at each end on the saddle-node bifurcation curve at a Bogdanov-Takens points (green star) and consists of both subcritical (dash-dotted) and supercritical (solid) bifurcations, separated by generalized Hopf bifurcations (black stars). While the location of the latter is accurate to within tolerance, the change in the sign of the Lyapunov coefficient is not detected until beyond this point, as evidenced by the crossing of the top-most generalized Hopf bifurcation by the dash-dotted curve.

20	...	3		8.7535e-01	3.3895e-01	2.1957e+00
30	...	4		8.5733e-01	3.5279e-01	2.0892e+00
40	...	5		8.3976e-01	3.6731e-01	1.9884e+00
50	...	6	EP	8.2262e-01	3.8254e-01	1.8928e+00
STEP	...	LABEL	TYPE	p2	p7	p1
0	...	7	EP	9.2427e-01	3.0587e-01	2.5001e+00
10	...	8		9.5437e-01	2.8827e-01	2.6987e+00
20	...	9		9.7499e-01	2.7722e-01	2.8399e+00
30	...	10		9.9613e-01	2.6666e-01	2.9890e+00
40	...	11		1.0178e+00	2.5655e-01	3.1463e+00
50	...	12	EP	1.0400e+00	2.4687e-01	3.3125e+00

Alternatively, embed the `lyapunov` function in the continuation problem as a *zero function*, as in the following:

```
>> labs = coco_bd_labs('HB-curve', 'GH');
>> prob = coco_prob;
>> prob = ode_HB2HB(prob, '', 'HB-curve', labs(2));
>> data = struct('dfdxhan', F('x'), 'Dfdxdxhan', F({'x*v', 'x*v'}), ...
    'Dfdxdxdxhan', F({'x*v', 'x*v', 'x*v'}), 'nanflag', 1);
```

```

>> prob = ep_HB_add_func(prob, '', 'lyap', @lyapunov, data, 'zero');
>> prob = coco_set(prob, 'cont', 'PtMX', 50);
>> coco(prob, 'GH-curve', [], {'p2' 'p7' 'p1'}, {[0.4 3], [0 8]});
```

STEP	DAMPING	IT	SIT	NORMS			COMPUTATION TIMES		
				GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	$F(x)$	$DF(x)$
0					1.30e-05	1.08e+01	0.0	0.0	0.0
1	1	1.00e+00	5.25e-05		9.57e-07	1.08e+01	0.0	0.1	0.0
2	1	1.00e+00	3.74e-10		1.75e-12	1.08e+01	0.0	0.2	0.0
STEP	...	LABEL	TYPE		p2	p7		p1	
0	...	1	EP		8.9132e-01	2.3248e-01		2.5000e+00	
10	...	2			8.6719e-01	2.4549e-01		2.3487e+00	
20	...	3			8.5291e-01	2.5400e-01		2.2608e+00	
30	...	4			8.3885e-01	2.6307e-01		2.1753e+00	
40	...	5			8.2504e-01	2.7274e-01		2.0923e+00	
50	...	6	EP		8.1150e-01	2.8305e-01		2.0121e+00	
STEP	...	LABEL	TYPE		p2	p7		p1	
0	...	7	EP		8.9132e-01	2.3248e-01		2.5000e+00	
10	...	8			9.1572e-01	2.2083e-01		2.6564e+00	
20	...	9			9.3067e-01	2.1432e-01		2.7539e+00	
30	...	10			9.4573e-01	2.0819e-01		2.8535e+00	
40	...	11			9.6089e-01	2.0240e-01		2.9550e+00	
50	...	12	EP		9.7614e-01	1.9693e-01		3.0585e+00	

7.3 Monitoring properties of periodic orbits

A similar construction as in Sec. 7.1 applies to the monitoring of properties of periodic orbits using the `po_add_bddat` constructor. Suppose, for example, that you want to monitor the Fourier integral

$$\mathcal{I}_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-i2n\pi t/T) dt$$

for each member $x(t)$ of a family of periodic orbits of a vector field $f(x, p)$, associated period T , and some integer n .

COCO represents such an orbit in terms of a *partition* of the interval $[-T/2, T/2]$ into N possibly unequal intervals and, on each interval, a set of vectors at $m + 1$ uniformly spaced instants of time, including the interval end points. The interval lengths are tracked by the sequence $\{\kappa_j\}_{j=1}^N$, such that the j -th interval extends from t_j to $t_{j+1} = t_j + T\kappa_j/N$ and $\sum_{j=1}^N \kappa_j = N$. The j -th interval is associated with a predefined set of m *collocation nodes* $t_j + (1 + z_i)T\kappa_j/2N$ with $z_i \in [-1, 1]$ and associated *quadrature weights* $T\kappa_j w_i/2N$, so that

$$\mathcal{I}_n \approx \frac{1}{2N} \sum_{j=1}^N \kappa_j \sum_{i=1}^m w_i x(t_j + (1 + z_i)T\kappa_j/2N) \exp(-i2n\pi(t_j + (1 + z_i)T\kappa_j/2N)/T)$$

If $x(t)$ is approximated by an m -th order polynomial on the j -th interval, then the values $x(t_j + (1 + z_i)T\kappa_j/2N)$ may be obtained from predefined linear combinations of the values of $x(t)$ at the $m + 1$ uniformly spaced instants of time on that interval.

The function `fourier` below uses the fields of the `data` argument to compute this double-sum approximation of the Fourier integral.

```
function y = fourier(data, xbp, T0, T, p)

mps = data.coll_seg.maps;
msh = data.coll_seg.mesh;
xcn = reshape(mps.W*xbp, mps.x_shp);
ecn = exp(-2i*data.n*pi*msh.tcn);
y = (msh.fka.*msh.fwt.*xcn).*ecn/2/mps.NTST;

end
```

Here, `xcn` and `ecn` contain a rectangular array with columns given by $x(t_j + (1+z_i)T\kappa_j/2N)$ and a column array with entries given by $\exp(-i2n\pi(t_j + (1+z_i)T\kappa_j/2N)/T)$, respectively, for $j = 1, \dots, N$, $i = 1, \dots, m$. A verification of this encoding is given by the following application to the family of periodic orbits emanating from a subcritical Hopf bifurcation, as well as the corresponding curve of saddle-node bifurcations of periodic orbits computed in Sec. 4.1.

```
>> syms r2 x1 x2 p1 p2
>> r2 = x1^2+x2^2;
>> F = sco_sym2funcs(... 
    [x1*(p1+p2*r2-r2^2)-x2; x2*(p1+p2*x2-r2^2)+x1], ...
    {[x1; x2], [p1; p2]}, {'x', 'p'}, 'filename', 'sys_hopf');
>> f = struct('f', F(''), 'dfdx', F('x'), 'dfdp', F('p'), ...
    'dfdx_dx', F('x*v'), 'dfdxdx_dx', F({'x', 'x*v'}), ...
    'dfdpdx_dx', F({'p', 'x*v'}));
>> prob = coco_prob;
>> prob = ode_isol2ep(prob, '', f, [0; 0], {'p1', 'p2'}, [-1; 1]);
>> coco(prob, 'ep_run', [], 'p1', [-1 1]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)  DF(x)  SOLVE
        0           0.00e+00  1.73e+00      0.0       0.0       0.0
      STEP      TIME      ||U||  LABEL  TYPE          p1
        0  00:00:00  1.7321e+00      1  EP  -1.0000e+00
        5  00:00:00  1.0000e+00      2  HB  1.5899e-07
        8  00:00:00  1.7321e+00      3  EP  1.0000e+00

>> HB = coco_bd_labs('ep_run', 'HB');
>> prob = coco_prob;
>> prob = ode_HB2po(prob, '', 'ep_run', HB);
>> prob = coco_set(prob, 'cont', 'PtMX', [50 0]);
>> prob = po_add_bddat(prob, '', 'cx1', @fourier, 'data', struct('n', 1));
>> coco(prob, 'po_run', [], {'p1' 'po.period'}, [-1 1]);

      STEP      DAMPING          NORMS          COMPUTATION TIMES
      IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)  DF(x)  SOLVE
        0           6.75e-06  8.94e+00      0.0       0.0       0.0
        1      1  1.00e+00  1.96e-06  2.68e-13  8.94e+00      0.0       0.0       0.0
        2      1  1.00e+00  1.14e-12  2.84e-18  8.94e+00      0.0       0.0       0.0
```

STEP	...	LABEL	TYPE	p1	p2	cx1
0	...	1	EP	-9.9933e-07	1.0000e+00	4.9983e-04
10	...	2		-2.2549e-01	1.0000e+00	2.9302e-01
12	...	3	FP	-2.5000e-01	1.0000e+00	3.5355e-01
12	...	4	SN	-2.5000e-01	1.0000e+00	3.5355e-01
20	...	5		4.9782e-01	1.0000e+00	5.8412e-01
23	...	6	EP	1.0000e+00	1.0000e+00	6.3601e-01

```

>> SN = coco_bd_labs('po_run', 'SN');
>> prob = coco_prob;
>> prob = ode_po2SN(prob, '', 'po_run', SN(1));
>> prob = po_add_bddat(prob, '', 'cx1', @fourier, 'data', struct('n', 1));
>> coco(prob, 'po_SN_run', [], {'p1', 'p2'}, {[[-1 1] [0.001 3]});

      STEP    DAMPING          NORMS          COMPUTATION TIMES
      IT SIT    GAMMA    ||d||    ||f||    ||U||    F(x)   DF(x)   SOLVE
      0        1.21e-05  1.25e+01  0.0       0.0       0.0
      1        1.00e+00  4.97e-05  2.06e-11 1.25e+01  0.0       0.0       0.0
      2        1.00e+00  9.84e-11  3.19e-15 1.25e+01  0.0       0.0       0.0

```

STEP	...	LABEL	TYPE	p1	p2	cx1
0	...	1	EP	-2.5000e-01	1.0000e+00	3.5355e-01
8	...	2	EP	-1.0000e+00	2.0000e+00	5.0000e-01

STEP	...	LABEL	TYPE	p1	p2	cx1
0	...	3	EP	-2.5000e-01	1.0000e+00	3.5355e-01
10	...	4		-6.3961e-04	5.0581e-02	7.9515e-02
12	...	5	EP	-2.5000e-07	1.0000e-03	1.1180e-02

Since $x_1(t)$ is known to equal $r \cos t$, we compute r as twice the real part of the Fourier coefficient for $n = 1$. In the visualization generated by the following commands (cf. Fig. 17), the results of the computation are compared to the predicted relationships $p_1 = r^4 - p_2 r^2$ along the family of periodic orbits and $p_1 = -r^4$, $p_2 = 2r^2$ along the family of saddle-node bifurcations.

```

>> figure(1)
>> clf
>> hold on
>> coco_plot_bd('po_run', 'p1', 'p2', 'cx1', @(x) 2*real(x(1,:)))
>> thm = struct();
>> thm.lsspec = {'ro', 'MarkerFaceColor', 'r'}, {'ro'}};
>> coco_plot_bd(thm, 'po_run', ...
    {'cx1', 'p2'}, @(x,y) (2*real(x(1,:))).^4-y.* (2*real(x(1,:))).^2, ...
    'p2', 'cx1', @(x) 2*real(x(1,:)));
>> coco_plot_bd('po_SN_run', 'p1', 'p2', 'cx1', @(x) 2*real(x(1,:)))
>> thm.lsspec = {'ko', 'MarkerFaceColor', 'k'};
>> thm.xlab = 'p1';
>> thm.ylab = 'p2';
>> thm.zlab = 'r';
>> coco_plot_bd(thm, 'po_SN_run', ...
    'cx1', @(x) -(2*real(x(1,:))).^4, ...
    'cx1', @(x) 2*(2*real(x(1,:))).^2, ...
    'cx1', @(x) 2*real(x(1,:)));

```

```
>> hold off
>> grid on
>> view(3)
```

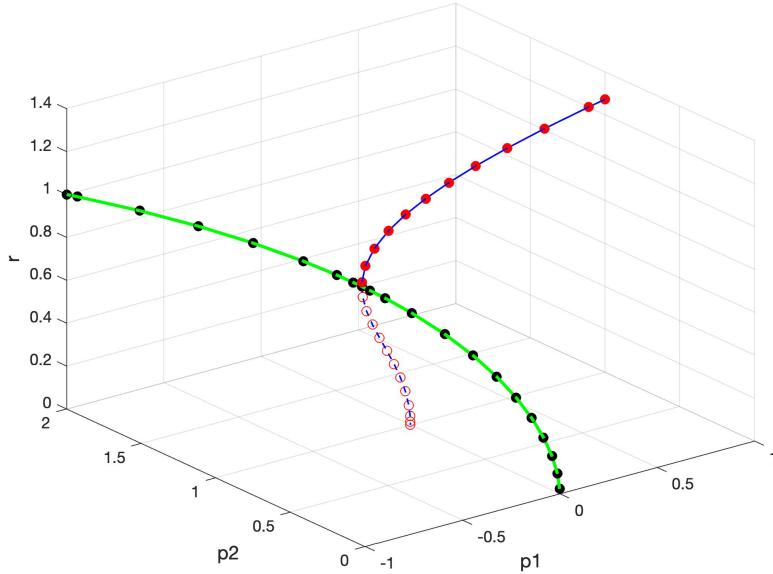


Figure 17: Comparison between theoretical predictions (red and black balls) and the results of continuation along families of periodic orbits (blue solid) and saddle-node bifurcations of periodic orbits (green solid) associated with the Hopf normal form in Sec. 4.1. The vertical coordinate r represents the projection onto the Fourier coefficient associated with $\cos t$.

7.4 Tracking and constraining orbit maxima

The `po_add_func` constructor mirrors the functionality of `ep_add_func` by allowing you to monitor properties of periodic orbits during continuation, automatically detect special points associated with particular numerical thresholds, and impose constraints on such properties.

To illustrate this functionality, suppose that you have performed continuation along a family of equilibria, including detection of a Hopf bifurcation, as shown in the following sequence of commands.

```
>> syms x1 x2 x3 p1 p2
>> F = sco_sym2funcs(..., ...
    [p1*x1+x2+p2*x1^2; -x1+p1*x2+x2*x3; (p1^2-1)*x2-x1-x3+x1^2], ...
    {[x1; x2; x3], [p1; p2]}, {'x', 'p'}, 'filename', 'sys_marsden');
>> f = struct('f', F()), 'dx', F('x'), 'dp', F('p'));
>> prob = coco_prob;
>> prob = ode_isol2ep(prob, '', f, [0; 0; 0], {'p1', 'p2'}, [-1; 6]);
>> coco(prob, 'ep_run', [], 'p1', [-1 1]);
```

STEP	DAMPING	NORMS	COMPUTATION TIMES		
IT SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	$F(x)$ $DF(x)$ SOLVE

0		0.00e+00	6.16e+00	0.0	0.0	0.0
STEP	TIME	U	LABEL	TYPE	p1	
0	00:00:00	6.1644e+00	1	EP	-1.0000e+00	
5	00:00:00	6.0000e+00	2	HB	1.5899e-07	
8	00:00:00	6.1644e+00	3	EP	1.0000e+00	

You then proceed to continue along a family of periodic orbits emanating from the Hopf bifurcation as shown below.

>> HB = coco_bd_labs('ep_run', 'HB');
>> prob = coco_prob;
>> prob = ode_HB2po(prob, '', 'ep_run', HB);
>> prob = coco_set(prob, 'cont', 'PtMX', [50 0], 'NAdapt', 1);
>> coco(prob, 'po_run', [], {'p1' 'po.period'}, [-1 1]);
STEP DAMPING NORMS COMPUTATION TIMES
IT SIT GAMMA d f U F(x) DF(x) SOLVE
0 0 1.00e+00 3.53e-05 7.39e-06 1.07e+01 0.0 0.0 0.0
1 1 1.00e+00 2.48e-10 9.58e-12 1.07e+01 0.0 0.0 0.0
2 1 1.00e+00 2.48e-10 5.00e-18 1.07e+01 0.0 0.0 0.0
STEP TIME U LABEL TYPE p1 po.period
0 00:00:00 1.0722e+01 1 EP -7.8687e-07 6.2832e+00
10 00:00:00 1.0975e+01 2 -6.5370e-03 6.4878e+00
18 00:00:01 1.2676e+01 3 SN -2.0251e-02 7.7362e+00
18 00:00:01 1.2676e+01 4 FP -2.0252e-02 7.7365e+00
20 00:00:02 1.3777e+01 5 -1.8542e-02 8.5358e+00
30 00:00:02 1.8325e+01 6 -1.3582e-02 1.1805e+01
40 00:00:03 2.3076e+01 7 -1.3193e-02 1.5189e+01
50 00:00:04 2.7721e+01 8 EP -1.3160e-02 1.8625e+01

By executing the command

```
>> coco_plot_sol('po_run', '', 't', 'x')
```

you note from the visualization (cf. Fig. 18) that the periodic function associated with label 5 achieves a maximum in x_1 near $t = 5.17$.

It now occurs to you that it could be interesting to track the maximum value of x_1 along the family of periodic orbits. To this end, you define the function `slope` shown below.

```
function y = slope(data, xbp, T0, T, p, t)

m      = data.col1_seg.int.NCOL;
xdim  = data.col1_seg.int.dim;
N      = data.col1_seg.maps.NTST;
tmi   = data.col1_seg.mesh.tmi;
tbp   = data.col1_seg.mesh.tbp;

% find interval
trs    = (t-T0)/T;
J = min(N, find(N*trs>=tmi, 1, 'last'));
xbpint = xbp((J-1)*xdim*(m+1)+(1:xdim*(m+1)));
```

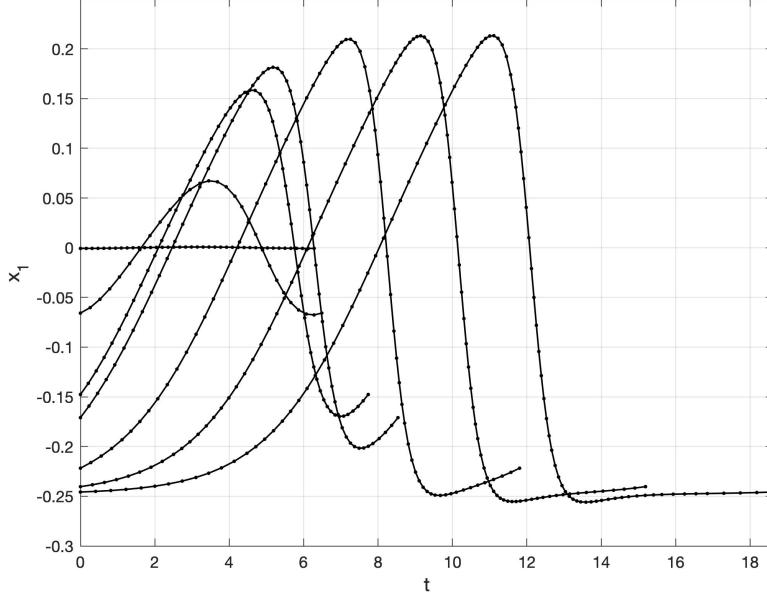


Figure 18: Time histories for labeled periodic orbits found during continuation using the code in Sect. 7.4.

```

tint    = tbp((J-1)*(m+1)+(1:m+1));
ts      = linspace(-1, 1, m+1)';

% interpolated point
s = repmat(2*(trs-tint(1))/(tint(end)-tint(1))-1, [1 m+1 m+1]);
sj = repmat(reshape(ts, [1 m+1 1]), [1 1 m+1]);
sk = repmat(reshape(ts, [1 1 m+1]), [1 m+1 1]);

t1 = s-sk;
t2 = sj-sk;
idx = find(abs(t2)<=eps);
t1(idx) = 1;
t2(idx) = 1;

x = kron(prod(t1./t2, 3), eye(xdim))*xbpint;

% interpolated derivative
s = repmat(2*(trs-tint(1))/(tint(end)-tint(1))-1, [1 m+1 m+1 m+1]);
sj = repmat(reshape(ts, [1 m+1 1 1]), [1 1 m+1 m+1]);
sk = repmat(reshape(ts, [1 1 m+1 1]), [1 m+1 1 m+1]);
sl = repmat(reshape(ts, [1 1 1 m+1]), [1 m+1 m+1 1]);

t3 = sj(:,:,:,:1)-sk(:,:,:,:1);
t4 = s-sl;
t5 = sj-sl;
idx1 = find(abs(t5)<=eps);
idx2 = find(abs(t3)<=eps);
idx3 = find(abs(sk-sl)<=eps);
t5(union(idx1, idx3)) = 1;
t4(union(idx1, idx3)) = 1;

```

```

t3(idx2) = 1;
t3      = 1.0./t3;
t3(idx2) = 0;

xt = kron(sum(t3.*prod(t4./t5, 4), 3), eye(xdim))*xbpint;

y = [t; x(data.idx); xt(data.idx)];

end

```

This uses Lagrange interpolation (as described in Section 6.2.2 of *Recipes for Continuation*) to compute numerical values for a component of the state and the corresponding time derivative at a given instant of time and returns all three of these quantities in the output argument. The function is introduced to the continuation problem in the sequence of commands shown below, where the time instant is held fixed at 5.17, while $x_1(5.17)$ and $\dot{x}_1(5.17)$ vary.

```

>> prob = coco_prob;
>> prob = ode_po2po(prob, '', 'po_run', 5);
>> prob = coco_set(prob, 'cont', 'PtMX', [0 50], 'NAdapt', 1);
>> prob = po_add_func(prob, '', 'max', @slope, struct('idx', 1), ...
    {'trs', 'xrs', 'xtrs'}, 'inactive', 'u0', 5.17);
>> coco(prob, 'new_po_run1', [], {'p1' 'po.period', 'xrs', 'xtrs'}, [-1 1]);

```

STEP	DAMPING	NORMS			COMPUTATION TIMES			
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	F(x)	DF(x)	SOLVE
0			2.01e-15	1.47e+01		0.0	0.0	0.0

STEP	...	LABEL	TYPE	p1	po.period	xrs	xtrs
0	...	1	EP	-1.8542e-02	8.5358e+00	1.8148e-01	9.0100e-04
10	...	2		-1.3822e-02	1.1232e+01	1.0514e-01	4.1433e-02
20	...	3		-1.3211e-02	1.4606e+01	-4.9327e-02	3.0655e-02
30	...	4		-1.3161e-02	1.8032e+01	-1.7123e-01	2.0405e-02
40	...	5		-1.3157e-02	2.1499e+01	-2.2523e-01	1.2060e-02
47	...	6	SN	-1.3156e-02	2.3711e+01	-2.3799e-01	5.3241e-03
50	...	7	EP	-1.3156e-02	2.4987e+01	-2.4176e-01	3.3737e-03

The results of the computation are visualized using the following commands (cf. Fig. 19).

```

>> figure(1)
>> clf
>> hold on
>> coco_plot_sol('new_po_run1', '', 't', 'x')
>> tx = coco_bd_vals('new_po_run1', 'all', {'trs', 'xrs'});
>> plot(tx(1,:), tx(2,:), 'ro', 'MarkerFaceColor', 'r');
>> hold off
>> grid on
>> box on
>> axis([0 inf -0.3 0.25])

```

To track a local maximum in x_1 , you execute the commands

```

>> prob = coco_set_parival(prob, 'xtrs', 0);

```

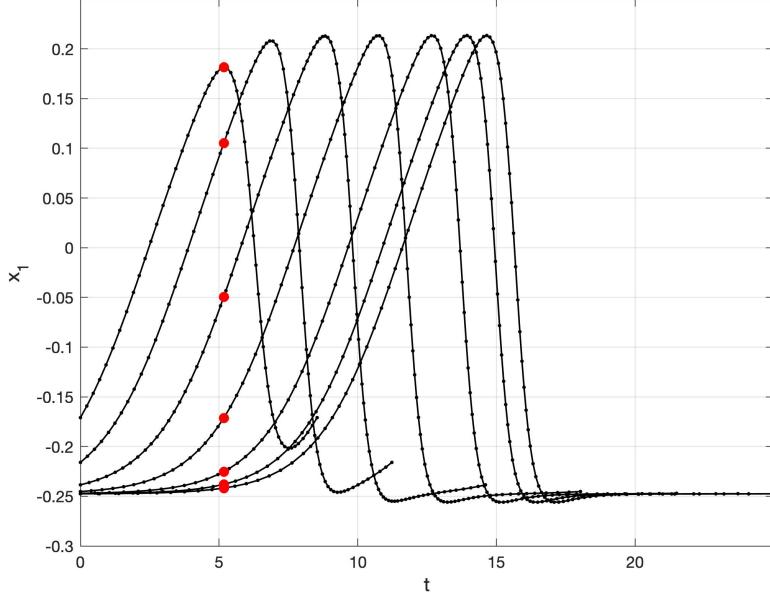


Figure 19: Time histories for labeled periodic orbits found during continuation using the code in Sect. 7.4 with points at $t = 5.17$ highlighted using red disks.

```
>> coco(prob, 'new_po_run2', [], {'p1' 'po.period', 'trs', 'xrs'}, [-1 1]);

STEP      DAMPING          NORMS          COMPUTATION TIMES
IT SIT      GAMMA      ||d||      ||f||      ||U||      F(x)    DF(x)    SOLVE
0           1.00e+00  9.01e-04  1.56e+01  0.0       0.0       0.0
1     1   1.00e+00  2.71e-02  4.67e-05  1.56e+01  0.0       0.1       0.0
2     1   1.00e+00  6.03e-04  2.37e-08  1.56e+01  0.0       0.2       0.0
3     1   1.00e+00  3.05e-07  6.08e-15  1.56e+01  0.0       0.2       0.0

STEP      ...      LABEL      TYPE      p1      po.period      trs      xrs
0       ...      1      EP      -1.8570e-02  8.5271e+00  5.1859e+00  1.8133e-01
10      ...      2      EP      -1.4056e-02  1.0854e+01  6.6562e+00  2.0665e-01
20      ...      3      EP      -1.3254e-02  1.3805e+01  8.3568e+00  2.1255e-01
30      ...      4      EP      -1.3168e-02  1.6801e+01  1.0046e+01  2.1332e-01
40      ...      5      EP      -1.3158e-02  1.9828e+01  1.1748e+01  2.1342e-01
50      ...      6      EP      -1.3156e-02  2.2872e+01  1.3458e+01  2.1343e-01
```

The results of the computation are visualized using the following commands (cf. Fig. 20).

```
>> figure(2)
>> clf
>> hold on
>> coco_plot_sol('new_po_run2', '', 't', 'x')
>> tx = coco_bd_vals('new_po_run2', 'all', {'trs', 'xrs'});
>> plot(tx(1,:), tx(2,:), 'ro', 'MarkerFaceColor', 'r');
>> hold off
>> grid on
>> box on
>> axis([0 inf -0.3 0.25])
```

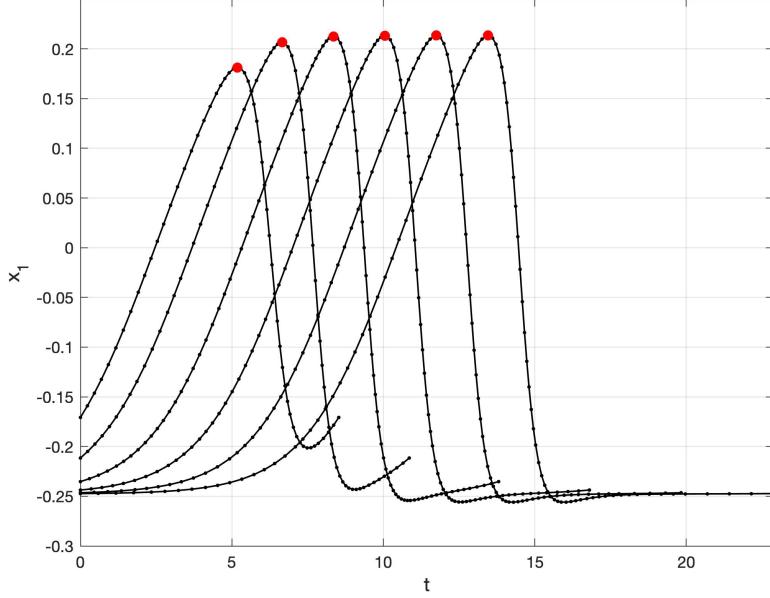


Figure 20: Time histories for labeled periodic orbits found during continuation using the code in Sect. 7.4 with a family of local maxima highlighted using red disks.

Finally, you decide to continue along a family of periodic orbits for which the local maximum in x_1 is held fixed at its initial value of x_1 . This is accomplished using the following command, in which p_2 is also allowed to vary.

```
>> coco(prob, 'new_po_run3', [], {'p1' 'po.period', 'trs', 'p2'}, [-1 1]);
```

STEP		DAMPING	NORMS			COMPUTATION TIMES		
IT	SIT	GAMMA	$\ d\ $	$\ f\ $	$\ U\ $	$F(x)$	$DF(x)$	SOLVE
0				$9.01e-04$	$1.67e+01$	0.0	0.0	0.0
1	1	$1.00e+00$	$2.72e-02$	$4.70e-05$	$1.67e+01$	0.0	0.1	0.0
2	1	$1.00e+00$	$2.18e-03$	$2.98e-08$	$1.67e+01$	0.0	0.2	0.0
3	1	$1.00e+00$	$1.29e-06$	$8.48e-15$	$1.67e+01$	0.0	0.2	0.0
4	1	$1.00e+00$	$1.36e-13$	$1.22e-15$	$1.67e+01$	0.0	0.3	0.0

STEP	...	LABEL	TYPE	p1	po.period	trs	p2
0	...	1	EP	$-1.8581e-02$	$8.5288e+00$	$5.1868e+00$	$5.9951e+00$
10	...	2		$-1.1092e-02$	$1.0616e+01$	$6.6250e+00$	$7.3292e+00$
20	...	3		$-9.2076e-03$	$1.3509e+01$	$8.4001e+00$	$7.9542e+00$
30	...	4		$-8.9274e-03$	$1.6502e+01$	$1.0138e+01$	$8.0823e+00$
40	...	5		$-8.8852e-03$	$1.9524e+01$	$1.1870e+01$	$8.1027e+00$
50	...	6	EP	$-8.8790e-03$	$2.2560e+01$	$1.3608e+01$	$8.1058e+00$

The results of this computation are visualized using the following commands (cf. Fig. 21).

```
>> figure(3)
>> clf
>> hold on
>> coco_plot_sol('new_po_run3', '', 't', 'x')
>> tx = coco_bd_vals('new_po_run3', 'all', {'trs', 'xrs'});
>> plot(tx(1,:), tx(2,:), 'ro', 'MarkerFaceColor', 'r');
```

```

>> hold off
>> grid on
>> box on
>> axis([0 inf -0.3 0.25])

```

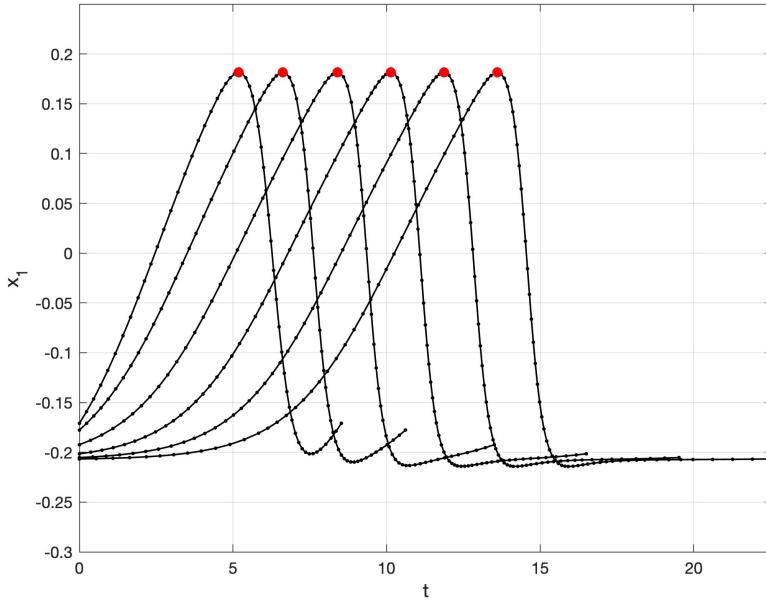


Figure 21: Time histories for labeled periodic orbits found during continuation using the code in Sect. 7.4 with a family of local maxima highlighted using red disks.

8 Outlook

As demonstrated in the previous section on user-initiated enhancements, COCO is expandable and not bound by a fixed set of predefined functionalities. The examples have barely scratched the surface of what is possible. They have also purposely stuck to a tight syntax, in the course of which some of the more exciting features may have been hidden from view.

Chances are that you have already thought of things you would like to do that don't appear straightforward within the framework presented in this tutorial. You may be inclined to ask whether a solution already exists. You may wonder how onerous an effort it might be to implement a solution when one does not appear available.

As the saying goes about a free lunch, be prepared for a demanding but highly rewarding ride. There is no straight path that fits all. Perhaps you can find examples that resemble your use case among the demos included with each of the '`ep`', '`coll`', and '`po`' toolboxes (see the `help` folder for the corresponding tutorial documentation). For a more systematic but longer journey, consider the textbook *Recipes for Continuation* by Dankowicz and Schilder.