# Functional differential equations

$$M\dot{x}(t) = f(x(t), \underbrace{x(t-\tau_1), \dots, x(t-\tau_m)}, p)$$

many discrete delays approximate distributed delay

$M \in \mathbb{R}^{n\times n}$, possibly singular

→ neutral DDEs possible

$$\frac{d}{dt}\left[x + g(x(t), x(t-\tau_1), \dots, x(t-\tau_m), p)\right] = f(x(t), x(t-\tau_1), \dots, p)$$

$\rightsquigarrow$

$$\dot{y} = f(x(t), x(t-\tau_1), \dots, p)$$

$$0 = x(t) + g(x(t), x(t-\tau_1), \dots, p) - y(t)$$

$\tau$ can depend on state: $\tau_1(x(t), p)$, $\tau_2(x(t), x(t-\tau_1), p), \dots$

→ implicitly defined delays are possible (H.-O. Walther, echo control)

$$\dot{x} = k\left[x_{ref} - \frac{c}{2} s(t-\tau_0)\right]$$

$$c\, s(t) = x(t - s(t)) + x(t)$$

Equilibria simple, solve $0 = f(x, x, \dots, p)$

Periodic orbits: only addition is non-local terms:

$M\dot{x} = Tf(x(t), x(t - \frac{\tau_1}{T}), \dots, p)$ for $t \in (0,1)$, wrapping around for $t \notin (0,1)$ on r.h.s.

Stability of equilibria: pseudo spectral method by Dimitri

infinitely many eigenvalues relevant

$$M \partial_t x(t) = A_0 x(t,0) + A_1 x(t, -\tau_1) + \dots A_m x(t, -\tau_m)$$

$$\partial_t x(t, \vartheta) = \partial_\vartheta x(t, \vartheta)$$

$$\leadsto \text{replace } \begin{bmatrix} M & 0 \dots 0 \\ & I \end{bmatrix} \lambda x = \begin{bmatrix} A_1 p(-\tau_1) + \dots + A_m p(-\tau_m) \\ \partial_\vartheta \end{bmatrix}$$

by (e.g.) Chebyshev polynomials

Hopf bifurcations: in $0 = f(x, x, \dots, p)$

variables: $x \in \mathbb{R}^n, p \in \mathbb{R}^{n_p}$ in $0 = -Mi\omega v + \sum_{k=1}^{m} \partial_{x_k} f(x, x, \dots, p) e^{-i\omega\tau_k}$

$\omega \in \mathbb{R}, v \in \mathbb{C}^n$ | $0 = \bar{v}^T v - 1$

| $0 = \operatorname{Im}(\bar{v}_{ref}^T v)$

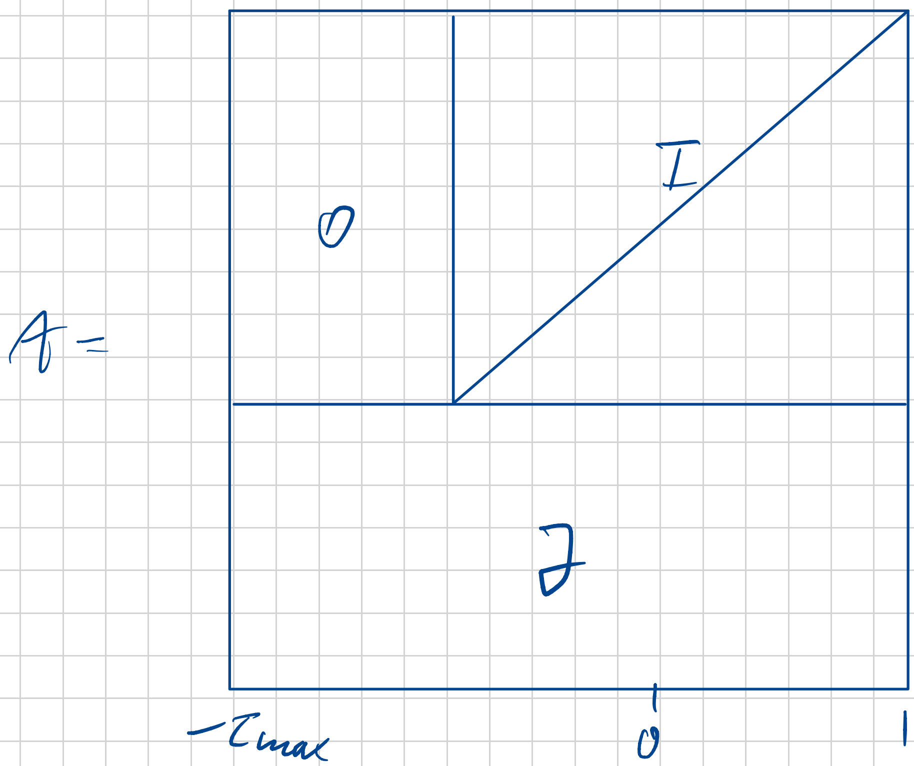
$\overline{3n+2}$

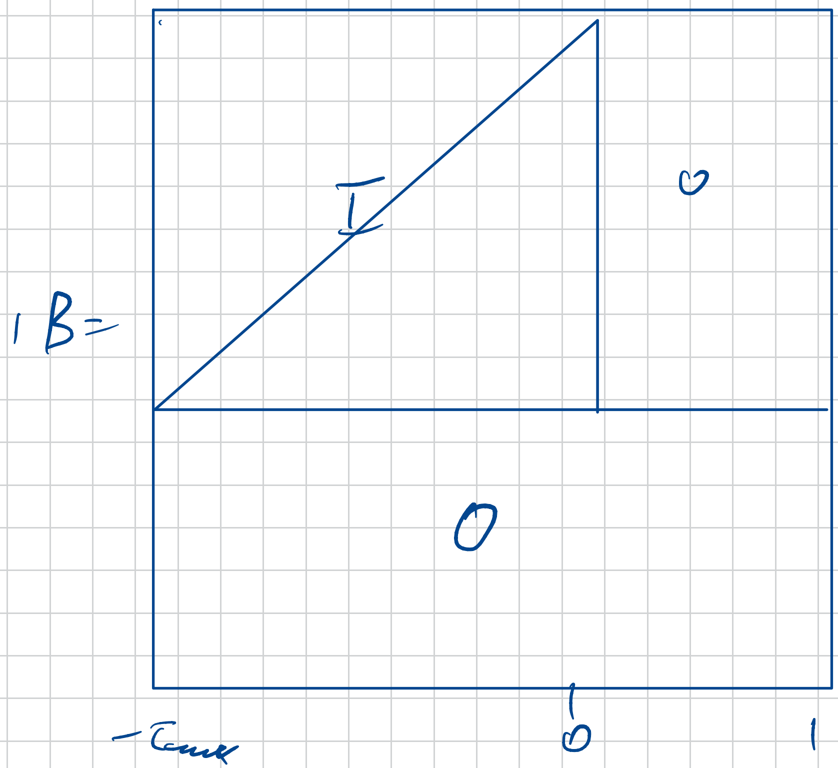
## Stability of periodic orbits

determine matrix

$$\partial : y \mapsto -MDy + \sum_{k=1}^{m} \overset{\text{projection}}{P_N} \, \partial_{x_k} f(\ldots)\Big[y(t-\tau_k) - \sum_{j=1}^{k-1} x'(t-\tau_k)\, \partial_{x_j}\tau_k \, y(t-\tau_j)\Big]$$

$n_{TST} \cdot \deg n) \times (n_{TST} \deg n)\lceil 1 + \frac{\tau_{max}}{T} \rceil$ matrix




& solve generalized sparse eigenvalue problem

$$Au = \mu B u$$

eigenvalues accumulate only at 0 if M invertible.

interface identical to ODE case => see demo, e.g. Mackey-Glass

difficulties: spectra for neutral DDEs, for renewal equations

Example: $$x(t) = \frac{\gamma}{2} \int_{\tau_2}^{\tau_1+\tau_2} x(s)\,(1-x(s))\,ds$$

=> demo in dde-problems