Bifurcations & continuation

(Kuznetsov '04: Elements of Applied Bifurcation Theory)

"Bifurcation" occurs in two different contexts:

(I) $0=f(u)$: $f:\mathbb{R}^n\to\mathbb{R}^m$, $n>m$ or $0=f(x,p)$, $f:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}^n$

(II) $\dot{x}=f(x,p)$ $f:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}^n$

?? What is the connection?

Def: (I) sol. u is not regular i.e. $f(u)=0$, $\partial f(u)$ has rank $<m$

(x,p) is not regular: $0=f(x,p)$, $\partial_x f(x,p)$ has rank $<n$

} implicit function theorem fails for f at u or (x,p)

$\Rightarrow$ detected and handled in atlas algorithms

Example for x,p: $0=x^2-p$ $(x,p)=0$

for u: $0=u_1^2-u_2^2$ $(u_1,u_2)=0$ but not $u_1^2-u_2=0$

$$\boxed{\begin{aligned}&0=f(u) \quad \text{FP wrt to } u_\ell\\ &0=\det\begin{bmatrix}\partial f(u)\\ 0\ 0\ 1\ 0\ 0\end{bmatrix} \text{?? continuation of fold}\end{aligned}}$$

(II) Vector field is not generic (often formulated for open subsets of $\mathbb{R}^n$), restrict to changes of p

$\dot{x}=f(x,p)$ has sol. $X_t(x_0;p)$. $X_t(x,p)=g(x,p,\varepsilon)\circ X_s(x,p+\varepsilon)\circ g^{-1}(x;p,\varepsilon)$ for all small ε, g continuous in x

What is the connection?

If defining system of any equilibrium or periodic orbit of a vector field has bifurcation then vector field has a bifurcation.

$0=f(x,p)$ $\begin{cases}X_T(x;p)=x\\ f(x_0,p_0)^T(x-x_0)=0\end{cases}$ for $(x,p)\approx(x_0,p_0)$

$F\left(\binom{x}{T};p\right)$

but this is only a fraction.

List of bifurcations:
- saddle-node of p.o., eq.
- Hopf bifurcation
- period doubling, torus bifurcation
- connecting orbits
- ???

$\boxed{\text{Some bifurcations of vector fields can be found and tracked in } p \text{ by solving nonlinear systems.}}$

Codimension: ?? Suppose $\dot{x} = f(x,p)$ has bifurcation at $p=0$, $p \in \mathbb{R}^r$

also occurs for $\dot{x} = f(x,p) + \varepsilon g(x)$ for small ε, with a small p

$\Rightarrow$ bifurcation has codimension $\leq r$

- " expected to see when varying one parameter.
- defining system is regular, dimension deficit 0 if r system parameters are free

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Bifurcation of $f(u)$ in case of rank-one deficit, for $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$

$0 = f(u_0)$, $\operatorname{rk} \partial f(u_0) = n-1$

Lyapunov-Schmidt Reduction: nullspace $V: \mathbb{R}^{(n+1)\times 2} = [V_1, V_2]$ nullspace of $\partial f(u_0)^T$ $W \in \mathbb{R}^{n \times 1}$

$\partial f(u_0) V = 0$, $W^T \partial f(u_0) = 0$, $\begin{array}{c} \begin{bmatrix} \partial f(u_0) & W \\ V^T & 0 \end{bmatrix} \begin{matrix} \}n \\ \}2 \end{matrix} \\ \begin{matrix} \underbrace{\qquad\quad}_{n+1} & \underbrace{\quad}_{1} \end{matrix} \end{array}$ is regular ??

$\Rightarrow$ (implicit function theorem) $\begin{bmatrix} f(u_0 + \beta + V\alpha) + W\gamma = 0 \\ V^T \beta = 0 \end{bmatrix}$, $\alpha \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}^{n+1}$, $f(u_0) = 0$ has unique solution (β, γ) for all small α.

this defines functions $\beta(\alpha_1, \alpha_2)$, $\gamma(\alpha_1, \alpha_2)$

$\boxed{0 = \gamma(\alpha_1, \alpha_2)}$ is called the bifurcation equation, (one eq., two variables)

We may expand β, γ in $(\alpha_1, \alpha_2) = 0$ by implicit differentiation:

What is $\left.\dfrac{\partial \gamma}{\partial \alpha}, \dfrac{\partial \beta}{\partial \alpha}\right|_{\alpha=(0,0)}$?

$$\begin{aligned} \partial f(u_0)\beta' + W\gamma' &= 0 \\ V^T \beta' &= 0 \end{aligned} \Rightarrow \begin{aligned} \partial\beta/\partial_\alpha &= 0 \text{ in } \alpha = 0 \\ \partial\gamma/\partial_\alpha &= 0 \end{aligned}$$

What is $(\gamma_{ij}, \beta_{ij}) = \left.\dfrac{\partial^2 \gamma}{\partial \alpha_i \partial \alpha_j} \quad \dfrac{\partial^2 \beta}{\partial \alpha_i \partial \alpha_j}\right|_{\alpha=(0,0)}$

$$\begin{aligned} \partial^2 f(u_0)[\cancel{\beta_i'} + V_i][\cancel{\beta_j'} + V_j] + \partial f(u_0)\beta_{ij} + W\gamma_{ij} &= 0 \\ V^T \beta_{ij} &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \partial f(u_0) & W \\ V^T & 0 \end{bmatrix} \begin{bmatrix} \beta_{ij} \\ \gamma_{ij} \end{bmatrix} = \begin{bmatrix} -\partial^2 f(u_0) V_i V_j \\ 0 \end{bmatrix}$$

$\Rightarrow$ easy to compute, if $\partial^2 f(u_0) V_i V_j$ is available

$\Rightarrow$ bifurcation equation has the form: $0 = \frac{\gamma_{11}}{2}\alpha_1^2 + \gamma_{12}\alpha_1\alpha_2 + \frac{\gamma_{22}}{2}\alpha_2^2 + O(\alpha^3)$

$$0 = \frac{1}{2}(\alpha_1,\alpha_2)\underbrace{\begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{12} & \gamma_{22}\end{bmatrix}}_{\Gamma = [b_1,b_2]^T\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}[b_1,b_2]}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} + O(\alpha^3)$$

(λ_1,λ_2 eigenvalues, b_1,b_2 orthogonal eigenvectors)

let $\begin{pmatrix}\tilde\alpha_1\\ \tilde\alpha_2\end{pmatrix} = [b_1,b_2]\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}$, γ is zero if $0 = \tilde\alpha_1^2\lambda_1 + \tilde\alpha_2^2\lambda_2 + O(\tilde\alpha^3)$

Assuming $\lambda_1,\lambda_2 \neq 0$ $\Rightarrow$ if $\lambda_1\lambda_2 > 0$ $\Rightarrow$ no solution near u_0

if $\lambda_1\lambda_2 < 0$ $\Rightarrow$ o.d.s.l. $s_1^2 = \lambda_1 > 0 > \lambda_2 = -s_2^2$ $(s_1, s_2 > 0)$

$\Rightarrow$ $0 = (\tilde\alpha_1 s_1 - \tilde\alpha_2 s_2)(\tilde\alpha_1 s_1 + \tilde\alpha_2 s_2) + O(\tilde\alpha^3)$

$\Rightarrow$ $(\tilde\alpha_1, \tilde\alpha_2)$ are on $\begin{pmatrix}s_2\\ s_1\end{pmatrix}\cdot t$ or $\begin{pmatrix}-s_2\\ s_1\end{pmatrix}\cdot t$ $(|t| \ll 1)$

$\Rightarrow$ branches are tangential to

$$[V_1,V_2]\begin{bmatrix}b_1^T\\ b_2^T\end{bmatrix}\begin{bmatrix}s_2 & -s_2\\ s_1 & s_1\end{bmatrix}t \quad \text{for small } t$$

What are eigenvalues λ_i for bifurcation equation of pitchfork?

e.g. $\gamma(x,p) = xp(-x^3)$

$\frac{1}{2}(x,p)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix} \rightarrow \lambda_1 = 1, \lambda_2 = -1$

Other bifurcations are analysed in a similar way

$F: \mathbb{R}^n \to \mathbb{R}^m$, $n > m$, $F(u_0) = 0$, rank $\partial F(u_0) = m - r$, Nullspace of $\partial F(u_0)$: $V \in \mathbb{R}^{n\times(n-m+r)}$

Nullspace of $\partial F(u_0)^T$, $W \in \mathbb{R}^{m\times r}$

$\Rightarrow \begin{bmatrix}\partial F(u_0) & W\\ V^T & 0\end{bmatrix}$ reg. $\in \mathbb{R}^{(n+r)\times(n+r)}$, $\begin{cases} F(u_0+\beta+V\alpha) + W\gamma = 0\\ V^T\beta = 0\end{cases}$ has unique solution $\beta(\alpha), \gamma(\alpha)$ for small α.

$\left.\frac{\partial\beta}{\partial\alpha}\right|_{\alpha=0} = 0$, $\left.\frac{\partial\gamma}{\partial\alpha}\right|_{\alpha=0} = 0$ $\quad \boxed{0 = \gamma(\alpha) \text{ is Lyapunov-Schmidt reduction of } F(u) = 0}$

e.g. for **Hopf bifurcation**

$\dot x = T f(x,p)$, x in space of functions: $[0,1] \to \mathbb{R}^n$, near $x(t) = x_0$, $T = T_0$, $p = p_0$ with

$0 = \int_0^1 \sin(2\pi t)V_r^T x(t)\,dt$ $\quad f(x_0,p_0) = 0$, $\partial_x f(x_0,p_0)$ has eigenvalues $\pm i\frac{2\pi}{T_0}$ with eigenvector $v_r + iv_i$

$x(0) = x(1)$ $\quad$ + genericity conditions

$u = (x(), T, p)$, $\quad$ dim null $\partial F = 2$, codim Im $\partial F = 1$

Tracking & detection of bifurcations

Local bifurcations $\Rightarrow$ determined by stability of equilibrium or periodic orbit

Codimension 1:

$\dot{x} = f(x,p)$

Equilibria: $0 = f(x_0,p_0)$ "saddle-node" $\partial_x f(x_0,p_0)$ is singular + genericity conditions

"Hopf bif" $\partial_x f(x_0,p_0)$ has purely imaginary eigenvalue $i\omega_0$ + genericity conditions

periodic orbit: $\dot{x} = T f(x,p)$

$x(0) = x(1)$

phase condition $\Rightarrow 0 = \int_0^1 \dot{x}_{rep}(t)^T x(t)\,dt$

How do we determine stability?

Solve variational problem:

Variational problem (part of coll toolbox)

previously defined segment (x, T_0, T, p), with equations $\dot{x} = T f(T_0 + tT, x(T_0+tT), p)$

define system $\dot{y} = T \partial_x f(T_0+tT, x(T_0+tT), p)\, y$

initialize with $y(T_0) \in \mathbb{R}^{n\times k}$, then $y(t_i)$ are internally computed

note that nk variables need to be fixed to maintain dimensional deficit

revisit linode

Eigenvalues of periodic orbits (autonomous)

Solve $\dot{y} = T \partial_x f(x(tT),p)\, y$, $y(0) = M_0$, $M_1 := y(1)$

Floquet multipliers $FM = eig(M_1, M_0) = eig(M_0 \backslash M_1)$

- PO toolbox monitors the variational problem (built in) for bifurcation detection
- to switch on monitoring of Floquet multipliers,

prob = po_mult_add(prob, 'po.orb')

Check marsden demo in PO toolbox: check bd2, column eigs

- add po_mult_add_prob
- check bd2, columns po_multipliers

Theory: one Floquet multiplier is equal to 1

- Practice: increase PtMx to 100
- observe what Floquet multipliers do

Conclusion: monitored FM's are less accurate than embedded FM's

Bifurcation tracking embeds one variational problem for critical FM.

- check coll_construct_var,
- stop at end of function var
- spy(J)), observe structure, M is variational solution

Global bifurcations

example marsden: end of branch of periodic orbits =

homoclinic bifurcations (connecting orbit to saddle)

generally: connecting orbit between hyperbolic equilibria is codim-1 bifurcation

$(e_1 \mapsto e_2)$ if $\#u(e_1) = \#u(e_2)$

connecting orbit from hyperbolic equilibrium to hyperbolic periodic orbit is codim-1 bifurcation

$(e_1 \mapsto p_2)$ if $\#u(e_1) = \#u(p_2)$

($\#u$ = number of unstable eigenvalues)

Example by Thorsten Riess (demo vascoll_v2_demo)

(1) $\dot{x} = T_p f(x, r_x)$, $x(0) = x(1)$, $\int_0^1 \dot{x}_{ref}^T(s)\, x(s)\, ds = 0$ ⇐ periodic orbit

(2) $\dot{u} = T_p \partial f(x, r_u)\, u$, $u(1) = \lambda_s u(0)$, $u(0)^T u(0) = 1$ ⇐ stable eigenvector of p.o. x

(3) $\dot{y} = T_y f(y, r_y)$, $y(0) = \varepsilon_1 v_0(r_y)$, $n_\Sigma^T (y(1) - b_\Sigma) = \sigma_y$ ⇐ unstable manifold of 0

(4) $\dot{z} = T_z f(z, r_z)$, $z(1) = x(0) + \varepsilon_2 u(0)$, $n_\Sigma^T (z(0) - b_\Sigma) = \sigma_z$ ⇐ stable manifold of $(x(0))$

$r_x = r_y$, $r_x = r_z$, $r_x = r_u$

Run Hopf: (1) only, T_p, r_x, x free, monitor (2)

determine λ_s, $u(0)$, add (3), (4) with $T_y, T_z = 0$, $\varepsilon_1, \varepsilon_2 = 0.1$, σ_y, σ_z parameters

glue $r_x = r_y = r_z = r_y$

(2) free T_y, σ_y, fix $r_x, \sigma_z, T_z, \varepsilon_1, \varepsilon_2$ until $\sigma_y = 0$ (event)

(3) free T_z, σ_z, T_y, fix $r_x, \sigma_y = 0, \varepsilon_1, \varepsilon_2$ until $\sigma_z = 0$ (event)

(4) free ε_2, T_z, T_y, fix $r, \sigma_y, \sigma_z, \varepsilon_1$

determine ε_2, T_z, z that has $z(0)$ closest to $y(1)$

set $\zeta = y(1) - z(0)$, $g = 1$, $\perp$ to ζ in Σ

(5) free g, r, T_z, T_y, fix $h^T(y(1) - z(0)) = 0$ until $g = 0$