Graduate School – Udine 15 – 19 Dec 25

Continuation methods for nonlinear problems

- Mutual Introduction
- website: https://github.com/jansieber/udine25-coco-gradschool
- central piece of software (Matlab based) COCO

  https://sourceforge.net/projects/cocotools/

  (copy included into gradschool site)

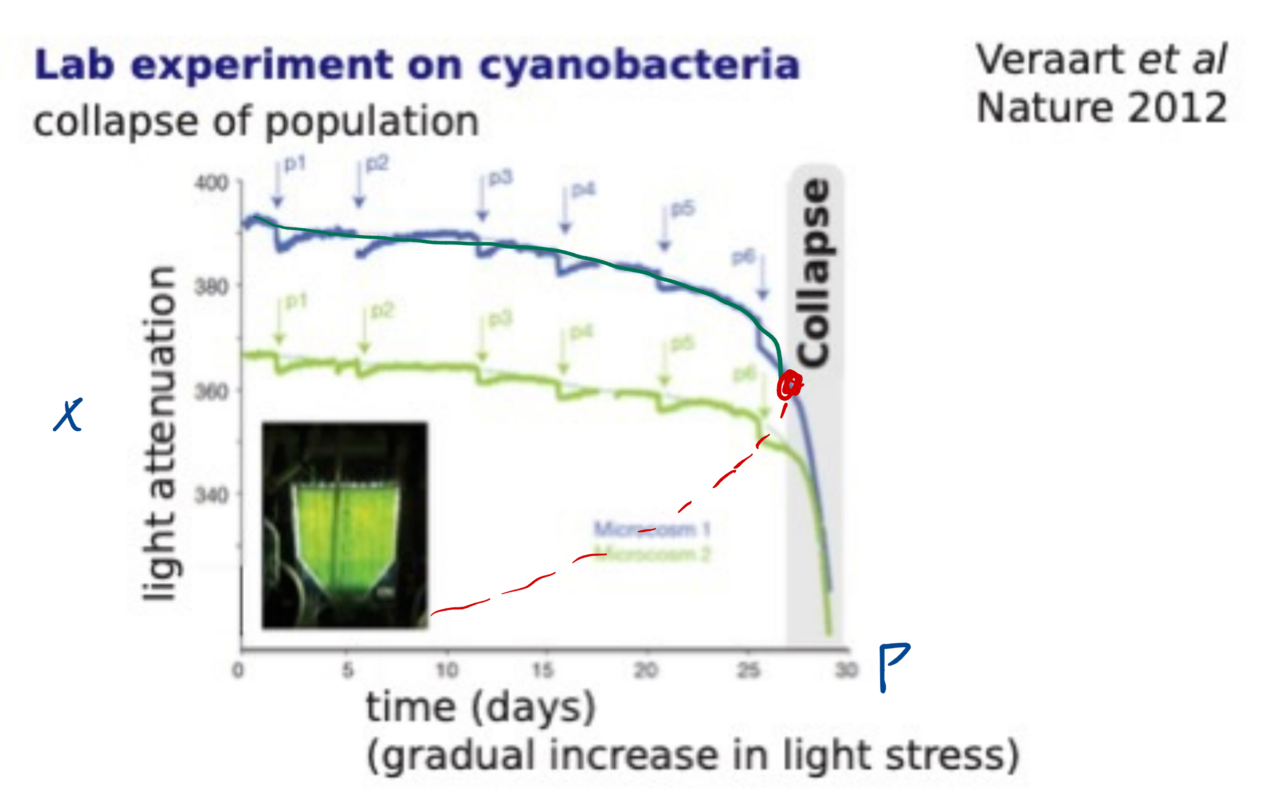
  original authors: Harry Dankowicz, Frank Schilder

- Plan for Monday
  - case study: rate-induced tipping
  - basic bifurcation analysis using COCO

Rate-induced tipping

tipping ~ small change in input, disproportional and irreversible change of output

Prototype ecology of lakes: tipping into



algae are stressed by light (causing increased mortality)

but shade each other

=> positive feedback effect

$$\dot{x} = p_0 - p - (x - x_0)^2$$

$$p(t) = p_0 + \varepsilon (t - t_0)$$

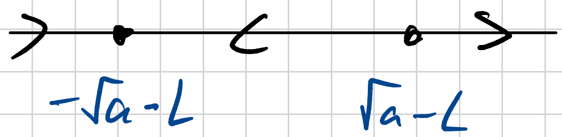
+ gradual increase of p through saddle-node bifurcation

Observation by Wieczorek et al '2011 of curious phenomenon (compost bomb)

(*) $\boxed{\dot{x} = (x+L)^2 - a}$, $a>0$ fixed, L depends on time

for all fixed L (*) has stable equilibrium at $-\sqrt{a} - L$

unstable equilibrium at $\sqrt{a} - L$



What happens if L increases rapidly?

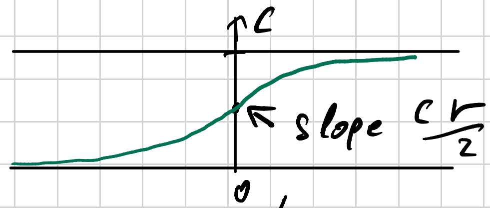
- $L(t) = c \cdot t \Rightarrow \dot{x} = (x+ct)^2 - a$, $y(t) = x(t) + ct$

  $\Rightarrow \dot{y} = \dot{x} + c = y^2 - a + c$

So if $c>a$: $(\dot{y} = y^2 + c - a)$ no stable equilibrium

$$(*)\quad \boxed{\dot{x} = (x+L(t))^2 - a}$$

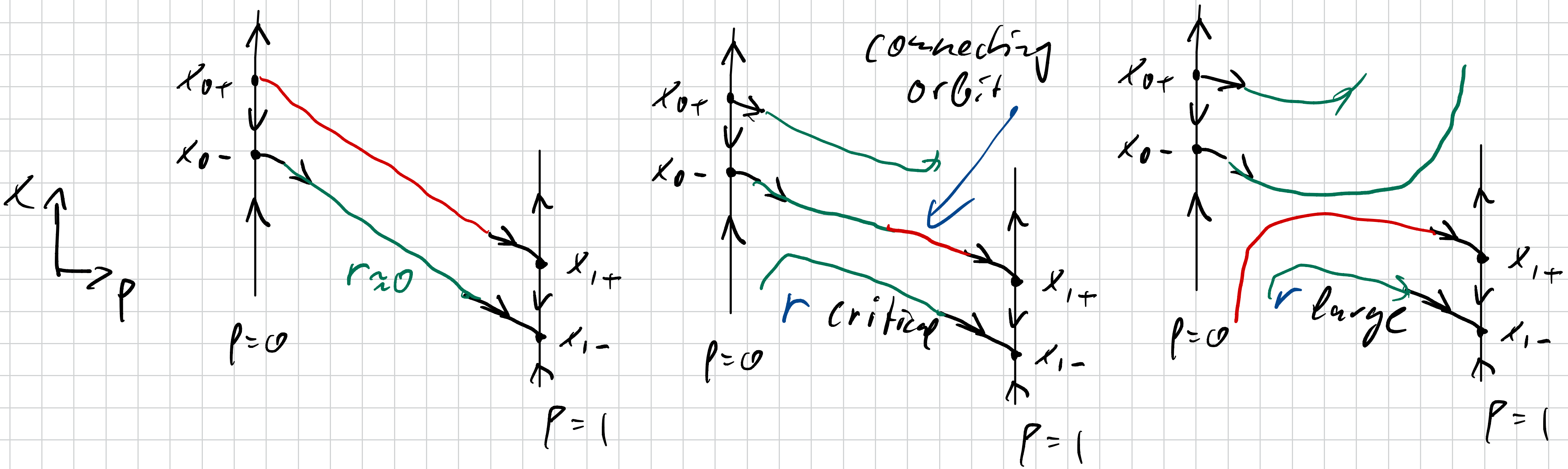
- (ramp) eg. $L(t) = \dfrac{c}{1+\exp(-2rt)}$



introduce $p(t)$, $\dot{p} = 2rp(1-p) \to p(t) = \dfrac{1}{1+\exp(-2rt)}$ is sol.

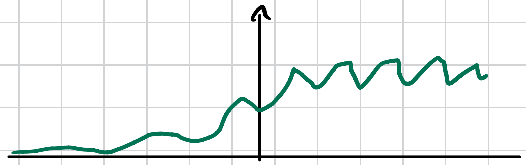
Overall $\quad \begin{aligned} \dot{p} &= 2rp(1-p) \\ \dot{x} &= (x+cp)^2 - a \end{aligned}$ in phase plane equilibria $p=0,\ x_{0,\pm} = \pm\sqrt{a}$

$p=1,\ x_{1,\pm} = \pm\sqrt{a} - c$



$\Rightarrow$ there is critical rate r at which

- ramp and growing oscillations $L(t)$



(*) $\boxed{\dot{x} = (x+L(t))^2 - a}$ $\quad L(t) = \dfrac{c - \beta \sin(\omega t - \varphi)}{1+\exp(-2rt)}$

$\Rightarrow$ demo_escape.m $\Rightarrow$ escape depends on phase φ

Use continuation to find precise boundaries and track them in other system parameters: connecting orbit from equilibrium to

Steps:

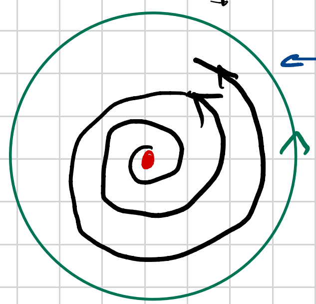
- Write autonomous ODE for (*)

$$(1)\begin{cases}\dot{p} = pr - q\omega - p(p^2+q^2)\\ \dot{q} = p\omega + qr - q(p^2+q^2)\end{cases} \qquad L = c(p^2+q^2) + \beta q\sqrt{p^2+q^2} \qquad (1) \text{ drives } (2)$$

$$(2)\quad \dot{x} = (x+L)^2 - a$$

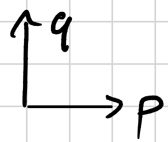
$\underline{\ }$ variables $\quad \underline{\ }$ = parameters

Phase portrait of (1)



radius $\sqrt{s} = 1$

use spiral as driver



φ does not enter (1-2).

$L(0) = \frac{1}{2}c + \beta \sin\varphi$

$p(0)^2 + q(0)^2 = \frac{1}{2} \rightarrow q(0) = \sin\varphi/\sqrt{2}$

$\rightarrow \varphi$ enters boundary condition at $t=0$

- at $t = -\infty$ $(p = q = 0)$, $L = 0 \Rightarrow x = 0$. Equilibrium $(x, p, q) = (0, 0, 0)$
- at $t = +\infty$ $p^2 + q^2 = 1$, $\begin{array}{l} p(t) = \cos(\varphi - \omega t) \\ q(t) = \sin(\varphi - \omega t) \end{array} \to L(t) = c - \beta \sin(\omega t - \varphi)$

$x(t)$ is unstable solution of $\dot{x}(t) = (x + L(t))^2 - a$, $\varphi = 0$ initially (find by integrating backwards).

set Boundary conditions
$$\begin{aligned} x_\infty(0) &= x_\infty(T_\infty) \\ p_\infty(0) &= p_\infty(T_\infty) \\ q_\infty(0) &= q_\infty(T_\infty) \end{aligned} \quad \leftarrow T_\infty = 2\pi/\omega$$

$p(0) \sin\varphi = q(0) \cos\varphi$ $\leftarrow$ fixes phase at $t = 0$ to φ

- Solution of (1–2) on $(-\infty, 0)$; introduce $T_- > 0$,

set Boundary condition
$$\begin{bmatrix} x_-(-T_-)^2 - a = 0 \\ \sqrt{2}\, p_-(0) - \cos\varphi = 0 \\ \sqrt{2}\, q_-(0) - \sin\varphi = 0 \end{bmatrix}$$

initially $T_- \ll 1$, $\left.\begin{array}{l} x_- = -\sqrt{a} \\ p_- = \cos\varphi / \sqrt{2} \\ q_- = \sin\varphi / \sqrt{2} \end{array}\right\}$ constants

continue T to large value

- Solution of (1–2) on $(0, \infty)$: introduce $T_+ > 0$,

set B.C. $p_+(0) = \cos\varphi / \sqrt{2}$
$q_+(0) = \sin\varphi / \sqrt{2}$
$x_+(T_+) = x_\infty(T_\infty)$ ← matches x_+ and x_∞
$p_+(T_+)\, q_\infty(T_\infty) = q_+(T_+)\, p_\infty(T_\infty)$ ← matches phase of $(p_+, q_+)(T_+)$ and $(p_\infty, q_\infty)(T_\infty)$

increase T to large integer multiple of $2\pi/\omega$

- we have gap between $x_-(0)$ and $x_+(0)$ "Lin gap"
  close gap: $x_-(0) - x_+(0) = \eta$, continue η in r
- now we have connecting orbit $(x, p, q) \sim (0, 0, 0)$ at $t = -T_-$
  $\sim (x_\infty, p_\infty, q_\infty)$ at $t = T_+$
  $T_-, T_+ \gg 1$
- continue in φ find local extrema of r wrt φ
- continue local extrema of r in r and a (adjusting φ)
  we adjust T_+, T_- along branches