A comment about n-d continuation

major part of algorithm is tracking of boundary and "closing" the manifold after growing it along different paths

$\Rightarrow$ Problem occurring for continuation of (e.g.) Hopf or periodic orbits

defining systems: Hopf

$$0 = f(x,p) \qquad (n)$$
$$0 = \partial_x f(x,p)\, v_1 - w_1 \qquad (n)$$
$$0 = \partial_x f(x,p)\, v_2 - w_2 \qquad (n)$$
$$0 = w_1 - v_2 \qquad (n)$$
$$0 = \kappa w_2 - v_1 \qquad (n)$$
$$0 = v_1^T v_1 - 1 \qquad 1$$
$$PC \rightarrow 0 = \underline{v_{ref}}^T v_1 \qquad 1$$

auton. periodic orbit

$$0 = \dot{x}(t) - T f(x(t), p)$$
$$0 = x(0) - x(1)$$
$$PC \rightarrow 0 = \int_0^1 \underline{x_{ref}'(t)}^T x(t)\, dt$$

x_{ref} = time profile of reference point

PC = phase condition

vector in plane (v_1, v_2) of reference point, orthogonal to $v_{1,ref}$

reference point = previously computed point along branch in 1d continuation
= originating neighbor chart in n-d continuation

Trouble: when growing manifold in different directions, "phase" may not align when the paths meet.

Solution in coco: recall that problem has form

$\Phi(u) = 0$ $\quad\quad u$ = variables

$\Psi(u) - \mu = 0$ $\quad\quad \mu$ = continuation parameters (if 'active')

1) grow manifold in $\{\mu_a \in \mathbb{R}^{n_{\mu a}}\}$, the set of active continuation parameters

2) introduce parameters for solution measures, e.g.

$$\mu_{j,k} = \left|\int_0^1 \cos(2\pi j t)\, x_k(t)\, dt\right|^2 + \left|\int_0^1 \sin(2\pi j t)\, x_k(t)\, dt\right|^2$$

for $j = 1, \ldots, n_f$ (low-order Fourier modes)

Remarks:
- tangent to manifold and prediction are still done in full dimension n
- many variables in u are "unsafe" (change meaning when remeshing)
- for periodic orbits with $\mu_{j,k}$ Hopf bifurcation is a singularity

- coco has 2 mechanisms for changing r.h.s.
  - 'remesh' -> variables u change meaning
  - 'update' -> equation changes but u stays same (eg to update reference point.)
- Reference: Dankowicz et al J. Comp. Nonlin. Dyn 2020 (ASME)

demo: base-until / demo_bvc2d.m