exercise 4

complex numbers, quaternions, and Kronecker products

solutions due

until **December 10, 2024** at **23:30** via **ecampus**

⚠

Note: Carefully read and follow the solution submission instructions which are detailed in task 4.11.

If your submission does not adhere to the guidelines in task 4.11, your solutions will not be accepted / graded.

general remarks

These exercises are supposed to *provoke* your brains. The didactic goal is for you to gain much broader perspectives on elementary material you will have seen during your bachelor studies.

If all goes well, you should realize that familiar concepts such as complex numbers or Boolean logic can be viewed from many different angles. Such changes of perspective often allow us to apply known concepts in settings where preconception and narrow minded thinking would insist they are not applicable . . .

task 4.1 [no points (immediate from Wikipedia)]

the group of fourth roots of unity and its friends

The numbers $+1$, $+i$, -1, and $-i$ are called the fourth roots of unity as they solve $x^4 = 1$ or, equivalently, $x^4 - 1 = 0$. To briefly verify this claim, we note

$$x^4 - 1 = x^4 - 1^4 = \left(x^2\right)^2 - \left(1^2\right)^2 = \left(x^2 - 1\right)\left(x^2 + 1\right)$$

For this product to evaluate to 0, one of the two factors must be 0. We thus have the following equations and solutions

$$\begin{aligned} \left(x^2 - 1\right) = 0 \quad &\Rightarrow \quad x = \pm 1 \\ \left(x^2 + 1\right) = 0 \quad &\Rightarrow \quad x = \pm i \end{aligned}$$

Moreover, together with the multiplication "$\cdot$" of complex numbers, the set

$$G_4 = \left\{+1, +i, -1, -i\right\} = \left\{i^0, i^1, i^2, i^3\right\}$$

forms a multiplicative group called the group of fourth roots of unity.

Next, convince yourself that the following claim is true: For any two integers $k, l \in \mathbb{Z}$, we have

$$i^k \cdot i^l = i^{k \oplus l}$$

where $\oplus$ denotes addition modulo 4, i.e. $k \oplus l = k + l \pmod 4$.

Since this claim holds true, we recognize that the above group is a cyclic group C_n of order $n = 4$. Moreover, we say the above group is generated by i, because

$$G_4 = \left\{i^k \mid k \in \mathbb{Z}\right\}$$

Finally, convince yourself that we could have also written the multiplicative group of fourth roots of unity as

$$G_4 = \left\{e^{i \frac{2 \pi k}{4}} \mid k \in \mathbb{Z}\right\} \tag{1}$$

Please turn page ...

The following are good exercises to test your understanding:

a) Fill in either (or both) of the following tables

$\cdot$	$+1$	$+i$	-1	$-i$
$+1$	...	...	...	...
$+i$	...	...	...	...
-1	...	...	...	...
$-i$	...	...	...	...

$\Leftrightarrow$

$\cdot$	i^0	i^1	i^2	i^3
i^0	...	...	...	...
i^1	...	...	...	...
i^2	...	...	...	...
i^3	...	...	...	...

Given your completed tables, would you say that G_4 is an Abelian group? Why or why not? Discuss this in your won words.

b) Equation (1) suggests a generalization of the fourth roots of unity, namely the n-th roots of unity

$$G_n = \left\{ e^{i\frac{2\pi k}{n}} \;\middle|\; k \in \mathbb{Z} \right\} = \left\{ \omega^k \;\middle|\; k \in \mathbb{Z} \right\}$$

where $n > 4 \in \mathbb{N}$. Note that we introduced $\omega = e^{i\frac{2\pi}{n}}$ because people really like this shorthand. Maybe you recognize ω as something you have seen before? If so, then where?

Observe that G_n forms yet another group under multiplication. Is it finite or infinite? Is it Abelian? Is it cyclic? If so, then what is the order of the cycle? Briefly discuss all this.

c) Finally, to wrap up the ideas brought forth in this task, we note that G_4 and G_n are subgroups of the circle group

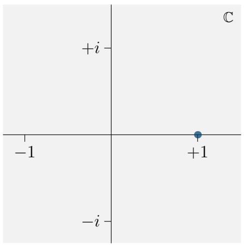
$$\mathbb{S}^1 = \left\{ c \in \mathbb{C} \;\middle|\; |c| = 1 \right\}$$

formed by the complex numbers of modulus 1. From this statement, it should be clear what it means for a group to be a subgroup of another group, right? Discuss this within your team.

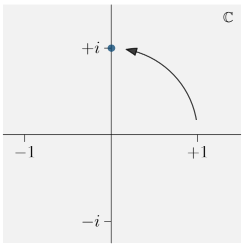
task 4.2 [5 points]

rotations in the complex plane

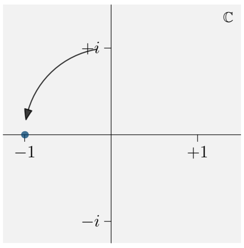
If we consider the complex numbers as points in the complex plane, then what you did in the previous tasks translates to the statement: “to multiply by i is to rotate by $90°$ to the left.” For example



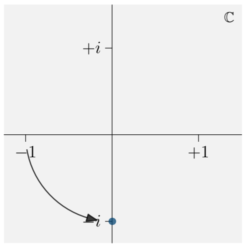
consider the number $+1 \in \mathbb{C}$



multiplying by i yields $+i$



multiplying by i yields -1



multiplying by i yields $-i$

Now, if $i^k, k \in \mathbb{Z}$, is a rotation by an integer multiple of $90°$, then $i^x, x \in \mathbb{R}$, should be a rotation by a real valued angle, right?

But this begs the question: What is i^x ? Put differently, if we write

$$i^x = \left(e^{\ln i}\right)^x = e^{x \ln i}$$

we need to figure out which complex number corresponds to $\ln i$.

Here is your task:

Symbolically compute $\ln i$. That is, do not just look it up on the Web but present a sequence of algebraic manipulations / mathematical arguments which gives its value.

task 4.3 [5 points]

even more cyclic groups

In quantum computing, groups of the following form are of considerable interest

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}^k \middle| \, k \in \mathbb{Z} \right\}$$

where $\omega = e^{i\frac{2\pi}{n}}$ for some $n \in \mathbb{N}$. For the case where $n = 4$, the generator of the corresponding matrix group is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

and frequently referred to as the phase gate. Reasons behind this name will become apparent later on.

Here are your tasks:

a) **[2 points]** Would you say that matrix S is unitary? Why or why not?

b) **[3 points]** Even though we do not yet know what quantum gates are, we can still have a first look at how S acts on a very specific vector, namely

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $k \in \{0, 1, 2, 3, 4\}$ compute the following vectors $S^k|1\rangle$. Based on your work above, it should be fairly obvious how they will look like, shouldn't it? Nevertheless, take a moment to ponder your results. What do you observe?

task 4.4 [5 points]

complex numbers as matrices

python has an in-built data type for complex numbers. For instance, the two numbers $c_1 = 3 \cdot 1 + 4 \cdot i$ and $c_2 = 2 \cdot 1 - 2 \cdot i$ can be implemented as

```
c1 = complex(3, +4)
c2 = complex(2, -2)
```

or simply as

```
c1 = 3 + 4j
c2 = 2 - 2j
```

Note: For people other than electrical engineers, it may be confusing that *python* refers to the imaginary unit i as `j`. But it is what it is.

Here are your tasks:

a) **[1 point]** Using *python*, compute the values of the following four simple expressions $c_1 + c_2,\ c_1 \cdot c_2,\ c_1^*,$ and $|c_1|$.

b) **[4 points]** Complex numbers can also be represented in terms of real valued 2×2 matrices. To see this, consider the two "basis" matrices

$$\mathbf{1} = \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}$$

$$\boldsymbol{i} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

and let

$$\boldsymbol{c}_1 = 3\,\mathbf{1} + 4\,\boldsymbol{i}$$
$$\boldsymbol{c}_2 = 2\,\mathbf{1} - 2\,\boldsymbol{i}$$

Implement the matrices $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{c}_1$, and $\boldsymbol{c}_2$ in *numpy* and then compute the following expressions $\boldsymbol{c}_1 + \boldsymbol{c}_2,\ \boldsymbol{c}_1 \cdot \boldsymbol{c}_2,\ \boldsymbol{c}_1^\intercal,$ and $\sqrt{\det \boldsymbol{c}_1}$ (where $+$ and $\cdot$ denote matrix addition and multiplication).

Print the matrices $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ as well as the results of your computations. What do you observe? How do your results relate to the results you got in subtask a)? Briefly discuss this.

task 4.5 [no points (immediate from Wikipedia)]

the quaternion group Q_8

Recall that any complex number $c \in \mathbb{C}$ can be written as $c = a \cdot 1 + b \cdot i$ where $1, a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. Also recall that complex numbers can be thought of as living in a 2D space called the complex plane.

In 1843, William R. Hamilton famously extended the ideas behind complex numbers to 4D and introduced the quaternions $q \in \mathbb{H}$ where

$$q = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$$

Here, $a, b, c, d \in \mathbb{R}$ and the unit quaternions i, j, k are defined by their properties / multiplication rules

$$i^2 = -1 \qquad j^2 = -1 \qquad k^2 = -1 \qquad ijk = -1$$

$$\begin{aligned} ij &= +k & jk &= +i & ki &= +j \\ ji &= -k & kj &= -i & ik &= -j \end{aligned}$$

The following is a good exercise to test your understanding:

The set

$$Q_8 = \{+1, -1, +i, -1, +j, -j, +k, -k\}$$

forms a multiplicative group. Use the above information to complete the Cayley table for this group:

$\cdot$	$+1$	-1	$+i$	$-i$	$+j$	$-j$	$+k$	$-k$
$+1$	...	...	...	...	...	...	...	...
-1	...	...	...	...	...	...	...	...
$+i$	...	...	...	...	...	...	...	...
$-i$	...	...	...	...	...	...	...	...
$+j$	...	...	...	...	...	...	...	...
$-j$	...	...	...	...	...	...	...	...
$+k$	...	...	...	...	...	...	...	...
$-k$	...	...	...	...	...	...	...	...

Does your result ($\Leftrightarrow$ the structure of the completed table) suggest that Q_8 is an Abelian group or not? Briefly discuss this.

task 4.6 [5 points]

quaternions as matrices

Just as complex numbers $z \in \mathbb{C}$, quaternions $q \in \mathbb{H}$ can be represented in terms of matrices, too. For these, we actually have several choices ...

Here are your tasks:

a) **[2 points]** Construct a set of 4×4 real valued matrices $\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ which represent the unit quaternions $1, i, j, k$ so that a quaternion

$$\boldsymbol{q} = a\,\mathbf{1} + b\,\boldsymbol{i} + c\,\boldsymbol{j} + d\,\boldsymbol{k}$$

is a matrix $\boldsymbol{q} \in \mathbb{R}^{4\times4}$. Implement your basis matrices in `numpy` and compute the product $\boldsymbol{ijk}$. Print your results and discuss what you observe.

b) **[2 points]** Construct a set of 2×2 complex valued matrices $\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ which represent the unit quaternions $1, i, j, k$ so that a quaternion

$$\boldsymbol{q} = a\,\mathbf{1} + b\,\boldsymbol{i} + c\,\boldsymbol{j} + d\,\boldsymbol{k}$$

is a matrix $\boldsymbol{q} \in \mathbb{C}^{2\times2}$. Implement your basis matrices in `numpy` and compute the product $\boldsymbol{ijk}$. Print your results and discuss what you observe.

c) **[1 point]** Use your code to automatically fill in the table in task 4.5.

Remark: Why are we interested in quaternions? Actually, for no important reason, but note the following curious parallelism:

The algebra of quaternions is completely defined through the equations

$$i^2 = j^2 = k^2 = i\,j\,k = -1$$

The algebra of Pauli matrices is completely defined through the equations

$$X^2 = Y^2 = Z^2 = -i\,X\,Y\,Z = I$$

task 4.7 [10 points]

vector logic

Vector logic was developed by the Uruguayan mathematician Eduardo Mizraji. Following his ideas, we represent the two truth values `false` and `true` in terms of orthonormal vectors, for instance

$$\texttt{false} \equiv \boldsymbol{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{("no")}$$

$$\texttt{true} \equiv \boldsymbol{s} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{("si")}$$

Now, to further explore Mizraji's framework, we first of all recall that the outer product $\boldsymbol{x}\boldsymbol{y}^\intercal$ of two vectors $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^n$ produces a matrix $\boldsymbol{Z} \in \mathbb{R}^{m \times n}$ such that

$$\boldsymbol{x}\boldsymbol{y}^\intercal = \boldsymbol{Z} = \begin{bmatrix} x_1\, y_1 & x_1\, y_2 & \cdots & x_1\, y_n \\ x_2\, y_1 & x_2\, y_2 & \cdots & x_2\, y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m\, y_1 & x_m\, y_2 & \cdots & x_m\, y_n \end{bmatrix}$$

Second of all, we recall that the Kronecker product $\boldsymbol{x} \otimes \boldsymbol{y}$ of two vectors $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^n$ produces a vector $\boldsymbol{z} \in \mathbb{R}^{mn}$ such that

$$\boldsymbol{x} \otimes \boldsymbol{y} = \boldsymbol{z} = \begin{bmatrix} x_1\, \boldsymbol{y} \\ x_2\, \boldsymbol{y} \\ \vdots \\ x_m\, \boldsymbol{y} \end{bmatrix}$$

Here are your tasks:

a) **[4 points]** Compute the two matrices

$$\boldsymbol{C} = \boldsymbol{n}\big[\boldsymbol{n} \otimes \boldsymbol{n}\big]^\intercal + \boldsymbol{n}\big[\boldsymbol{n} \otimes \boldsymbol{s}\big]^\intercal + \boldsymbol{n}\big[\boldsymbol{s} \otimes \boldsymbol{n}\big]^\intercal + \boldsymbol{s}\big[\boldsymbol{s} \otimes \boldsymbol{s}\big]^\intercal$$

$$\boldsymbol{D} = \boldsymbol{n}\big[\boldsymbol{n} \otimes \boldsymbol{n}\big]^\intercal + \boldsymbol{s}\big[\boldsymbol{n} \otimes \boldsymbol{s}\big]^\intercal + \boldsymbol{s}\big[\boldsymbol{s} \otimes \boldsymbol{n}\big]^\intercal + \boldsymbol{s}\big[\boldsymbol{s} \otimes \boldsymbol{s}\big]^\intercal$$

b) **[4 points]** Given these matrices, consider two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \{\boldsymbol{n}, \boldsymbol{s}\}$ and feed them into the functions

$$f_{\boldsymbol{C}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{C}\big[\boldsymbol{x}_1 \otimes \boldsymbol{x}_2\big]$$
$$f_{\boldsymbol{D}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{D}\big[\boldsymbol{x}_1 \otimes \boldsymbol{x}_2\big]$$

Looking at your results, which Boolean (logic) functions would you say do $f_{\boldsymbol{C}}$ and $f_{\boldsymbol{D}}$ represent?

c) **[2 points]** In order to explore whether your results are dependent on our particular choices for $\boldsymbol{n}$ and $\boldsymbol{s}$, repeat the above procedure but this time with the vectors

$$\boldsymbol{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{s} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For the fun of it, repeat the above procedure two more times, first using

$$\boldsymbol{n} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{s} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

and then

$$\boldsymbol{n} = \frac{1}{\sqrt{2}} \begin{bmatrix} +i \\ -i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{s} = \frac{1}{\sqrt{2}} \begin{bmatrix} +i \\ +i \end{bmatrix}$$

What do you observe? Does the representation of `false` / `no` and `true` / `si` impact the logic of the results produced by this method? What would happen if $\boldsymbol{n}$ and $\boldsymbol{s}$ were *not* orthonormal?

task 4.8 [no points]

Kronecker products

Above, we have already seen the Kronecker product $\boldsymbol{x} \otimes \boldsymbol{y}$ of two vectors $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^n$ which produces a vector $\boldsymbol{z} \in \mathbb{R}^{mn}$ such that

$$\boldsymbol{x} \otimes \boldsymbol{y} = \boldsymbol{z} = \begin{bmatrix} x_1\, \boldsymbol{y} \\ x_2\, \boldsymbol{y} \\ \vdots \\ x_m\, \boldsymbol{y} \end{bmatrix}$$

This idea idea immediately generalizes to the Kronecker product $\boldsymbol{X} \otimes \boldsymbol{Y}$ of two matrices $\boldsymbol{X} \in \mathbb{R}^{k \times l}$ and $\boldsymbol{Y} \in \mathbb{R}^{m \times n}$ which results in a matrix $\boldsymbol{Z} \in \mathbb{R}^{km \times ln}$ such that

$$\boldsymbol{X} \otimes \boldsymbol{Y} = \boldsymbol{Z} = \begin{bmatrix} X_{11}\, \boldsymbol{Y} & X_{12}\, \boldsymbol{Y} & \cdots & X_{1m}\, \boldsymbol{Y} \\ X_{21}\, \boldsymbol{Y} & X_{22}\, \boldsymbol{Y} & \cdots & X_{2m}\, \boldsymbol{Y} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1}\, \boldsymbol{Y} & X_{m2}\, \boldsymbol{Y} & \cdots & X_{mm}\, \boldsymbol{Y} \end{bmatrix}$$

In fact, the Kronecker product of two vectors $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^n$ can be understood as the special case of a Kronecker product of two single column matrices $\boldsymbol{x} \in \mathbb{R}^{m \times 1}$ and $\boldsymbol{y} \in \mathbb{R}^{n \times 1}$.

Kronecker products are *associative* and *distribute over addition*, that is we have the following

$$\boldsymbol{X} \otimes \big[\boldsymbol{Y} \otimes \boldsymbol{Z}\big] = \big[\boldsymbol{X} \otimes \boldsymbol{Y}\big] \otimes \boldsymbol{Z}$$

$$\boldsymbol{X} + \big[\boldsymbol{Y} \otimes \boldsymbol{Z}\big] = \boldsymbol{X} \otimes \boldsymbol{Y} + \boldsymbol{X} \otimes \boldsymbol{Z}$$

$$\big[\boldsymbol{X} + \boldsymbol{Y}\big] \otimes \boldsymbol{Z} = \boldsymbol{X} \otimes \boldsymbol{Z} + \boldsymbol{Y} \otimes \boldsymbol{Z}$$

However, Kronecker products are generally *not commutative*. That is, we typically have $\boldsymbol{X} \otimes \boldsymbol{Y} \neq \boldsymbol{Y} \otimes \boldsymbol{X}$ even though there exist cases where equality holds. (Can you think of such cases?)

Moreover, if $\boldsymbol{W}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ are matrices of commensurate sizes, we have the *mixed product product property*

$$\big[\boldsymbol{W} \otimes \boldsymbol{X}\big]\big[\boldsymbol{Y} \otimes \boldsymbol{Z}\big] = \big[\boldsymbol{W}\boldsymbol{Y}\big] \otimes \big[\boldsymbol{X}\boldsymbol{Z}\big]$$

Transposition and conjugate transposition also distribute over Kronecker products. That is

$$\big[\boldsymbol{X} \otimes \boldsymbol{Y}\big]^\intercal = \boldsymbol{X}^\intercal \otimes \boldsymbol{Y}^\intercal \qquad \text{and} \qquad \big[\boldsymbol{X} \otimes \boldsymbol{Y}\big]^\dagger = \boldsymbol{X}^\dagger \otimes \boldsymbol{Y}^\dagger$$

Here is yet another interesting property of Kronecker products that occurs in the context of outer products. We have

$$\boldsymbol{x}\big[\boldsymbol{x} \otimes \boldsymbol{y}\big]^\intercal = \boldsymbol{x}\big[\boldsymbol{x}^\intercal \otimes \boldsymbol{y}^\intercal\big] = \boldsymbol{x}\boldsymbol{x}^\intercal \otimes \boldsymbol{y}^\intercal$$

Now, all of the above seems dry and stale but it actually is of considerable importance quantum computing.

We therefore suggest that you convince yourself of these algebraic properties of Kronecker products. To this end, you may either prove the above claims using direct yet tedious computations or you could proceed empirically and use computer implementations to test the validity of what we claimed for a couple of exemplary matrices.

task 4.9 [10 points]

more vector logic

In task 4.7, we represented the truth values `false` and `true` in terms of orthonormal vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ and then considered bivariate Boolean (logic) functions $f : \{\boldsymbol{n}, \boldsymbol{s}\}^2 \to \{\boldsymbol{n}, \boldsymbol{s}\}$.

In this task, we will take a step back and work with univariate Boolean functions $f : \{\boldsymbol{n}, \boldsymbol{s}\} \to \{\boldsymbol{n}, \boldsymbol{s}\}$. To this end, consider the following matrices

$$
\begin{aligned}
\boldsymbol{I} &= \boldsymbol{n}\boldsymbol{n}^\intercal + \boldsymbol{s}\boldsymbol{s}^\intercal \\
\boldsymbol{N} &= \boldsymbol{n}\boldsymbol{s}^\intercal + \boldsymbol{s}\boldsymbol{n}^\intercal \\
\boldsymbol{F} &= \boldsymbol{n}\boldsymbol{n}^\intercal + \boldsymbol{n}\boldsymbol{s}^\intercal \\
\boldsymbol{T} &= \boldsymbol{s}\boldsymbol{n}^\intercal + \boldsymbol{s}\boldsymbol{s}^\intercal
\end{aligned}
$$

Here are your tasks:

a) **[2 points]** Given the above matrices, consider a vector $\boldsymbol{x} \in \{\boldsymbol{n}, \boldsymbol{s}\}$ and feed it into the functions

$$
\begin{aligned}
f_0(\boldsymbol{x}) &= \boldsymbol{F}\boldsymbol{x} \\
f_1(\boldsymbol{x}) &= \boldsymbol{I}\boldsymbol{x} \\
f_2(\boldsymbol{x}) &= \boldsymbol{N}\boldsymbol{x} \\
f_3(\boldsymbol{x}) &= \boldsymbol{T}\boldsymbol{x}
\end{aligned}
$$

What do you observe? Briefly discuss which Boolean functions the $f_j(\boldsymbol{x})$, $j \in \{0, 1, 2, 3\}$ actually compute.

b) **[6 points]** Given what you saw in (the voluntary) task 4.8, show only via symbolic manipulations (!) that matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ from task 4.7 can also be computed as

$$
\begin{aligned}
\boldsymbol{C} &= \boldsymbol{F} \otimes \boldsymbol{n}^\intercal + \boldsymbol{I} \otimes \boldsymbol{s}^\intercal \\
\boldsymbol{D} &= \boldsymbol{I} \otimes \boldsymbol{n}^\intercal + \boldsymbol{T} \otimes \boldsymbol{s}^\intercal
\end{aligned}
$$

c) **[2 points]** Prove the following: If matrices $\boldsymbol{N}$ and $\boldsymbol{D}$ are given, we can compute matrix $\boldsymbol{C}$ as

$$\boldsymbol{C} = \boldsymbol{N}\boldsymbol{D}\left[\boldsymbol{N} \otimes \boldsymbol{N}\right]$$

and, if matrices $\boldsymbol{N}$ and $\boldsymbol{C}$ are given, we can compute matrix $\boldsymbol{D}$ as

$$\boldsymbol{D} = \boldsymbol{N}\boldsymbol{C}\left[\boldsymbol{N} \otimes \boldsymbol{N}\right]$$

Note: Symbolically, these proofs require very careful thinking so you may just do them computationally.

task 4.10 [10 points]

first steps towards quantum computing

Still using concepts (symbols and terminology) from vector logic, it's finally time that we meet an operator of pivotal importance in quantum computing. At this point, we will call it $\boldsymbol{M}$ for "mystery" and define it as

$$\boldsymbol{M} = \boldsymbol{n}\boldsymbol{n}^\intercal \otimes \boldsymbol{I} + \boldsymbol{s}\boldsymbol{s}^\intercal \otimes \boldsymbol{N}$$

However, to be able to integrate this operator into our current context, we also need the following matrix which does not occur in quantum computing

$$\boldsymbol{\Pi}_2 = \bigl[\boldsymbol{n} + \boldsymbol{s}\bigr]^\intercal \otimes \boldsymbol{I}$$

Here are your tasks:

a) **[8 points]** Given these matrices, consider two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \bigl\{\boldsymbol{n}, \boldsymbol{s}\bigr\}$ and feed them into the function

$$f_{\boldsymbol{\Pi}_2 \boldsymbol{M}}\bigl(\boldsymbol{x}_1, \boldsymbol{x}_2\bigr) = \boldsymbol{\Pi}_2\, \boldsymbol{M}\bigl[\boldsymbol{x}_1 \otimes \boldsymbol{x}_2\bigr]$$

What does this function do? In order to answer this question, it might be helpful to complete the following table:

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_1 \otimes \boldsymbol{x}_2$	$\boldsymbol{M}\bigl[\boldsymbol{x}_1 \otimes \boldsymbol{x}_2\bigr]$	$\boldsymbol{\Pi}_2\, \boldsymbol{M}\bigl[\boldsymbol{x}_1 \otimes \boldsymbol{x}_2\bigr]$
$\boldsymbol{n}$	$\boldsymbol{n}$	...	...	...
$\boldsymbol{n}$	$\boldsymbol{s}$	...	...	...
$\boldsymbol{s}$	$\boldsymbol{n}$	...	...	...
$\boldsymbol{s}$	$\boldsymbol{s}$	...	...	...

b) **[2 points]** What happens if you work with the following matrix and function?

$$\boldsymbol{\Pi}_1 = \boldsymbol{I} \otimes \bigl[\boldsymbol{n} + \boldsymbol{s}\bigr]^\intercal$$

$$f_{\boldsymbol{\Pi}_1 \boldsymbol{M}}\bigl(\boldsymbol{x}_1, \boldsymbol{x}_2\bigr) = \boldsymbol{\Pi}_1\, \boldsymbol{M}\bigl[\boldsymbol{x}_1 \otimes \boldsymbol{x}_2\bigr]$$

task 4.11

submission of presentation and code

Prepare a presentation / set of slides about your solutions and results. **Submissions of only jupyter notebooks will not be accepted.**

Your slides should help you to give a scientific presentation of your work (i.e. to give a short talk in front of your fellow students and instructors and answer any questions they may have).

W.r.t. to formalities, please make sure that

- your presentation contains a title slide which lists the **names and matriculation numbers** of everybody in your team who contributed to the solutions.

W.r.t. content, please make sure that

- your presentation contains about 12 to 15 content slides but not more
- your presentation is concise and clearly structured
- your presentation answers questions such as
  - "what was the task / problem we considered?"
  - "what difficulties (if any) did we encounter?"
  - "how did we solve them?"
  - "what were our results?"
  - "what did we learn?"

Save / export your slides as a PDF file and upload it to eCampus.

Furthermore, please name all your code files in a manner that indicates which task they solve (e.g. `task-1-5.py`) and put them in an archive or a ZIP file.

Upload this archive / ZIP file with your code snippets to eCampus.