exercise 4

Kronecker products and Pauli matrices

solutions due

until **January 14, 2025** at **23:30** via **ecampus**

Note: Carefully read and follow the solution submission instructions which are detailed in task 5.11.

If your submission does not adhere to the guidelines in task 5.11, your solutions will not be accepted / graded.

general remarks

The first 4 tasks of this exercise sheet are voluntary and intended to allow you to test and apply knowledge you should have by now. You do not have to report their results in your solution submission.

Exercises 5.5 through 5.10 are crucial if you want to develop a solid understanding of quantum computing. We thus encourage every student to work on each of these exercises. There might come a point in your lives where you will be glad you did . . .

task 5.1 [no points]

tensor products generalize familiar products

If we want to master quantum computing, we must master tensor products of vectors and operators. To this end, let us take a step back and consider properties of products of numbers.

Let $\mathbb{F}$ be a field of numbers. We know what it means to add or multiply numbers in such a field. We also know that there are algebraic laws which dictate the behavior or interplay of multiplication and addition. For instance:

Letting $\lambda, x, y, z \in \mathbb{F}$, we have the following distributive laws which tell us how multiplication distributes over addition

$$\big(x + y\big) \cdot z = x \cdot z + y \cdot z \tag{D1}$$

$$x \cdot \big(y + z\big) = x \cdot y + x \cdot z \tag{D2}$$

We also have the following law for multiplying a scalar with a product

$$\lambda \cdot \big(x \cdot y\big) = \big(\lambda \cdot x\big) \cdot y = x \cdot \big(\lambda \cdot y\big) \tag{SM}$$

This last statement may seem strange because λ, x, y are all the same kind of object, namely numbers in $\mathbb{F}$. Why then would we refer to λ as a scalar and to $x \cdot y$ as a product? This becomes clear once we generalize things a bit! So, here we go . . .

Next, let λ be a *scalar* in $\mathbb{F}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ be *vectors* in $\mathbb{F}^n$. Furthermore, let $\otimes : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n \otimes \mathbb{F}^n$ be a tensor product. In the lectures, we stated what kind of axioms a tensor product has to obey. For our current special case where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are all n-dimensional vectors over $\mathbb{F}$, these axioms read

$$\big[\boldsymbol{x} + \boldsymbol{y}\big] \otimes \boldsymbol{z} = \boldsymbol{x} \otimes \boldsymbol{z} + \boldsymbol{y} \otimes \boldsymbol{z} \tag{A1}$$

$$\boldsymbol{x} \otimes \big[\boldsymbol{y} + \boldsymbol{z}\big] = \boldsymbol{x} \otimes \boldsymbol{y} + \boldsymbol{x} \otimes \boldsymbol{z} \tag{A2}$$

and

$$\lambda \cdot \big[\boldsymbol{x} \otimes \boldsymbol{y}\big] = \big[\lambda \cdot \boldsymbol{x}\big] \otimes \boldsymbol{y} = \boldsymbol{x} \otimes \big[\lambda \cdot \boldsymbol{y}\big] \tag{A3}$$

Do you see any similarities between (D1), (D2), (SM) and (A1), (A2), (A3)?

Discuss your observations.

Turn page for further remarks.

Remarks:

- Despite of the structural similarity between (D1), (D2), (SM) and (A1), (A2), (A3), we must keep in mind that the product of numbers $x, y \in \mathbb{F}$ is commutative

$$x \cdot y = y \cdot x$$

  but the tensor product of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ is generally not

$$\boldsymbol{x} \otimes \boldsymbol{y} \neq \boldsymbol{y} \otimes \boldsymbol{x}$$

- In the lectures, we stated the axioms a tensor product must obey in a more complicated manner, namely

$$\big[\boldsymbol{u}_1 + \boldsymbol{u}_2\big] \otimes \boldsymbol{v} = \boldsymbol{u}_1 \otimes \boldsymbol{v} + \boldsymbol{u}_2 \otimes \boldsymbol{v} \tag{A1}$$

$$\boldsymbol{u} \otimes \big[\boldsymbol{v}_1 + \boldsymbol{v}_2\big] = \boldsymbol{u} \otimes \boldsymbol{v}_1 + \boldsymbol{u} \otimes \boldsymbol{v}_2 \tag{A2}$$

$$\lambda \big[\boldsymbol{u} \otimes \boldsymbol{v}\big] = \big[\lambda\, \boldsymbol{u}\big] \otimes \boldsymbol{v} = \boldsymbol{u} \otimes \big[\lambda\, \boldsymbol{v}\big] \tag{A3}$$

  This is because, in the lectures, we considered two arbitrary vector spaces U and V. Above, we were dealing with the special case where $U = V = \mathbb{F}^n$.

task 5.2 [no points]

tensor products of standard basis vectors

Consider the two $\mathbb{R}^2$ standard basis vectors

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

and compute the tensor products $|j\rangle \otimes |k\rangle$ for all pairs $(j, k) \in \{0, 1\}^2$. What do you observe? Is there a certain pattern or regularity recognizable for the above tensor products? Discuss this.

Now, proceed as above but compute the tensor products $|j\rangle \otimes |k\rangle \otimes |l\rangle$ for all triples $(j, k, l) \in \{0, 1\}^3$. What do you observe? Does the above pattern or regularity extend to this more general case? Discuss this.

Assume that $|j\rangle, |k\rangle, |l\rangle \in \big\{|0\rangle, |1\rangle\big\}$. Also assume you were given a vector $|\psi\rangle = |j\rangle \otimes |k\rangle \otimes |l\rangle$. Can you think of a way of recovering the factors $|j\rangle, |k\rangle, |l\rangle$ from $|\psi\rangle$? Test your ideas on the above results.

Remark regarding notation: In the quantum computing literature, tensor products such as

$$|\psi\rangle = |j\rangle \otimes |k\rangle \otimes |l\rangle \tag{1}$$

are often written in a shorter form, namely

$$|\psi\rangle = |j\rangle|k\rangle|l\rangle \tag{2}$$

At least as often if not more frequently, they are written in an even shorter form, namely

$$|\psi\rangle = |jkl\rangle \tag{3}$$

The important take home message is that equations (2) and (3) have to be understood as a very lazy way of writing equation (1). They all express the same thing.

task 5.3 [no points]

tensor products and their algebra

The following exercises ask for *algebraic proofs*. These are pen-and-paper proofs involving axioms (A1), (A2), and (A3) from task 5.1

Show algebraically that

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = ac \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + ad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + bc \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + bd \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Show algebraically that

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} ac \\ ad \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} bc \\ bd \end{bmatrix}$$

Let U be an m-dimensional vector space with basis vectors $\{|u_1\rangle, \ldots, |u_m\rangle\}$ and V be an n-dimensional vector space with basis vectors $\{|v_1\rangle, \ldots, |v_n\rangle\}$. Note that $|x\rangle \in U$ and $|y\rangle \in V$ can be written as

$$|x\rangle = \sum_{j=1}^{m} x_j \, |u_j\rangle$$
$$|y\rangle = \sum_{k=1}^{n} y_k \, |v_k\rangle$$

Show algebraically that $|x\rangle \otimes |y\rangle = |z\rangle$ can be written as

$$|z\rangle = \sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk} \, |u_j\rangle \otimes |v_k\rangle$$

and provide specific expression for the coefficients z_{jk}.

task 5.4 [no points]

tensor products, combinatorics, and joint probabilities

Assume you were given a coin with two sides $\{H, T\}$ and die with six sides $\{1, 2, 3, 4, 5, 6\}$. If you perform an experiment where you first throw the coin and then roll the die, there are 12 possible outcomes as listed in this table:

coin \ die	1	2	3	4	5	6
H	$(H, 1)$	$(H, 2)$	$(H, 3)$	$(H, 4)$	$(H, 5)$	$(H, 6)$
T	$(T, 1)$	$(T, 2)$	$(T, 3)$	$(T, 4)$	$(T, 5)$	$(T, 6)$

In another experiment where you first roll the die and then throw the coin, there are again 12 different outcomes but, in a strict sense, they differ from the outcomes in you first experiments. The following table illustrates this:

die \ coin	H	T
1	$(1, H)$	$(1, T)$
2	$(2, H)$	$(2, T)$
3	$(3, H)$	$(3, T)$
4	$(4, H)$	$(4, T)$
5	$(5, H)$	$(5, T)$
6	$(6, H)$	$(6, T)$

Now, consider a 2D vector space C and a 6D vector space D. Let the two basis vectors

$$|H\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |T\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

of C represent the possible outcomes of throwing the coin and the six basis vectors

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad |6\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

of D represent the different possible outcomes of rolling the die.

Please turn page.

What is the dimension of the space $C \otimes D$? What is the dimension of the space $D \otimes C$? Are both spaces the same? What are the basis vectors of $C \otimes D$? What do they signify? What are the basis vectors of $D \otimes C$? What do they signify? Discuss this in your own words.

Is there also a significance to the vector space, say, $D \otimes D \otimes D = D^{\otimes 3}$? What is the dimension of this space? What do its basis vectors represent? Discuss this in your own words.

Now represent the *classical* probabilities $p(H)$ and $p(T)$ for a coin throw to result in heads or tails, respectively, as follows

$$p(H) = \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} \qquad \text{and} \qquad p(T) = \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}$$

By the same token, let

$$p(1) = \frac{1}{6}\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \qquad \cdots \qquad p(6) = \frac{1}{6}\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}$$

represent the probabilities for a roll of the die to end up on either of its six sides.

Could you use these representations to determine the joint probability $p(T, 3)$ of throwing T *and* rolling 3? What about the joint probability $p(3, T)$?

task 5.5 [5 points]

tensor products of complex unit vectors

Let $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$ be two unit vectors

$$|\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$
$$|\beta\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

whose coefficients obey the Born rule

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$
$$|\beta_0|^2 + |\beta_1|^2 = 1$$

Also let $|\alpha\rangle \otimes |\beta\rangle = |\gamma\rangle$ and recall that

$$|\gamma\rangle = \sum_{j=0}^{1} \sum_{k=0}^{1} \alpha_j \beta_k |j\rangle \otimes |k\rangle = \sum_{j=0}^{1} \sum_{k=0}^{1} \gamma_{jk} |jk\rangle$$

Show that, in our specific situation with $|\alpha_0|^2 + |\alpha_1|^2 = 1$ and $|\beta_0|^2 + |\beta_1|^2 = 1$, we have

$$\sum_{j=0}^{1} \sum_{k=0}^{1} |\gamma_{jk}|^2 = 1$$

task 5.6 [5 points]

effects of the Born rule

Let $|\alpha\rangle \in \mathbb{C}^2$ be given by

$$|\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

such that

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

Here are your tasks:

a) **[1 point]** If $\alpha_0 = \frac{1}{\sqrt{2}}$, then what is α_1 ?

b) **[1 point]** If $\alpha_0 = \frac{1}{2}$, then what is α_1 ?

c) **[1 point]** If $\alpha_1 = \frac{1}{4}$, then what is α_0 ?

d) **[1 point]** If $\alpha_1 = i$, then what is α_0 ?

e) **[1 point]** If $\alpha_1 = \frac{i}{\sqrt{2}}$, then what is α_0 ?

task 5.7 [10 points]

revisiting Pauli- and Hadamard matrices

Recall that the four Pauli matrices are given by

$$
\begin{aligned}
I &= +1\cdot|0\rangle\langle 0| + 1\cdot|1\rangle\langle 1| \\
Z &= +1\cdot|0\rangle\langle 0| - 1\cdot|1\rangle\langle 1| \\
X &= +1\cdot|1\rangle\langle 0| + 1\cdot|0\rangle\langle 1| \\
Y &= +i\cdot|1\rangle\langle 0| - i\cdot|0\rangle\langle 1|
\end{aligned}
$$

and observe that the 2×2 Hadamard matrix can be computed as

$$
H = \frac{1}{\sqrt{2}}\big[X + Z\big]
$$

Here are your tasks:

a) **[3 points]** Use the above to show that

$$
X^2 = Y^2 = Z^2 = I
$$

b) **[2 points]** Use the above to show that

$$
-i\,XYZ = I
$$

c) **[3 points]** Use everything so far to show that

$$
\begin{aligned}
XY &= -YX = i\,Z \\
YZ &= -ZY \;= i\,X \\
ZX &= -XZ = i\,Y
\end{aligned}
$$

d) **[2 points]** Use everything so far to show that

$$
HXH = Z
$$

task 5.8 [10 points]

getting used to CNOT gates (1)

In the lecture 09, we already saw that the truth table for a `controlled NOT` or `CNOT` gate is

x_1	x_2	y_1	y_2
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

and that the corresponding matrix representation of this `CNOT` gate reads

$$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here are your tasks:

a) **[8 points]** Prove that the following holds true

$$\begin{aligned} CX &= \big|0\big\rangle\big\langle 0\big| \otimes I + \big|1\big\rangle\big\langle 1\big| \otimes X \\ &= I \otimes I + \big|1\big\rangle\big\langle 1\big| \otimes \big[X - I\big] \end{aligned}$$

b) **[2 points]** Consider this idea: If we represent 0 as qubit $|0\rangle$ and 1 as qubit $|1\rangle$ and introduce the following vectors

$$\big|\psi\big\rangle = \big|x_1\big\rangle \otimes \big|x_2\big\rangle = \big|x_1\, x_2\big\rangle$$

$$\big|\phi\big\rangle = \big|y_1\big\rangle \otimes \big|y_2\big\rangle = \big|y_1\, y_2\big\rangle$$

we may equivalently write the `CNOT` truth table as

$\|\psi\rangle$	$\|\phi\rangle$
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Now compute

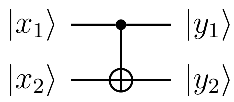
$$\left|\phi\right\rangle = CX \left|\psi\right\rangle$$

for all instances of $|\psi\rangle$ and report your results. Do you recover the above truth table?

task 5.9 [10 points]

getting used to CNOT gates (2)

Working with inputs $|x_1\rangle, |x_2\rangle \in \{|0\rangle, |1\rangle\}$, we may graphically represent the CNOT gate in terms of the following circuit diagram



In task 5.8, you already saw that this circuit implements the this truth table

$\lvert x_1x_2\rangle$	$\lvert y_1y_2\rangle$
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$

Looking at this table, we can understand the effect of the CNOT gate as follows

$$|y_2\rangle = \begin{cases} X\,|x_2\rangle & \text{if } |x_1\rangle = |1\rangle \\ |x_2\rangle & \text{otherwise} \end{cases}$$

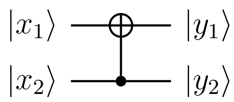
In other words, the CNOT gate can be understood as flipping qubit $|x_2\rangle$ if and only if qubit $|x_1\rangle$ is in state $|1\rangle$. We therefore also say that qubit $|x_1\rangle$ is the control qubit and qubit $|x_2\rangle$ is the target qubit of the CNOT gate. To make this notationally explicit, we henceforth write $\mathtt{CNOT}_{1\to2}$.

Finally, recall that the matrix representation of the $\mathtt{CNOT}_{1\to2}$ is given by

$$CNOT_{1\to2} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here is your task:

Say we wanted to reverse the roles of control and target qubit and consider the $\texttt{CNOT}_{2\to1}$ gate



Write down and compute the matrix representation of this operation.

task 5.10 [10 points]

getting used to CNOT gates (3)

Consider the following operator

$$SWAP = CNOT_{1\to 2}\, CNOT_{2\to 1}\, CNOT_{1\to 2}$$

and apply it to all $|x_1 x_2\rangle$ where $x_1, x_2 \in \{0, 1\}$. Report your results. What do you observe? What does this operator do?

task 5.11

submission of presentation and code

Prepare a presentation / set of slides about your solutions and results. **Submissions of only jupyter notebooks will not be accepted.**

Your slides should help you to give a scientific presentation of your work (i.e. to give a short talk in front of your fellow students and instructors and answer any questions they may have).

W.r.t. to formalities, please make sure that

- your presentation contains a title slide which lists the **names and matriculation numbers** of everybody in your team who contributed to the solutions.

W.r.t. content, please make sure that

- your presentation contains about 12 to 15 content slides but not more
- your presentation is concise and clearly structured
- your presentation answers questions such as
  - "what was the task / problem we considered?"
  - "what difficulties (if any) did we encounter?"
  - "how did we solve them?"
  - "what were our results?"
  - "what did we learn?"

Save / export your slides as a PDF file and upload it to eCampus.

Furthermore, please name all your code files in a manner that indicates which task they solve (e.g. `task-1-5.py`) and put them in an archive or a ZIP file.

Upload this archive / ZIP file with your code snippets to eCampus.