Lecture Assignment 8

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ECSE 403

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Question 1:

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% Question 1

%Case 1

l1 = 1/4;

T1 = 10;

plot_ricc_traj(l1,T1)


%Case 2

l1 = 1/4;

T2 = 1;

plot_ricc_traj(l1,T2)


%Case 3

l2 = 3/4;

T1 = 10;

plot_ricc_traj(l2,T1)


%Case 1

l2 = 3/4;

T2 = 1;

plot_ricc_traj(l2,T2)
```

```
function plot_ricc_traj(l,T)


e = sqrt(1-l);


kt = [];

u = [];
```

```
i = 1;
dt = 0.01;
x0 = 1;
for t = 0:dt:T
    kt(i) = e*((1 + e)+(e-1)*exp(2*(t-T)/e))/((1 + e)-(e-1)*exp(2*(t-T)/
e)) - e^2;
    u(i) = -1/(1-l)*kt(i);
    x(i) = exp(-t)*x0 - exp(-t)*u(i);
    i = i+1;
end

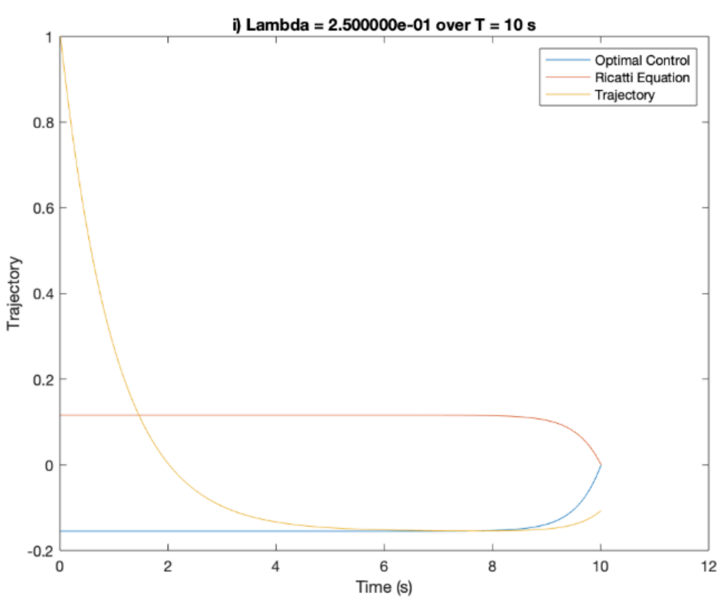


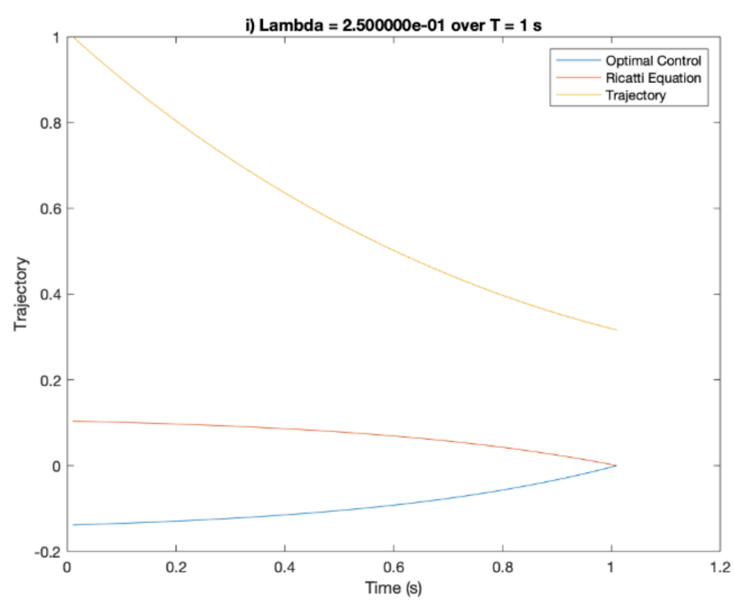
A = -1;
B = 1;
C = 1;
sys = ss(A, B, C, 0);
x = lsim(sys,u,dt:dt:(T+dt),x0);

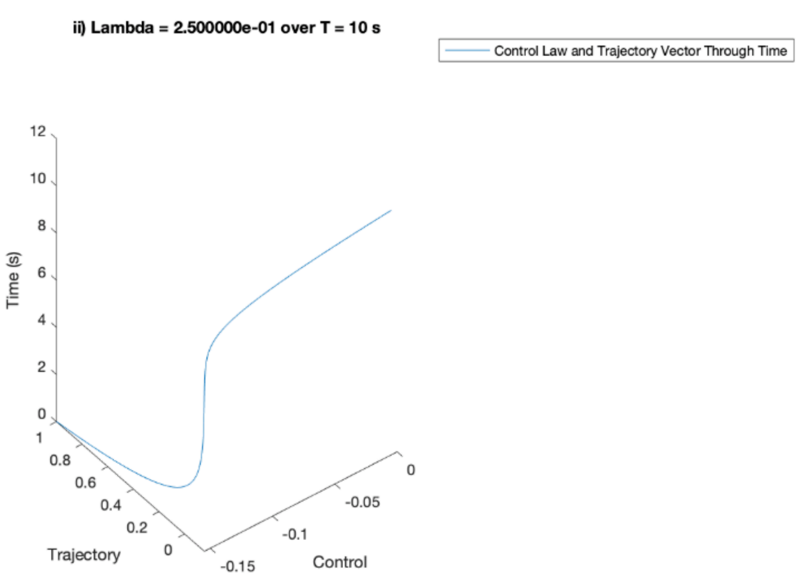
figure(1)
plot(dt:dt:(T+dt), u);
hold on;
plot(dt:dt:(T+dt), kt)
hold on;
plot(dt:dt:(T+dt), x)
hold off;
legend('Optimal Control','Ricatti Equation','Trajectory');
tit = 'i) Lambda = %d over T = %d s';
title(sprintf(tit,l,T));
xlabel('Time (s)')
ylabel('Trajectory')
```

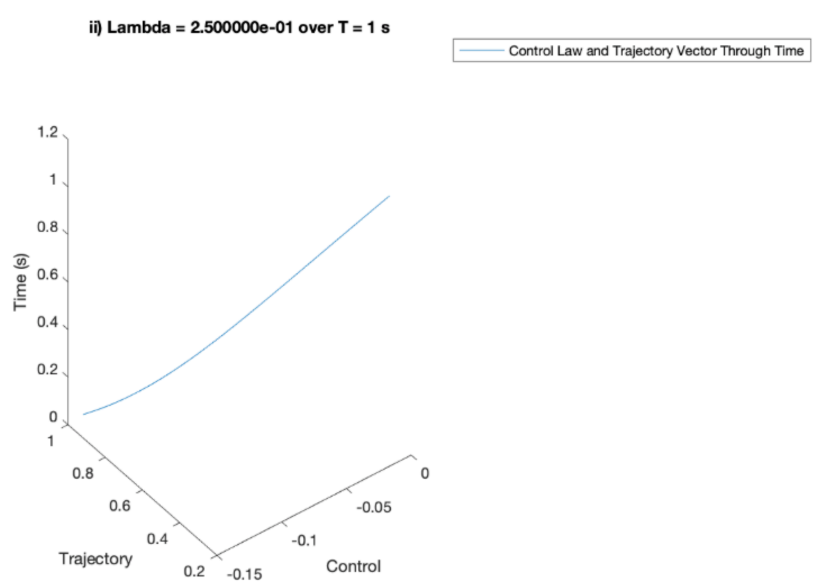
```
figure(5)
plot3(u,x,dt:dt:(T+dt))
legend('Control Law and Trajectory Vector Through Time');
tit = 'ii) Lambda = %d over T = %d s';
title(sprintf(tit,l,T));
xlabel('Control')
ylabel('Trajectory')
zlabel('Time (s)')

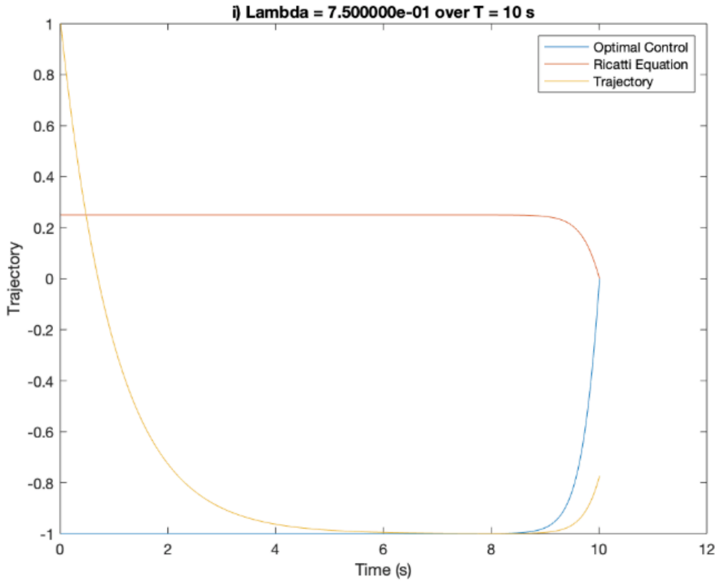

end
```

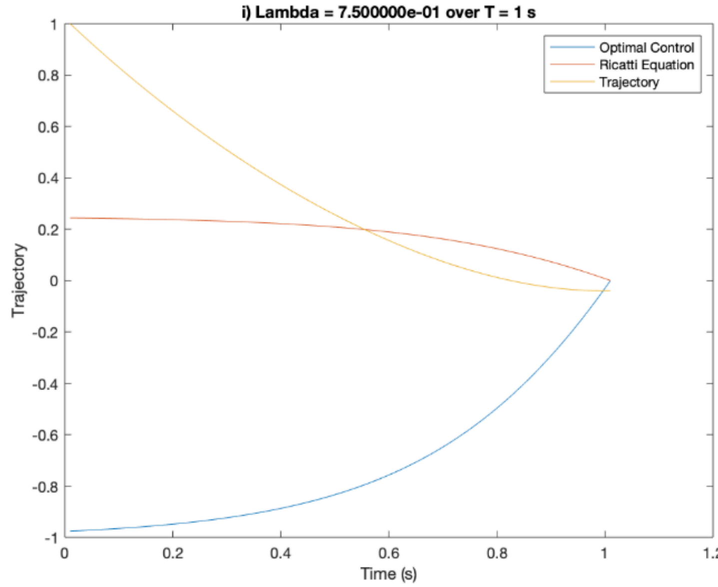


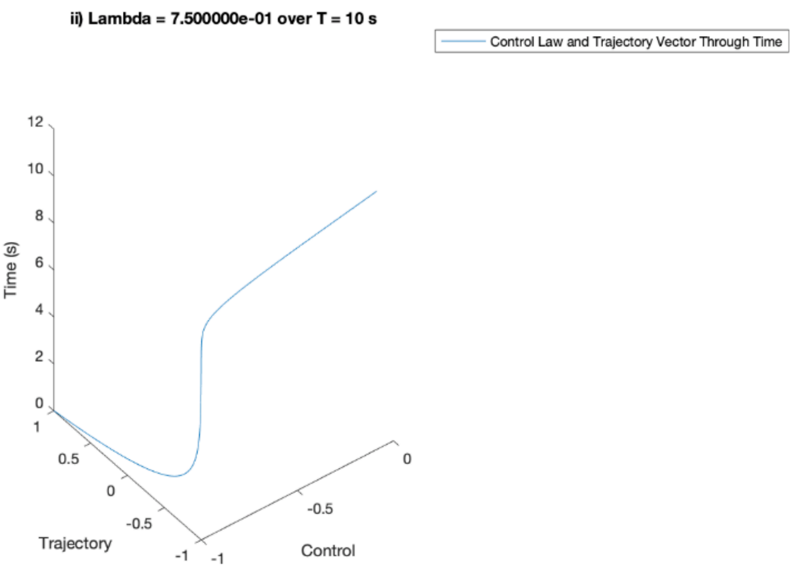


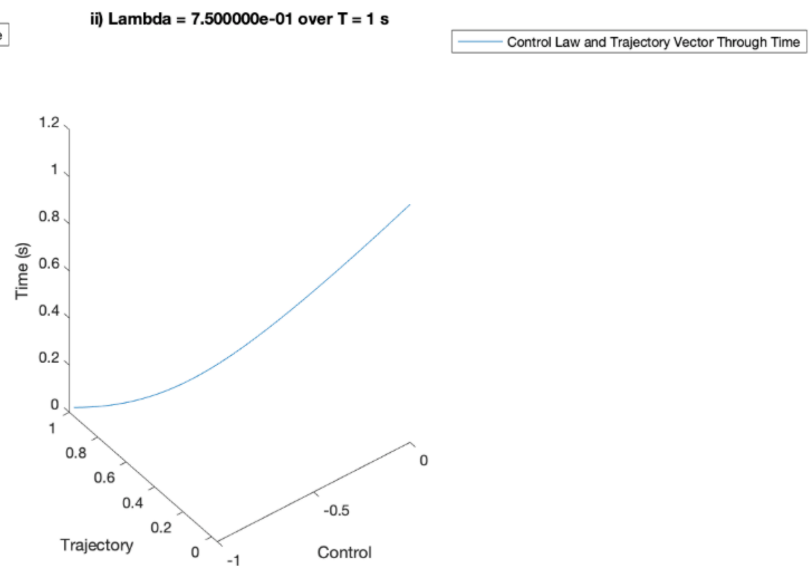








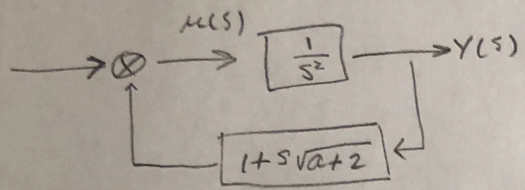




If we increase lambda, the trajectory of the Ricatti equation will be initialized at a greater value. Likewise, the trajectory will start at a lower value. If we increase T, the trajectories and controls will taper towards each other at a faster rate.

2.

a)



$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{\alpha+2} \end{bmatrix} x(t)$$

Characteristic polynomial:

$$(sI-A) = \begin{pmatrix} s & -1 \\ 1 & s+\sqrt{\alpha+2} \end{pmatrix} = s^2 + s\sqrt{\alpha+2} + 1$$

(i) $$\text{Roots} = \frac{-\sqrt{\alpha+2} \pm \sqrt{(\alpha+2)-4}}{2} = \frac{-\sqrt{\alpha+2}}{2} \pm \frac{\sqrt{\alpha-2}}{2}$$

if $\alpha = 0$,

$\lambda_1, \lambda_2 = -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$ → Real parts of both roots are negative ∴ Stable

$\alpha > 2$, $\lambda_1, \lambda_2 = -\frac{\sqrt{\alpha+2}}{2} \pm \frac{1}{2}\sqrt{\alpha-2}$ → Real part of both roots are negative ∴ Stable

(ii) For overdamping, roots are strictly real: $(\alpha-2) > 0$

(and non-repeating) $\alpha > 2$

If overdamping increase, ζ should increase, and decay rate increase.

as seen in $-\frac{\sqrt{b\alpha+2}}{2} - \frac{1}{2}\sqrt{b\alpha-2}$, if $b>0$, then both terms increase in magnitude so decay rate increases.

and in $-\frac{\sqrt{b\alpha+2}}{2} + \frac{1}{2}\sqrt{b\alpha-2}$, $b>0$, the term

$$\frac{\sqrt{b\alpha+2}}{2} > \frac{1}{2}\sqrt{b\alpha-2} \longrightarrow (b\alpha+2) > (b\alpha-2)$$
$$2 > -2$$

clearly, if we increase α by factor of b, decay rate increase, so overdamping increase

(iii) For underdamping, complex roots are needed.

$$\lambda_1, \lambda_2 = -\frac{\sqrt{\alpha+2}}{2} \pm \frac{\sqrt{\alpha-2}}{2}$$

$$= -\frac{\sqrt{\alpha+2}}{2} \pm \frac{i}{2}\sqrt{2-\alpha}$$

If $\alpha < 2$, system is underdamped. If α is decreased, frequency of sinusoidal response increases.

If $b > 0$

$$-\frac{\sqrt{-b\alpha+2}}{2} \pm \frac{\sqrt{-b\alpha-2}}{2} = \frac{-\sqrt{-b\alpha+2}}{2} \pm \frac{i}{2}\sqrt{2-b}$$

has complex roots. as b increase, $2 < b\alpha$, too right t

$-b\alpha+2 >$

(iv) $\lambda_1, \lambda_2 = -\frac{\sqrt{4}}{2} \pm 0 = -1$ has roots of multiplicity 2 $\longrightarrow$ Critically damped roots are needed.

(b) $\lambda = 10$

$$K_{n} = \sqrt{(\alpha + 2\sqrt{10}} \geq 0$$

$$K_{i2} = \sqrt{10}$$

$$TF = \frac{\frac{1}{s^2}}{\frac{1}{s^2}\left(\frac{1}{\sqrt{10}} + s\sqrt{\alpha + 2\sqrt{10}}\right) + 1} = \frac{1}{s^2 + s\sqrt{\alpha+2\sqrt{10}} + \frac{1}{\sqrt{10}}}$$

(i) Roots: $\lambda_1, \lambda_2 = -\sqrt{\alpha + 2\sqrt{10}} \pm \sqrt{(\alpha + 2\sqrt{10}) - \frac{4}{\sqrt{10}}}$

If $\alpha = 0$, $-\sqrt{2\sqrt{10}} \pm \sqrt{2\sqrt{10} - \frac{4}{\sqrt{10}}} = -\sqrt{2\sqrt{10}} \pm \sqrt{\frac{20\sqrt{10}}{10} - \frac{4\sqrt{10}}{10}}$

$$= -\sqrt{2\sqrt{10}} \pm \sqrt{\frac{8}{5}\sqrt{10}}$$

Since $\sqrt{2\sqrt{10}} > \sqrt{\frac{8}{5}(\sqrt{10}}$, then real part of roots are strictly negative

$2 > 8/5$ $\hookrightarrow$ Stable

If $\alpha > 0$, $-\sqrt{\alpha + 2\sqrt{10}} \pm \sqrt{\alpha + \frac{8}{5}\sqrt{10}}$. If $\alpha > 0$,

$\sqrt{\alpha + 2\sqrt{10}} > \sqrt{\alpha + 8/5\sqrt{10}}$ so real parts of roots are strictly negative

$\hookrightarrow$ Stable

(ii) For overdamping, we need strictly real roots, which requires

$\alpha + \frac{8}{5}\sqrt{10} > 0$, $\alpha > -\frac{8}{5}\sqrt{10}$

If α is increased past this point, magnitude of decay rate increases, so system is more overdamped.

(iii) Likewise for underdamping

$$\alpha + \frac{8\sqrt{10}}{5} < 0$$

$$\alpha < -\frac{8}{5}\sqrt{10}$$

If α decrease past this point, magnitude of imaginary part increases:

$$\sqrt{\left(-b\left(\frac{8}{5}\sqrt{10}\right) + \frac{8}{5}\sqrt{10}\right)}$$

$$= \sqrt{\left(\frac{8}{5}\sqrt{10}\right)(-1)(-b+1)}$$ ~~which will incre~~

$$= i\sqrt{b-1}\sqrt{\frac{8}{5}\sqrt{10}}$$ so coefficient of i will increase

as b increase $\rightarrow$ underdamping increases.

(iv) $\alpha = \frac{8}{5}\sqrt{10}$

$$\lambda_1, \lambda_2 = -\sqrt{2\sqrt{10}} \pm \sqrt{\alpha - \frac{8}{5}\sqrt{10}} = -\sqrt{2\sqrt{10}}$$ of multiplicity 2

so system is critically damped.

2. iii)

```
l = 1;
plot_control(1)

l = 10;
plot_control(10)


function plot_control(l)
alpha = 1;
x0 = [1; 1];
T = 20;
t = 0;
A = [0 1; -1 -sqrt(alpha + 2)];
B = [0; 1];
C = [1 0];

e = sqrt(1-l);
e = sqrt(1-l);
i = 1;
kt = [];
u = [];
Gramian = 0;
sqrt(1-10) - (1-10);
dt = 0.01;
for t=0:dt:20
    kt(i) = e*((1 + e)+(e-1)*exp(2*(t-20)/e))/((1 + e)-
(e-1)*exp(2*(t-20)/e)) - e^2;
    u(i) = -1/(1-l)*kt(i);
```

```
    i = i + 1;

    Gramian = Gramian + expm(A * (T - i*dt)) * B * transpose(B) *
expm(transpose(A) * (T - i * dt))*dt;

end

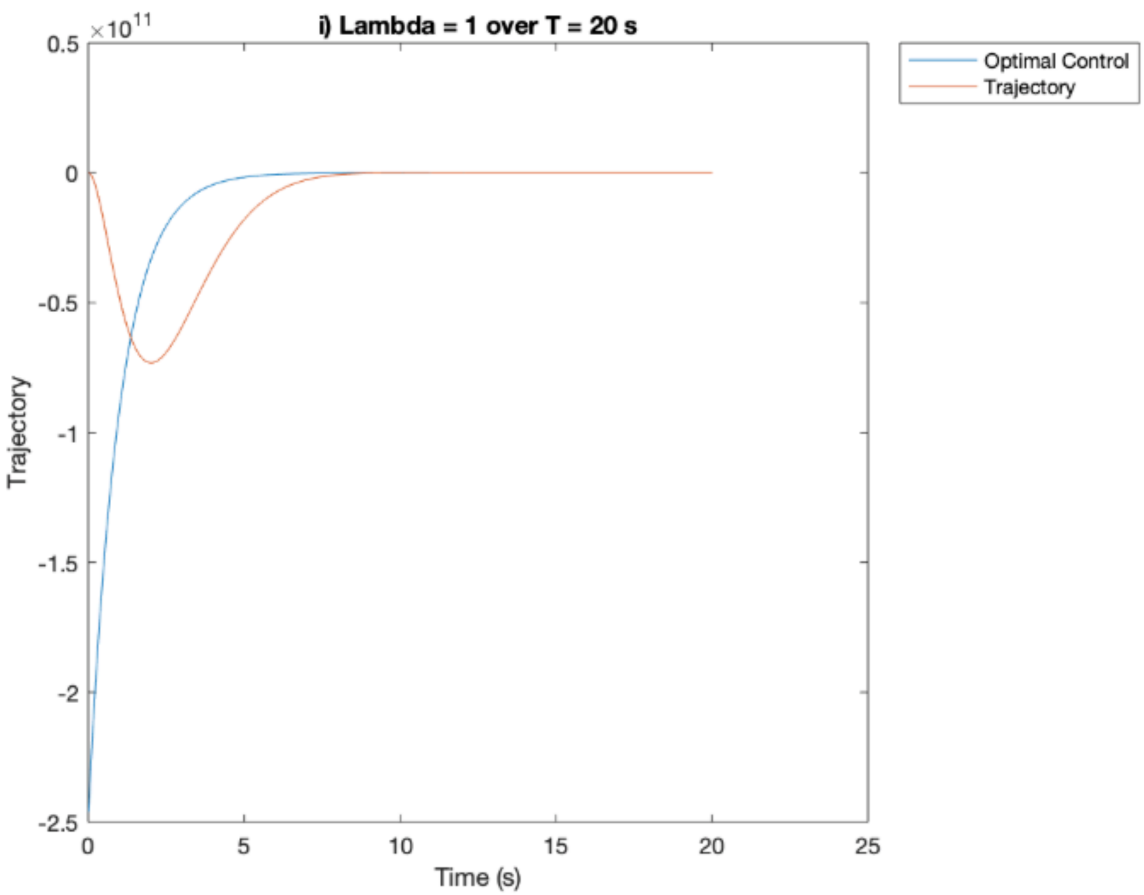

i = 1;

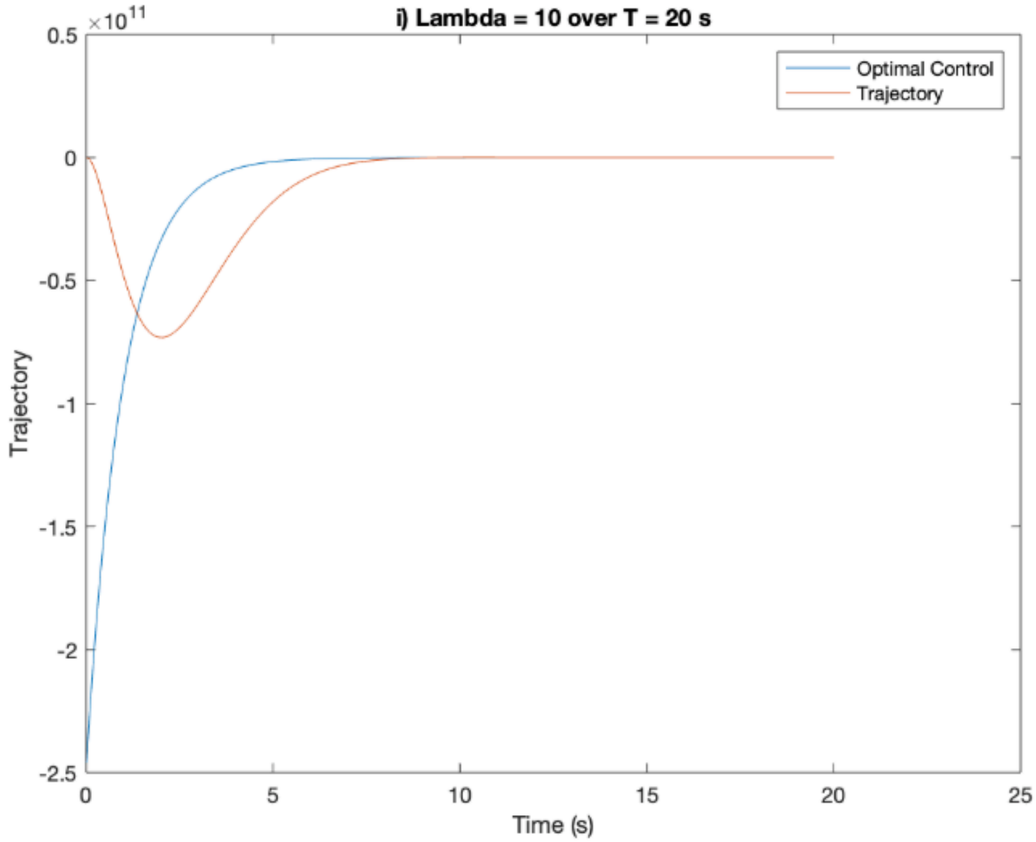
for t=0:dt:20

    u(i) = [exp(T-t) exp(-2*(T-t))] * inv(Gramian) * [2-exp(T); -1-
exp(-2*(T))];

    i = i + 1;

end
```





i) Lambda = 10 over T = 20 s
×10^11
Optimal Control
Trajectory
0.5
0
-0.5
-1
-1.5
-2
-2.5
Trajectory
0
5
10
15
20
25
Time (s)