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Lab Assignment 2

ECSE 403 - Prof. Caines

October 3rd, 2020

The final goal of this lab is to model and control an inverted pendulum system. In that system the input is applied to a DC motor connected to a cart-pole. The first step to model the whole system is to model the DC motor. We saw the (approximate) model of the DC motor in lab 1 as:

Jm Ø'' + (b + KtKe/Ra)Ø' = (Kt/Ra)Va

where θ *is the shaft angle (in radians) of the motor and va is the applied voltage. System's parameters are as following:*

```
%Constants
Jm = 0.01; %kgm^2
b  = 0.001;%Nmsec
Ke = 0.02; %Vsec
Kt = 0.02; %Nm/A
Ra = 10;    %Ω
```

4.1 Proportional Controller

1. Consider the transfer function θ(s)/*Va(s) in lab assignment 1 as open loop system. Suppose we are using a proportional gain with a unity feedback loop. Using the standard form of second order systems:*

ω^2 / (s^2 + 2ζωs + ω^2)

find the parameters ωn, ζ *of the closed loop system as a function of proportional controller(gain) K.*

```
%Coefficients of transfer function
b0 = Kt / (Ra * Jm);
a1 = (b + (Kt*Ke)/Ra)/Jm;
%Define transfer function
G1 = tf([b0],[1 a1 0])
```

```
G1 =
 
       0.2
  -------------
  s^2 + 0.104 s
 
Continuous-time transfer function.
```

```
%Redefine using symbols
syms G(K)
syms s
G(K) = simplify((0.2*K) / (s^2 + 0.104*s + 0.2*K))
```

```
G(K) =
```

$$\frac{25\,K}{125\,s^2 + 13\,s + 25\,K}$$

```
%By inspection of G(K), we can solve for
% wn and ζ
wn = sqrt((25/125)*K)
```

```
wn =
```

$$\sqrt{\frac{K}{5}}$$

```
damping = (13/104) / (2*wn)
```

```
damping =
```

$$\frac{1}{16\,\sqrt{\frac{K}{5}}}$$

2. For the closed loop system in Question 1, plot the the roots of the closed loop system for proportional gains K ∈ [0.01, 0.1, 1, 10, 100, 103] in one plot to see the movement of poles as a function of proportional gain K.

```
k = [0.01, 0.1, 1, 10, 100, 1000];

%Proportional feedback with various K values
G_k1 = feedback(G1*k(1,1),1);
G_k2 = feedback(G1*k(1,2),1);
G_k3 = feedback(G1*k(1,3),1);
G_k4 = feedback(G1*k(1,4),1);
G_k5 = feedback(G1*k(1,5),1);
G_k6 = feedback(G1*k(1,6),1);

%Plotting pole zero plot
pzplot(G_k1,G_k2,G_k3,G_k4,G_k5,G_k6);
legend('K = 0.01','K = 0.1','K = 1','K = 10','K = 100','K = 1000');
```

3. Explain what root locus diagram describes and plot the root locus diagram for θ(s) *va(s) as open loop function. [5 marks]*

The root locus plot shows the 'weight' of each open loop pole or zero on the locations of the poles and zeros of the closed loop system. By varying the gain, we can understand how to modify the open loop system to meet desired closed loop system dynamics.

```
rlocus(G1)
```

4. Using the time-domain step response of second order systems find a value of proportional gain K such that closed loop step response has approximately 20% overshoot. Verify your proportional controller using step response of closed loop system. [10 marks]

```
K = 0.001;
G_K = feedback(G1*K,1);
x = stepinfo(G_K);
overshoot = x.Overshoot;

while overshoot < 19.9
    K = K + 0.0005;
    G_K = feedback(G1*K,1);
    x = stepinfo(G_K);
    overshoot = x.Overshoot;
end

K
```

```
K = 0.0650
```

```
overshoot
```

```
overshoot = 19.9891
```

```
stepplot(G_K)
```

5. Ignoring the over shoot constraint in Question 4, find the proportional gain K such that closed loop step response has peak time of approximately 4s. Verify your proportional controller using step response of closed loop system. [10 marks]

```
K = 0.001;
G_K = feedback(G1*K,1);
x = stepinfo(G_K);
peak_time = x.PeakTime;

while peak_time > 4.1 || peak_time < 3.9
    K = K + 0.005;
    G_K = feedback(G1*K,1);
    x = stepinfo(G_K);
    peak_time = x.PeakTime;
end

K
```

```
K = 2.9360
```

```
peak_time
```

```
peak_time = 4.0997
```

```
stepplot(G_K)
```

6. Suppose a controller with transfer function (s+0.9/s+0.75) is added to the system. Is there a 2 proportional gain for this system which makes it unstable? [5 marks]

```
G6 = G1 * tf([1,0.9],[1,0.75]);
rlocus(G6)
```

A proportional gain around 7.5 makes the system unstable.

4.2 State Space Representation

7. The electormechanical equations of the DC motor can be described as following: Jmθ'' + bθ' = *Kti La* ˙i + *Rai = va − Ke* ˙θ, *where i and La = 0.5H are the current and inductance of the armature. The transfer function in Lab 1, is an approximation of the model which is described here. We can find the exact model of a DC motor by state space representation.*

```
La = 0.5; %(H)
D = 0;
```

Describe the state space representation of the DC motor in the standard from of: x˙(t) = *Ax(t) + Bu(t) y(t) = Cx(t) + Du(t) D = 0 with two different setups: (a) Variables* (θ',i) *are assumed to be states of the system and variable* θ' *is assumed to be the output.*

```
%See derivation
A1 = [-b/Jm, Kt/Jm ; -Ke/La, -Ra/La];
B1 = [0 ; 1/La];
C1 = [1  0];
```

(b) Variables (θ, ˙θ, *ia) are assumed to be states of the system and variable* θ *is assumed to be the output.*

```
%See derivation
A2 = [0, 1, 0; 0, -b/Jm, Kt/Jm; 0, -Ke/La, -Ra/La];
B2 = [0 ; 0 ; 1/La];
C2 = [1  0  0];
```

8. Consider the state space model (A1, B1, C1). Find the transfer function(denote it by H1(s)) and plot the step response. Consider transfer function of ˙θ(s)/*va(s) in Lab 1(denote it by G1(s)). Are transfer functions H1(S) and G1(s) exactly the same? Compare the step responses of H1(s) and G1(s), are they exactly the same?*

```
[n1,d1] = ss2tf(A1,B1,C1,D);
H1 = tf(n1, d1)
```

```
H1 =

          4
  -------------------
  s^2 + 20.1 s + 2.08
```

```
Continuous-time transfer function.
```

```
G1 = tf([b0], [1 a1])
```

```
G1 =

     0.2
  ---------
  s + 0.104

Continuous-time transfer function.
```

```
figure(3);
stepplot(H1);
hold on;
stepplot(G1);
legend('H1','G1')
hold off;
```

```
stepinfo(H1)
```

```
ans = struct with fields:
        RiseTime: 21.1215
    SettlingTime: 37.6588
     SettlingMin: 1.7385
     SettlingMax: 1.9230
       Overshoot: 0
      Undershoot: 0
            Peak: 1.9230
        PeakTime: 101.3819
```

```
stepinfo(G1)
```

```
ans = struct with fields:
        RiseTime: 21.1251
    SettlingTime: 37.6161
     SettlingMin: 1.7394
     SettlingMax: 1.9230
       Overshoot: 0
      Undershoot: 0
            Peak: 1.9230
        PeakTime: 101.4023
```

Though the transfer functions are not the same, the system's exhibit the same dynamic behaviour as evidence by the step plots and the step info.

9. Graph the pole-zero plot for both H1(s) and G1(s). Based on these Graph explain why the approximation in Lab 1 was a ”good” approximation. [5 marks]

Both systems have a pole around -0.104. H1 has a pole close to -20, but is non dominant in the dynamics of the system as it is far from the origin, and the other pole. So, even though G1 is of higher order, it displays the same dynamic behaviour as H1, and can be considered a good approximation.

```
pzplot(H1)
```

```
legend('H1')
```

```
pzplot(G1)
legend('G1')
```

10. Consider the state space model (A2, B2, C2). Find the transfer function(denote it by H2(s)) and plot the step response.[5 marks] Consider transfer function of θ(s)/*va(s) in Lab 1(denote it by G2(s)). Are transfer functions H2(S) and G2(s) exactly the same? Compare the step responses of H2(s) and G2(s), are they exactly the same? [5 marks]*

Though the transfer functions are not the same, the system's exhibit the same dynamic behaviour as evidence by the step plots.

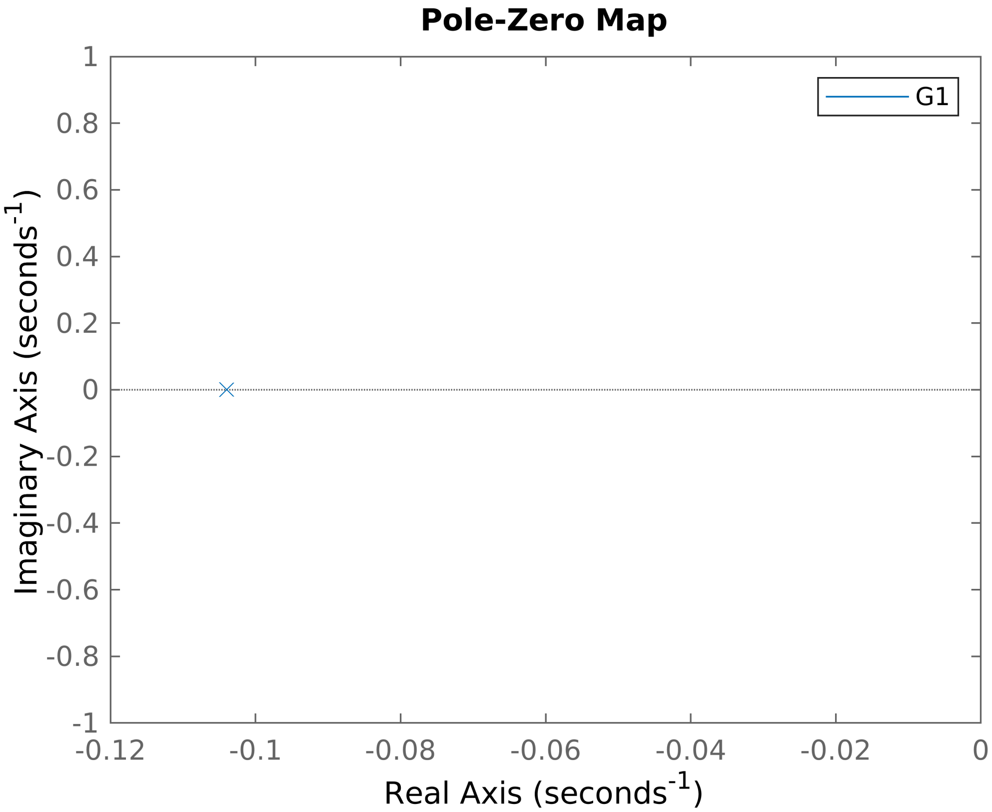
```
[n2,d2] = ss2tf(A2,B2,C2,D);
H2 = tf(n2, d2)
```

```
H2 =
 
               4
  -----------------------
  s^3 + 20.1 s^2 + 2.08 s
 
Continuous-time transfer function.
```

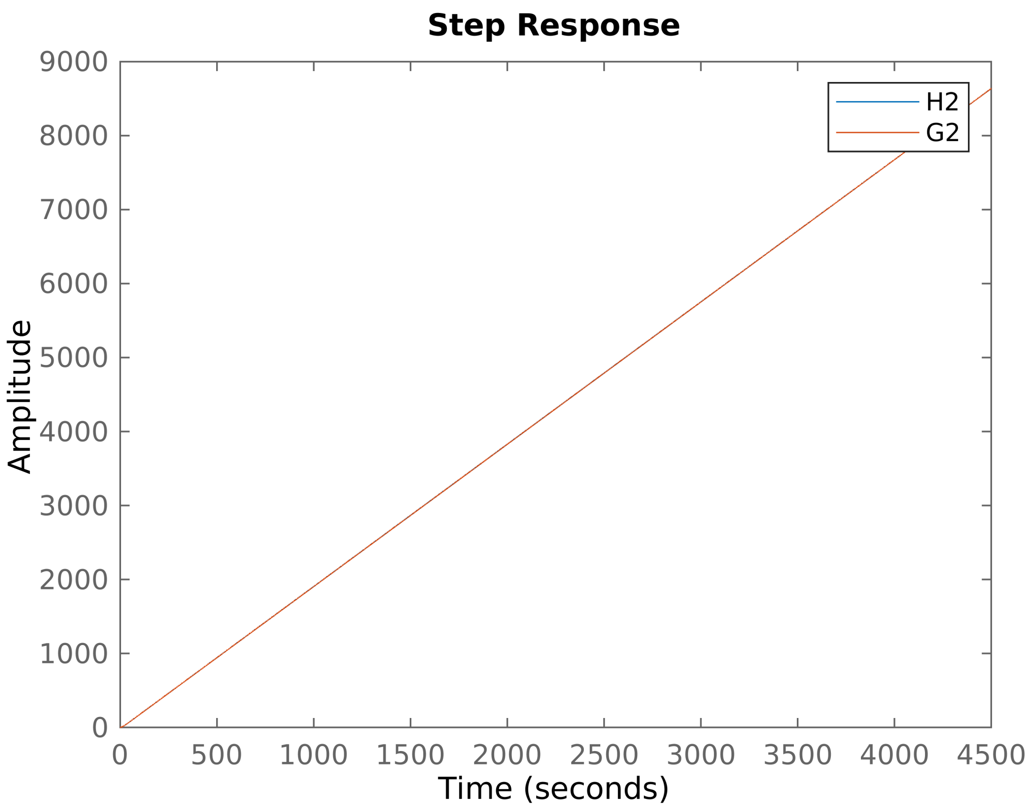
```
G2 = tf([b0], [1 a1 0])
```

```
G2 =
 
       0.2
  -------------
  s^2 + 0.104 s
 
Continuous-time transfer function.
```

```
figure(3);
```

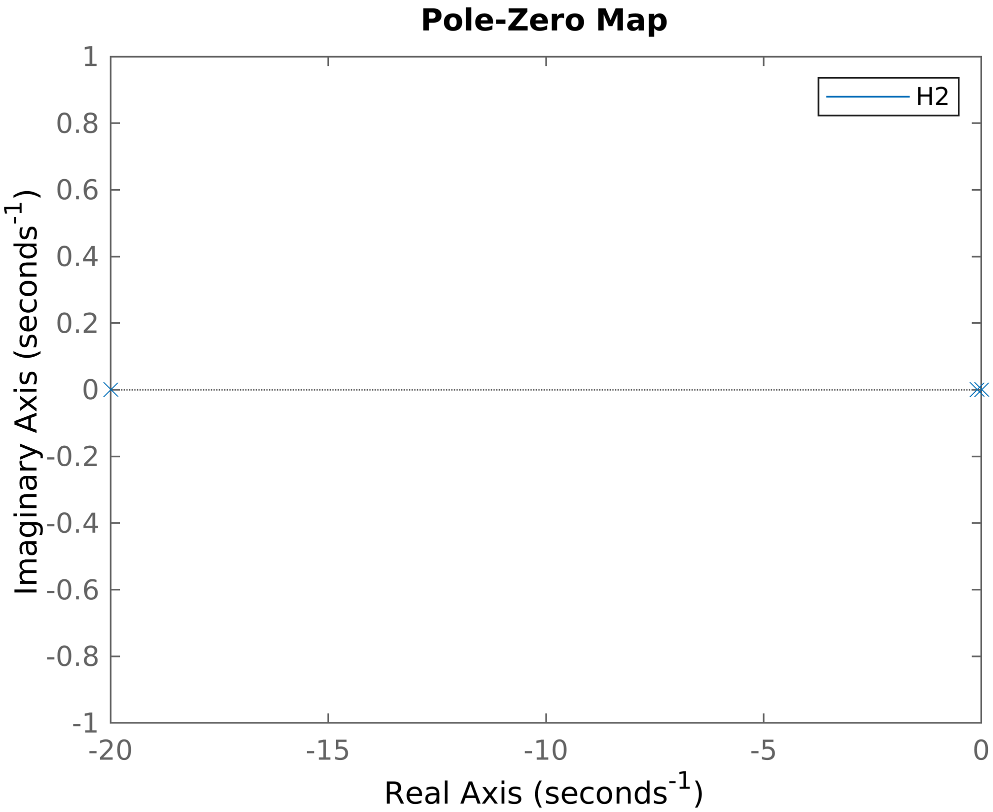



```
stepplot(H2);
hold on;
stepplot(G2);
legend('H2','G2')
hold off;
```

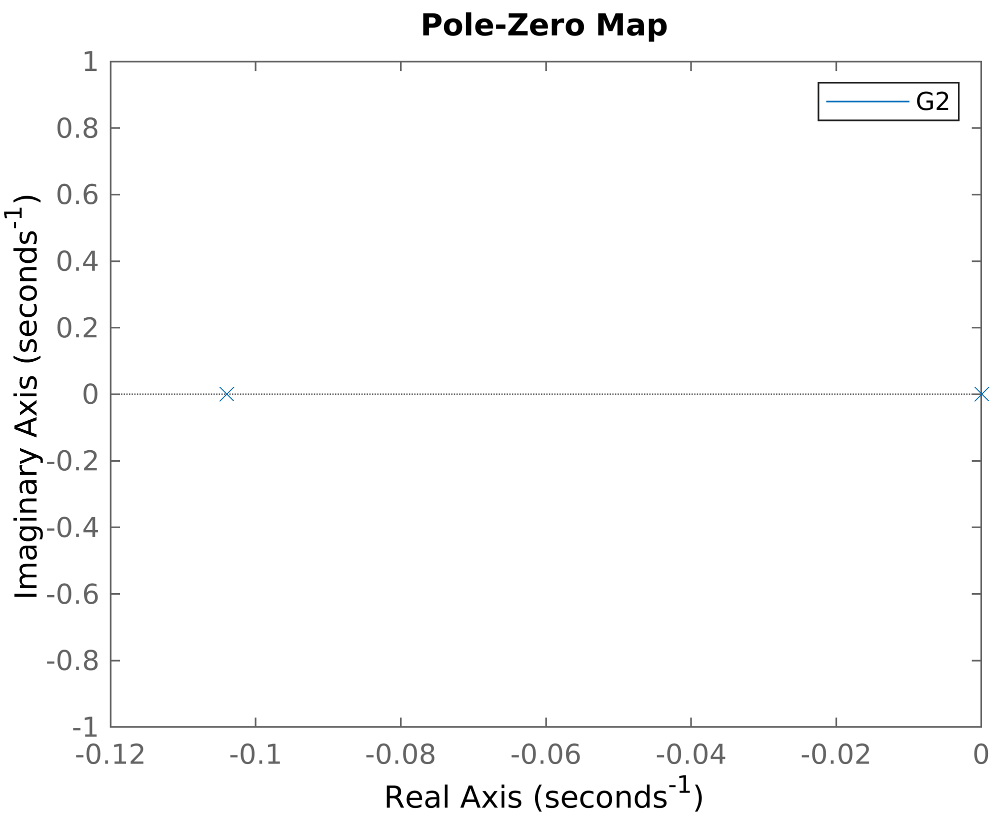


11. Graph the pole-zero plot for both H2(s) and G2(s). Based on these Graph explain why the approximation in Lab 1 was a "good" approximation. [5 marks]

```
pzplot(H2)
legend('H2')
```




```
pzplot(G2)
legend('G2')
```



Both systems have a poles around -0.0104 and 0. H2 has another pole close to -20, but is non dominant in the dynamics of the system as it is far from the origin, and the other poles. So, even though G1 is of higher order, it displays the same dynamic behaviour as H2, and can be considered a good approximation.

12. Consider the uncontrolled dynamics(or zero input dynamics): x' = A1x(t) Find the eigen-values and eigen-vectors of A1. Find the response of the uncontrolled system to the eigen-vectors as initial conditions. Is there an initial condition which results in non exponentially stable response?

```
D = [];
state_space1 = ss(A1,B1,C1,D);

[V1,D1] = eig(A1);
eigenvalues1 = [D1(1,1); D1(2,2)]
```

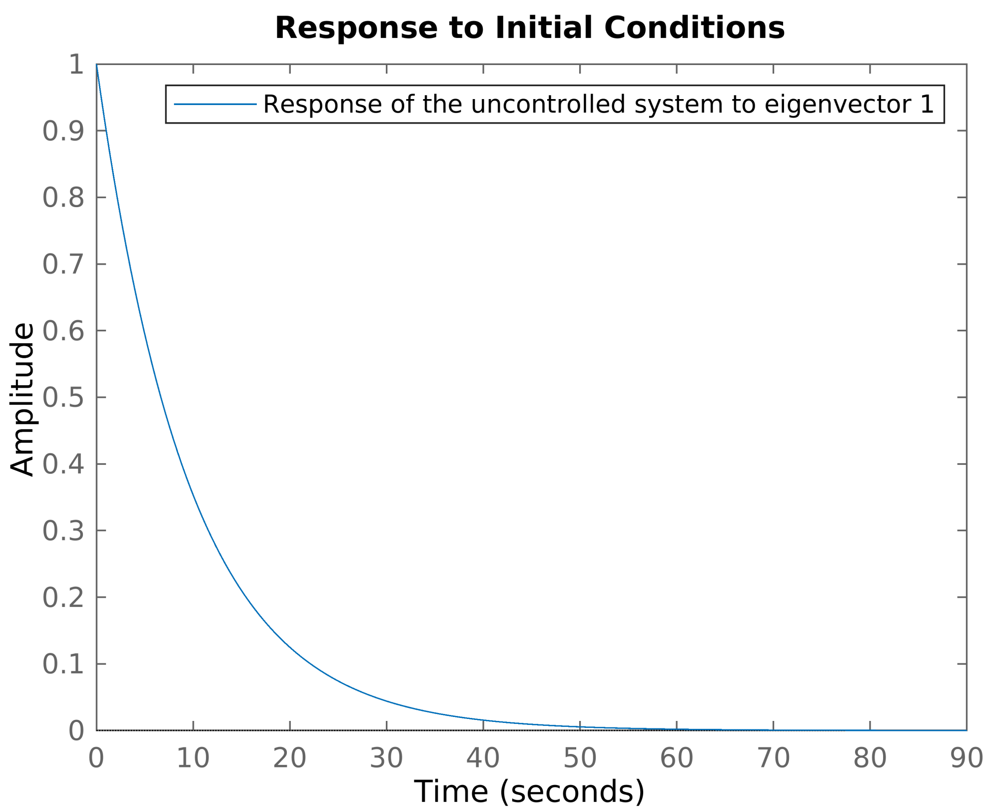
```
eigenvalues1 = 2×1
   -0.1040
  -19.9960
```

```
eig1_vec1 = [V1(1,1) ; V1(2,1)]
```

```
eig1_vec1 = 2×1
    1.0000
   -0.0020
```

```
initial(state_space1, eig1_vec1)
```

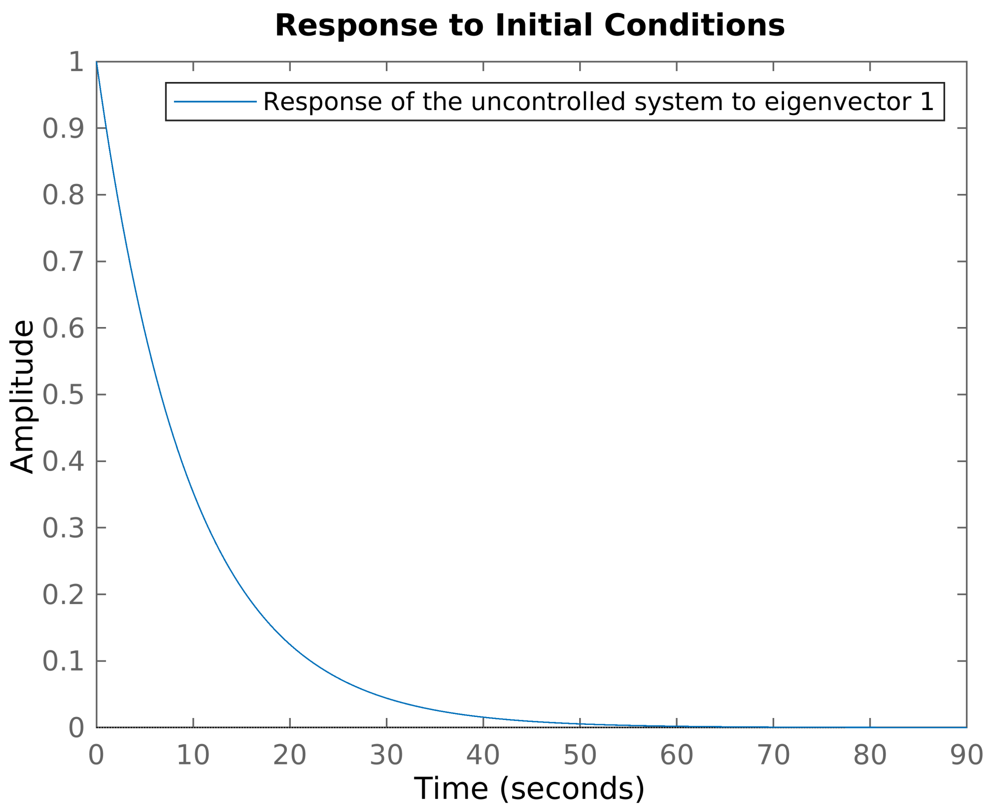
```
legend('Response of the uncontrolled system to eigenvector 1')
```




```
eig1_vec2 = [V1(1,2) ; V1(2,2)]
```

```
eig1_vec2 = 2×1
   -0.1000
    0.9950
```

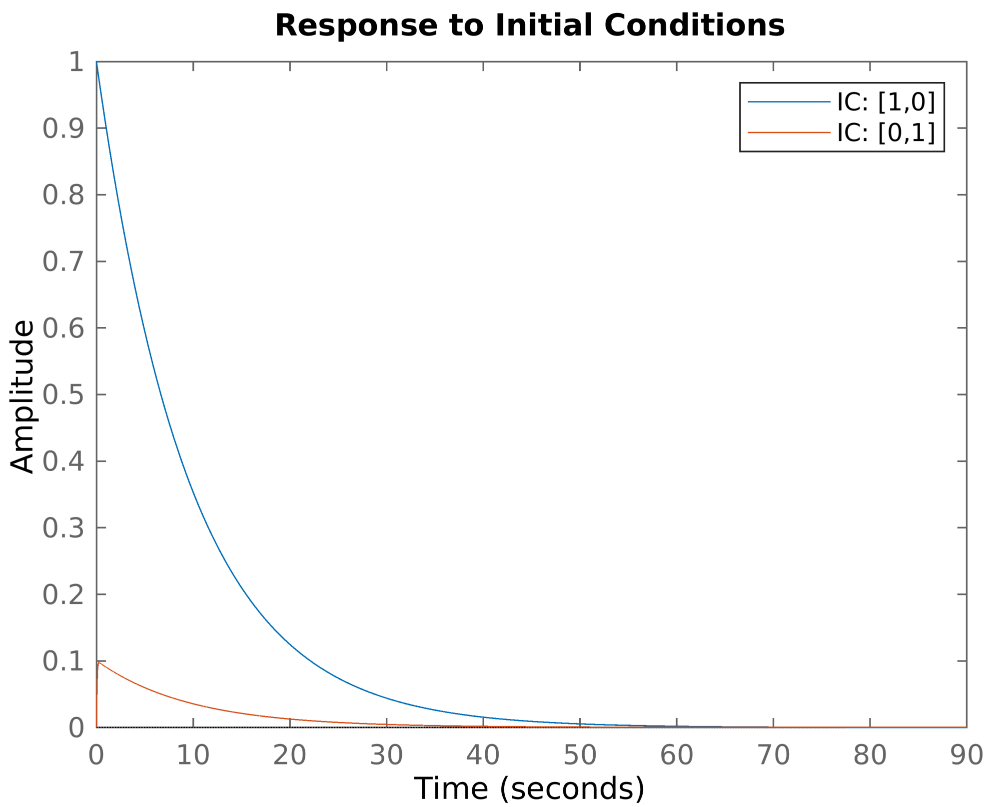
```
initial(state_space1,eig1_vec1)
legend('Response of the uncontrolled system to eigenvector 1')
```



No, the step responses are stable as time tends to infinity, as evidenced by the plots.

13. Plot the system's response to initial conditions [1; 0] and [0; 1]. Explain intuitively what is the physical interpretation of these initial conditions. Explain physical interpretation of the final value of the system response to each of these initial conditions. (for example if system starts with an initial speed and zero initial current, intuitively why this speed approaches 0?) [10 marks]

```
figure(6)
initial(state_space1,[1;0])
hold on;
initial(state_space1,[0;1])
legend('IC: [1,0]', 'IC: [0,1]')
hold off;
```



The initial conditions [1 ; 0] indicates that the system is initialized with a non-zero angular velocity, but a zero armature current.

Conversely, the initial condition [0 ; 1] signifies that the system is initialized at rest but with a non-zero armature current.

In the system equations defined, it is clear that angular acceleration and velocity are positively related to armature current and that current and current changes are negatively related to angular velocity. So, if the system is intialized with only angular velocity, the output (angular velocity) will naturally tend to 0 as time goes to infinity. Likewise, the angular velocity willl also tend to 0 if current is initialized to non-zero.

14. Consider uncontrolled dynamics x' = A2x(t) Find the eigen-values and eigen-vectors of A2. Find the response of the uncontrolled system to the eigen-vectors as initial conditions. Is there an initial condition which results in non exponentially stable response?

```
[V2,D2] = eig(A2)
```

```
V2 = 3×3
    1.0000   -0.9946    0.0050
         0    0.1035   -0.1000
         0   -0.0002    0.9950
D2 = 3×3
         0         0         0
         0   -0.1040         0
         0         0  -19.9960
```

```
eigenvalues2 = [D2(1,1);D2(2,2);D2(3,3)]
```

```
eigenvalues2 = 3×1
         0
   -0.1040
  -19.9960
```

```
eig2_vec1 = [V2(1,1) ; V2(2,1) ; V2(3,1)]
```

```
eig2_vec1 = 3×1
     1
     0
     0
```

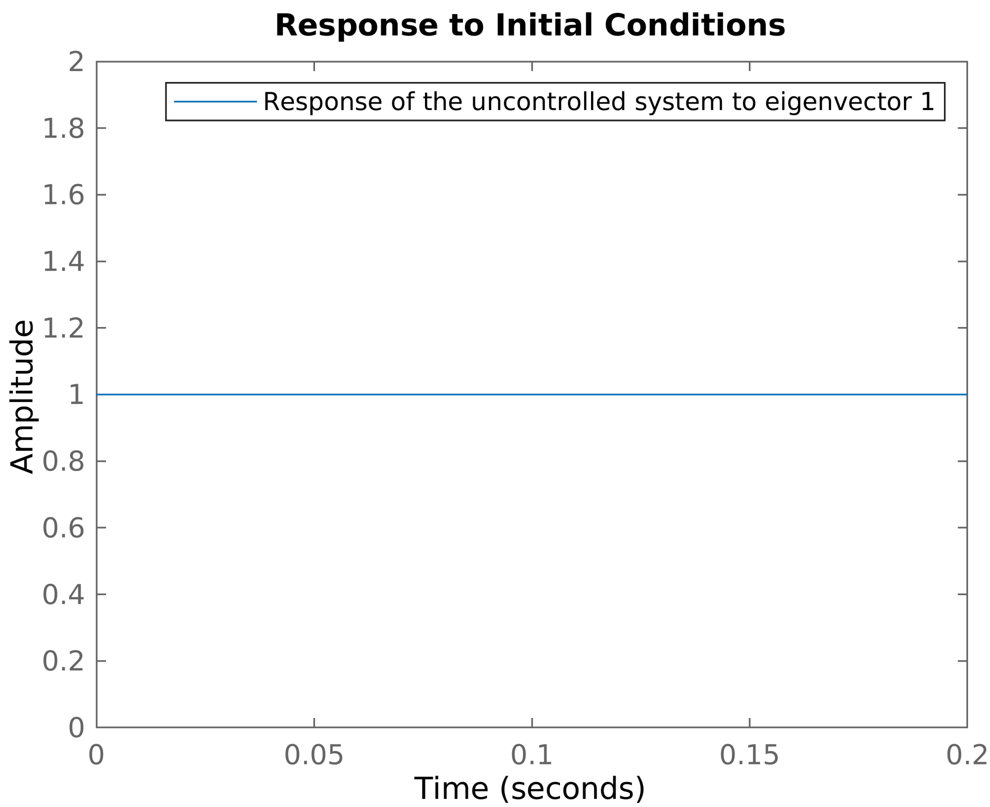
```
eig2_vec2 = [V2(1,2) ; V2(2,2) ; V2(3,2)]
```

```
eig2_vec2 = 3×1
   -0.9946
    0.1035
   -0.0002
```

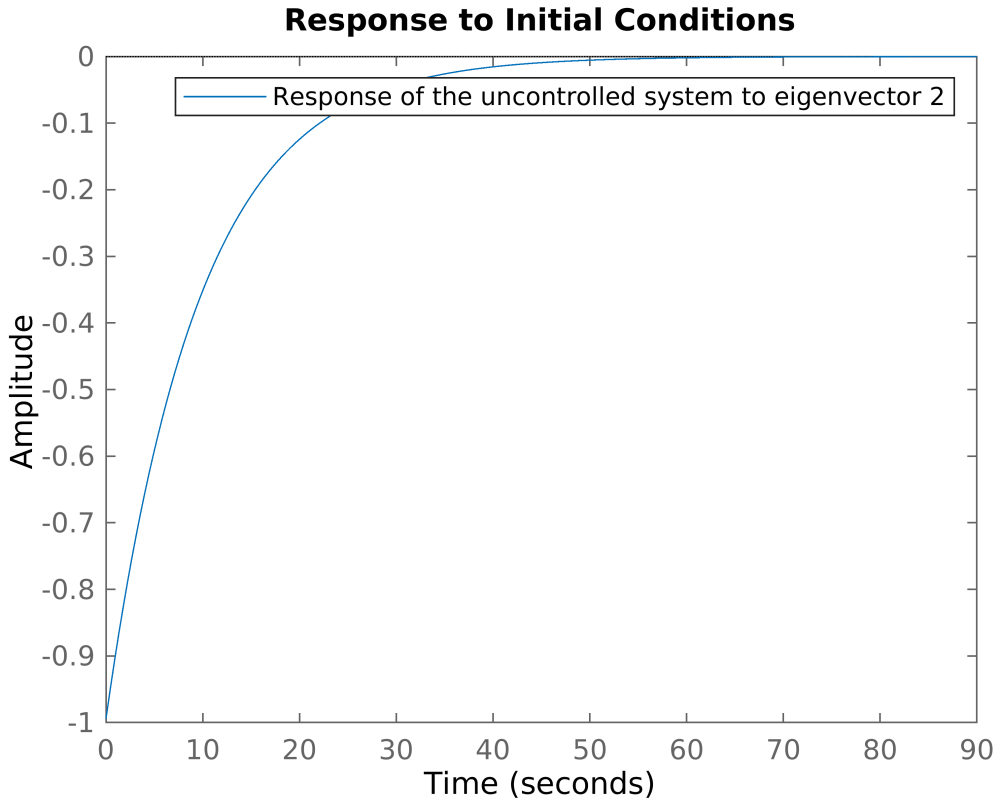
```
eig2_vec3 = [V2(1,3) ; V2(2,3) ; V2(3,3)]
```

```
eig2_vec3 = 3×1
    0.0050
   -0.1000
    0.9950
```

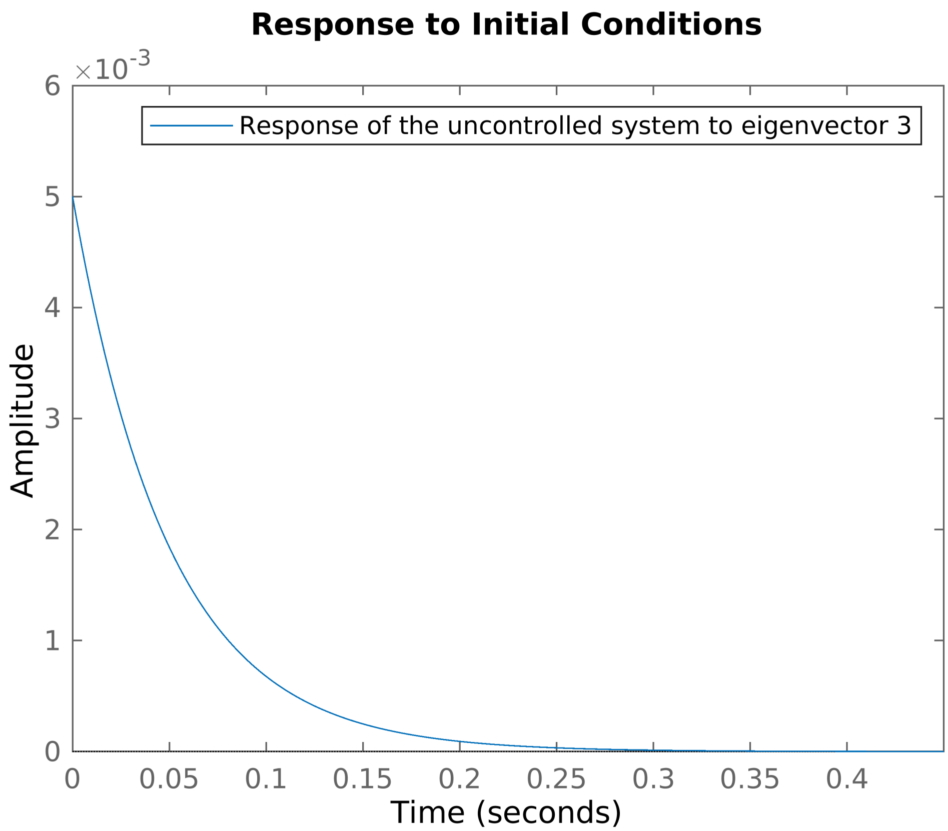
```
state_space2 = ss(A2,B2,C2,D);

initial(state_space2,eig2_vec1)
legend('Response of the uncontrolled system to eigenvector 1')
```




```
initial(state_space2,eig2_vec2)
legend('Response of the uncontrolled system to eigenvector 2')
```




```
initial(state_space2,eig2_vec3)
legend('Response of the uncontrolled system to eigenvector 3')
```



Only the intitial conditions [1 ; 0 ; 0] results in an exponentially unstable step response, as evidenced by its plot - the response remains constant as time goes to infinity.

Derivation of state space systems:

System of Differential Equations:

$$Jm\frac{d^2}{dt^2}\theta + b\frac{d}{dt}\theta = Kt \cdot i$$

$$La\frac{d}{dt}i + Ra \cdot i = Va - Ke\frac{d}{dt}\theta$$

with state variables: $\frac{d\theta}{dt}, i$

Solving for 1st derivatives of state variables:

$$\frac{d^2\theta}{dt^2} = \frac{Kt}{Jm}i - \left(\frac{b}{Jm}\right)\frac{d\theta}{dt}$$

$$\frac{di}{dt} = \frac{Va}{La} - \frac{Ke}{La}\frac{d\theta}{dt} - \frac{Ra}{La}i$$

We can set up our fundamental state space equation by inspection:

$$\begin{bmatrix} \frac{d^2\theta}{dt^2} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{b}{Jm} & \frac{Kt}{Jm} \\ -\frac{Ke}{La} & -\frac{Ra}{La} \end{bmatrix} \begin{bmatrix} \frac{d\theta}{dt} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{La} \end{bmatrix} Va$$

and declare $\frac{d\theta}{dt}$ to be the output variable:

$$\frac{d\theta}{dt} = [1 \quad 0] \begin{bmatrix} \frac{d\theta}{dt} \\ i \end{bmatrix}$$

Similarly, we declare state variables $\theta, \frac{d\theta}{dt}, i$

and construct fundamental state space equation as above:

$$\begin{bmatrix} \theta \\ \frac{d\theta}{dt} \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-b}{Jm} & \frac{Kt}{Jm} \\ 0 & \frac{-Ke}{La} & \frac{-Re}{La} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{La} \end{bmatrix} Va$$

and declare θ to be the output variable:

$$\theta = [1 \quad 0 \quad 0] \begin{bmatrix} \theta \\ \frac{d\theta}{dt} \\ i \end{bmatrix}$$