<u>Lab 7</u>

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1. The pendulum system along all the forces is depicted in the Figure 1.[25 marks]

The nonlinear equations of the inverted pendulum are given as following:

$$\begin{bmatrix} (m_c + m_p) & m_p l_p cos(\theta_p) \\ m_p l_p cos(\theta_p) & m_p l_p^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta}_p \end{bmatrix} + \begin{bmatrix} -m_p l_p sin(\theta_p)\dot{\theta}_p^{\,2} \\ -m_p g l_p sin(\theta_p) \end{bmatrix} = \begin{bmatrix} F_c \\ 0 \end{bmatrix},$$

where in these equations m_c is the mass of cart, x_c is the position of the cart, m_p is the mass of pendulum, θ_p is the pendulum's angle as shown in the figure, l_p is length of the pendulum, g is gravity, and F_c is the input force. These dynamics are nonlinear. Our goal in this question is to reach a linearized approximation for this system around $\dot{x}_c = x_c = \theta_p = \dot{\theta}_p = 0$.

(a) Simplify these equations by using the approximation $\sin(\theta_p) \approx \theta_p$, $\cos(\theta_p) \approx 1$, and setting any order of θ_p and $\dot{\theta}_p$ greater than 1 equal zero($\dot{\theta}_p^{\,2} \approx 0$).

(b) Solve the matrix equation and find $\begin{bmatrix} \ddot{x}_c \\ \ddot{\theta}_p \end{bmatrix}$ as a function of F_c and θ_p.

(c) Use the fact that $m_c \gg m_p$ and simplify the expression.

1. $$\begin{pmatrix} m_c + m_p & m_p l_p \cos(\theta_p) \\ m_p l_p \cos(\theta_p) & m_p l_p^2 \end{pmatrix} \begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_p \end{pmatrix} + \begin{pmatrix} -m_p l_p \sin(\theta_p)(\dot{\theta}_p)^2 \\ -m_p l_p g \sin(\theta_p) \end{pmatrix} = \begin{pmatrix} F_c \\ 0 \end{pmatrix}$$

$\sin\theta_p \approx \theta_p$, $\cos\theta_p \approx 1$, $\dot{\theta}_p^2 \approx 0$, $\theta_p \approx 0$, $m_c >> m_p$

$$\begin{pmatrix} m_c + m_p & m_p l_p \\ m_p l_p & m_p l_p^2 \end{pmatrix} \begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_p \end{pmatrix} + \begin{pmatrix} 0 \\ -m_p l_p g \theta_p \end{pmatrix} = \begin{pmatrix} F_c \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_p \end{pmatrix} = \begin{pmatrix} m_c + m_p & m_p l_p \\ m_p l_p & m_p l_p^2 \end{pmatrix}^{-1} \begin{pmatrix} F_c \\ m_p l_p g \theta_p \end{pmatrix} \approx \begin{pmatrix} m_c & m_p l_p \\ m_p l_p & m_p l_p^2 \end{pmatrix}^{-1} \begin{pmatrix} F_c \\ m_p l_p g \theta_p \end{pmatrix}$$

$$= \begin{pmatrix} m_p l_p^2 & -m_p l_p \\ -m_p l_p & m_c \end{pmatrix} \begin{pmatrix} F_c \\ m_p l_p g \theta_p \end{pmatrix} \Big/ \left(m_c m_p l_p^2 - m_p^2 l_p^2\right)$$

$\hookrightarrow m_p l_p^2 (m_c - m_p)$
$= m_p l_p^2 m_c$

$$= \begin{pmatrix} \frac{1}{m_c} & -\frac{1}{l_p m_c} \\ -\frac{1}{l_p m_c} & \frac{1}{m_p l_p^2} \end{pmatrix} \begin{pmatrix} F_c \\ m_p l_p g \theta_p \end{pmatrix}$$

$$= \begin{pmatrix} \frac{F_c}{m_c} - \frac{m_p l_p g}{m_c l_p} \theta_p \\ -\frac{F_c}{l_p m_c} + \frac{m_p l_p g \theta_p}{m_p l_p^2} \end{pmatrix} = \begin{pmatrix} \frac{F_c}{m_c} - \frac{m_p g \theta_p}{m_c} \\ -\frac{F_c}{m_c l_p} + \frac{g \theta_p}{l_p} \end{pmatrix}$$

2. Using the linear approximated dynamic in Question 1, find the state space representation of the system with state vector $\begin{bmatrix} x_c \\ \dot{x}_c \\ \theta_p \\ \dot{\theta}_p \end{bmatrix}$ and the input F_c.[15 marks]

2.
$$\begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_p \end{pmatrix} = \begin{pmatrix} \frac{F_c}{m_c} - \frac{m_p g}{m_c}\theta_p \\ -\frac{F_c}{m_c l_p} + \frac{g}{l_p}\theta_p \end{pmatrix} = \begin{pmatrix} 0 & -\frac{m_p g}{m_c} \\ 0 & \frac{g}{l_p}\theta_p \end{pmatrix} \begin{pmatrix} \dot{x}_c \\ \theta_p \end{pmatrix} \begin{pmatrix} \frac{1}{m_c} \\ -\frac{1}{m_c l_p} \end{pmatrix} F_c$$

$$\begin{pmatrix} \dot{x}_c \\ \ddot{x}_c \\ \dot{\theta}_p \\ \ddot{\theta}_p \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_p g}{m_c} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l_p} & 0 \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x}_c \\ \theta_p \\ \dot{\theta}_p \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m_c} \\ 0 \\ -\frac{1}{m_c l_p} \end{pmatrix} F_c$$

3 . In the lab, the system is equipped with two sensors. One sensor measures the position of the cart and another measures the angle of the pendulum. Find the C matrix corresponding to these measurements.[10 marks]

3. Two sensors (x_c , θ_p)

$$y = C \begin{pmatrix} x_c \\ \dot{x}_c \\ \theta_p \\ \dot{\theta}_p \end{pmatrix} \rightarrow C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
km = 0.0077; %N.m amp

ke = 0.0077; %N.m amp

mp = 0.106;  %kg

mc = 0.526;  %kg

lp = 0.168;  %m
```

```
Ra = 2.6;     %ohms
rg = 0.0064; %m
kg = 3.7;     %kg
g  = 9.8;      %m/s2



A = [0 1 0 0 ; 0 0 -mp*g/mc 0 ; 0 0 0 1 ; 0 0 g/lp 0]
B = [0 ; 1/mc ; 0 ; -1/mc*lp]
C = [1 0 0 0 ; 0 0 1 0]
```

```
A = 4×4
         0    1.0000         0         0
         0  -14.4899   -1.9749         0
         0         0         0    1.0000
         0   86.2495   58.3333         0
B = 4×1
         0
    3.2550
         0
  -19.3751
C = 2×4
     1     0     0     0
     0     0     1     0
```

4. Check the observability and controllability rank conditions.

```
rank(obsv(A, C))
rank(ctrb(A, B))
```

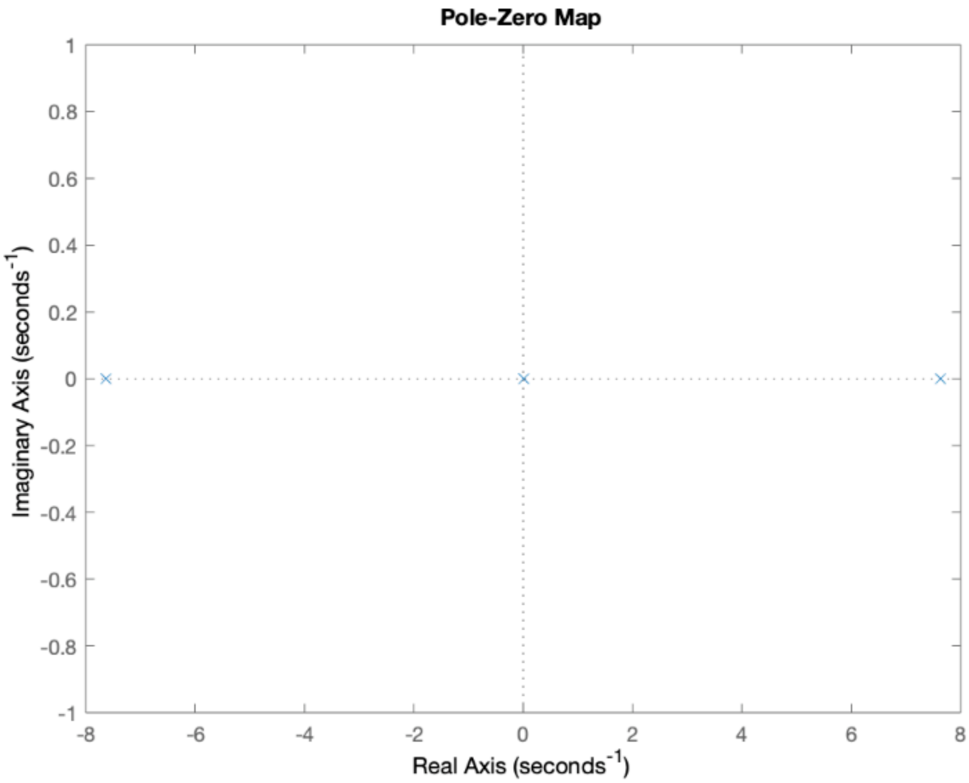
```
ans = 4
ans = 4
```

The observability and controllability matrices are full rank, so the system is controllable and observable.

5. Plot the poles of the system. Is the system stable?

```
eig(A)
G1 = ss(A, B, C, 0);
pzplot(G1)
```

```
ans = 4×1
         0
         0
    7.6376
   -7.6376
```



There is a pole in the RHP so the system is unstable.

6. The force F_c in the derivation and Question 1-5, physically comes from the torque produced by the DC motor in the lab. The DC motor is connected to a gear and through the gear cart system moves. The overall force(F_c) to the voltage of DC motor(v_a), can be expressed as following:

$$F_c = \frac{k_g k_m}{r_g R_a}\left(v_a - \frac{k_e k_g}{r_g}\dot{x}_c\right)$$

where new variables k_g and r_g, are internal gear ratio and motor gear ration. Calculate the state space representation of the system with same states and outputs and v_a as input.[15 marks]

6. $F_c = \frac{K_g K_m}{r_g R_a}\left(V_a - \frac{K_e K_g \dot{x}_c}{r_g}\right) = \frac{K_g K_m}{r_g R_a} V_a - \frac{K_g^2 K_m K_e}{r_g^2 R_a}\dot{x}_c$

$$\begin{pmatrix} \dot{x}_c \\ \ddot{x}_c \\ \dot{\theta}_p \\ \ddot{\theta}_p \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_p g}{m_c} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l_p} & 0 \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x}_c \\ \theta_p \\ \dot{\theta}_p \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m_c} \\ 0 \\ -\frac{1}{m_c l_p} \end{pmatrix} \left(\frac{K_g K_m}{r_g R_a} V_a - \frac{K_g^2 K_m K_e}{r_g^2 R_a} \dot{x}_c \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{K_g^2 K_m K_e}{m_c r_g^2 R_a} & -\frac{m_p g}{m_c} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{K_g^2 K_m K_e}{m_c r_g^2 R_a} & \frac{g}{l_p} & 0 \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x}_c \\ \theta_p \\ \dot{\theta}_p \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{K_g K_m}{m_c r_g R_a} \\ 0 \\ -\frac{K_g K_m}{r_g R_a m_c l_p} \end{pmatrix} V_a$$

7. Check the observability and controllability rank conditions.

```
K = -(kg*kg*km*ke)/(mc*rg*rg*Ra);
A = [0 1 0 0 ; 0 K -mp*g/mc 0 ; 0 0 0 1 ; 0  -K/lp g/lp 0]
B = [0 ; (kg*km)/(mc*rg*Ra) ; 0 ; -(kg*km)/(rg*Ra*mc*lp)]
C = [1 0 0 0 ; 0 0 1 0]
rank(obsv(A, C))
rank(ctrb(A, B))
```

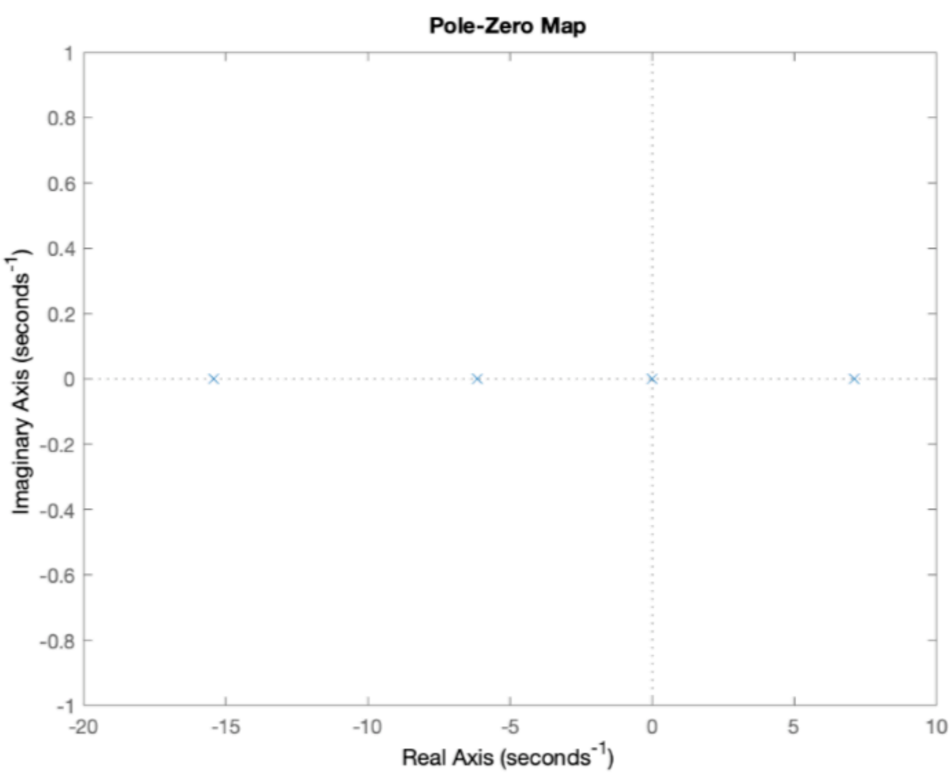
```
A = 4×4
         0    1.0000         0         0
         0  -14.4899   -1.9749         0
         0         0         0    1.0000
         0   86.2495   58.3333         0
B = 4×1
         0
    3.2550
         0
  -19.3751
C = 2×4
     1     0     0     0
     0     0     1     0
ans = 4
ans = 4
```

The observability and controllability matrices are full rank, so the system is controllable and observable.

8. Plot the poles of the system. Is the system stable?

```
eig(A)
G1 = ss(A, B, C, 0);
pzplot(G1)
```

```
ans = 4×1
          0
   -15.4365
    -6.1559
     7.1024
```



There is a pole in the RHP so the system is unstable.

9. Explain in what range of parameters (θp,xc) this approximation is an accurate approximation. [5 marks]

We used the approximation $\sin(\theta p) = \theta p$ and $\cos(\theta p) = 1$. As this is the small angle apronThis small angle approximation is only valid for $|\theta p| < 10$ degrees. We also approximated orders of $\theta p'$ and θp greater than 1 as 0. An acceptable limit for $\theta p'$ and θp is < 0.25, as $0.25^2 = 0.0625$.