Decomposition of complete graphs into unicyclic graphs with nine edges

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Abstract

Let G be a disconnected unicyclic graph with $n = 9$ edges such that the only cycle in G has exactly four edges and there are exactly two trees attached to the cycle at non-adjacent vertices. In this paper, we use standard techniques based on Rosa type labelings to prove that G decomposes complete graphs K_{2nk} and K_{2nk+1} for every non-negative integer k.

Keywords: Graph decomposition, G-design, ρ-labeling, σ-labeling, σ^+-labeling, 1-rotational σ^+-labeling.

1 Introduction

Let U be a graph on m vertices. A decomposition of graph U is a family of pairwise edge disjoint subgraphs $\mathcal{D} = \{G_0, G_1, \ldots, G_s\}$ such that every edge of U belongs to exactly one member of $\mathcal{D}$. In case each member of $\mathcal{D}$ is isomorphic to a given graph G, we refer to $\mathcal{D}$ as a G-decomposition of U or alternatively we call $\mathcal{D}$ a G-design of U if U is isomorphic to a complete graph on m vertices. A G-decomposition of the complete graph K_n is cyclic if there exists an ordering $(x_0, x_1, \ldots, x_{n-1})$ of the vertices of K_n and a permutation π of the vertices of K_n defined by $\pi(x_j) = x_{j+1}$ for $j = 0, 1, \ldots, n-1$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo n.
Graph decomposition has been an extensively studied topic in graph theory for decades. Most of this attention has been directed towards finding isomorphic decompositions of complete graphs. The most popular method for finding such decompositions was introduced by A. Rosa in [3] and further developed by S. I. El-Zanati, C. Vanden Eynden, and N. Punnim in [1], and J. Fahnenstiel and D. Froncek in [2].
Recently, a progress on graph decomposition into unicyclic, connected graphs with nine edges, was made by several UMD students D. Baker, C. Schwieder, and G. Aspenson [4].

In this paper we will be considering a similar class of unicyclic graphs with $n = 9$ edges but disconnected and such that the cycle is C_4 and there are exactly two trees attached to the cycle at non-adjacent vertices. We prove that every such graph G decomposes the complete graphs K_{2nk} and K_{2nk+1} for every non-negative integer k.

2 Definitions and tools

In [3], Rosa introduced different graph labelings (valuations as he called them), as a tool to decompose complete graphs.

Definition 2.1 (ρ-labeling)**.** Let G be a graph with n edges. A ρ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, \dots, 2n\}$ such that

$$\{\min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\} : \{u, v\} \in E(G)\} = \{1, 2, \dots, n\}.$$

Rosa, also in [3], defined a more restrictive version of ρ-labeling, the σ-labelings.

Definition 2.2 (σ-labeling)**.** A ρ-labeling f of a graph G with n edges is a σ-labeling if

$$\{|f(u) - f(v)| : \{u, v\} \in E(G)\} = \{1, 2, \dots, n\}.$$

Then, El-Zanati, Vanden Eynden, and Punnim [1] introduced ordered labelings and showed that they can be used as a tool to decompose complete graphs K_{2nk} and K_{2nk+1} into bipartite graphs for every non-negative integer k. Notice that our graph G is, in fact, bipartite.

Definition 2.3 (Ordered labelings)**.** A σ- or ρ-labeling f of a bipartite graph G with n edges and bipartition (A, B) is *ordered* if $f(a) < f(b)$ for each edge $\{a, b\}$ with $a \in A$ and $b \in B$. Ordered σ- and ρ-labelings are called σ^+- and ρ^+-labelings, respectively.

We use the following theorem to find a G-design of K_{2nk+1}.

Theorem 2.4 (El-Zanati, Vanden Eynden, Punnim [1])**.** *Let G be a bipartite graph with n edges. If G admits a ρ^+-labeling, then there exists a cyclic G-design of K_{2nk+1}, for all non-negative integers k.*

Another theorem we use in this paper was proved by Fahnenstiel and Froncek [2]. The result shows that a G-design of the complete graph K_{2nk} can be found if G admits σ^+-labeling.

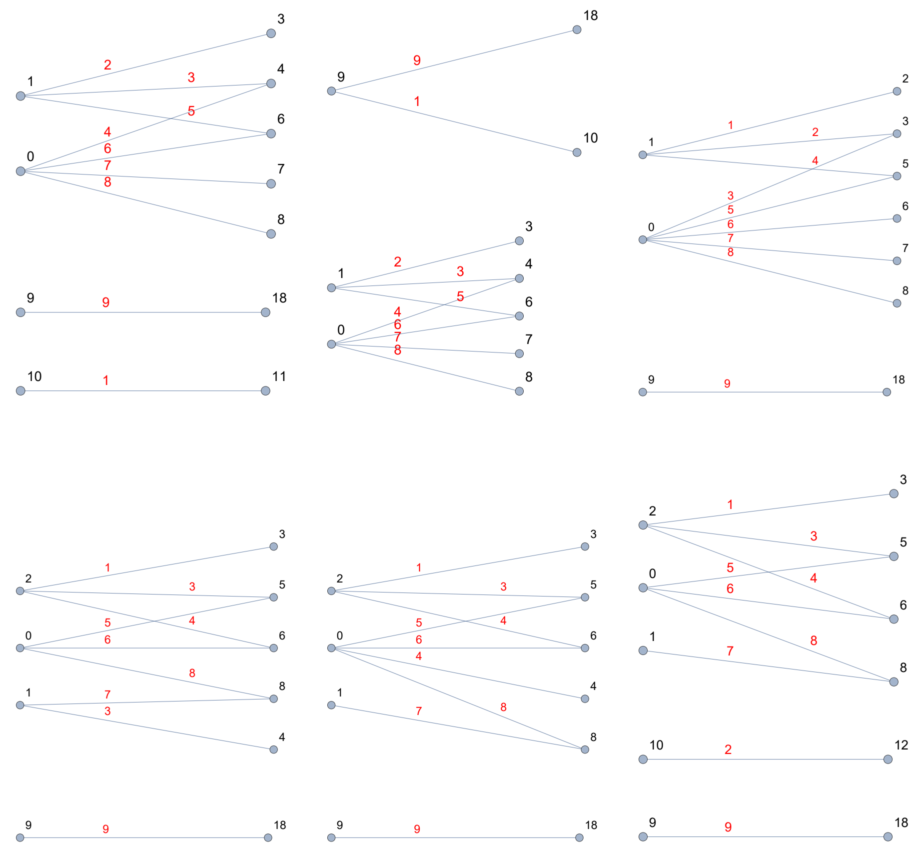
Theorem 2.5 (Fahnenstiel, Froncek [2])**.** *Let G be a graph with a σ^+-labeling on n edges such that the edge of length n is a pendant edge. Then there exists a G-decomposition of K_{2nk} for any positive integer k.*

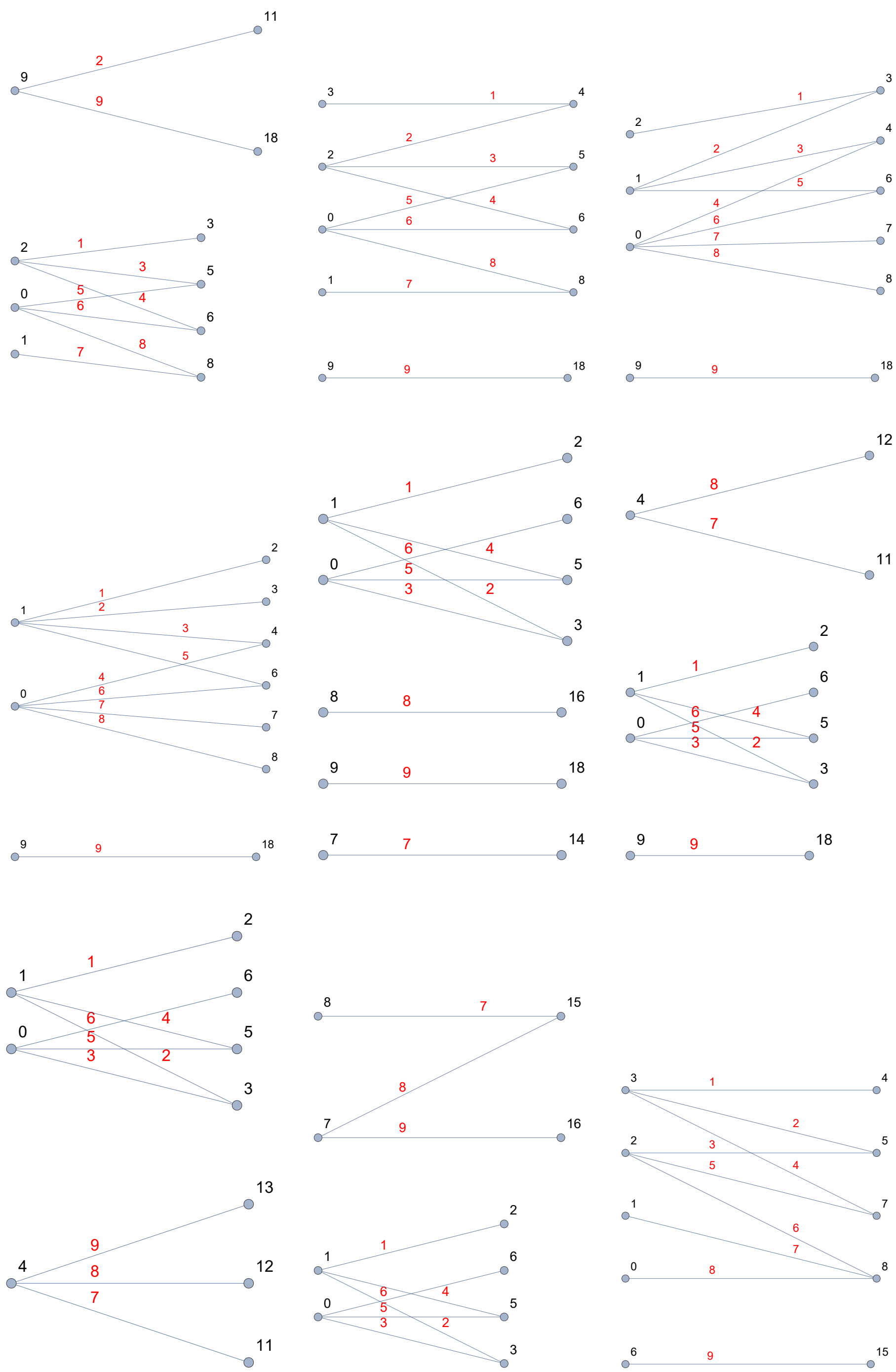
3 Results

In order to show that the graphs in the aforementioned class decompose the graphs K_{2nk} and K_{2nk+1}, we found σ^+-labelings for them. Then, by Definition 2.2 and Theorem 2.4, we see that these graphs decompose K_{2nk+1}. The labelings can be converted to 1-rotational σ^+-labelings by relabeling the pendant vertices incident to edges of length 9 with ∞, thus showing that these graphs decompose K_{2nk} by Theorem 2.5. We summarize our findings in the following theorem.

Theorem 3.1. *Let G be a disconnected unicyclic graph with $n = 9$ edges such that the only cycle in G has exactly four edges and there are exactly two trees attached to the cycle at non-adjacent vertices. Then G decomposes complete graphs K_{2nk} and K_{2nk+1} for every non-negative integer k.*

The figures that follow show the σ^+-labelings for the 15 graphs in this class.





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