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Tigran Arsenyan

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Abstract: Use the most precise, shortest sentences that state what problem the thesis addresses, how it is approached, pinpoint the exact result achieved, and describe the applications and significance of the results. Highlight anything novel that was discovered or improved by the thesis. Maximum length is 200 words, but try to fit into 120. Abstracts are often used for deciding if a reviewer will be suitable for the thesis; a well-written abstract thus increases the probability of getting a reviewer who will like the thesis.

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Contents

Introduction

1 Kempe chains and Routing graphs

The following proofs are reformulations of those presented in [1].

Definition 1 (H-certificate)**.** *Graph H is a minor of G if there exists $c := (V_t)_{t \in V(H)}$ of pairwise disjoint $V(G)$, called bags, such that $\forall t \in V(H)\ V_t \neq \emptyset$ and $G[V_t]$ is connected, and $\forall (u, v) \in E(H)$, $\exists$ edge connecting V_u and V_v, any such c is called H-certificate*

Definition 2 (rooted H-certificate)**.** *H-certificate is a rooted if $V(H) \subset V(G)$ and $t \in V_t\ \forall t \in V(H)$. If there is a rooted H-certificate in graph G, then H is a rooted minor of G*

Definition 3 (Routing Graph)**.** *Let $\mathcal{C}$ be a coloring of graph G, let T be the transversal of coloring $\mathcal{C}$, then routing graph $H(G, \mathcal{C}, T)$ is defined as the graph on vertices of T, such that $\forall (i, j) \in V(H)(i \neq j)$, $(i, j) \in E(H)$ if and only if $\exists$ Kempe chain between vertices i and j in graph G*

Definition 4 (Property (*))**.** *All graphs H which are routing graph of some G with some coloring $\mathcal{C}$ and transversal T such that G has a rooted H-certificate, are said to have property (*)*

Theorem 1. *K has property (*) if and only if every component of K has it.*

Proof. **(⇒)** If K has property (*) then all components of K have property (*)

Let K be a graph with property (*) and K' is a component of K, let's do case analysis, we have 2 cases:
Case 1: $|V(K)| = |V(K')|$
Case 2: $|V(K)| > |V(K')|$

Case 1: The component K' of K is a spanning subgraph of K, which is same as $|V(K)| = |V(K')|$.
Let K have the property (*), take spanning subgraph K' of K. Now take graph G' with coloring $\mathcal{C}$ such that $|\mathcal{C}| = |V(K')|$ and a transversal T such that K' is isomorphic to the spanning subgraph of routing graph $H(G', \mathcal{C}, T)$. Now, $\forall (u, v) \in E(K) \setminus E(K')$ add (u, v) edge to graph G', we can do so without breaking the coloring because the edge taken from $E(K) \setminus E(K')$ is only between vertices which have different colors, now we obtain graph G, then K is isomorphic to spanning subgraph of $H(G, \mathcal{C}, T)$, since K has property (*) then there is a rooted H-certificate c in G, and c is also and H'-certificate for G'.

Case 2: $|V(K)| > |V(K')|$
Take graph G' with coloring $\mathcal{C}'$ such that $|\mathcal{C}'| = |V(K')|$ and a transversal T' such that K' is isomorphic to the spanning subgraph of routing graph $H(G', \mathcal{C}', T')$, take set $S := V(K) \setminus V(K')$ and construct graph G as disjoint union of G and K_S(Here K_S is complete graph on vertex set S), let coloring

for G be $\mathcal{C} := \mathcal{C}' \cup \{\{s\}|s \in S\}$ and $T := T' \cup S$, now by construction K is isomorphic to the spanning subgraph H of routing graph $H(G, \mathcal{C}, T)$, and since K has property (*) it also has rooted H-certificate in G let's denote it as c and by definition of rooted H-certificate it's defined as $c := (V_t)_{t \in V(K)}$, then let $c' := (V_t)_{t \in V(K')}$ is a rooted H'-certificate in G'.

($\Leftarrow$) If every component of K has property (*), then K also has property (*). Take graph G with coloring $\mathcal{C}$ such that $|\mathcal{C}| = |V(K)|$ and a transversal T such that K is isomorphic to the spanning subgraph of routing graph $H(G, \mathcal{C}, T)$, then for every component K_i of K there is G_i a subgraph of G(all G_i's are disjoint), coloring $\mathcal{C}_i$ and T_i such that K_i is a spanning subgraph of $H(G_i, \mathcal{C}_i, T_i)$. Since every K_i has property (*), then there is c_i-certificate in G_i, hence by the union of all those (c_1, c_2, ... c_n) certificates, we get a rooted K-certificate in G, hence K has property (*) $\square$

Theorem 2. *If K has property (*), then all subgraphs of K have property (*)*

Proof. Let K have the property (*), and K' be subgraph of K, let L be the edgeless graph on vertices $V(K) \setminus V(K')$, $L \cup K'$ is a spanning subgraph of K, hence it has property (*) (Shown in forward direction of the proof of Theorem 1), since K' is a component of $K' \cup L$, by Theorem 1 it also has property (*) $\square$

Lemma 3. *Let K be a graph, if $\exists q \in V(K)$ such that $\deg(q) = 1$ and $K - q$ has property (*), then K has property (*)*

Proof. Let K be a graph such that $\exists q \in V(K)$ such that $\deg(q) = 1$ and $K - q$ has property (*), but let's assume for contradiction that K doesn't have property (*). This means there exists graph G (with minimal $V(G) + E(G)$) with coloring $\mathcal{C}$ such that $|\mathcal{C}| = |V(K)|$ and a transversal T such that K is isomorphic to the spanning subgraph of routing graph $H(G, \mathcal{C}, T)$, but there is no rooted H-certificate in G. Since G is minimal it means $\forall A, B \in \mathcal{C} (A \neq B)$ $G[A \cup B]$ has at most one component which is not a single vertex, which means if $\exists (u, v) \in E(H)$ and $u \in A \cap T$, $v \in B \cap T$ then there is a 2-colored path from u to v in $G[A \cup B]$, on the other hand if there is no edge (u, v) in $E(H)$ which such property then $G[A \cup B] = \emptyset$, this induced that $H = H(G, \mathcal{C}, T)$. Let r be the incident vertex of q, let $Q, R \in \mathcal{C}$ be the respective color classes of r and q, so $r \in R$, $q \in Q$, $R \neq Q$. Here we have 2 cases:

Case 1: $Q = \{q\}$

Since $K - q$ has property (*), it means $K - q$ it means $G - q$ has rooted $H(G - q, \mathcal{C} \setminus Q, T - q)$-certificate, hence by adding $Q = \{q\}$ bag to it, we would get rooted H-certificate for G(Contradiction)

Case 2: $\exists x \in Q \setminus \{q\}$

Then because of minimality of G and the construction of it having 2 colored paths it has degree of 2, and it's in the 2-colored path between r and q, hence it has 2 neighbors which are from R color class, let's denote them y and z, let's contract yxz to w and give color R to w, and we would obtain

graph G' with following coloring and transversal defined as follows:
For $A \in C$, define A' as follows:

$$A' := \begin{cases} (A \setminus \{y, z\}) \cup \{w\} & \text{if } A = R, \\ A \setminus \{x\} & \text{if } A = Q, \\ A & \text{otherwise.} \end{cases}$$

For $z \in T$, define z' as follows:

$$z' := \begin{cases} w & \text{if } z \in \{y, z\}, \\ z & \text{otherwise.} \end{cases}$$

For T' we don't consider cases concerning x because it already had representative from color class Q in it (q), so removal of x doesn't affect T'.

Then $C' := \{A' : A \in C\}$ is a coloring of G' and $T' := \{t' : t \in T\}$ is a transversal of C'.
Now, we show that $H = H(G, \mathcal{C}, T)$ is isomorphic to $H(G', \mathcal{C}', T')$. Let's consider all $(s, t) \in E(H)$:

- $\{s, t\} \neq \{q, r\}$ Then yxz don't lay on any path from s to t, hence any s, t-path from G is a s', t'-path in G'
- $s \in T \setminus \{q, r\}$ and $t = r$ and $r \in \{y, z\}$:
  If $\{y, z\} \not\subseteq V(P_{s,r})$, then the s, r-path is s', r'-path, otherwise if $\{y, z\} \subseteq V(P_{s,r})$, we can obtain new s', r'-path from s, r-path by replacing the subpath between y and z by w.
- $s \in T \setminus \{q, r\}$ and $t = r$ and $r \notin \{y, z\}$:
  If $\{y, z\} \not\subseteq V(P_{s,r})$, then s, r-path is s', r'-path in G', otherwise we replace the subpath between y and z with w, and obtain new s', r'-path in G'

And yxz lies on $P_{q,r}$, then replacing yxz by w results a new q', r'-path. By considering all cases, we showed that H is isomorphic to $H(G', \mathcal{C}', T')$. Since by choice of G and G', G' has rooted $H(G', \mathcal{C}', T')$ certificate, if w is in some bag B, by replacing B with B $\{w\} \cup \{x, y, z\}$, we would obtain rooted H-certificate for G (Contradiction).

Since for both cases we got contradiction, this implies that K indeed has (*) property as well. □

1.1 Kempe chains and rooted K7-minors

To find the graphs which don't have property (*), we need to construct a graph G such that it has all necessary paths between transversal vertices, to construct a routing graph, but not enough edges incident to transversal vertices so that it's impossible to have rooted minor as the routing graph.

Definition 5 (Z(G))**.** *For a graph G $Z(G)$ is defined as follows:*

1. $V(Z(G)) := V(G) \times \{1, 2\}$
   For example if $V(G) := \{a, b\}$, *then* $V(Z(G)) := \{\{a, 1\}, \{b, 1\}, \{a, 2\}, \{b, 2\}\}$, *in other words we are duplicating the vertices of* G *into* $Z(G)$

2. $E(Z(G)) := \{(x, i)(y, j) : xy \in E(G) \wedge (i \neq 1 \vee j \neq 1)\}$
   Here we keep all the edges from original graph G *in the component of* $Z(G)$ *which have label* 2*, We remove all the edges between the vertices which have label* 1*, which induces an anticlique between those vertices. Note:* $\forall xy \in E(G)$*, we have the following edges in* $Z(E(G))$*:* $\{\{(x, 2), (y, 2)\}, \{(x, 1), (y, 2)\}, \{(x, 2), (y, 1)\}\}$

Let coloring for G be $\mathcal{C} := \{(x, 1)(x, 2) \ \in V(Z(G))\}$ is a coloring of $Z(G)$ and $T := V(G) \times \{1\}$ Is transversal of the coloring.

Example. An example of $Z(G)$ given G is cycle of 7

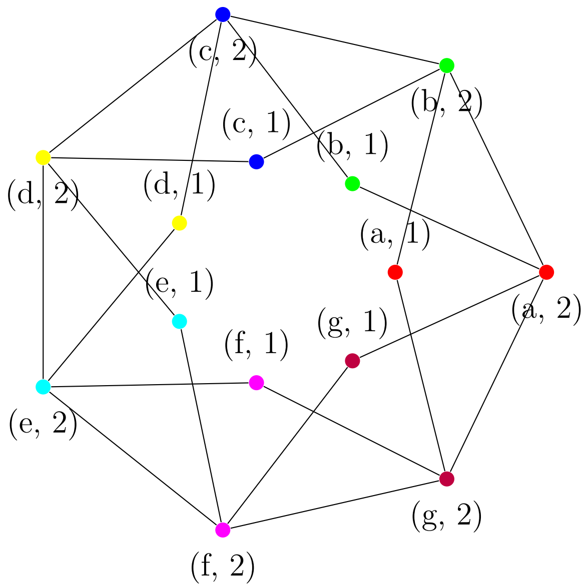

Figure 1.1 Example of $Z(G)$ given G is C_7

We can observe that routing graph $H := (Z(G), \mathcal{C}, T)$ is isomorphic to G via $((x, 1) \to x)$, and we also have copy of G in induced subgraph of $Z[V(G) \times \{2\}]$. Now we study if there exists rooted H-certificate in $Z(G)$ for different graphs of G.

Claim 4. *The bags of any H-certificate $c = (V_t)_{t \in T}$ in $Z(G)$ have average order at most 2.*

Proof. $|T| = |V(H)| = |V(G)|$ and $|V(Z(G))| = 2|V(G)|$, all V_t's are pairwise disjoint, hence the average size of any bag is:

$$\frac{1}{|T|} \sum_{t \in T} |V_t| \leq \frac{|V(Z(G))|}{|T|} = \frac{2|V(G)|}{|V(G)|} = 2 \tag{1.1}$$

□

This means if we have a bag with an order 3, then there is also a bag with order 1. And locally the inverse implications sounds almost the same.

Claim 5. *If $st \in E(H)$ is not on any triangle of H, then $|V_s| = 1 \implies |V_t| \geq 3$*

Proof. Let $st \in E(H)$, and suppose $V_s = \{s\}$, hence $|V_s| = 1$, $|V_t| \geq 2$ because s, t are not adjacent in $Z(G)$. If $|V_t| = 2$, then for $u \in V(Z(G))$, $V_t = \{t, u\}$, this means there is an edge $su \in E(Z(G))$ as well, at the same time the corresponding u' of u in $V(H)$ should be adjacent with t, but since s, t are not in a triangle of H, there is no edge between s and u', which means there is no su edge as well, which is a contradiction. Hence $|V_t| \geq 3$ □

If all the bags of the certificate have order 2, then we can look at a function $f : V(G) \to V(G)$, which for a bag $V_{(x,1)} = \{(x, 1), (y, 2)\}$ is defined as $f(x) := y$. Since the bags are disjoint, f is an injection and therefore a permutation of $V(G)$. Since all the elements of each bag are connected we can observe that $xf(x) \in E(G)$, and we can represent f as a partial orientation of G, where xy is oriented from x to y if and only if $y = f(x)$. For a rooted H-certificate c in $Z(G)$ any $xy \in E(G)$ implies that $V_{(x,1)}$ and $V_{(y,1)}$ are adjacent, which is equivalent to say $f(y)$ is adjacent to $f(x)$ or x, or $f(x)$ is adjacent to $f(y)$ or y. Conversely, if f is a permutation of $V(G)$ with the following properties:

1. $(\forall x \in V(G))(xf(x) \in E(G))$
2. $xy \in G$ implies that $f(x)$ is adjacent to either y or $f(y)$, or $f(y)$ is adjacent to either x or $f(x)$

Then $V_{(x,1)} = \{(x, 1), (f(x), 2)\}$ defines an H-vertificate in $Z(G)$. Let's call such permutation as a 'good permutation' throughout this chapter

Claim 6. *If G has a good permutation, then every vertex of degree at least 3 in G is on a cycle of length at most 4 in G*

Proof. Let f be good permutation and w be a vertex of degree 3, let x, y, z be w's neighbors, WLOG $f(w) = x$ and $f(y) \neq w$. Let $u := f(y) \neq w$, if u is adjacent to w, then wyu form a triangle, and we are done.
Otherwise, let's assume u, w are not adjacent, by (**2**) condition of a 'good' permutation, $f(w) = x$ is adjacent to either y or $u = f(y)$, or $f(y) = u$ is adjacent to $f(w) = x$ or w, in any case w will be either on 4-cycle or 3-cycle. □

Theorem 7. *K_7 doesn't have property (*)*

Proof. Let graph G be a graph on 7 vertices, which is obtained by adding vertex x to C_6 and adding 2 edges to x such that the endpoints of the edges are at distance 3 from each other in C_6.

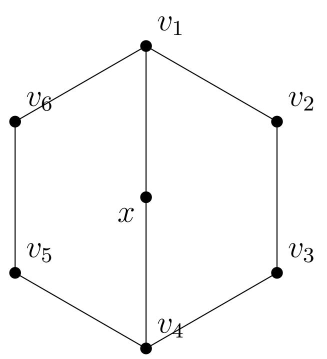

Figure 1.2 Graph G obtained from C_6

For contradiction assume that $Z(G)$ has an H-certificate $(V_t)_{t\in T}$ with $T = V(G) \times \{1\}$. Let A be the set of vertices $t \in T$ such that $|V_t| = 1$. Observe that there are 2 vertices of degree 3 in G, v_1 and v_4, and both of them are in cycle of 5, and hence by **Claim 6** G doesn't have a good permutation, which means $|A| \geq 1$. Since $|V_t| = 1$ for every element of A, it means A is an anticlique in H, hence $|A| \leq 3$. By **Claim 5**, $|V_s| \geq 3$ for every s in $N_H(A)$. For each case of $|A| = 1$, $|A| = 2$, $|A| = 3$, it can be seen that $N_H(A) \geq |A| + 1$.

- $3(|A| + 1)$ is the lower bound of the number of vertices in the bags of the neighborhood of A, because each bag has size ≥ 3 and there are at least $|A| + 1$ neighbors for A
- $1|A|$ is the number of vertices of the bags of A, because by definition each of A has size 1
- $2(7 - (|A| + 1) - |A|)$ is the number of vertices in the rest of the bags. Which all have $|V_t| = 2$

Let's denote q the number of vertices in the bags of $Z(G)$ which form H. And let's count it.

$$q = \sum_{t\in T} |V_t| \geq 3(|A| + 1) + 2(7 - (|A| + 1) - |A|) + 1|A| = 15 \tag{1.2}$$

At the same time, $q \leq |V(Z(G))| = 14$, causing a contradiction. Hence, the graph G doesn't have property (*), and since all the subgraphs of a graph with property (*) have the proprety as well, this implies that K_7 doesn't have property (*). $\square$

2 Connected transversals of 5-colorings

Kriesell and Mohr [1] showed that all graphs on five vertices and, at most, six edges have (*) properties. We will show properties of a minimal counter-example for any graph on five vertices.

Let's denote the rooted minor graph as H, Let G be the minimal graph such that graph H is isomorphic to $H(G, \mathcal{C}, T)$, where transversal $T := \{1, 2, 3, 4, 5\}$, and $\mathcal{C}$ is the coloring of G. The minimality implies that $(\forall A, B \in \mathcal{C} : A \neq B)$ $G[A \cup B]$ is a single path between the transversal vertices of corresponding colors from T.

Lemma 8. *A minimal counter-example such that any graph on five vertices and at least seven edges does not have the property (*) is 2-connected.*

Proof. Assume that G is 1-connected for contradiction. Then, a cut vertex exists in G; let us denote it as x. We have six cases to consider. In the following figure, the black vertices are the transversal vertices T on which we cannot build a rooted minor isomorphic to H, and if a vertex is with a hole then it is a non-transversal vertex.

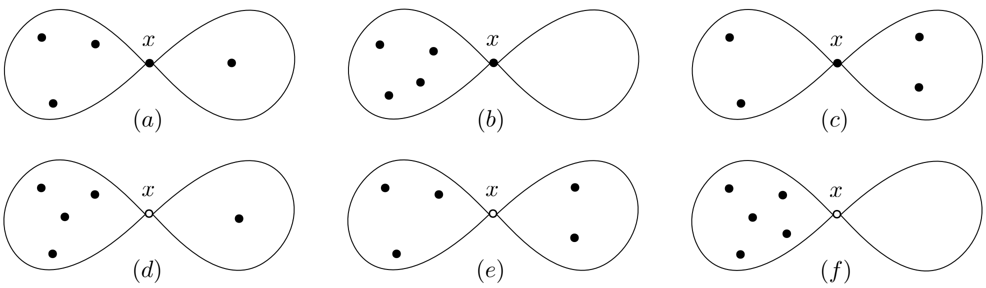

Figure 2.1 All cases of cut vertices in a 1-connected graph

Notice that for the cases (a), (b), and (c), the cut vertex x is a transversal vertex.

H does not have a vertex of degree one, otherwise By theorem 4 in [1] we have that K_4 has property (*) and $K_4 + 1$ vertex with degree one also has property (*) (By lemma 1 in [1]), and since property (*) is inhereted to the subgraphs of the graph (By theorem 1 in [1]), then H would have property (*), which is a contradiction.

We have that all vertices in H have degrees at least two, and if all of them are exactly two, then H is a cycle, and By theorem 4 in [1] we have that all cycles have property (*), hence H would have property (*), which is a contradiction.

So we have that H has at least one vertex of degree at least three, if it's exactly three and the rest are two, then by handshake lemma we have at least two vertices of degree three, and if its' degree is four and the rest of vertices have degree two, then we get that the graph H is the hourglass graph, which by Theorem 7 in [1] has property (*), hence H would have property (*), which is a contradiction. So we have that at least two vertices in H have degree at least three and the rest of vertices have degree at least two.

Let R be the subgraph of the right side of the cut and L be the subgraph of the left side of the cut. For case (a), let us denote the transversal vertex in R as y, and since all vertices in H have degrees at least two, then we have to have at least two Kempe chains between y and two other transversal vertices in G. All paths from y to other transversal vertices may only pass through x because except x, all other transversals are in L; since x is a transversal vertex itself, we can have a kempe chain from y to x, but since y and x have diferent colors, we cannot have more Kempe chains going from y through x to other transversal vertices. Hence, we have only one Kempe chain from y to other transversal vertices, bringing us to a contradiction. For case (b), unlike case (a), we can realize all necessary Kempe chains. If no Kempe chain between transversals has edges in R, we can contract all vertices of R into x and preserve all Kempe chains between transversals; hence, G was not a minimal graph, a contradiction. So let us assume there is some Kempe chain which has a subpath in R; since all Kempe chains start from L, such a Kempe chain has to pass through x; this means that if G is a counter-example; after contracting all edges of the R into x, we still have left Kempe chains between all necessary transversals in L, but we got a smaller graph which still should be a counter-example; if the contracted version is not a counter-example, then it would mean G was not a counter-example in the beginning since we could get a rooted minor H from G. Hence, the graph G is not minimal, which is a contradiction.

For case (c), if both vertices with degree at least three are in R, let us denote them as y, z, then we can only have kempe chanins from y to x and z, and from z to x and y, but $deg(y) >= 3, deg(z) >= 3$ in H, which is a contradiction since they can have at most two Kempe chains in G in this case. Then assume only one vertex of degree at least three is in R, let it be y, similarly we get that y can have at most two Kempe chains, which is a contradiction. Symmetrically same holds if x and y were in L or one of them was in L, hence in all cases we get a contradiction. For case (d), the vertex in R has a degree of at least two. Hence, the color of x is the color of the vertex itself, so we can contract the paths from it to x and still get the counter-example, which is a contradiction since the graph G was not minimal. For case (e) let vertices in L be denoted as a, b, c and the vertices in R as y, z, if we have atleast one vertex with degree at least three in R, let it be y, then we can have one Kempe chain between y and z, and at least two Kempe chains from y to L, hence x has the color of y, but now since z has degree at least two, at least one Kempe chain from z should go to L and it can pass only through x, since the color of x is same color of y, the Kempe chain from z even if includes x it can only be connected to y, hence it can't pass to L, which is a contradiction. So we have that both vertices with degree at least three are in L, in L we have three vertices, and if all of them are maximally connected to each other, we get the abc cycle, and since two vertices in L have degree at least three, at least two kempe chains should pass from L to R, both of them pass through x, so both of them should be connected to the same transveral vertex in R, WLOG it's y, but then z in R can only be connected to y, if it has Kempe chain to any other transversal than y, it has to pass through x but x has color of y, so the only Kempe chain from z including x can go to y which is a contradiction since in H $deg(z) >= 2$, but we only can have one Kempe chain in G from z, For case (f), we could contract all the vertices from R into x and still get the counter-example because all the Kempe chains still exist between the transversals, it is a contradiction to the minimality of G. Thus, G is 2-connected. $\square$

Conclusion

Bibliography

1. Matthias Kriesell, Samuel Mohr. Kempe Chains and Rooted Minors. 2022.

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A Attachments

A.1 First Attachment