FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

**BACHELOR THESIS**

Tigran Arsenyan

# Thesis title

Computer Science Institute of Charles University

Supervisor of the bachelor thesis: Supername Supersurname
Study programme: study programme

Prague 2025

I declare that I carried out this bachelor thesis on my own, and only with the cited sources, literature and other professional sources. I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ............. date .............                    .....................................

Author's signature

Dedication.

Title: Thesis title

Author: Tigran Arsenyan

Department: Computer Science Institute of Charles University

Supervisor: Supername Supersurname, department

Abstract: Use the most precise, shortest sentences that state what problem the thesis addresses, how it is approached, pinpoint the exact result achieved, and describe the applications and significance of the results. Highlight anything novel that was discovered or improved by the thesis. Maximum length is 200 words, but try to fit into 120. Abstracts are often used for deciding if a reviewer will be suitable for the thesis; a well-written abstract thus increases the probability of getting a reviewer who will like the thesis.

Keywords: keyword, key phrase

Název práce: Název práce česky

Autor: Tigran Arsenyan

Katedra: Název katedry česky

Vedoucí bakalářské práce: Supername Supersurname, katedra vedoucího

Abstrakt: Abstrakt práce přeložte také do češtiny.

Klíčová slova: klíčová slova, klíčové fráze

# Contents

# Introduction

The four-color theorem, that every planar graph is four-colorable, was one of the central problems in graph theory for over a century. Appel and Haken proved it in 1976 [1] using a computer program. The proof was controversial because it was the first major theorem to be proved using a computer. The theorem was later proved in 1997 by Robertson, Sanders, Seymour, and Thomas [2], still using a computer, but with more straightforward configurations than Appel and Haken's in several aspects.

Hadwiger's conjecture [3] suggests a generalization of the four-color theorem. It is considered one of the most challenging problems in graph theory. The conjecture is the following:

**Conjecture 1** (Hadwiger 1943 [3] (Strong version))**.** *For every integer $k \geq 0$, every graph $G$ with no $K_{k+1}$ minor can be colored with $k$ colors.*

It was proved by Hadwiger himself for cases $k \leq 3$. The case $k = 4$ is implied by the four-color theorem, since by Wagner's theorem [4] a graph is planar if and only if it has no $K_5$ or $K_{3,3}$ minor. Robertson, Seymour, and Thomas proved the case $k = 5$ in 1993 [5], where they did not use a computer to prove it. However, they proved that a minimal counter-example to the case $k = 5$ should have a vertex whose removal results in a planar graph, reducing the problem to the four-color theorem. $k = 6$ is still open, and there are some result in this direction is that of Albar and Gonçalves [6].

**Theorem 1.** *Every graph with no $K_7$ minor is 8-colorable, and every graph with no $K_8$ minor is 10-colorable.*

And Kawarabayashi and Toft [7] proved that every 7-coloarable graph has $K_7$ or $K_{4,4}$ as minor.

Bollobás, Catlin, and Erdös [8] showed that the conjecture is true for almost all graphs using probabilistic arguments. In general, the cases $k \geq 6$ remain open.

Because of the current state of the conjecture, there is an interest in a weaker version of the conjecture.

**Conjecture 2.** *(Hadwiger's Conjecture weak version) There exists a constant $c$ such that, for every $k \geq 0$, every graph with no $K_{k+1}$ minor can be colored with $ck$ colors.*

The best-known result for the weak version is that for every graph with no $K_{t+1}$ minor, there exists a constant $c$ such that the graph is $ck\sqrt{\log k}$-colorable. This was proved forty years ago by Kostochka [9] and Thomason [10] independently. Moreover, there has been no significant improvement since then.

Since the conjecture is hard to prove in general, it is interesting to study the conjecture for specific classes of graphs. Hadwiger suggested looking into the graphs with a bounded number of optimal colorings [3], one particular class is the uniquely optimally colorable graphs.

**Claim 2.** *Let $x_1, x_2, ..., x_k$ be differently colored vertices of a uniquely $k-$colorable graph $G$. Then there is system of edge-disjoint $x_i, x_j$ paths ($\forall i, j \in [n], i \neq j$), by taking only the edges between the corresponding two color classes.*

*Proof.* For any given $i, j \in [n], i \neq j$, the graph $G[V_i \cup V_j]$ is a connected component, where $V_i$ and $V_j$ are the color classes of $x_i$ and $x_j$ respectively, if this wasn't the case then, we could color one of the disconnected components by swapping the colors $i$ and $j$, resulting to another coloring. However, $G$ is uniquely k-colorable, hence a contradiction. $\square$

The question is whether a $K_k$ minor exists rooted at $x_1, ..., x_k$. Kriesell showed that the answer is positive for cases $k \leq 10$ [11], if the graph is antitriangle-free [12], Moreover, with Mohr, they showed it is true for line graphs [11].

Taking a vertex set of the graph containing all colors seems a natural approach to look for a $K_t$ minor, since it contains all the colors of the graph, it's a place which is 'hard to color', hence it's a good place to hope for finding a minor. And this is exactly the Halroyd's conjecture.

First let's define what is a colorful set.

**Definition 1.** *For graph $G$, A set of vertices $S \subseteq V(G)$ is called colorful if for every proper coloring of $G$, $S$ contains atleast one vertex for each color of the coloring.*

**Conjecture 3.** *(Halroyd's conjecture) Let $G$ be a graph colored with $k$ colors, let $S$ be a colorful set in $G$, then $G$ has an $S$-rooted $K_k$ minor.*

Holroyd called it the Strong Hadwiger Conjecture since if we take $S = V(G)$, it implies Hadwiger's conjecture.

Holroyd himself has proven the conjecture for cases $k \leq 3$ [13], and recently it has been proved for the case $k = 4$ by Martinsson and Steiner [14].

Usually, in the context of Hadwiger's conjecture, only clique minors were considered; however, Kriesell and Mohr [15] considered the following question, which does not necessarily look for clique minors. Let $G$ be a graph with a proper coloring $\mathcal{C}$, let $k = |\mathcal{C}|$, and let $x_1, ..., x_k$ be a vertex set of $G$ with different colors. Then there is a system of esdge-disjoint $(x_i, x_j)$-paths for some pairs $(x_i, x_j)$. They examined weather there is a rooted minor $H$ of $G$ on vertices $x_1, ..., x_k$, where $H$ has edges between $x_i, x_j$ if and only if there is an edge-disjoint path between $x_i$ and $x_j$ in $G$.

The answer to this question is affirmative for the cases of $k \leq 4$; for $k = 5$, it is true for graphs with at most six edges, but in general open, and for $k = 6$, it's open, and it is negative for $k \geq 7$.

This paper will characterize minimal counterexample graphs for $k = 5$ such that the answer is negative. We also made a computer enumeration for the case $k = 6$, checking all the graphs with at most sixteen vertices, and we found that the answer was positive for all of them.

# 1 Prelimenaries

**Definition 2** (H-certificate)**.** *Graph $H$ is a minor of $G$ if there exists $c := (V_t)_{t\in V(H)}$ of pairwise disjoint $V(G)$, called bags, such that $\forall t \in V(H)$ $V_t \neq \emptyset$ and $G[V_t]$ is connected, and $\forall (u, v) \in E(H)$, $\exists$ edge connecting $V_u$ and $V_v$, any such $c$ is called $H$-certificate*

**Definition 3** (rooted H-certificate)**.** *$H$-certificate is a rooted if $V(H) \subset V(G)$ and $t \in V_t$ $\forall t \in V(H)$. If there is a rooted H-certificate in graph $G$, then $H$ is a rooted minor of $G$*

**Definition 4** (Routing Graph)**.** *Let $\mathcal{C}$ be a coloring of graph $G$, let $T$ be the transversal of coloring $\mathcal{C}$, then routing graph $H(G, \mathcal{C}, T)$ is defined as the graph on vertices of $T$, such that $\forall(i, j) \in V(H)(i \neq j)$, $(i, j) \in E(H)$ if and only if $\exists$ Kempe chain between vertices $i$ and $j$ in graph $G$*

**Definition 5** (Property (*))**.** *All graphs $H$ which are routing graph of some $G$ with some coloring $\mathcal{C}$ and transversal $T$ such that $G$ has a rooted $H$-certificate, are said to have property (*)*

**Theorem 3.** *Property (*) inherits to subgraphs of $K$ and $K$ has property (*) if and only if every component of $K$ has it.*

*Proof.* Proved in [15] Theorem 1 □

**Theorem 4.** *Let $K$ be a graph and $q \in V(K)$ and $deg(1) = 1$. If $K - q$ has property (*), then $K$ has property (*).*

*Proof.* Proved in [15] Lemma 1 □

**Theorem 5.** *$K_7$ does not have property (*)*

*Proof.* Proved in [15] Theorem 2

Since by Theorem 3 property (*) inherits to subgraphs of $K_7$, they found a subgraph of $K_7$, $H$, and a graph $G$ with a coloring $\mathcal{C}$ and a transversal $T$ such that $H$ is isomorphic to $H(G, \mathcal{C}, T)$, but $H$ is not a rooted minor of $K_7$.

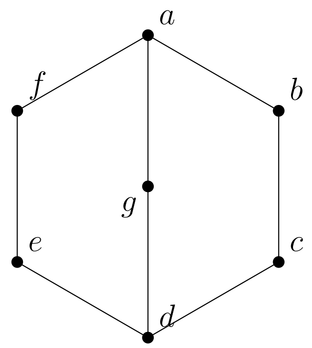


**Figure 1.1** The subgraph of $K_7$ without (*) property

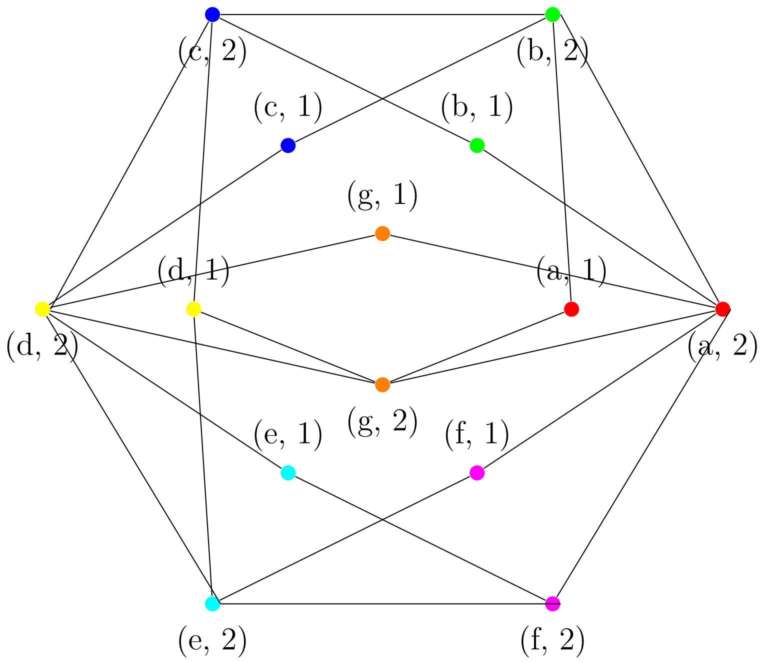


**Figure 1.2** Example of $Z(G)$ given $G$ is $C_7$ with additional connections to $(g, 1)$ and $(g, 2)$.

In Figure 1.2, the graph $G$ is shown with coloring $\mathcal{C}$ and transversal $T := \{(a, 1), (b, 1), (c, 1), (d, 1), (e, 1), (f, 1), (g, 1)\}$. □

**Theorem 6.** *Every graph on at most four vertices has property (*).*

*Proof.* Proved in [15] Theorem 4 □

**Theorem 7.** *Every connected graph with at most one cycle has property (*)*

*Proof.* Proved in [15] Theorem 5 □

**Theorem 8.** *Every graph on five vertices and at most six edges has property (*)*

*Proof.* Proved in [15] Theorem 7 □

# 2 Connected transversals of 5-colorings

By Theorem 8, all graphs on five vertices and, at most, six edges have (*) properties. We will show the properties that a counter-example must satisfy for a graph with five vertices and at least seven edges to fail to have property (*).

**Lemma 9.** *Let $H$ be a graph on five vertices, Let $G$ be a graph with coloring $\mathcal{C}$ and transversal $T$ such that $H$ is isomorphic to $H(G, \mathcal{C}, T)$; if $H$ is not a rooted minor of $G$, then $G$ is 2-connected.*

*Proof.* Let $G$ be the minimal such graph, the minimality of $G$ implies that $(\forall A, B \in \mathcal{C} : A \neq B)$ $G[A \cup B]$ is a single path between the transversal vertices of corresponding colors from $T$.

Assume that $G$ is 1-connected for contradiction. Then, a cut vertex exists in $G$; let us denote it as $x$.

We have six cases to consider. In the following figure, the black vertices are the transversal vertices $T$ on which we cannot build a rooted minor isomorphic to $H$, and if a vertex is with a hole, then it is a non-transversal vertex.

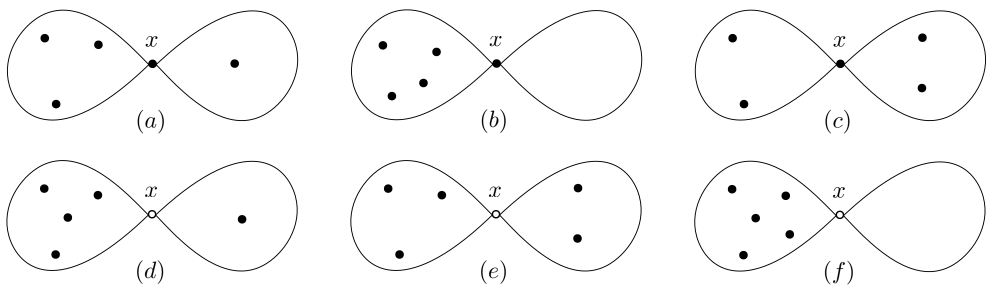


**Figure 2.1** All cases of cut vertices in a 1-connected graph

Notice that for the cases (a), (b), and (c), the cut vertex $x$ is a transversal vertex.

$H$ does not have a vertex of degree one, otherwise By Theorem 6, we have that $K_4$ has property (*) and $K_4 + 1$ vertex with degree one also has property (*) (By Theorem 4), and since property (*) is inhereted to the subgraphs of the graph (By Theorem 3), then $H$ would have property (*), which is a contradiction.

We have that all vertices in $H$ have degrees at least two, and if all of them are exactly two, then $H$ is a cycle, and By Theorem 4 in [15], we have that all cycles have property (*), hence $H$ would have property (*), which is a contradiction.

So we have that $H$ has at least one vertex of degree at least three; if it is exactly three and the rest are two, then by handshake lemma, we have at least two vertices of degree three. If its' degree is four and the rest of the vertices have degree two, then we get that the graph $H$ is the hourglass graph, which by Theorem 8 has property (*), hence $H$ would have property (*), which is a contradiction. So we have at least two vertices in $H$ that have a degree of at least three, and the rest have a degree of at least two.

Let $R$ be the subgraph of the right side of the cut and $L$ be the subgraph of the left side of the cut.

- **Case (a)**: Let us denote the transversal vertex in $R$ as $y$, and since all vertices in $H$ have degrees at least two, then we have to have at least two Kempe chains between $y$ and two other transversal vertices in $G$. All paths from $y$ to other transversal vertices may only pass through $x$ because except $x$, all other transversals are in $L$; since $x$ is a transversal vertex itself, we can have a kempe chain from $y$ to $x$, but since $y$ and $x$ have different colors, we cannot have more Kempe chains going from $y$ through $x$ to other transversal vertices. Hence, we have only one Kempe chain from $y$ to other transversal vertices, bringing us to a contradiction.

- **Case (b)**: Unlike case (a), we can realize all necessary Kempe chains. If no Kempe chain between transversals has edges in $R$, we can contract all vertices of $R$ into $x$ and preserve all Kempe chains between transversals; hence, $G$ was not a minimal graph, a contradiction. So let us assume there is some Kempe chain which has a subpath in $R$; since all Kempe chains start from $L$, such a Kempe chain has to pass through $x$; this means that if $G$ is a counter-example; after contracting all edges of the $R$ into $x$, we still have left Kempe chains between all necessary transversals in $L$, but we got a smaller graph which still should be a counter-example; if the contracted version is not a counter-example, then it would mean $G$ was not a counter-example in the beginning since we could get a rooted minor $H$ from G. Hence, the graph $G$ is not minimal, which is a contradiction.

- **Case (c)**: If both vertices with degrees at least three are in $R$, let us denote them as $y, z$, then we can only have Kempe chains from $y$ to $x$ and $z$, and from $z$ to $x$ and $y$, but $deg(y) >= 3, deg(z) >= 3$ in $H$, which is a contradiction since they can have at most two Kempe chains in $G$ in this case. Then, assume only one vertex of degree, at least three, is in $R$, and let it be $y$. Similarly, we get that $y$ can have at most two Kempe chains, which is a contradiction. Symmetrically, the same holds if $x$ and $y$ were in $L$ or one of them was in $L$; hence, in all cases, we get a contradiction.

- **Case (d)**: The vertex in $R$ has a degree of at least two. Hence, the color of $x$ is the color of the vertex itself, so we can contract the paths from it to $x$ and still get the counter-example, which is a contradiction since the graph $G$ was not minimal.

- **Case (e)**: Let vertices in $L$ be denoted as $a, b, c$ and the vertices in $R$ as $y, z$, if we have at least one vertex with degree at least three in $R$, let it be $y$, then we can have one Kempe chain between $y$ and $z$, and at least two Kempe chains from $y$ to $L$, hence $x$ has the color of $y$, but now since $z$ has degree at least two, at least one Kempe chain from $z$ should go to $L$, and it can pass only through $x$, since the color of $x$ is the same color as $y$, the Kempe chain from $z$, even if it includes $x$, can only be connected to $y$; hence, it cannot pass to $L$, which is a contradiction. So we have that both vertices with degrees at least three are in $L$; in $L$ we have three vertices, and if all of them are maximally connected to each other, we get the $abc$ cycle, and since two vertices in $L$ have a degree of at least three, at least two Kempe chains should pass from $L$ to $R$, both of them pass through $x$, so both of them should be connected to the same transversal vertex in $R$, WLOG it is $y$, but then $z$ in $R$ can only be connected to $y$ if it has Kempe chain to any other transversal than $y$, it has to pass through $x$ but $x$ has color of $y$, so the only Kempe chain from $z$ including $x$ can go to $y$ which is a contradiction since in $H$ $deg(z) >= 2$, we only can have one Kempe chain in $G$ from $z$

- **Case (f)**: We could contract all the vertices from $R$ into $x$ and still get the counter-example because all the Kempe chains still exist between the transversals, it contradicts the minimality of $G$.

Thus, $G$ is 2-connected. □

# Conclusion

# Bibliography


1. KENNETH, Appel; HAKEN, Wolfgang. Every Planar Map is Four Colorable. *Illinois Journal of Mathematics*. 1977, vol. 21, pp. 429–567.
2. ROBERTSON, Neil; SANDERS, Daniel; SEYMOUR, Paul; THOMAS, Robin. The Four-Colour Theorem. *Journal of Combinatorial Theory, Series B*. 1997, vol. 70, no. 1, pp. 2–44. ISSN 0095-8956. Available from DOI: https://doi.org/10.1006/jctb.1997.1750.
3. HADWIGER, Hugo. Uber eine Klassifikation der Streckenkomplexe. *Vierte Jahresbericht der Deutschen Mathematiker-Vereinigung*. 1943.
4. WAGNER, K. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.* 1937, vol. 114, pp. 570–590. Available from DOI: https://doi.org/10.1007/BF01594196.
5. ROBERTSON, Neil; SEYMOUR, Paul; THOMAS, Robin. Hadwiger's conjecture for K6-free graphs. *Combinatorica*. 1993, vol. 13, pp. 279–361. Available from DOI: https://doi.org/10.1007/BF01202354.
6. ALBAR, B.; GONÇALVES, D. On triangles in $K_r$-minor free graphs. 2013. Available also from: http://arXiv.org/abs/1304.5468.
7. KAWARABAYASHI, K.; TOFT, B. Any 7-chromatic graph has K7 or K4,4 as a minor. *Combinatorica*. 2005, vol. 25, no. 3, pp. 327–353. Available from DOI: https://doi.org/10.1007/s00493-005-0019-1.
8. BOLLOBÁS, B.; CATLIN, P.A.; ERDÖS, P. Hadwiger's Conjecture is True for Almost Every Graph. *European Journal of Combinatorics*. 1980, vol. 1, no. 3, pp. 195–199. ISSN 0195-6698. Available from DOI: https://doi.org/10.1016/S0195-6698(80)80001-1.
9. KOSTOCHKA, A. V. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*. 1984, vol. 4, no. 4, pp. 307–316. Available from DOI: https://doi.org/10.1007/BF02579141.
10. THOMASON, Andrew. An extremal function for contractions of graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*. 1984, vol. 95, no. 2, pp. 261–265. Available from DOI: 10.1017/S0305004100061521.
11. KRIESELL, M. A note on uniquely 10-colorable graphs. *Journal of Graph Theory*. 2021, vol. 98, no. 1, pp. 24–26.
12. KRIESELL, M. Unique Colorability and Clique Minors. *Journal of Graph Theory*. 2017, vol. 85, no. 1, pp. 207–216.
13. HOLROYD, Fred. Strengthening Hadwiger's Conjecture. *Bulletin of the London Mathematical Society*. 1997, vol. 29, no. 2, pp. 139–144. ISSN 0024-6093. Available from DOI: 10.1112/S0024609396002159.
14. MARTINSSON, Anders; STEINER, Raphael. Strengthening Hadwiger's conjecture for 4- and 5-chromatic graphs. *Journal of Combinatorial Theory, Series B*. 2024, vol. 164, pp. 1–16. ISSN 0095-8956. Available from DOI: https://doi.org/10.1016/j.jctb.2023.08.009.

15. Matthias Kriesell, Samuel Mohr. Kempe Chains and Rooted Minors. 2022.

# List of Figures

# List of Tables

# List of Abbreviations

# A Attachments

## A.1 First Attachment