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1 Introduction

The four-color theorem, that every planar graph is four-colorable, was one of the central problems in graph theory for over a century. Appel and Haken proved it in 1976 [1] using a computer program. The proof was controversial because it was the first major theorem to be proved using a computer. The theorem was later proved in 1997 by Robertson, Sanders, Seymour, and Thomas [2], still using a computer but with more straightforward configurations than Appel and Haken's in several aspects.

Hadwiger's conjecture [3] suggests a generalization of the four-color theorem. It is considered one of the most challenging problems in graph theory. Before stating the conjecture, we define the notion of a *graph minor*.

Definition 1 (Minor)**.** *The G contains a graph H as a minor if there exist a system $\{B_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ such that:*

- *For every vertex v from $V(H)$, the subgraph of G induced by vertices B_v, denoted $G[B_v]$, is connected.*
- *For every edge u, v in $E(H)$, there is an edge of G with one end in $G[B_u]$ and the other end in $G[B_v]$.*

We say that the system $\{B_v : v \in V(H)\}$ is a model of H in G.

Conjecture 1 (Hadwiger 1943 [3])**.** *For every integer $k \geq 0$, every graph G with no K_{k+1} minor can be colored with k colors.*

Equivalently, it can be stated that every graph with chromatic number at least k contains K_k minor.

It was proved by Hadwiger himself for cases $k \leq 3$. Wagner [4] showed that the case $k = 4$ is equivalent to the four-color theorem.

1. *Forward direction:* The case $k = 4$ implies the four color theorem since by Wagner [4], every planar graph has no K_5 or $K_{3,3}$ as minors. Hence all planar graphs do not have K_5 as minor. Hadwiger's conjecture for case $k = 4$ claims if graph doesn't contain K_5 as minor, then it is 4-colorable.
2. *Reverse direction:* Wagner [4] showed that a graph G is K_5-minor free if it is obtained by clique sums of planar graphs and a Wagner graph W, where W is a 3-colorable non-planar graph on eight vertices. For any two four-colorable graphs, their clique sum is also four-colorable. Hence, the four-color theorem implies Hadwiger's conjecture for the case $k = 4$.

Robertson, Seymour, and Thomas proved the case $k = 5$ in 1993 [5], where they did not use a computer to prove it. However, they proved that a minimal counter-example to the case $k = 5$ should have a vertex whose removal results in a planar graph, reducing the problem to the four-color theorem. The case $k = 6$ is still open, and there are some results in this direction as that of Albar and Gonçalves [6].

Theorem 1. *Every graph with no K_7 minor is 8-colorable, and every graph with no K_8 minor is 10-colorable.*

Moreover, Kawarabayashi and Toft [7] proved that every 7-chromatic graph has K_7 or $K_{4,4}$ as minor. Bollobás, Catlin, and Erdös [8] showed that the conjecture is true for almost all graphs using probabilistic arguments. In general, the cases $k \geq 6$ remain open. Because of the conjecture's current state, there is interest in a weaker version known as the Linear Hadwiger's Conjecture.

Conjecture 2. *(Linear Hadwiger's Conjecture) There exists a constant c such that, for every $k \geq 0$, every graph with no K_{k+1} minor can be colored with ck colors.*

For more than forty years, the best-known result for the linear version was that every graph with no K_{t+1} minor is $O(t\sqrt{\log t})$-colorable. This was proved in the 1980s by Kostochka [9] and Thomason [10] independently. Recently, in 2025, Delcours and Postle [11] lowered this bound to $O(k \log \log(k))$.

Holroyd[12] tryied to strengthen Hadwiger's conjecture by looking specific regions in a graph where a minor of a complete graph is likely to appear, based on the coloring of the region. Before stating his conjecture, we need to define the concepts of rooted minors and colorful sets.

Definition 2 (Rooted minor)**.** *Let G and H be graphs and let $S = \{X_v : v \in V(H)\}$ be a system of distinct vertices of G. Then H is an S-rooted minor of G if exists a model $\{B_v : v \in V(H)\}$ of H in G such that for every vertex $v \in V(H)$ $X_v \in B_v$.*

Definition 3 (Chromatic coloring)**.** *Let G be a graph. The chromatic number of G denoted as $\chi(G)$ is the smallest integer k such that G is properly colored with k colors. We say that a coloring $\mathfrak{C}$ is chromatic coloring of the graph G if it has exactly $\chi(G)$ color classes.*

Definition 4 (Colorful set)**.** *For a graph G, a set of vertices $S \subseteq V(G)$ is called colorful in G if for every chromatic coloring of G, S contains at least one vertex from each color of the coloring.*

The colorful sets of the graph are places that are 'hard to color'. Hence, we might hope to find the K_t minors rooted in those sets. And this is exactly the Halroyd's conjecture.

Conjecture 3. *(Halroyd's conjecture) Let G be a graph with chromatic number k, let S be a colorful set in G. Then there exists a subset $S' \subseteq S$ of size k such that G has an S'-rooted K_k minor.*

(comment:) In notes I have: Formulate that the system doesn't matter for complete graphs for colorful sets. I am not sure what we wanted to say there

Holroyd called it the Strong Hadwiger Conjecture because it generalizes Hadwiger's conjecture. If we take $S = V(G)$, then the Halroyd's conjecture states that there exists an $S' \subseteq S$ of size k such that the graph G has S'-rooted K_k minor. Since the set S is the all vertices of graph G, it means G contains K_k as a minor which is exactly the statement of Hadwiger's conjecture.

Holroyd himself proved conjecture for cases $k \leq 3$ [12], and in 2024 the case $k = 4$ was proved by Martinsson and Steiner [13].

A classical tool in studying Hadwiger's conjecture is the notion of Kempe chains. It was first introduced by Kempe in an attempt to prove the four-color theorem. Even though his proof was not correct, Kempe chains were shown to be very useful in problems related to Hadwiger's conjecture. They were used for proving the four-color and five-color theorems as well.

1.1 Kempe chains

Definition 5 (Proper coloring)**.** *A proper (vertex) coloring of a graph G is a coloring of the vertices of G such that no two adjacent vertices share the same color.*

(comment:) Should we define proper coloring earlier, and also say that we assume that all colorings are proper? Because by default we assume it throughout the paper

Definition 6 (Kempe chain)**.** *Let G be a graph with proper coloring. Then for two distinct color classes i and j, the Kempe chain in colors i and j is the maximal connected subgraph of G where vertices have only colors of i or j.*

Claim 2. *Let G be a graph with proper coloring and let S be a colorful set in G. For a color class i, let S_i be the set of vertices from S that have color i. Then for every distinct color classes i and j There exists a Kempe chain between some of the vertices of S_i and S_j.*

Proof. Assume for contradiction that there is no Kempe chain between S_i and S_j. We will show that we can recolor the vertices of S_i with color j and still maintain a proper coloring, contradicting the assumption that S is colorful (since in the new coloring, no vertex of S has color i).

1. Let $v \in S_i$.
   - If v is not contained in any Kempe chain of colors i and j.
   - Otherwise, v lies in some Kempe chain of colors i and j. By our assumption, that Kempe chain contains no vertices of S_j we can switch the colors $i \leftrightarrow j$ along that chain. The resulting coloring is proper, where v now has color j.
2. Repeat this for each vertex of S_i. At each step, properness is maintained, and no vertex of S_j is ever touched (since there are no i–j chains reaching S_j). After recoloring all of S_i, the colorful set S no longer contains any vertex of color i, a contradiction.

Hence there exists a Kempe chain connecting some vertex in S_i to some vertex in S_j. □

We can observe that the vertices of colorful sets from Halroyd's conjecture must be connected by Kempe chains. Suppose, for contradiction, that there exist two vertices u, v in a colorful set, colored i and j respectively, but no Kempe chain of colors i and j connects them. Then, consider the Kempe chain of colors i and j that contains u. Swapping the colors within this chain—replacing every i with j and vice versa—produces a new proper coloring of the graph. However, this would result in u taking the same color as v, contradicting the assumption that the set is colorful. Thus, every pair of vertices in a colorful set must be connected by a Kempe chain.

Since Hadwiger's conjecture is difficult to prove in general, it is interesting to study it for specific classes of graphs. Hadwiger suggested looking into the graphs with a bounded number of optimal colorings [3], one particular class is the uniquely optimally colorable graphs.

Claim 3. *Let G be a uniquely k-colorable graph with colors $\{1, 2, \ldots, k\}$ Let $v_1, v_2, \ldots, v_k$ be differently colored vertices of the graph G, where v_i is has color i. Then there are Kempe chains between all pairs of v_i and v_j from $\{v_1, v_2, \ldots, v_k\}$.*

Proof. Let $S := \{v_1, v_2, \ldots, v_k\}$, then S is a colorful set. By the claim 2 for every distinct color classes i and j there exists a Kempe chain between S_i and S_j. Since each color appears only once in S, we have that each S_i has size of 1. Hence, there is a Kempe chain between each distinct pair of the vertices of the colorful set. □

This claim suggests a question: whether the existence of the Kempe chains forces an existence of K_k minor rooted at $\{v_1, v_2, \ldots, v_k\}$.

Kriesell proved the Hadwiger's conjecture for uniquely k-colorable graphs where $k \leq 10$ [14], if the graph is antitriangle-free [15]. Moreover, with Mohr, they proved it is true for line graphs [16].

1.2 Rooted minors

One of the central problems in Graph Theory is to find a minor in a given graph. There has been significant progress in this direction, one of which is the structure theorem of Robertson and Seymour, which says that if a graph G does not have K_t minor, then G is "almost embeddable" on a surface of low Euler genus relative to t [17]. This result was developed as part of their proof of Wagner's Conjecture—now known as the Robertson–Seymour Theorem —which says that the class of finite undirected graphs is well-quasi-ordered under the graph minor relation. In the proof, they use the following theorems, which are proved in [18]

Theorem 4. *let G be a 3-connected graph and let v_1, v_2, v_3 be three distinct vertices. Then either G has five connected disjoint subgraphs $X_1, X_2, \ldots, X_5$ such that X_i contains v_i for every $i = 1, 2, 3$ and for every $j = 4, 5$ X_j has neighbour in each X_i for all $i = 1, 2, 3$ or G is planar such that v_1, v_2, v_3 are on boundary.*

Theorem 5. *Let G be a 4-connected graph and v_1, v_2, v_3, v_4 be four distinct vertices. Then either G has K_4 minor rooted at $\{v_1, v_2, v_3, v_4\}$ or G is planar such that v_1, v_2, v_3, v_4 are on the boundary.*

Those two results were the starting points for rooted minor problems.
It turns out that rooted minors are not only useful for the proof of Graph Minor theorem, but also for some structure theorems which are used to prove some existence of graph minor, some of which are presented below:

Robertson, Seymour, and Thomas [5] used rooted K_4–minors to prove the case $k = 5$ of Hadwiger's conjecture. Kawarabayashi and Toft [7] used rooted minors to prove that every 7-chromatic graph has K_7 or $K_{4,4}$ as minor.

In graphs embedded on surfaces, the concept of a rooted minor extends naturally to problems involving face covers. A recent paper [19] shows that in a 3-connected graph embedded in a surface of Euler genus g, if the graph has no rooted $K_{2,t}$ minor, then there exists a face cover whose size is bounded by a function of g and t. In the planar case, they got $O(t^4)$ upper bound, which improved the result of Böhme and Mohar [20].

Kempe chains and rooted minors

Usually, in the context of Hadwiger's conjecture, only clique minors were considered. However, Kriesell and Mohr [21] considered the following question, which does not necessarily look for clique minors. Let G be a graph with a proper coloring $\mathfrak{C}$, let $k = |\mathfrak{C}|$, and let $v_1, ..., v_k$ be a vertex set of G with different colors. Then, there is a system of Kempe chains for some pairs (v_i, v_j). They examined whether there is a rooted minor H of G on vertices $v_1, ..., v_k$, where H has edges between v_i, v_j if and only if there is a Kempe chain between v_i and v_j in G.

The answer to this question is affirmative for the case $k \leq 4$. For $k = 5$, it holds for graphs with at most six edges but remains open in general. The case $k = 6$ is also open, while counterexamples exist for $k \geq 7$.

In this paper, we investigate properties of minimal counterexample graphs for the case $k = 5$. Additionally, we perform a computational enumeration for $k = 6$, examining all graphs with at most sixteen vertices. Our results show that the answer remains positive for all graphs in this range.

2 Preliminaries

In this chapter, we will present definitions and primary results from the paper of Kriesell and Mohr [21], which we will use to build up our investigations.

They introduced the concept of *routing graphs* to more formally characterize the problem from the previous chapter, as we saw. First, let us define the transversal of a set partition, and then we can define the routing graph.

Definition 7 (Transversal of a partition)**.** *A (minimal) transversal of a partition is a set containing exactly one element from each partition member and nothing else.*

Example. Coloring $\mathfrak{C}$ of a graph partitions its vertices into color classes. A transversal T of this partition would contain exactly one vertex of each color from $\mathfrak{C}$.

Definition 8 (Routing Graph)**.** *Let $\mathfrak{C}$ be a coloring of graph G, let T be the transversal of coloring $\mathfrak{C}$, then the routing graph $H(G, \mathfrak{C}, T)$ is the graph with vertex set T, where for every pair of vertices u, v from T, an edge (u, v) exists if and only if there is a Kempe chain between u and v in G.*

Now, we can define the problem in a more compact way, which is as follows:

Which graphs H have the property that, if H is a routing graph of some graph G with coloring $\mathfrak{C}$ and a transversal T, then G has H as T-rooted minor? We say those graphs are *KM-forcing.*

2.1 KM-forcing graphs

Definition 9 (KM-forcing)**.** *A graph H is KM-forcing if for every graph G, coloring $\mathfrak{C}$ and transversal T such that H is isomorphic to the routing graph $H(G, \mathfrak{C}, T)$, graph G has H as T-rooted minor.*

Remark. The term *KM-forcing* can be interpreted in two ways: as *Kriesell–Mohr forcing* or as *Kempe chain rooted minor forcing.* We leave the choice of interpretation to the reader's imagination.

Example. K_1 is KM-forcing. If K_1 is a routing graph $H(G, \mathfrak{C}, T)$, then $|T| = 1$, and G can be colored with only one color; hence, it has no Kempe chains to other colored vertices, and K_1 is a rooted minor of G.

Example. The complete graph K_2 is KM-forcing. For any graph G with coloring $\mathfrak{C}$ and transversal $T = \{u, v\}$ where the routing graph $H(G, \mathfrak{C}, T)$ is isomorphic to K_2, by definition there exists a Kempe chain between u and v in G. Contracting all internal vertices of this chain while keeping u and v results in an edge (u, v). In the contracted graph of G, removing all unnecessary edges and vertices would result in K_2 as a rooted minor of G.

We will list several results from [21], which capture properties of KM-forcing graphs. Moreover, they are helpful for later proofs. All the theorems in this chapter are proved in [21].

Theorem 6. *If graph K is KM-forcing, all its subgraphs are also KM-forcing.*

Theorem 6 is crucial for later showing that K_7 is not KM-forcing. It is sufficient to find a subgraph of K_7 which is not KM-forcing, and by theoerm 6 this would imply that K_7 is not KM-forcing as well.

Now, we will see a characterization of KM-forcing graphs, which helps to show that K_4 is KM-forcing.

Theorem 7. *Graph K is KM-forcing if and only if every component of K is KM-forcing.*

Another valuable result for further investigations is that a KM-forcing graph is still KM-forcing if we attach a pending edge to it.

Theorem 8. *Let K be a graph and q be a vertex with a degree of one. If $K - q$ is KM-forcing, then K is KM-forcing as well.*

2.2 Kempe chains and rooted K_7-minors

Definition 10. *A coloring $\mathfrak{C}$ is a Kempe coloring if any two vertices from distinct color classes belong to the same Kempe chain.*

Hadwiger [3] asked whether for a given Kempe coloring $\mathfrak{C}$ of a graph G and transversal T, the graph $H := H(G, \mathfrak{C}, T)$ is a complete graph and whether G contains a T-rooted H minor. This would follow if every complete graph were KM-forcing. By Theorem 6, this would imply that *every* graph is KM-forcing. It would prove Hadwiger's conjecture for graphs admitting Kempe colorings if true. However, as we will see, KM-forcing is too restrictive a property—and it already fails for K_7.

Theorem 9. *K_7 is not KM-forcing.*

By Theorem 6, if K_7 were KM-forcing, all its subgraphs would also be KM-forcing. Thus, finding a subgraph of K_7 that is not KM-forcing is enough.

To construct graphs that are not KM-forcing, we need a graph G with:

1. Enough paths between transversal vertices to form a routing graph
2. Not many edges incident to transversal vertices so that the construction of a rooted minor fails.

A good construction with these properties is the $Z(G)$ graph:

Definition 11 ($Z(G)$)**.** *For a graph G, $Z(G)$ is defined as:*

1. ***Vertices:*** $V(Z(G)) := V(G) \times \{1, 2\}$
   Example: If $V(G) := \{a, b\}$, then $V(Z(G)) := \{(a, 1), (b, 1), (a, 2), (b, 2)\}$
2. ***Edges:*** *For each edge (x, y) from $E(G)$, the graph $Z(G)$ contains edges $((x, 2)(y, 2)), ((x, 1)(y, 2)), ((x, 2)(y, 1))$.*
   Formally:

$$E(Z(G)) := \{(x, i)(y, j) : xy \in E(G) \text{ and } (i \neq 1 \text{ or } j \neq 1)\}$$

Z(G) has coloring $\mathfrak{C} := \{\{(x,1),(x,2)\} : x \in V(G)\}$*, and transversal* $T := V(G) \times \{1\}$.

Kriesell and Mohr identified:

- A subgraph G of K_7 (Figure 2.1)
- The graph $Z(G)$ (Figure 2.2) with coloring $\mathfrak{C}$ and transversal T

such that:

- G is isomorphic to $H(G, \mathfrak{C}, T)$, but
- G is not a rooted minor of $Z(G)$

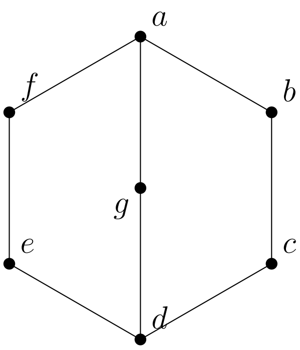

Figure 2.1 The subgraph of K_7 which is not KM-forcing

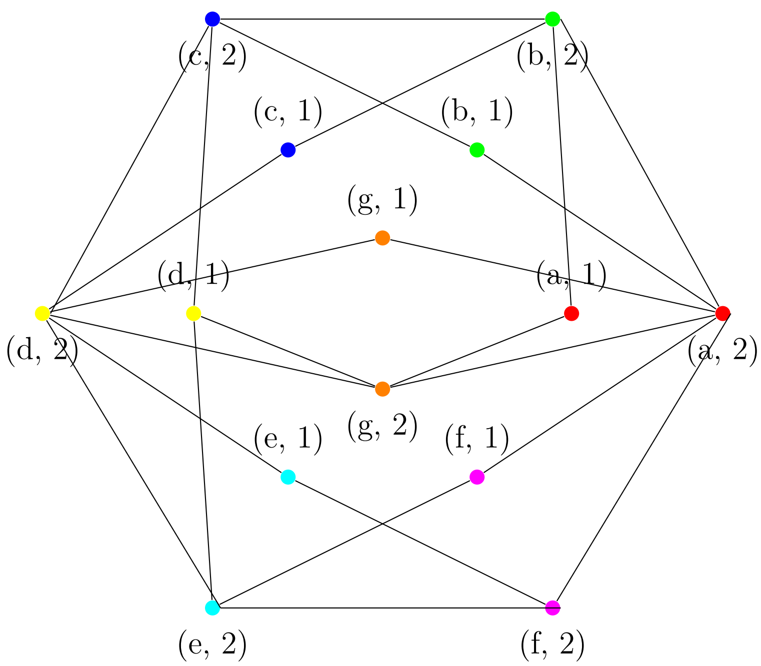

Figure 2.2 The graph $Z(G)$ given G is the graph from 2.1

In Figure 2.2, the graph $Z(G)$ is shown with coloring and transversal described in the definition of Z(G) 11. For clarity, the transversal T is the following

$T := \{(a, 1), (b, 1), (c, 1), (d, 1), (e, 1), (f, 1), (g, 1)\}$.

We may be tempted to use the construction of $Z(G)$ to check whether any graph is *km*-forcing. However, Kriesell and Mohr showed that for any graph G with at most six vertices, $Z(G)$ always contains a G-rooted minor.

Theorem 10. *Let G be any graph with at most six vertices. Consider $Z(G)$ with coloring $\mathfrak{C}$ and the transversal T, as it's defined in 11. Then $Z(G)$ has a rooted $H(Z(G), \mathfrak{C}, T)$-minor.*

There are also positive results, one of which is that K_4 is KM-forcing.

Theorem 11. *Every graph on at most four vertices is KM-forcing.*

One might ask whether the class of KM-forcing graphs is bounded. This is not the case, as implied by:

Theorem 12. *Every connected graph with at most one cycle is KM-forcing.*

As we can see, there is a gap between K_4 and K_7. We know that K_4 is KM-forcing and K_7 is not. What about the K_5 and K_6? The question for both of them is open, but there is a partial result on graphs with five vertices, which is the following:

Theorem 13. *Every graph on five vertices with at most six edges is KM-forcing.*

This result naturally leads us to the question of what happens when we consider graphs beyond this bound. In the next chapter, we will look into graphs G, colroings $\mathfrak{C}$ and transversals T where some graph H which has at least five vertices and seven edges appears as the routing graph $H(G, \mathfrak{C}, T)$ but is not a rooted minor of G.

3 Structural Properties of Non-KM-forcing Graphs

Earlier, in Theorem 9, we saw a construction called $Z(G)$, which generates graphs containing the necessary Kempe chains without introducing additional edges that would force a G-rooted minor. This suggests that connectivity plays a role in determining whether a graph is KM-forcing.

By Theorem 13, all graphs with five vertices and at most six edges are KM-forcing. Our goal is to investigate the connectivity properties of graphs G in which some graph H is not a rooted minor of G but appears as the routing graph $H(G, \mathfrak{C}, T)$. We call such graphs H as non-KM-forcing in G. By the following lemma, which proves a stronger statement, we will show that if H has at least seven edges and five vertices, and there exists some G in which H is non-KM-forcing, then smallest such G is 2-connected.

Lemma 14. *Let H and G be graphs, let $\mathfrak{C}$ be a coloring of G, and let T be the corresponding transversal such that H is isomorphic to $H(G, \mathfrak{C}, T)$..*

Suppose the following hold:

1. *H is not a rooted minor in G.*
2. *Every proper subgraph H' of H is KM-forcing.*
3. *If H is non-KM-forcing in any graph $G' \ncong G$, then*

$$|V(G)| + |E(G)| \leq |V(G')| + |E(G')|.$$

Then G is 2-connected.

Proof. The condition (3) forces G to be the minimal counter-example. By minimality of G, for each pair of distinct color classes $A, B \in \mathfrak{C}$, the subgraph of G induced by $A \cup B$ is a single path connecting the corresponding transversal vertices in T.

Assume, for contradiction, that G is 1-connected. Then, there exists a cut vertex x splitting G into two subgraphs L and R that intersect only at x (see Figure 3.1).

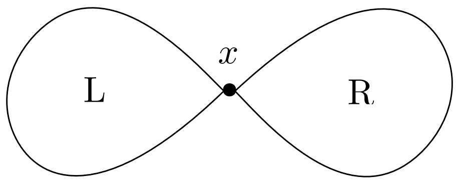

Figure 3.1 A cut vertex x splitting G into subgraphs L and R.

Case 1: x is a transversal vertex.

In this case, every Kempe chain passing from L to R must pass through x. Since is a transversal, such a chain would connect x to another transversal vertex, contradicting the minimal-path property above. Hence no transversal in L can connect to one in R via a Kempe chain.

Let H_L be the subgraph of H induced by the transversals in $L \cup \{x\}$, and let H_R be defined similarly for R. By condition (2), each is KM-forcing in the corresponding subgraph of G, so there are rooted minors of H_L in L and H_R in R. Since these contractions occur on disjoint vertex sets (except x) and respect the transversal roots, combining them results a rooted minor of H in G, contradicting condition (1).

Case 2: x is not a transversal vertex.

If no Kempe chain passes through x between L and R, the same argument as in Case 1 applies. Otherwise, without loss of generality, suppose there is a Kempe chain in color class A passing through x from some transversal in L to one in R. Then x also has color A.

Let H_L be the subgraph of H induced by the transversals in L together with the color-A transversal. Construct L' by replacing x in L with that transversal. Since all relevant Kempe chains remain, by condition (2), H_L is KM-forcing in L'.

Similarly, let H_R be the subgraph of H induced by the transversals in R plus one transversal in L connected by the A-Kempe chain. Form R' by adding a single edge from that transversal in L to R by creating an edge with x. This preserves the Kempe chains corresponding to edges of H_R. Again, by condition (2), H_R is KM-forcing in R'.

Since the contractions realizing the minors of H_L in L' and of H_R in R' act on disjoint parts of G (except the cut vertex), they combine to give a rooted minor of H in G, contradicting condition (1).

In all cases, we reach a contradiction. Therefore, G must be 2-connected. □

Corollary. For every graph H on five vertices and at least seven edges, if there is a graph G with coloring $\mathfrak{C}$ and transversal T such that H is isomorphic to $H(G, \mathfrak{C}, T)$ but there is no H rooted minor in G. Then G is 2-connected

Proof. By Theorem 13, all graphs with five vertices and, at most, six edges are KM-forcing. Hence, we can apply the Lemma 14 and get that G is 2-connected. □

4 Computer Enumeration for Finding Counter-Examples in K_6

Since it remains open whether K_5 and K_6 are km-forcing, it is natural to attempt to find a counter-example. A single counter-example would show that the graph is not km-forcing. We performed a computer enumeration of the process of generating graphs G with a coloring $\mathfrak{C}$ and a transversal T for a given minor H, where H is isomorphic to $H(G, \mathfrak{C}, T)$. We then checked whether H is a rooted minor in G.

The idea is similar to the previous chapter's construction of $Z(G)$. We want to have all necessary Kempe chains between transversal vertices while ensuring that these vertices have no more than the necessary degrees. However, since Kriesell and Mohr [21] showed that for every graph G with at most six vertices, the construction $Z(G)$ does not provide a counter-example 10 (i.e., G remains a rooted minor of $Z(G)$), for looking for counter-examples for K_6, we must consider different constructions.

At a high level, our algorithm for finding possible counter-examples works as follows. We begin with a given rooted minor H and construct a set of colorings. The coloring is chosen so that the vertices of H are the transversals of those colorings; in other words, the transversal set corresponds to the vertex set of H.

Next, we construct graphs for each coloring where each edge of H corresponds to a Kempe chain between the corresponding transversal vertices. After the Kempe chains are built, we know that H is the routing graph of all those graphs with their coloring and the transversal $V(H)$. Then, we check whether those graphs contain H as a rooted minor. If we find a graph G where H is not a rooted minor, we have found a counter-example.

The algorithm follows these three main steps:

1. Given a graph H, generate set of colorings such that the transversal $T = V(H)$.

2. For each such coloring, construct graphs G where each edge of H corresponds to a Kempe chain in G between the transversal vertices.

3. For each constructed graph G, check whether G contains H as a rooted minor.

4.1 Step 1: Generating Colorings

In the first step, we generate colorings, where each coloring has n vertices.

Each transversal of a coloring should be mapped to the vertices of H. Therefore, each coloring would have $|V(H)|$ colors. First, we assign the colors to the transversal vertices and map them with $V(H)$, then for the rest of $n - |V(H)|$ vertices, we assign all possible combinations with the replacement of the colors $1, \ldots |V(H)|$

Example. Let $H = K_3$ (so $|V(H)| = 3$) and let $n = 8$. First, we select three vertices and make them the transversals. We give each of them a distinct color from $1, 2, 3$. Then, we have $\binom{7}{3} = 35$ ways to assign colors for the five remaining vertices. Hence, we get 35 different colorings.

Algorithm 1 GenerateColorings(H, n):

Require: Graph H all differently colored vertices, with $k = |V(H)|$, number of vertices n

Ensure: Set of colorings $\mathcal{A}$

Initialize empty coloring $\mathfrak{C} \leftarrow \{\}$
for each vertex $v \in V(H)$ **do**
    Create new vertex v'
    Set $v'.isTransversal \leftarrow True$
    Set $v'.color \leftarrow v.color$
    Add v' to $\mathfrak{C}$
end for
$R \leftarrow$ all combinations with replacement of $|V(H)|$ elements from $n - |V(H)|$
for each combination $r \in R$ **do**
    $\mathfrak{C}' \leftarrow$ copy of $\mathfrak{C}$
    for each $v \in r$ **do**
        Create new vertex v'
        Set $v'.isTransversal \leftarrow False$
        Set $v'.color \leftarrow v.color$
        Add v' to $\mathfrak{C}'$
    end for
    Add $\mathfrak{C}'$ to $\mathcal{A}$
end for
return $\mathcal{A}$

Complexity Analysis. Let $k = |V(H)|$ be the number of colors and n the number of vertices to generate.

- In the beginning the algorithm creates $|V(H)|$ colors.
- Then we generate all combinations with replacement.

$$\binom{k+n-1}{k} \in \mathcal{O}(k^{n+k-1})$$

- For each combination, we create a new coloring, assuming the creation of the coloring, creating a new vertex, and adding it to the coloring takes constant time. Creating a new coloring takes $\mathcal{O}(k)$ time.
- Overall, since the dominating step is the generation of all combinations with replacement, and for each such combination, we use $\mathcal{O}(k)$ time to create a new coloring, we get the total time complexity $\mathcal{O}(k^{n+k})$.

4.2 Step 2: Constructing Kempe Chains

For each coloring $\mathfrak{C}$, we build graphs such that H is the routing graph of those graphs. After finishing the construction of We check whether H is a rooted minor in those graphs.

Below is an overview of the algorithm. At any point, we have a graph G with all vertices colored and a list of remaining edge–pairs (chains) from H to construct. For each edge (s_i, t_i) from $E(H)$, we attempt to connect the corresponding s_i to t_i from G by growing a Kempe chain:

- Start at the current vertex s_i and identify the next color needed (either that of t_i or the current vertex if we have just matched t_i's color).
- Among all unvisited vertices of that color, pick one, add an edge to extend the chain, mark it visited, and recurse from this new vertex.
- Repeat until we reach t_i, at which point the chain for (s_i, t_i) is complete.

Once one chain is complete, we move on to the next pair in the list. After all Kempe chains have been built, we perform a minor check (see Section 4.3) to confirm that H appears as a rooted minor of G.

Algorithm 2 BUILDKEMPECHAINS(G, *chains*, s, t, *visited*, *available*, H)

Input:

- G — adjacency map of the graph
- *chains* — list of pairs (s_i, t_i) to connect
- s — current chain start vertex
- t — current chain end vertex
- *visited* — set of vertices in the current chain
- *available* — list of available vertices to extend chain
- H — minor to test existence against

Procedure:

```
if s = t then
    if chains = ∅ then
        if TestMinor(H, edges(G)) then return
        else
            report counterexample
        end if
    else                                                ▷ Building new chain
        (s′, t′) ← chains.extract()
        c ← color(t′)
        available ← {v ∈ V(G) : color(v) = c}
        BuildKempeChains(G, chains, s′, t′, {s′}, available, M)
    end if
    return
end if
for all v ∈ available do
    G′ ← copy(G)
    add edge (s, v) in G′
    c ← { color(s),   if color(v) = color(t)
        { color(t),   otherwise
    visited′ ← visited ∪ {v}
    available′ ← {w ∈ V(G′) : w ∉ visited′ ∧ color(w) = c}
    BuildKempeChains(G′, chains, v, t, visited′, available′, M, S)
end for
```

Complexity Analysis. Let $n := |V(G)|, m := |E(H)|$

1. **Single chain** At each recursive step, we choose at most $n-1$ unvisited vertices of a single color and never revisit a vertex. Hence, we have recursion depth up to n in the worst case. So:

$$\text{calls per chain} = 1 + (n-1) + (n-1)(n-2) + \cdots = \mathcal{O}\Big((n-1)!\Big) \;=\; O(n!)$$

2. **All chains.** We build m chains, so overall we get.

$$\mathcal{O}\Big(m \cdot n!\Big).$$

3. **Minor check.** We will see in 4.3 that testing for the rooted minor is done in $\mathcal{O}(n_H n_G e_H(e_G + n_G \log n_G))$, where n_G is the number of vertices of the larger graph, n_H the number of vertices of the minor, and e_H the number of edges of the minor. This runtime is insignificant compared to $O(n!)$.

Hence, the overall worst-case time complexity is

$$\mathcal{O}\Big(m \cdot n!\Big).$$

4.3 Step 3: Checking for Rooted Minors

Finally, for each constructed graph G, we check whether it contains H as a rooted minor. If any such G fails to contain H, we have found a counter-example.

For this step, we use a heuristic tool for finding minor embeddings [22]. The primary function we rely on is `find_embedding()`, which implements the heuristic algorithm described in [23]. Since `find_embedding()` is a heuristic, we can be sure it exists when it returns an embedding of H in G. However, when it returns no embedding, we cannot immediately conclude that one does not exist.

To reduce the false negatives, if no embedding is found, we rerun the function 10 additional times with different random seeds. While the authors of the algorithm do not provide an estimate for the probability of false negatives (returning no embedding when one exists) or any probabilistic quantity that we can rely on in our practice in most cases, the function returns an embedding on the first try and succeeds within two or three retries.

The algorithm is the following:

Algorithm 3 TestIfMinorExists(H, G, S)

Input:

- H — the target minor graph
- G — the parent graph to check

Output: True if H is a rooted minor of G, otherwise False.

$embedding \leftarrow find\_embedding(H, G)$
if $embedding \neq \emptyset$ **then**
  return True
end if
for $i \leftarrow 1$ **to** 10 **do**
  $seed \leftarrow$ random integer
  $embedding \leftarrow find\_embedding(H, G, \text{random\_seed} = seed)$
  if $embedding \neq \emptyset$ **then**
    return True
  end if
end for
return False

Complexity Analysis. The `find_embedding` function runs in

$$\mathcal{O}\Big(n_H\, n_G\, e_H\, (e_G + n_G \log n_G)\Big),$$

as described in [23], where n_G is the number of vertices of the larger graph, n_H the number of vertices of the minor, and e_H the number of edges of the minor.

Since, in the worst case, we perform this operation 10 times, the total complexity still is:

$$\mathcal{O}\Big(n_H\, n_G\, e_H\, (e_G + n_G \log n_G)\Big).$$

4.4 Bringing everything together

We combine all the subroutines discussed in the previous paragraphs for the main algorithm. For a given rooted minor H and a number of vertices n in the parent graphs, we first generate all possible colorings for n and H, where the transversal vertices of each coloring are mapped to the vertices of H. Then, for each such coloring, we build graphs consisting only of Kempe chains, such that there is a Kempe chain between two transversal vertices of the coloring if and only if those two transversals form an edge in H. After constructing all Kempe chains, we test whether H is a rooted minor of that graph. If it is not a rooted minor, then we know that H is non-km-forcing and that any graph such that H is a subgraph of it is also non-km-forcing.

Algorithm 4 Searches for counter-examples

Input:

- H — rooted minor we look for.
- n — number of vertices in the parent graph G

Output: Graph G where H is a routing graph of G with its corresponding coloring and transversal but not a rooted minor in G, or **None** if no such pair exists.

$colorings \leftarrow$ GenerateColorings(H, n)
for all coloring $\mathfrak{C}$ **in** *colorings* **do**
    $G \leftarrow$ empty graph on $\mathfrak{C}$
    $result \leftarrow$ BuildKempeChains$(G, E(H), s, t, \emptyset, available, H)$
    if $result \neq$ **None then**
        return $result$ ▷ Found a counter-example
    end if
end for
return None ▷ No counter-examples found

Complexity Analysis. Let $k = |V(H)|$, let $m = |E(H)|$.

1. GenerateColorings takes $\mathcal{O}(k^{n+k})$ time
2. We have $\mathcal{O}(k^{n+k})$ colorings and for each coloring BuildKempeChains takes $\mathcal{O}\Big(m \cdot n!\Big)$ time. So overall the main for loop takes $\mathcal{O}\Big(k^{n+k} \cdot m \cdot n!\Big)$ time

Since the main loop dominates the time complexity, we get the overall time complexity of the algorithm as follows:

$$\mathcal{O}\Big(k^{n+k} \cdot m \cdot n!\Big)$$

4.5 Results

First, with the subroutine TestIfMinorExists, we verified the result of [21] that K_7 is non-km-forcing.

We used the main algorithm from 4.4 to primarily test whether K_6 is non-km-forcing or not. By 6, finding any non-km-forcing subgraph of K_6 is sufficient to prove that K_6 is non-km-forcing. Since we also know that K_4 is km-forcing 9 and that graphs with five vertices and at most six edges are km-forcing, the subgraphs of K_6 we are looking for are in between dense graphs on five vertices and graphs on six vertices. Moreover, we know that all cycles are km-forcing 12, so we do not need to consider C_6. Adding a pending edge to a km-forcing graph still keeps this property 8. So, we have a limited amount of subgraphs of K_6 to consider.

Because the algorithm has super-exponential time complexity, we could test for all candidate subgraphs of K_6 We found no counter-examples on parent graphs with at most 13 vertices.

Conclusion

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A Attachments

A.1 First Attachment