FACULTY
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BACHELOR THESIS

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Abstract: Use the most precise, shortest sentences that state what problem the thesis addresses, how it is approached, pinpoint the exact result achieved, and describe the applications and significance of the results. Highlight anything novel that was discovered or improved by the thesis. Maximum length is 200 words, but try to fit into 120. Abstracts are often used for deciding if a reviewer will be suitable for the thesis; a well-written abstract thus increases the probability of getting a reviewer who will like the thesis.

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Klíčová slova: klíčová slova, klíčové fráze

Contents

Introduction

1 Kempe chains and Routing graphs

The following proofs are reformulations of those presented in [1].

Definition 1 (H-certificate)**.** *Graph H is a minor of G if there exists $c := (V_t)_{t \in V(H)}$ of pairwise disjoint $V(G)$, called bags, such that $\forall t \in V(H)$ $V_t \neq \emptyset$ and $G[V_t]$ is connected, and $\forall (u,v) \in E(H)$, $\exists$ edge connecting V_u and V_v, any such c is called H-certificate*

Definition 2 (rooted H-certificate)**.** *H-certificate is a rooted if $V(H) \subset V(G)$ and $t \in V_t$ $\forall t \in V(H)$. If there is a rooted H-certificate in graph G, then H is a rooted minor of G*

Definition 3 (Routing Graph)**.** *Let $\mathcal{C}$ be a coloring of graph G, let T be the transversal of coloring $\mathcal{C}$, then routing graph $H(G, \mathcal{C}, T)$ is defined as the graph on vertices of T, such that $\forall (i,j) \in V(H)(i \neq j)$, $(i,j) \in E(H)$ if and only if $\exists$ Kempe chain between vertices i and j in graph G*

Definition 4 (Property (*))**.** *All graphs H which are routing graph of some G with some coloring $\mathcal{C}$ and transversal T such that G has a rooted H-certificate, are said to have property (*)*

Theorem 1. *K has property (*) if and only if every component of K has it.*

Proof. **(⇒)** If K has property (*) then all components of K have property (*)

Let K be a graph with property (*) and K' is a component of K, let's do case analysis, we have 2 cases:
Case 1: $|V(K)| = |V(K')|$
Case 2: $|V(K)| > |V(K')|$

Case 1: The component K' of K is a spanning subgraph of K, which is same as $|V(K)| = |V(K')|$.
Let K have the property (*), take spanning subgraph K' of K. Now take graph G' with coloring $\mathcal{C}$ such that $|\mathcal{C}| = |V(K')|$ and a transversal T such that K' is isomorphic to the spanning subgraph of routing graph $H(G', \mathcal{C}, T)$. Now, $\forall (u,v) \in E(K) \setminus E(K')$ add (u,v) edge to graph G', we can do so without breaking the coloring because the edge taken from $E(K) \setminus E(K')$ is only between vertices which have different colors, now we obtain graph G, then K is isomorphic to spanning subgraph of $H(G, \mathcal{C}, T)$, since K has property (*) then there is a rooted H-certificate c in G, and c is also and H'-certificate for G'.

Case 2: $|V(K)| > |V(K')|$
Take graph G' with coloring $\mathcal{C}'$ such that $|\mathcal{C}'| = |V(K')|$ and a transversal T' such that K' is isomorphic to the spanning subgraph of routing graph $H(G', \mathcal{C}', T')$, take set $S := V(K) \setminus V(K')$ and construct graph G as disjoint union of G and K_S(Here K_S is complete graph on vertex set S), let coloring

for G be $\mathcal{C} := \mathcal{C}' \cup \{\{s\}|s \in S\}$ and $T := T' \cup S$, now by construction K is isomorphic to the spanning subgraph H of routing graph $H(G, \mathcal{C}, T)$, and since K has property (*) it also has rooted H-certificate in G let's denote it as c and by definition of rooted H-certificate it's defined as $c := (V_t)_{t \in V(K)}$, then let $c' := (V_t)_{t \in V(K')}$ is a rooted H'-certificate in G'.

($\Leftarrow$) If every component of K has property (*), then K also has property (*). Take graph G with coloring $\mathcal{C}$ such that $|\mathcal{C}| = |V(K)|$ and a transversal T such that K is isomorphic to the spanning subgraph of routing graph $H(G, \mathcal{C}, T)$, then for every component K_i of K there is G_i a subgraph of G(all G_i's are disjoint), coloring $\mathcal{C}_i$ and T_i such that K_i is a spanning subgraph of $H(G_i, \mathcal{C}_i, T_i)$. Since every K_i has property (*), then there is c_i-certificate in G_i, hence by the union of all those (c_1, c_2, ... c_n) certificates, we get a rooted K-certificate in G, hence K has property (*) □

Theorem 2. *If K has property (*), then all subgraphs of K have property (*)*

Proof. Let K have the property (*), and K' be subgraph of K, let L be the edgeless graph on vertices $V(K) \setminus V(K')$, $L \cup K'$ is a spanning subgraph of K, hence it has property (*) (Shown in forward direction of the proof of Theorem 1), since K' is a component of $K' \cup L$, by Theorem 1 it also has property (*) □

Lemma 3. *Let K be a graph, if $\exists q \in V(K)$ such that $\deg(q) = 1$ and $K - q$ has property (*), then K has property (*)*

Proof. Let K be a graph such that $\exists q \in V(K)$ such that $\deg(q) = 1$ and $K - q$ has property (*), but let's assume for contradiction that K doesn't have property (*). This means there exists graph G (with minimal $V(G) + E(G)$) with coloring $\mathcal{C}$ such that $|\mathcal{C}| = |V(K)|$ and a transversal T such that K is isomorphic to the spanning subgraph of routing graph $H(G, \mathcal{C}, T)$, but there is no rooted H-certificate in G. Since G is minimal it means $\forall A, B \in \mathcal{C} (A \neq B)$ $G[A \cup B]$ has at most one component which is not a single vertex, which means if $\exists (u, v) \in E(H)$ and $u \in A \cap T$, $v \in B \cap T$ then there is a 2-colored path from u to v in $G[A \cup B]$, on the other hand if there is no edge (u, v) in $E(H)$ which such property then $G[A \cup B] = \emptyset$, this induced that $H = H(G, \mathcal{C}, T)$. Let r be the incident vertex of q, let $Q, R \in \mathcal{C}$ be the respective color classes of r and q, so $r \in R$, $q \in Q$, $R \neq Q$. Here we have 2 cases:

Case 1: $Q = \{q\}$

Since $K - q$ has property (*), it means $K - q$ it means $G - q$ has rooted $H(G - q, \mathcal{C} \setminus Q, T - q)$-certificate, hence by adding $Q = \{q\}$ bag to it, we would get rooted H-certificate for G(Contradiction)

Case 2: $\exists x \in Q \setminus \{q\}$

Then because of minimality of G and the construction of it having 2 colored paths it has degree of 2, and it's in the 2-colored path between r and q, hence it has 2 neighbors which are from R color class, let's denote them y and z, let's contract yxz to w and give color R to w, and we would obtain

graph G' with following coloring and transversal defined as follows:
For $A \in C$, define A' as follows:

$$A' := \begin{cases} (A \setminus \{y,z\}) \cup \{w\} & \text{if } A = R, \\ A \setminus \{x\} & \text{if } A = Q, \\ A & \text{otherwise.} \end{cases}$$

For $z \in T$, define z' as follows:

$$z' := \begin{cases} w & \text{if } z \in \{y,z\}, \\ z & \text{otherwise.} \end{cases}$$

For T' we don't consider cases concerning x because it already had representative from color class Q in it (q), so removal of x doesn't affect T'.

Then $C' := \{A' : A \in C\}$ is a coloring of G' and $T' := \{t' : t \in T\}$ is a transversal of C'.
Now, we show that $H = H(G, \mathcal{C}, T)$ is isomorphic to $H(G', \mathcal{C}', T')$. Let's consider all $(s,t) \in E(H)$:

- $\{s,t\} \neq \{q,r\}$ Then yxz don't lay on any path from s to t, hence any s,t-path from G is a s',t'-path in G'
- $s \in T \setminus \{q,r\}$ and $t = r$ and $r \in \{y,z\}$:
  If $\{y,z\} \not\subseteq V(P_{s,r})$, then the s,r-path is s',r'-path, otherwise if $\{y,z\} \subseteq V(P_{s,r})$, we can obtain new s',r'-path from s,r-path by replacing the subpath between y and z by w.
- $s \in T \setminus \{q,r\}$ and $t = r$ and $r \notin \{y,z\}$:
  If $\{y,z\} \not\subseteq V(P_{s,r})$, then s,r-path is s',r'-path in G', otherwise we replace the subpath between y and z with w, and obtain new s',r'-path in G'

And yxz lies on $P_{q,r}$, then replacing yxz by w results a new q',r'-path. By considering all cases, we showed that H is isomorphic to $H(G', \mathcal{C}', T')$. Since by choice of G and G', G' has rooted $H(G', \mathcal{C}', T')$ certificate, if w is in some bag B, by replacing B with B $\{w\} \cup \{x,y,z\}$, we would obtain rooted H-certificate for G (Contradiction).

Since for both cases we got contradiction, this implies that K indeed has (*) property as well. $\square$

1.1 Kempe chains and rooted K7-minors

To find the graphs which don't have property (*), we need to construct a graph G such that it has all necessary paths between transversal vertices, to construct a routing graph, but not enough edges incident to transversal vertices so that it's impossible to have rooted minor as the routing graph.

Definition 5 (Z(G)). *For a graph G $Z(G)$ is defined as follows:*

1. $V(Z(G)) := V(G) \times \{1, 2\}$
   For example if $V(G) := \{a, b\}$*, then* $V(Z(G)) := \{\{a, 1\}, \{b, 1\}, \{a, 2\}, \{b, 2\}\}$*, in other words we are duplicating the vertices of* G *into* $Z(G)$

2. $E(Z(G)) := \{(x, i)(y, j) : xy \in E(G) \wedge (i \neq 1 \vee j \neq 1)\}$
   Here we keep all the edges from original graph G *in the component of* $Z(G)$ *which have label* 2*, We remove all the edges between the vertices which have label* 1*, which induces an anticlique between those vertices. Note:* $\forall xy \in E(G)$*, we have the following edges in* $Z(E(G))$*:* $\{\{(x, 2), (y, 2)\}, \{(x, 1), (y, 2)\}, \{(x, 2), (y, 1)\}\}$

Let coloring for G be $\mathcal{C} := \{(x, 1)(x, 2) \ \in V(Z(G))\}$ is a coloring of $Z(G)$ and $T := V(G) \times \{1\}$ Is transversal of the coloring.

Example. An example of $Z(G)$ given G is cycle of 7

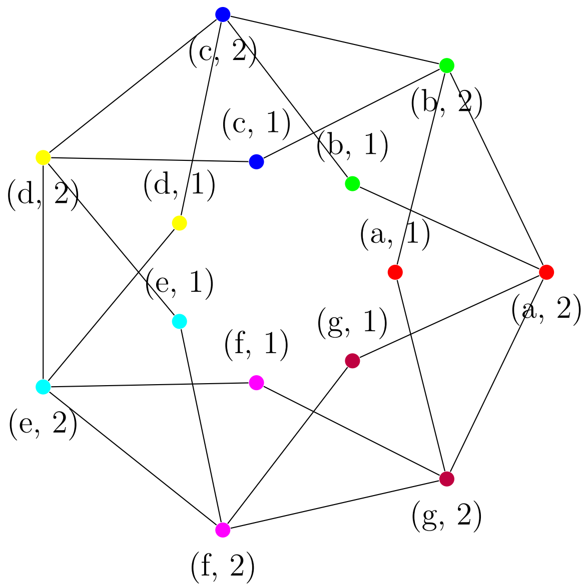


Figure 1.1 Example of $Z(G)$ given G is C_7

We can observe that routing graph $H := (Z(G), \mathcal{C}, T)$ is isomorphic to G via $((x, 1) \to x)$, and we also have copy of G in induced subgraph of $Z[V(G) \times \{2\}]$. Now we study if there exists rooted H-certificate in $Z(G)$ for different graphs of G.

Claim 4. *The bags of any* H*-certificate* $c = (V_t)_{t \in T}$ *in* $Z(G)$ *have average order at most 2.*

Proof. $|T| = |V(H)| = |V(G)|$ and $|V(Z(G))| = 2|V(G)|$, all V_t's are pairwise disjoint, hence the average size of any bag is:

$$\frac{1}{|T|} \sum_{t \in T} |V_t| \leq \frac{|V(Z(G))|}{|T|} = \frac{2|V(G)|}{|V(G)|} = 2 \tag{1.1}$$

□

This means if we have a bag with an order 3, then there is also a bag with order 1. And locally the inverse implications sounds almost the same.

Claim 5. *If $st \in E(H)$ is not on any triangle of H, then $|V_s| = 1 \implies |V_t| \geq 3$*

Proof. Let $st \in E(H)$, and suppose $V_s = \{s\}$, hence $|V_s| = 1$, $|V_t| \geq 2$ because s, t are not adjacent in $Z(G)$. If $|V_t| = 2$, then for $u \in V(Z(G))$, $V_t = \{t, u\}$, this means there is an edge $su \in E(Z(G))$ as well, at the same time the corresponding u' of u in $V(H)$ should be adjacent with t, but since s, t are not in a triangle of H, there is no edge between s and u', which means there is no su edge as well, which is a contradiction. Hence $|V_t| \geq 3$ □

If all the bags of the certificate have order 2, then we can look at a function $f : V(G) \to V(G)$, which for a bag $V_{(x,1)} = \{(x, 1), (y, 2)\}$ is defined as $f(x) := y$. Since the bags are disjoint, f is an injection and therefore a permutation of $V(G)$. Since all the elements of each bag are connected we can observe that $xf(x) \in E(G)$, and we can represent f as a partial orientation of G, where xy is oriented from x to y if and only if $y = f(x)$. For a rooted H-certificate c in $Z(G)$ any $xy \in E(G)$ implies that $V_{(x,1)}$ and $V_{(y,1)}$ are adjacent, which is equivalent to say $f(y)$ is adjacent to $f(x)$ or x, or $f(x)$ is adjacent to $f(y)$ or y. Conversely, if f is a permutation of $V(G)$ with the following properties:

1. $(\forall x \in V(G))(xf(x) \in E(G))$
2. $xy \in G$ implies that $f(x)$ is adjacent to either y or $f(y)$, or $f(y)$ is adjacent to either x or $f(x)$

Then $V_{(x,1)} = \{(x, 1), (f(x), 2)\}$ defines an H-vertificate in $Z(G)$. Let's call such permutation as a 'good permutation' throughout this chapter

Claim 6. *If G has a good permutation, then every vertex of degree at least 3 in G is on a cycle of length at most 4 in G*

Proof. Let f be good permutation and w be a vertex of degree 3, let x, y, z be w's neighbors, WLOG $f(w) = x$ and $f(y) \neq w$. Let $u := f(y) \neq w$, if u is adjacent to w, then wyu form a triangle, and we are done.
Otherwise, let's assume u, w are not adjacent, by (**2**) condition of a 'good' permutation, $f(w) = x$ is adjacent to either y or $u = f(y)$, or $f(y) = u$ is adjacent to $f(w) = x$ or w, in any case w will be either on 4-cycle or 3-cycle. □

Theorem 7. *K_7 doesn't have property (*)*

Proof. Let graph G be a graph on 7 vertices, which is obtained by adding vertex x to C_6 and adding 2 edges to x such that the endpoints of the edges are at distance 3 from each other in C_6.

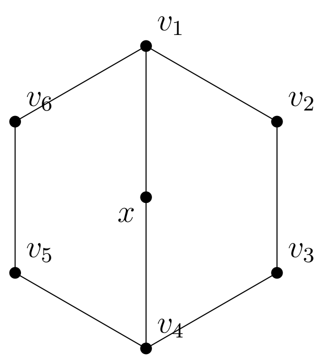


Figure 1.2 Graph G obtained from C_6

For contradiction assume that $Z(G)$ has an H-certificate $(V_t)_{t\in T}$ with $T = V(G) \times \{1\}$. Let A be the set of vertices $t \in T$ such that $|V_t| = 1$. Observe that there are 2 vertices of degree 3 in G, v_1 and v_4, and both of them are in cycle of 5, and hence by **Claim 6** G doesn't have a good permutation, which means $|A| \geq 1$. Since $|V_t| = 1$ for every element of A, it means A is an anticlique in H, hence $|A| \leq 3$. By **Claim 5**, $|V_s| \geq 3$ for every s in $N_H(A)$. For each case of $|A| = 1$, $|A| = 2$, $|A| = 3$, it can be seen that $N_H(A) \geq |A| + 1$.

- $3(|A| + 1)$ is the lower bound of the number of vertices in the bags of the neighborhood of A, because each bag has size ≥ 3 and there are at least $|A| + 1$ neighbors for A
- $1|A|$ is the number of vertices of the bags of A, because by definition each of A has size 1
- $2(7-(|A|+1)-|A|)$ is the number of vertices in the rest of the bags. Which all have $|V_t| = 2$

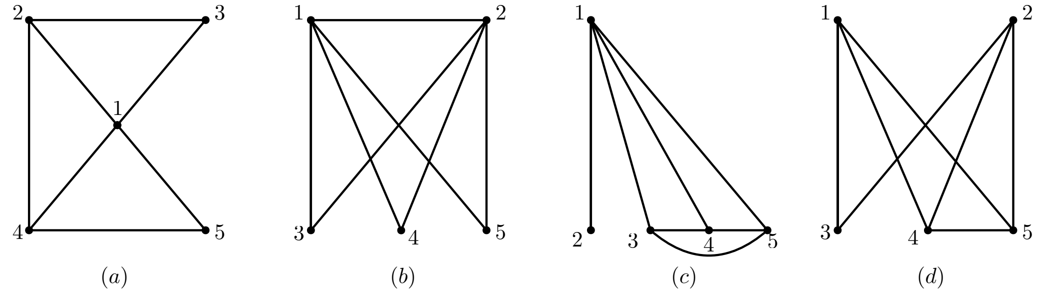
Let's denote q the number of vertices in the bags of $Z(G)$ which form H. And let's count it.

$$q = \sum_{t\in T} |V_t| \geq 3(|A| + 1) + 2(7 - (|A| + 1) - |A|) + 1|A| = 15 \tag{1.2}$$

At the same time, $q \leq |V(Z(G))| = 14$, causing a contradiction. Hence, the graph G doesn't have property (*), and since all the subgraphs of a graph with property (*) have the proprety as well, this implies that K_7 doesn't have property (*). $\square$

2 Connected transversals of 5-colorings

Kriesell and Mohr [1] showed that all graphs on five vertices and at most 6 edges has (*) property. We will try to extend this result to 7 edges. First let us see what are those graphs on five vertices and 7 edges. Let us assume the graphs are connected, otherwise by Theorem 1 in [1]. we have that the graph has (*) property. Now let's see those graphs.



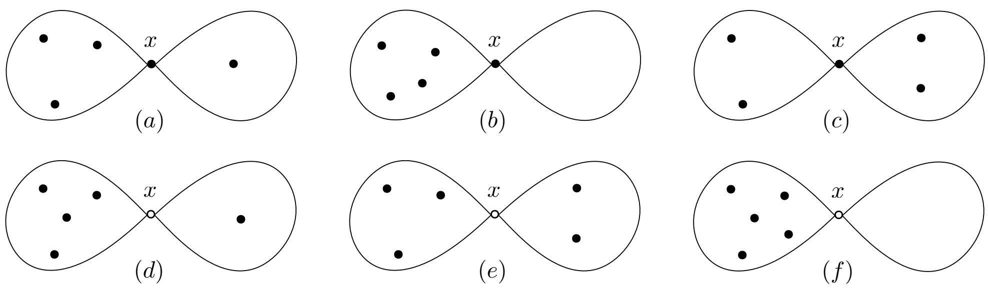
The (c) graph has (*) property since K_4 has (*) property (By theorem 4 in [1]), and $K_4 + 1$ vertex with degree 1 also has (*) property (By lemma 1 in [1]).

We will show that the minimal in terms of |V| + |E| counter example for graph (a) to not hold property (*) must be 3-connected.

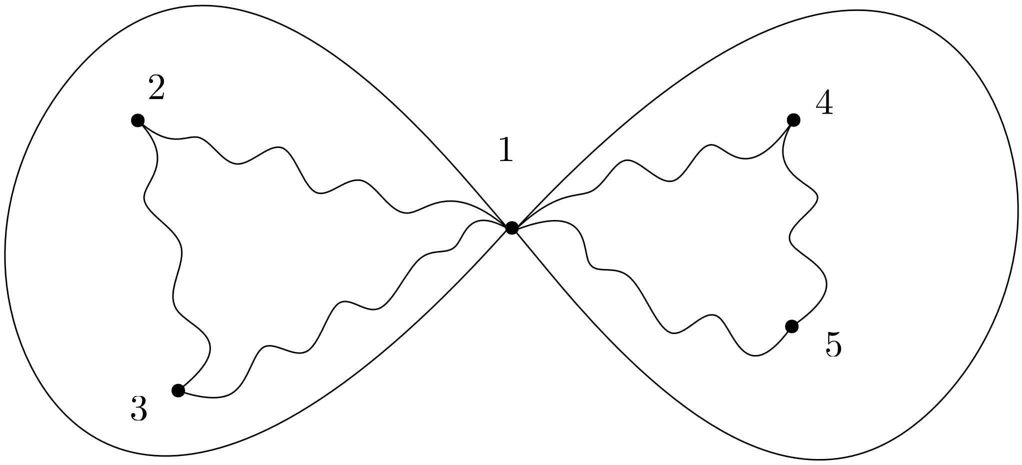
Lemma 8. *A minimal counter example such that graph (a) doesn't hold property (*) is 2-connected.*

Proof. Let's denote graph (a) as H, Let G be the minimal graph such that graph H is isomorphic to $H(G, \mathcal{C}, T)$, where transversal $T := \{1, 2, 3, 4, 5\}$, and $\mathcal{C}$ is the coloring of G The minimality implies that $(\forall A, B \in \mathcal{C} : A \neq B)$ $G[A \cup B]$ is a single path between the transversal vertices of corresponding colors from T.

Assume for contradiction that G is 1-connected. Then there exists a cut vertex in G, let us denote it as x. We have six cases to consider.



For case (a) we have that the vertex on the right side of the cut has degree at least 2, but it can have Kempe chain only to the x, hence it's a contradiction. For case (b) we can contract the edges of the right side of the cut and still get the counter example, hence the graph is not minimal, a contradiction. For case (c) the vertex x should have color 1, otherwise if it is the vertex with color 1 is on any side of the cut, we wouldn't be able to have Kempe chain to the 2 other transversal vertices of the opposite side of the cut. Now we have that x has color 1 and WLOG 2 is on the left side of the cut. And the other vertex in that same side of the cut must be of color 3 or 4, it's not 4 since it has degree 3, and it wouldn't be able to connect to vertex 5, hence we have the following graph.



But now we see that 2 and 4 can not have a Kempe chain between each other, hence they can not be connected in H, hence it's a contradiction. For the case (d) we have that the vertex on the right side of the cut has degree at least 2, hence the color of x is the color of the vertex itself, so we can contract the paths from it to x and still get the counter example, hence a contradiction since the graph wasn't minimal. For the case (e) on the right side of the cut we have that both of the vertices have degree at least 2, hence there should pass at least 3 paths from right side of the cut to the left side of the cut, in this case the color of the vertex x should be 1, and any combination of colors of vertices on right side still forces to not have a Kempe chain between at least one vertex on the right side and one vertex on the left side, hence it's a contradiction. For the case (f) we could just contract all the vertices on the right side of the cut to vertex x and still get the counter example, hence it's a contradiction. Hence the graph G is 2-connected. $\square$

Conclusion

Bibliography

1. Matthias Kriesell, Samuel Mohr. Kempe Chains and Rooted Minors. 2022.

List of Figures

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A Attachments

A.1 First Attachment