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Abstract: Use the most precise, shortest sentences that state what problem the thesis addresses, how it is approached, pinpoint the exact result achieved, and describe the applications and significance of the results. Highlight anything novel that was discovered or improved by the thesis. Maximum length is 200 words, but try to fit into 120. Abstracts are often used for deciding if a reviewer will be suitable for the thesis; a well-written abstract thus increases the probability of getting a reviewer who will like the thesis.

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Contents

1 Introduction

The four-color theorem, that every planar graph is four-colorable, was one of the central problems in graph theory for over a century. Appel and Haken proved it in 1976 [1] using a computer program. The proof was controversial because it was the first major theorem to be proved using a computer. The theorem was later proved in 1997 by Robertson, Sanders, Seymour, and Thomas [2], still using a computer but with more straightforward configurations than Appel and Haken's in several aspects.

Hadwiger's conjecture [3] suggests a generalization of the four-color theorem. It is considered one of the most challenging problems in graph theory. Before stating the conjecture, we define the notion of a *graph minor*.

Definition 1 (Minor)**.** *The G contains a graph H as a minor if there exist pairwise disjoint subsets $\{S_1, S_2, \ldots, S_{|V(H)|}\}$ of $V(G)$ such that:*

- *For each i from $1, 2, ...|V(H)|$, the subgraph of G induced by vertices S_i, denoted $G[S_i]$, is connected*
- *For every edge (u, v) in H, there is an edge of G with one end in S_u and the other end in S_v.*

Conjecture 1 (Hadwiger 1943 [3])**.** *For every integer $k \geq 0$, every graph G with no K_{k+1} minor can be colored with k colors.*

Equivalently, it can be stated that every graph with chromatic number at least k contains K_k minor.

It was proved by Hadwiger himself for cases $k \leq 3$. Wagner [4] showed that the case $k = 4$ is equivalent to the four-color theorem.

1. *Forward direction:* The case $k = 4$ implies the four color theorem since by Wagner [4], every planar graph has no K_5 or $K_{3,3}$ as minors, hence all planar graphs do not have K_5 as minor and Hadwiger's conjecture for case $k = 4$ claims if graph doesn't contain K_5 as minor, then it is 4-colorable.
2. *Reverse direction:* Wagner [4] showed that a graph G is K_5-minor free if it is obtained by clique sums of planar graphs and a Wagner graph W, where W is a 3-colorable non-planar graph on eight vertices. For any two four-colorable graphs, their clique sum is also four-colorable. Hence, the four-color theorem implies Hadwiger's conjecture for the case $k = 4$.

Robertson, Seymour, and Thomas proved the case $k = 5$ in 1993 [5], where they did not use a computer to prove it. However, they proved that a minimal counter-example to the case $k = 5$ should have a vertex whose removal results in a planar graph, reducing the problem to the four-color theorem. The case $k = 6$ is still open, and there are some results in this direction as that of Albar and Gonçalves [6].

Theorem 1. *Every graph with no K_7 minor is 8-colorable, and every graph with no K_8 minor is 10-colorable.*

Moreover, Kawarabayashi and Toft [7] proved that every 7-chromatic graph has K_7 or $K_{4,4}$ as minor.

Bollobás, Catlin, and Erdös [8] showed that the conjecture is true for almost all graphs using probabilistic arguments. In general, the cases $k \geq 6$ remain open.

Because of the conjecture's current state, there is interest in a weaker version known as the Linear Hadwiger's Conjecture.

Conjecture 2. *(Linear Hadwiger's Conjecture) There exists a constant c such that, for every $k \geq 0$, every graph with no K_{k+1} minor can be colored with ck colors.*

For more than forty years, the best-known result for the linear version was that every graph with no K_{t+1} minor is $O(t\sqrt{\log t})$-colorable. This was proved in the 1980s by Kostochka [9] and Thomason [10] independently. Recently, in 2025, Delcours and Postle [11] lowered this bound to $O(k \log\log(k))$.

Definition 2 (Rooted minor)**.** *H is a rooted minor in G if H is a minor of G and there exists a predeteremined set of vertices $\{x_i \in V(G) | i = 1, \ldots, |V(H)|\}$, called the roots, such that x_i is in S_i for all i*

Definition 3 (Colorful set)**.** *For a graph G, a set of vertices $S \subseteq V(G)$ is called colorful in G if for every chromatic coloring of G, S contains at least one vertex from each color of the coloring.*

The colorful sets of the graph are places that are 'hard to color'. Hence, we might hope to find the K_t minors rooted in those sets. And this is exactly the Halroyd's conjecture.

Conjecture 3. *(Halroyd's conjecture) Let G be a graph with chromatic number k, let S be a colorful set in G, then G has an S-rooted K_k minor.*

Holroyd called it the Strong Hadwiger Conjecture because it generalizes Hadwiger's conjecture. If we take $S = V(G)$, then the Halroyd's conjecture states that graph G has S-rooted K_k minor, but since set S is the all vertices of graph G, it means G contains K_k as a minor which is exactly the statement of Hadwiger's conjecture.

Holroyd himself proved conjecture for cases $k \leq 3$ [12], and in 2024 the case $k = 4$ was proved by Martinsson and Steiner [13].

A classical tool in studying Hadwiger's conjecture is the notion of Kempe chains. It was first introduced by Kempe in an attempt to prove the four-color theorem. Even though his proof was not correct, Kempe chains were shown to be very useful in problems related to graph coloring.

1.1 Kempe chains

Definition 4 (Kempe chain)**.** *Let G be a graph with proper coloring. For a vertex $v \in V(G)$ and two distinct colors i and j such that v has either color i or j, the Kempe chain containing v with respect to colors i and j is the maximal connected subgraph H of G such that:*

1. *Every vertex $u \in V(H)$ has either color i or j.*

2. *For every edge $(u, w) \in E(H)$, both u and w are colored only with colors i and j.*

We can observe that the vertices of colorful sets from Halroyd's conjecture must be connected by Kempe chains. Suppose, for contradiction, that there exist two vertices u, v in a colorful set, colored i and j respectively, but no Kempe chain of colors i and j connects them. Then, consider the Kempe chain of colors i and j that contains u. Swapping the colors within this chain—replacing every i with j and vice versa—produces a new proper coloring of the graph. However, this would result in u taking the same color as v, contradicting the assumption that the set is colorful. Thus, every pair of vertices in a colorful set must be connected by a Kempe chain.

Since Hadwiger's conjecture is difficult to prove in general, it is interesting to study it for specific classes of graphs. Hadwiger suggested looking into the graphs with a bounded number of optimal colorings [3], one particular class is the uniquely optimally colorable graphs.

Claim 2. *Let G be a uniquely k-colorable graph with colors $\{1, 2, \dots, k\}$ Let $v_1, v_2, \dots, v_k$ be differently colored vertices of the graph G, where v_i is has color i. Then there are Kempe chains between all pairs of v_i and v_j from $\{v_1, v_2, \dots, v_k\}$.*

Proof. For any given pair of differently colored vertices v_i and v_j, the induced subgraph of G consisting of the vertices of G with only colors of i and j is connected, if this wasn't the case then, we could color one of the disconnected components by swapping the colors i and j, resulting in another proper coloring. However, G is uniquely k-colorable, hence a contradiction. $\square$

This claim suggests a question: whether the existence of the Kempe chains forces an existence of K_k minor rooted at $\{v_1, v_2, \dots, v_k\}$.

Kriesell showed that there is a K_k minor for cases $k \leq 10$ [14], if the graph is antitriangle-free [15], Moreover, with Mohr, they showed it is true for line graphs [14].

1.2 Rooted minors

One of the central problems in Graph Theory is to find a minor in a given graph. There has been significant progress in this direction, one of which is the structure theorem of Robertson and Seymour, which says that if a graph G does not have K_t minor, then G is "almost embeddable" on a surface of low Euler genus relative to t [16]. This result leads Robertson and Seymour to prove Wagner's Conjecture. In the proof, they use the following theorems, which are proved in [17]

Theorem 3. *let G be a 3-connected graph and let v_1, v_2, v_3 be three distinct vertices. Then either G has five connected disjoint subgraphs $X_1, X_2, \dots, X_5$ such that X_i contains v_i for every $i = 1, 2, 3$ and for every $j = 4, 5$ X_j has neighbour in each X_i for all $i = 1, 2, 3$ or G is planar such that v_1, v_2, v_3 are on boundary.*

Theorem 4. *Let G be a 4-connected graph and v_1, v_2, v_3, v_4 be four distinct vertices. Then either G has four connected disjoint subgraphs $X_1, X_2, \dots, X_4$ such that X_i*

contains v_i for every $i = 1, 2, 3, 4$ and each X_i has a neighbour in X_j for every i, j such that $i \neq j$ or G is planar such that v_1, v_2, v_3, v_4 are on the boundary.

Those two results were the starting points for rooted minor problems. It turns out that rooted minors are not only useful for the proof of Graph Minor theorem, but also for some structure theorems which are used to prove some existence of graph minor, some of which are presented below:

Applications to Hadwiger's Conjecture

Robertson, Seymour, and Thomas [5] used rooted K_4–minors to prove the case $k = 5$ of Hadwiger's conjecture. Kawarabayashi and Toft [7] used rooted minors to prove that every 7-chromatic graph has K_7 or $K_{4,4}$ as minor.

Embedded graphs and face covers

In graphs embedded on surfaces, the concept of a rooted minor extends naturally to problems involving face covers. A recent paper [18] shows that in a 3-connected graph embedded in a surface of Euler genus g, if the graph has no rooted $K_{2,t}$ minor, then there exists a face cover whose size is bounded by a function of g and t. In the planar case, they got $O(t^4)$ upper bound, which improved the result of Böhme and Mohar [19].

Kempe chains and rooted minors

Usually, in the context of Hadwiger's conjecture, only clique minors were considered. However, Kriesell and Mohr [20] considered the following question, which does not necessarily look for clique minors. Let G be a graph with a proper coloring $\mathfrak{C}$, let $k = |\mathfrak{C}|$, and let $v_1, ..., v_k$ be a vertex set of G with different colors. Then, there is a system of Kempe chains for some pairs (v_i, v_j). They examined whether there is a rooted minor H of G on vertices $v_1, ..., v_k$, where H has edges between v_i, v_j if and only if there is a Kempe chain between v_i and v_j in G.

The answer to this question is affirmative for the case $k \leq 4$. For $k = 5$, it holds for graphs with at most six edges but remains open in general. The case $k = 6$ is also open, while counterexamples exist for $k \geq 7$.

In this paper, we investigate properties of minimal counterexample graphs for the case $k = 5$, where the answer is negative. Additionally, we perform a computational enumeration for $k = 6$, examining all graphs with at most sixteen vertices. Our results show that the answer remains positive for all graphs in this range.

2 Preliminaries

In this chapter, we will present definitions and primary results from the paper of Kriesell and Mohr [20], which we will use to build up our investigations.

They introduced the concept of *routing graphs* to more formally characterize the problem from the previous chapter, as we saw, was the following.

Let G be a graph with a coloring $\mathfrak{C}$, let $k = |\mathfrak{C}|$, and let $v_1, ..., v_k$ be a vertex set of G with different colors. Then, there is a system of Kempe chains for some pairs (v_i, v_j). They examined whether there is a rooted minor H of G on vertices $v_1, ..., v_k$, where H has edges between v_i, v_j if and only if there is a Kempe chain between v_i and v_j in G.

First, let us define the transversal of a set partition, and then we can define the routing graph.

Definition 5 (Transversal of a partition). *A (minimal) transversal of a partition is a set containing exactly one element from each partition member and nothing else.*

Example. Coloring $\mathfrak{C}$ of a graph partitions its vertices into color classes. A transversal T of this partition would contain exactly one vertex of each color from $\mathfrak{C}$.

Definition 6 (Routing Graph). *Let $\mathfrak{C}$ be a coloring of graph G, let T be the transversal of coloring $\mathfrak{C}$, then the routing graph $H(G, \mathfrak{C}, T)$ is the graph with vertex set T, where for every pair of vertices u, v from T, an edge (u, v) exists if and only if there is a Kempe chain between u and v in G.*

Now, we can define the problem in a more compact way, which is as follows:

Which graphs H have the property that, if H is a routing graph of some graph G with coloring $\mathfrak{C}$ and a transversal T, then G has a H-rooted minor? We say those graphs are *km-forcing.*

2.1 *km*-forcing graphs

Definition 7 (*km*-forcing). *A graph H is km-forcing if for every graph G, coloring $\mathfrak{C}$ and transversal T such that H is isomorphic to the routing graph $H(G, \mathfrak{C}, T)$, graph G has a H-rooted minor.*

Example. K_1 is *km*-forcing. If K_1 is a routing graph $H(G, \mathfrak{C}, T)$, then $|T| = 1$, and G can be colored with only one color; hence, it has no Kempe chains to other colored vertices, and K_1 is a rooted minor of G.

Example. The complete graph K_2 is *km*-forcing. For any graph G with coloring $\mathfrak{C}$ and transversal $T = \{u, v\}$ where the routing graph $H(G, \mathfrak{C}, T)$ is isomorphic to K_2, by definition there exists a Kempe chain between u and v in G. Contracting all internal vertices of this chain while keeping u and v results in an edge (u, v). In the contracted graph of G, removing all unnecessary edges and vertices would result in K_2 as a rooted minor of G.

We will list several results from [20], which capture properties of km-forcing graphs. Moreover, they are helpful for later proofs. All the theorems in this chapter are proved in [20].

Theorem 5. *If graph K is km-forcing, all its subgraphs are also km-forcing.*

Theorem 5 is crucial for later showing that K_7 is not km-forcing. It is sufficient to find a subgraph of K_7 which is not km-forcing, and by theoerm 5 this would imply that K_7 is not km-forcing as well.

Now, we will see a characterization of km-forcing graphs, which helps to show that K_4 is km-forcing.

Theorem 6. *Graph K is km-forcing if and only if every component of K is km-forcing.*

Another valuable result for further investigations is that a km-forcing graph is still km-forcing if we attach a pending edge to it.

Theorem 7. *Let K be a graph and q be a vertex with a degree of one. If $K - q$ is km-forcing, then K is km-forcing as well.*

2.2 Kempe chains and rooted K_7-minors

Definition 8. *A coloring $\mathfrak{C}$ is a Kempe coloring if any two vertices from distinct color classes belong to the same Kempe chain.*

Hadwiger [3] asked whether for a given Kempe coloring $\mathfrak{C}$ of a graph G and transversal T, the graph $H := H(G, \mathfrak{C}, T)$ is a complete graph and whether G contains a rooted H-minor. This would follow if every complete graph were km-forcing. By Theorem 5, this would imply that *every* graph is km-forcing. It would prove Hadwiger's conjecture for graphs admitting Kempe colorings if true. However, as we will see, km-forcing is too restrictive a property—and it already fails for K_7.

Theorem 8. *K_7 is not km-forcing.*

By Theorem 5, if K_7 were km-forcing, all its subgraphs would also be km-forcing. Thus, finding a subgraph of K_7 that is not km-forcing is enough.

To construct graphs that are not km-forcing, we need a graph G with:

1. Enough paths between transversal vertices to form a routing graph

2. Not many edges incident to transversal vertices so that the construction of a rooted minor fails.

A good construction with these properties is the $Z(G)$ graph:

Definition 9 ($Z(G)$)**.** *For a graph G, $Z(G)$ is defined as:*

1. ***Vertices:*** $V(Z(G)) := V(G) \times \{1, 2\}$
   Example: If $V(G) := \{a, b\}$, then $V(Z(G)) := \{(a, 1), (b, 1), (a, 2), (b, 2)\}$

2. ***Edges:*** *For each edge* (x, y) *from* $E(G)$, *the graph* $Z(G)$ *contains edges* $((x, 2)(y, 2)), ((x, 1)(y, 2)), ((x, 2)(y, 1))$.
   Formally:

$$E(Z(G)) := \{(x, i)(y, j) : xy \in E(G) \text{ and } (i \neq 1 \text{ or } j \neq 1)\}$$

Z(G) has coloring $\mathfrak{C} := \{\{(x, 1), (x, 2)\} : x \in V(G)\}$, *and transversal* $T := V(G) \times \{1\}$.

Kriesell and Mohr identified:

- A subgraph G of K_7 (Figure 2.1)
- The graph $Z(G)$ (Figure 2.2) with coloring $\mathfrak{C}$ and transversal T

such that:

- G is isomorphic to $H(G, \mathfrak{C}, T)$, but
- G is not a rooted minor of $Z(G)$

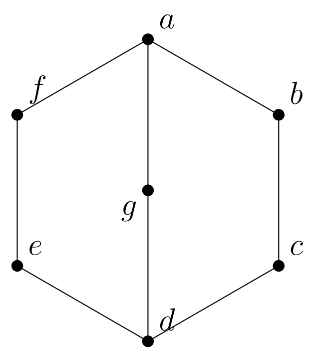

Figure 2.1 The subgraph of K_7 which is not *km*-forcing

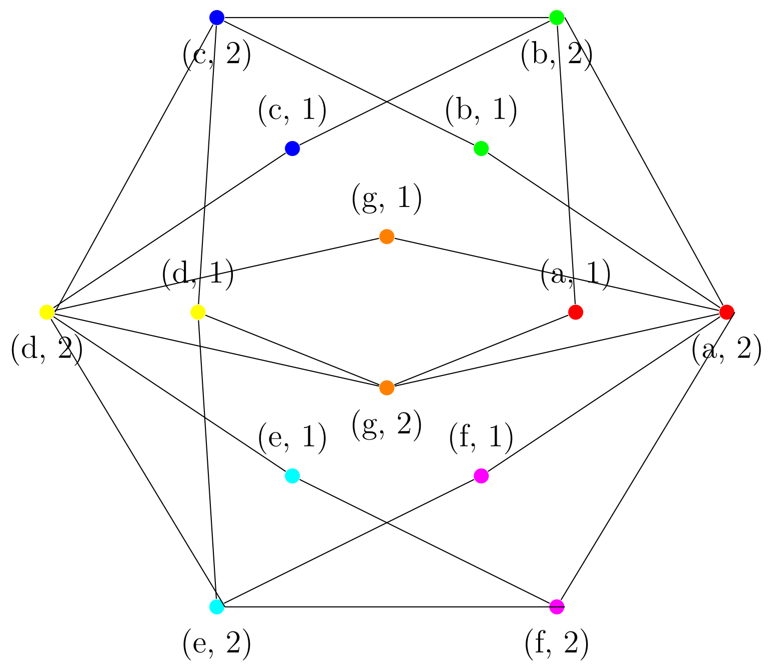

Figure 2.2 The graph $Z(G)$ given G is the graph from 2.1

In Figure 2.2, the graph $Z(G)$ is shown with coloring and transversal described in the definition of Z(G) 9. For clarity, the transversal T is the following $T := \{(a, 1), (b, 1), (c, 1), (d, 1), (e, 1), (f, 1), (g, 1)\}$.

We may be tempted to use the construction of $Z(G)$ to check whether any graph is km-forcing. However, Kriesell and Mohr showed that for any graph G with at most six vertices, $Z(G)$ always contains a G-rooted minor.

Theorem 9. *Let G be any graph with at most six vertices. Consider $Z(G)$ with coloring $\mathfrak{C}$ and the transversal T, as it's defined in 9. Then $Z(G)$ has a rooted $H(Z(G), \mathfrak{C}, T)$-minor.*

There are also positive results, one of which is that K_4 is *km*-forcing.

Theorem 10. *Every graph on at most four vertices is km-forcing.*

One might ask whether the class of *km*-forcing graphs is bounded. This is not the case, as implied by:

Theorem 11. *Every connected graph with at most one cycle is km-forcing.*

As we can see, there is a gap between K_4 and K_7. We know that K_4 is *km*-forcing and K_7 is not. What about the K_5 and K_6? The question for both of them is open, but there is a partial result on graphs with five vertices, which is the following:

Theorem 12. *Every graph on five vertices with at most six edges is km-forcing.*

This result naturally leads us to the question of what happens when we consider graphs beyond this bound. In the next chapter, we will look into graphs G, colroings $\mathfrak{C}$ and transversals T where some graph H which has at least five vertices and seven edges appears as the routing graph $H(G, \mathfrak{C}, T)$ but is not a rooted minor of G.

3 Structural Properties of Non-km-forcing Graphs

Earlier, in Theorem 8, we saw a construction called $Z(G)$, which generates graphs containing the necessary Kempe chains without introducing additional edges that would force a G-rooted minor. This suggests that connectivity plays a role in determining whether a graph is *km*-forcing.

By Theorem 12, all graphs with five vertices and at most six edges are *km*-forcing. Our goal is to investigate the connectivity properties of graphs G in which some graph H is non-*km*-forcing but appears as the routing graph $H(G, \mathfrak{C}, T)$. We show that if H has at least seven edges and five vertices, and there exists some G in which H is non-*km*-forcing, then G is 2-connected.

Lemma 13. *Let H be a graph on five vertices, Let G be a graph with coloring $\mathfrak{C}$ and transversal T such that H is isomorphic to $H(G, \mathfrak{C}, T)$; if H is not a rooted minor of G, then G is 2-connected.*

Proof. Let G be the minimal such graph, the minimality of G implies that for every distinct color class A and B from $\mathfrak{C}$ the graph G induced by A and B is a single path between the transversal vertices of corresponding colors from T.

Assume that G is 1-connected for contradiction. Then, a cut vertex exists in G; let us denote it as x.

We have six cases to consider. In the following figure, the black vertices are the transversal vertices T on which we cannot build a rooted minor isomorphic to H, and if a vertex is with a hole, then it is a non-transversal vertex.

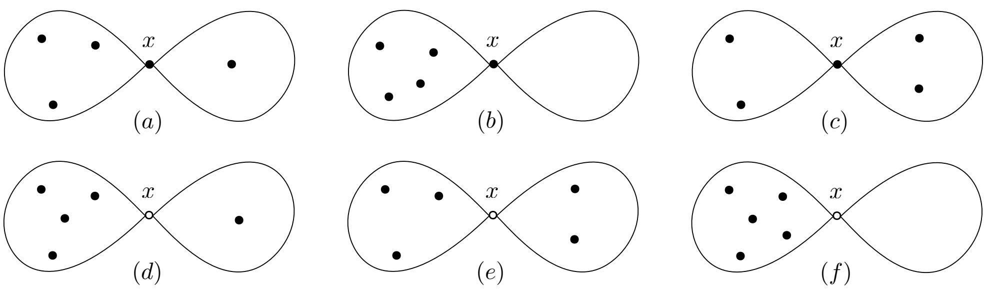

Figure 3.1 All cases of cut vertices in a 1-connected graph

Notice that for the cases (a), (b), and (c), the cut vertex x is a transversal vertex.

H does not have a vertex of degree one, otherwise By Theorem 10, we have that K_4 is *km*-forcing and $K_4 + 1$ vertex with degree one is also *km*-forcing (By Theorem 7). Since if graph is *km*-forcing then all its subgraphs are also *km*-forcing (By Theorem 5), then H would be *km*-forcing, which is a contradiction.

We have that all vertices in H have degrees at least two, and if all of them are exactly two, then H is a cycle, and By Theorem 4 in [20], we have that all cycles are *km*-forcing, hence H would be *km*-forcing, which is a contradiction.

So we have that H has at least one vertex of degree at least three; if it is exactly three and the rest are two, then by handshake lemma, we have at least two vertices of degree three. If its' degree is four and the rest of the vertices have degree two, then we get that the graph H is the hourglass graph, which by Theorem 12 is km-forcing, hence H would be km-forcing, which is a contradiction. So we have at least two vertices in H that have a degree of at least three, and the rest have a degree of at least two.

Let R be the subgraph of the right side of the cut and L be the subgraph of the left side of the cut.

- **Case (a)**: Let us denote the transversal vertex in R as y, and since all vertices in H have degrees at least two, then we have to have at least two Kempe chains between y and two other transversal vertices in G. All paths from y to other transversal vertices may only pass through x because except x, all other transversals are in L; since x is a transversal vertex itself, we can have a kempe chain from y to x, but since y and x have different colors, we cannot have more Kempe chains going from y through x to other transversal vertices. Hence, we have only one Kempe chain from y to other transversal vertices, bringing us to a contradiction.

- **Case (b)**: Unlike case (a), we can realize all necessary Kempe chains. If no Kempe chain between transversals has edges in R, we can contract all vertices of R into x and preserve all Kempe chains between transversals; hence, G was not a minimal graph, a contradiction. So let us assume there is some Kempe chain which has a subpath in R; since all Kempe chains start from L, such a Kempe chain has to pass through x; this means that if G is a counter-example; after contracting all edges of the R into x, we still have left Kempe chains between all necessary transversals in L, but we got a smaller graph which still should be a counter-example; if the contracted version is not a counter-example, then it would mean G was not a counter-example in the beginning since we could get a rooted minor H from G. Hence, the graph G is not minimal, which is a contradiction.

- **Case (c)**: If both vertices with degrees at least three are in R, let us denote them as y, z, then we can only have Kempe chains from y to x and z, and from z to x and y, but $deg(y) >= 3, deg(z) >= 3$ in H, which is a contradiction since they can have at most two Kempe chains in G in this case. Then, assume only one vertex of degree, at least three, is in R, and let it be y. Similarly, we get that y can have at most two Kempe chains, which is a contradiction. Symmetrically, the same holds if x and y were in L or one of them was in L; hence, in all cases, we get a contradiction.

- **Case (d)**: The vertex in R has a degree of at least two. Hence, the color of x is the color of the vertex itself, so we can contract the paths from it to x and still get the counter-example, which is a contradiction since the graph G was not minimal.

- **Case (e)**: Let vertices in L be denoted as a, b, c and the vertices in R as y, z, if we have at least one vertex with degree at least three in R, let it be y, then we can have one Kempe chain between y and z, and at least two Kempe chains from y to L, hence x has the color of y, but now since z has degree at least two, at least one Kempe chain from z should go to L, and it can pass only through x, since the color of x is the same color as y, the Kempe chain from z, even if it includes x, can only be connected to y; hence, it cannot pass to L, which is a contradiction. So we have that both vertices with degrees at least three are in L; in L we have three vertices, and if all of them are maximally connected to each other, we get the *abc* cycle, and since two vertices in L have a degree of at least three, at least two Kempe chains should pass from L to R, both of them pass through x, so both of them should be connected to the same transversal vertex in R, WLOG it is y, but then z in R can only be connected to y if it has Kempe chain to any other transversal than y, it has to pass through x but x has color of y, so the only Kempe chain from z including x can go to y which is a contradiction since in H $deg(z) >= 2$, we only can have one Kempe chain in G from z

- **Case (f)**: We could contract all the vertices from R into x and still get the counter-example because all the Kempe chains still exist between the transversals, it contradicts the minimality of G.

Thus, G is 2-connected. □

4 Computer Enumeration for Finding Counter-Examples in K_6

Since it remains open whether K_5 and K_6 are km-forcing, it is natural to attempt to find a counter-example. A single counter-example would show that the graph is not km-forcing. We performed a computer enumeration of the process of generating graphs G with a coloring $\mathfrak{C}$ and a transversal T for a given minor H, ensuring that H is always isomorphic to $H(G, \mathfrak{C}, T)$. We then checked whether H appears as a rooted minor in G.

The idea is similar to the previous chapter's construction of $Z(G)$. We aim to maintain all necessary Kempe chains between transversal vertices while ensuring that these vertices have no more than the necessary degrees. However, since Kriesell and Mohr [20] showed that for every graph G with at most six vertices, the construction $Z(G)$ does not provide a counter-example 9 (i.e., G remains a rooted minor of $Z(G)$), for looking for counter-examples for K_6, we must consider graphs with at least thirteen vertices.

At a high level, our algorithm for finding possible counter-examples works as follows. We begin with a given rooted minor H and construct a set of graphs with a predefined coloring but initially without edges. The coloring is chosen so that the vertices of H are rooted in these graphs; in other words, the transversal set corresponds to the graph H.

Next, for each edge of H, we construct the Kempe chain for each graph G in this set. After the Kempe chains are built, we know that H is the routing graph of G given its coloring and transversal. Then, we check whether G contains H as a rooted minor. If we find a graph G where H is not a rooted minor, we have found a counter-example.

The algorithm follows these three main steps:

1. Given a graph H, generate a set of graphs G, each with a coloring $\mathfrak{C}$ and a transversal T, such that $T = H$.
2. For each such empty graph G with its assigned coloring, construct the necessary Kempe chains.
3. For each constructed graph G, check whether G contains H as a rooted minor.

4.1 Step 1: Generating Candidate Graphs

In the first step, we generate candidate graphs G that may be counter-examples, where each graph has n vertices.

The coloring of each graph G should be such that we get graph H as a transversal of its coloring. Therefore, G will have the coloring of size $|V(H)|$. We color the vertices of each graph G using $|V(H)|$ colors in such a way that the colors are distributed as evenly as possible. The balanced distribution gives us more possibilities to create Kempe chains in the next step.

To achieve this, we divide the n vertices into layers, each complete layer containing exactly $|V(H)|$ vertices. The vertices are assigned distinct colors in every complete layer, so each color appears exactly once. The first layer will be the transversal of the coloring (layer 0).

Since m is not neccessarily an exact multiple of $|V(H)|$, there can be a set of vertices that do not form a complete layer. For these vertices, we consider all possible ways of assigning the colors. Each new coloring of these remaining vertices results in a different candidate graph.

Example. Let $H = K_3$ (so $|V(H)| = 3$) and $n = 8$. In this case, we can form two complete layers of three vertices each, each layer colored with the three colors in a fixed order. The remaining two vertices can then be colored by choosing two available colors. There are $\binom{3}{2}$ ways to select two distinct colors from three and three ways to assign the same color to both. Hence, this produces $\binom{3}{2} + 3 = 6$ distinct candidate graphs.

Algorithm 1 GenerateCandidateGraphs(H, n):

Require: Graph H all differently colored vertices, with $k = |V(H)|$, number of vertices n
Ensure: Set of candidate graphs $\mathcal{G}$

$L \leftarrow \lfloor n/k \rfloor$ ▷ Number of full layers
$R \leftarrow n \bmod k$ ▷ Number of leftover vertices
Initialize empty graph $G \leftarrow \{\}$
for $i \leftarrow 0$ to $L - 1$ **do**
  for each vertex $v \in V(H)$ **do**
    Create new vertex v'
    Set $v'.layer \leftarrow i$
    Set $v'.color \leftarrow v.color$
    Add v' to G
  end for
end for
Initialize empty set of graphs $\mathcal{G} \leftarrow \{\}$
if $R = 0$ **then**
  Add G to $\mathcal{G}$ ▷ We have only one generated graph
else
  $\mathcal{C} \leftarrow$ all combinations with replacement of R elements from $V(H)$
  for each combination $c \in \mathcal{C}$ **do**
    $G' \leftarrow$ copy of G
    for each $v \in c$ **do**
      Create new vertex v'
      Set $v'.layer \leftarrow L$
      Set $v'.color \leftarrow v.color$
      Add v' to G
    end for
    Add G' to $\mathcal{G}$
  end for
end if
return $\mathcal{G}$

Complexity Analysis. Let $k = |V(H)|$ be the number of colors and n the number of vertices to generate.

- In the beginning the algorithm creates $L = \lfloor n/k \rfloor$ full layers, each with size k. This takes $\mathcal{O}(n)$ time.
- If $R > 0$, we generate all combinations with replacement of R elements from k choices. The number of such combinations is:

$$\binom{k+R-1}{R} = \mathcal{O}(k^R)$$

  For each combination, we create a new graph with R additional nodes, which takes $\mathcal{O}(R)$ time for each graph. Hence, this step has time complexity of $\mathcal{O}(R \cdot k^R)$.
- Overall, the total time complexity is $\mathcal{O}(n + R \cdot k^R)$
- Since $R \leq k$, in the worst case we get time complexity of $\mathcal{O}(n + k^{k+1})$

4.2 Step 2: Constructing Kempe Chains

Now, for each candidate graph G and each edge of H, we construct a Kempe chain between the corresponding transversal vertices of G. After constructing all Kempe chains, we will have H as the routing graph of G, its coloring, and the transversal. First, let us have a subroutine that checks if there is a Kempe chain in the graph G for two given vertices.

Algorithm 2 EXISTSKEMPE(G, *current*, *beginning*, *end*, *visited* $= \emptyset$)

Require: graph G, current vertex, end vertex, beginning vertex, set of already visited vertices
Ensure: **True** if there exists a Kempe chain between *beginning* and *end*, **False** otherwise

if *current* = *end* **then**
    return True
end if
for all *neighbor* $\in$ *current.neighbors* **do**
    if *neighbor.color* = *current.color* or *neighbor* $\in$ *visited* **then**
        Skip iteration
    else if *neighbor.color* is the alternating color of the beginning or end **then**
        return EXISTSKEMPE(G, *neighbor*, *beginning*, *end*, *visited* $\cup \{$*neighbor*$\}$)
    end if
end for
return False

Complexity Analysis. This is a modified version of the DFS. Therefore, the time complexity is $O(m + n)$, where m is the number of edges of the graph G, and n is the number of vertices of the graph G.

Now, we can have the main algorithm, which is building a Kempe chain in a given graph G with colored vertices for two given vertices. This function also adds new variations of graph G into a common set in case there are multiple ways of building the Kempe chain between two vertices.

Algorithm 3 BuildKempeChain($G, u, v_0, w_0, \mathcal{G}, C$)

Input:

- G — Graph with colored and layered vertices
- u — Current vertex
- v_0 — Start vertex
- w_0 — End vertex
- $\mathcal{G}$ — Set of all generated graph variants
- C — the current Kempe chain

Output:

- The graph G, modified with an extended Kempe chain from v_0 to w_0 (If it exists).
- Additional graph variants, also containing the extended Kempe chain, added to the set $\mathcal{G}$

if C is empty **then**
    $C \leftarrow [u]$
end if
if $u = w_0$ **then**
    return ▷ The Kempe chain exists
end if
Determine color c to look for based on $v_0.color$ and $w_0.color$
$availableVertices \leftarrow$ all vertices with color c and not in C.
if $u.layer = 0$ **then**
    Filter $availableVertices$ to exclude vertices with $layer = 0$
end if
if $availableVertices = \emptyset$ **then**
    if ExistsKempe($G, v_0, w_0, \{v_0\}$) **then**
        return
    end if
    Remove G from $\mathcal{G}$ if present ▷ Removing degenerate cases
    return
end if
`alreadyExtended` $\leftarrow$ **false**
for all $v \in availableVertices$ **do**
    if `alreadyExtended` **then**
        $G' \leftarrow$ copy of G
        Add edge (u, v) to G'
        Add G' to $\mathcal{G}$
        BuildKempeChain($G', v, v_0, w_0, \mathcal{G}, C \cup \{w\}$)
    else
        Add edge (u, v) to G
        BuildKempeChain($G, v, v_0, w_0, \mathcal{G}, C \cup \{w\}$)
        `alreadyExtended` $\leftarrow$ **true**
    end if
end for

Complexity Analysis.

4.3 Step 3: Checking for Rooted Minors

Finally, for each constructed graph G, we check if it contains H as a rooted minor. If any such G does not contain H, we have successfully found a counter-example. For this step, we use a heuristic tool for finding minor embeddings [21]. The main utility function that we used, `find_embedding()`, is an implementation of the heuristic algorithm described in [22].

Algorithm 4 FindCounterExample(H, $\mathcal{G}$)

Input:

- H — the target minor graph
- $\mathcal{G}$ — a set of graphs to check

Output: A pair (G, H) where H is not a rooted minor of G.

for *kempeGraph* in $\mathcal{G}$ **do**
    $embedding \leftarrow find\_embedding(H, kempeGraph, suspend\_chains = H)$
    if $embedding = \emptyset$ **then**
        for $i \leftarrow 1$ **to** 10 **do**
            $embedding \leftarrow find\_embedding(H, kempeGraph, random\_seed = randomNumber, suspend\_chains = H)$
            if $embedding \neq \emptyset$ **then**
                continue the outer loop ▷ Minor exists. Skip the iteration
            end if
        end for
        return $(kempeGraph, H)$
    end if
end for

4.4 Bringing everything together

The following algorithm formalizes this process:

Algorithm 5 Searches for counter examples

Input:

- n — the number of vertices in the target minor H
- m — the number of vertices in the parent graph G

Output: A pair (G, H) where H is a routing graph of G with it's corresponding coloring and transversal but not a rooted minor in G, or **None** if no such pair exists.

$\textit{candidateMinors} \leftarrow \textsc{EnumerateNonKMForcingCandidates}(n)$
for each candidate minor $H \in \textit{candidateMinors}$ **do**
  $\textit{parentGraphs} \leftarrow \textsc{GenerateCandidateGraphs}(H, m)$
  for each parent graph $G \in \textit{parentGraphs}$ **do**
    $\textit{currentFilledGraphs} \leftarrow [G]$
    for each edge $e \in E(H)$ **do**
      $\textit{kempeChain} \leftarrow \textsc{BuildKempeChain}(G, e, \textit{currentFilledGraphs})$
    end for
    $\textit{result} \leftarrow \textsc{FindCounterExample}(H,\ \textit{currentFilledGraphs})$
    if $\textit{result} \neq$ None **then**
      return *result* ▷ Found a counter-example
    end if
  end for
end for
return None ▷ No counter-examples found

Conclusion

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List of Figures

A Attachments

A.1 First Attachment