FACULTY
OF MATHEMATICS
AND PHYSICS
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BACHELOR THESIS

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Kempe chains and rooted minors

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Study programme: Computer Science

Prague 2025

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First, I would like to thank my supervisor, Prof. Zdeněk Dvořák, for his endless patience and invaluable guidance, as well as for pointing out the mistakes I made along the way. I am also deeply grateful to my family for their support throughout my studies, and to my friends and loved ones for always being there for me. Finally, I would like to acknowledge the contributions of the authors whose works I have cited in this thesis.

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Abstract: Kempe chains are one of the basic tools used to prove variations of Hadwiger's conjecture. Thus, it is important to understand what the existence of particular Kempe chains guarantees in terms of (rooted) minors. In this thesis, we survey the known results on this topic and focus on the results of Kriesell and Mohr, who studied the following question: For any transversal T of a coloring $\mathfrak{C}$ of order k of a graph G, such that any pair of color classes induces a connected subgraph, does there exist a partition H of a subset of $V(G)$ into connected sets such that T is a transversal of H and the sets of H are pairwise adjacent whenever the corresponding vertices in T are connected by a 2-colored path? This is open for $k \geq 5$; they proved the case $k = 5$ when the subgraph induced by T is connected. They also showed that for $k \geq 7$, it is not sufficient to only guarantee 2-colored paths. We contribute by using computer-assisted enumeration to search for counterexamples for $k = 6$, and show that minimal counterexamples G for $k = 5$ must be 2-connected.

Keywords: Kempe Chain, Rooted Minor, Hadwiger's Conjecture

Contents

1 Introduction

The four-color theorem, that every planar graph is four-colorable, was one of the central problems in graph theory for over a century. Appel and Haken proved it in 1976 [10] using a computer program. The proof was controversial because it was the first major theorem to be proved using a computer. The theorem was later proved in 1997 by Robertson, Sanders, Seymour, and Thomas [17], still using a computer but with more straightforward configurations than Appel and Haken's in several aspects.

Hadwiger's conjecture [7] suggests a generalization of the four-color theorem. It is considered one of the most challenging problems in graph theory. Before stating the conjecture, we define the notion of a *graph minor*.

Definition 1 (Minor)**.** *The graph G contains a graph H as a* minor *if there exist a system $\{B_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ such that:*

- *For every vertex v from $V(H)$, the subgraph of G induced by vertices B_v, denoted $G[B_v]$, is connected.*
- *For every edge u, v in $E(H)$, there is an edge of G with one end in $G[B_u]$ and the other end in $G[B_v]$.*

We say that the system $\{B_v : v \in V(H)\}$ is a model of H in G.

Conjecture 1 (Hadwiger 1943 [7])**.** *For every integer $k \geq 0$, every graph G with no K_{k+1} minor can be colored with k colors.*

Equivalently, every graph with a chromatic number of at least k contains K_k minor.

Remark. Throughout this paper, all colorings are assumed to be proper, i.e., adjacent vertices receive different colors.

The conjecture was proved by Hadwiger himself for cases $k \leq 3$. Wagner [23] showed that the case $k = 4$ is equivalent to the four-color theorem.

1. *Forward direction:* The case $k = 4$ implies the four color theorem since by Wagner [23], every planar graph has no K_5 or $K_{3,3}$ as minors. Hence, all planar graphs do not have K_5 as minor. Hadwiger's conjecture for case $k = 4$ claims if the graph does not contain K_5 as minor, then it is 4-colorable.

2. *Reverse direction:* Wagner [23] showed that a graph G is K_5-minor free if it is obtained by clique sums of planar graphs and a Wagner graph W, where W is a 3-colorable non-planar graph on eight vertices. For any two four-colorable graphs, their clique sum is also four-colorable. Hence, the four-color theorem implies Hadwiger's conjecture for the case $k = 4$.

Robertson, Seymour, and Thomas proved the case $k = 5$ in 1993 [20], where they did not use a computer to prove it. However, they proved that a minimal counter-example to the case $k = 5$ should have a vertex whose removal results in a planar graph, reducing the problem to the four-color theorem. The case $k = 6$ is still open, and there are some results in this direction as that of Albar and Gonçalves [1].

Theorem 1. *Every graph with no K_7 minor is 8-colorable, and every graph with no K_8 minor is 10-colorable.*

Moreover, Kawarabayashi and Toft [9] proved that every 7-chromatic graph has K_7 or $K_{4,4}$ as minor. Bollobás, Catlin, and Erdös [3] showed that the conjecture is true for almost all graphs using probabilistic arguments. In general, the cases $k \geq 6$ remain open. Because of the conjecture's current state, there is interest in a weaker version known as the Linear Hadwiger's Conjecture.

Conjecture 2. *(Linear Hadwiger's Conjecture) There exists a constant c such that, for every $k \geq 0$, every graph with no K_{k+1} minor can be colored with ck colors.*

For more than forty years, the best-known result for the linear version was that every graph with no K_{t+1} minor is $O(t\sqrt{\log t})$-colorable. This was proved in the 1980s by Kostochka [11] and Thomason [22] independently. Recently, in 2025, Delcours and Postle [5] lowered this bound to $O(k \log\log(k))$.

Holroyd[8] tried to strengthen Hadwiger's conjecture by looking at specific regions in a graph where a minor of a complete graph is likely to appear based on the coloring of the region. Before stating his conjecture, we must define the concepts of rooted minors and colorful sets.

Definition 2 (Rooted minor)**.** *Let G and H be graphs and let $S = \{x_v : v \in V(H)\}$ be a system of distinct vertices of G. Then H is an S-*rooted minor *of G if exists a model $\{B_v : v \in V(H)\}$ of H in G such that for every vertex $v \in V(H)$ $x_v \in B_v$.*

Definition 3 (Chromatic coloring)**.** *Let G be a graph. The chromatic number of G denoted as $\chi(G)$ is the smallest integer k such that G is properly colored with k colors. We say that a coloring $\mathfrak{C}$ is* chromatic coloring *of the graph G if it has exactly $\chi(G)$ color classes.*

Definition 4 (Colorful set)**.** *For a graph G, a set of vertices $S \subseteq V(G)$ is called* colorful *in G if for every chromatic coloring of G, S contains at least one vertex from each color of the coloring.*

The colorful sets of the graph are places that are 'hard to color.' Hence, we might find the K_t minors rooted in those sets. And this is exactly the Holroyd's conjecture.

Conjecture 3. *(Holroyd's conjecture) Let G be a graph with chromatic number k, and let S be a colorful set in G. Then there exists a subset $S' \subseteq S$ of size k such that G contains an S'-rooted K_k minor.*

Remark. In the case of rooted clique-minors K_k, the labels of the vertices of K_k play no role, since any two bijections $S' \to V(K_k)$ differ by a permutation of the bags and hence give isomorphic models. Equivalently, once we have chosen the colourful set S' of size k, there is no need to specify which $x_v \in S'$ goes into which bag: any assignment results in an S'-rooted K_k-minor.

Holroyd called it the Strong Hadwiger Conjecture because it generalizes Hadwiger's conjecture. If we take $S = V(G)$, then the Holroyd's conjecture states that there exists an $S' \subseteq S$ of size k such that the graph G has S'-rooted K_k minor.

Since the set S is the all vertices of graph G, it means G contains K_k as a minor which is exactly the statement of Hadwiger's conjecture.

Holroyd himself proved the conjecture for cases $k \leq 3$ [8], and in 2024 the case $k = 4$ was proved by Martinsson and Steiner [15].

The notion of Kempe chains is a classical tool in studying Hadwiger's conjecture. Kempe first introduced it in an attempt to prove the four-color theorem. Even though his proof was incorrect, they were later used to prove the four-color and five-color theorems. Kempe chains show to be very useful in problems related to Hadwiger's conjecture.

1.1 Kempe chains

Definition 5 (Kempe chain)**.** *Let G be a graph with a coloring. Then, for two distinct color classes, i, and j, the* Kempe chain *in colors i and j is the maximal connected subgraph of G where vertices have only colors of i or j.*

Lemma 2. *Let G be a graph with a coloring and let S be a colorful set in G. For a color class i, let S_i be the set of vertices from S that have color i. Then, for every distinct color class i and j a Kempe chain exists between some vertices of S_i and S_j.*

Proof. Assume for contradiction that there is no Kempe chain between S_i and S_j. We will show that we can recolor the vertices of S_i with color j and still maintain a proper coloring, contradicting the assumption that S is colorful (since in the new coloring, no vertex of S has color i).

1. Let $v \in S_i$.
   - If v is not contained in any Kempe chain of colors i and j, switch the color of v to j.
   - Otherwise, v lies in some Kempe chain of colors i and j. Assuming the Kempe chain contains no vertices of S_j, we can switch the colors $i \leftrightarrow j$ along that chain. The resulting coloring is proper, where v now has color j.
2. Repeat this for each vertex of S_i. The coloring is maintained at each step, and no vertex of S_j is ever recolored (since there are no i–j chains reaching S_j). After recoloring all of S_i, the colorful set S no longer contains any vertex of color i, a contradiction.

Hence, a Kempe chain exists connecting some vertex in S_i to some vertex in S_j. □

Since Hadwiger's conjecture is difficult to prove in general, it is interesting to study it for specific classes of graphs.

Definition 6. *A graph G is called* uniquely optimally colorable *if it has only one chromatic coloring up to permutation of the colors.*

Hadwiger suggested looking into the graphs with a bounded number of optimal colorings [7]. One particular class is the uniquely optimally colorable graphs, which Kriesell proved for uniquely k-colorable graphs where $k \leq 10$ [13]. Kriesell also proved Hadwiger's conjecture for the antitriangle-free graphs [12]. Moreover, with Mohr, they proved it is true for line graphs [14].

Claim 3. *Let G be a uniquely k-colorable graph with colors $\{1, 2, \ldots, k\}$ Let $v_1, v_2, \ldots, v_k$ be differently colored vertices of the graph G, where v_i is has color i. Then there are Kempe chains between all pairs of v_i and v_j from $\{v_1, v_2, \ldots, v_k\}$.*

Proof. Let $S := \{v_1, v_2, \ldots, v_k\}$, then S is a colorful set. By the lemma 2 for every distinct color classes i and j there exists a Kempe chain between S_i and S_j. Since each color appears only once in S, each S_i has a size of 1. Hence, there is a Kempe chain between each distinct pair of the vertices of the colorful set. $\square$

This shows a structural connection between the colorful sets and the Kempe chains. It is a natural question to ask whether the existence of Kempe chains between all pairs of vertices of a colorful set S implies the existence of a K_k minor rooted at S. However, Kriesell and Mohr [16] showed that this is not true in general.

Usually, in the context of Hadwiger's conjecture, only clique minors were considered. However, Kriesell and Mohr [16] considered the following question, which does not necessarily look for clique minors. Let G be a graph with a coloring $\mathfrak{C}$, let $k = |\mathfrak{C}|$, and let $v_1, ..., v_k$ be a vertex set of G with different colors. Then, there is a system of Kempe chains for some pairs (v_i, v_j). Let H be a graph with vertices $v_1, ..., v_k$ and edges between v_i, v_j if and only if there is a Kempe chain between v_i and v_j in G. They questioned whether there is a rooted minor H of G on vertices $v_1, ..., v_k$.

The answer to this question is affirmative for the case $k \leq 4$. For $k = 5$, it holds for graphs with at most six edges but generally remains open. The case $k = 6$ is also open, while counter-examples exist for $k \geq 7$.

In the next chapter, 2, we present the main results of Kriesell and Mohr [16], where they investigate the above question. We discuss the main results from it and set up the necessary terminology for us to do our own investigations. In Chapter 3, we investigate connectivity properties of minimal counter-example graphs for the case $k = 5$. In chapter 4, we develop an algorithm which performs computer enumeration for finding counter-examples within graphs with at most 13 vertices, Our results show that there are no counter-examples for $k = 6$ within this range. We use the algorithm to verify the result of Kriesell and Mohr [16] for the case $k = 7$.

Before moving to the next chapter, we will present some results regarding the rooted minors and their applications in graph theory.

1.2 Rooted minors

One of the central problems in graph theory is finding a minor in a given graph. There has been significant progress in this direction, one of which is the structure theorem of Robertson and Seymour, which says that if a graph G does not have K_t minor, then G is "almost embeddable" on a surface of low Euler genus relative to t [18]. This result was developed as part of their proof of Wagner's Conjecture—now known as the Robertson–Seymour Theorem —which says that the class of finite undirected graphs is well-quasi-ordered under the graph minor relation. In the proof, they use the following theorems, which are proved in [19]

Theorem 4. *let G be a 3-connected graph and let v_1, v_2, v_3 be three distinct vertices. Then either G has five connected disjoint subgraphs $X_1, X_2, \dots, X_5$ such that X_i contains v_i for every $i = 1, 2, 3$ and for every $j = 4, 5$ X_j has neighbour in each X_i for all $i = 1, 2, 3$ or G is planar such that v_1, v_2, v_3 are on boundary.*

Theorem 5. *Let G be a 4-connected graph and v_1, v_2, v_3, v_4 be four distinct vertices. Then either G has K_4 minor rooted at $\{v_1, v_2, v_3, v_4\}$ or G is planar such that v_1, v_2, v_3, v_4 are on the boundary.*

Those two results were the starting points for rooted minor problems.
It turns out that rooted minors are not only useful for the proof of graph Minor theorem, but also for some structure theorems which are used to prove some existence of graph minor, some of which are presented below:

Robertson, Seymour, and Thomas [20] used rooted K_4–minors to prove the case $k = 5$ of Hadwiger's conjecture. Kawarabayashi and Toft [9] used rooted minors to prove that every 7-chromatic graph has K_7 or $K_{4,4}$ as minor.

In graphs embedded on surfaces, the concept of a rooted minor extends naturally to problems involving face covers. A recent paper [6] shows that in a 3-connected graph embedded in a surface of Euler genus g, if the graph has no rooted $K_{2,t}$ minor, then there exists a face cover whose size is bounded by a function of g and t. In the planar case, they got $O(t^4)$ upper bound, which improved the result of Böhme and Mohar [2].

2 Preliminaries

In this chapter, we will present definitions and primary results from the paper of Kriesell and Mohr [16], which we will use to build up our investigations.

They introduced the concept of *routing graphs* to more formally characterize the problem from the previous chapter, as we saw. First, let us define the transversal of a set partition, and then we can define the routing graph.

Definition 7 (Transversal of a partition)**.** *A (minimal)* transversal *of a* partition *is a set containing exactly one element from each partition member and nothing else.*

Example. Coloring $\mathfrak{C}$ of a graph partitions its vertices into color classes. A transversal T of this partition would contain exactly one vertex of each color from $\mathfrak{C}$.

Definition 8 (Routing Graph)**.** *Let $\mathfrak{C}$ be a coloring of graph G, let T be the transversal of coloring $\mathfrak{C}$, then the* routing graph *$H(G, \mathfrak{C}, T)$ is the graph with vertex set T, where for every pair of vertices u, v from T, uv edge exists if and only if there is a Kempe chain between u and v in G.*

Now, we can define the problem in a more compact way, which is as follows:

Which graphs H have the property that, if H is a routing graph of some graph G with coloring $\mathfrak{C}$ and a transversal T, then G has H as T-rooted minor? We say those graphs are *KM-forcing.*

2.1 KM-forcing graphs

Definition 9 (KM-forced)**.** *A graph H is* KM-forced *in graph G if for every coloring $\mathfrak{C}$ of G and transversal T such that H is isomorphic to the routing graph $H(G, \mathfrak{C}, T)$, the graph G has $H(G, \mathfrak{C}, T)$ as a T-rooted minor.*

Definition 10 (KM-forcing)**.** *A graph H is* KM-forcing *if H is KM-forced in every graph G.*

Remark. The term *KM-forcing* can be interpreted in two ways: as *Kriesell–Mohr forcing* or as *Kempe chain rooted minor forcing.* We leave the choice of interpretation to the reader's imagination.

Example. K_1 is KM-forcing. If K_1 is a routing graph $H(G, \mathfrak{C}, T)$, then $|T| = 1$, and G can be colored with only one color; hence, it has no Kempe chains to other colored vertices, and K_1 is a rooted minor of G.

Example. The complete graph K_2 is KM-forcing. For any graph G with coloring $\mathfrak{C}$ and transversal $T = \{u, v\}$ where the routing graph $H(G, \mathfrak{C}, T)$ is isomorphic to K_2, by definition there exists a Kempe chain between u and v in G. Contracting all internal vertices of this chain while keeping u and v results in uv edge. In the contracted graph of G, removing all unnecessary edges and vertices would result in K_2 as a rooted minor of G.

We will list several results from [16], which capture properties of KM-forcing graphs. Moreover, they are helpful for later proofs. All the theorems in this chapter are proved in [16].

Theorem 6. *If graph H is KM-forcing, all its subgraphs are also KM-forcing.*

Theorem 6 is crucial for later showing that K_7 is not KM-forcing. It is sufficient to find a subgraph of K_7 which is not KM-forcing, and by theoerm 6 this would imply that K_7 is not KM-forcing as well.

Now, we will see a characterization of KM-forcing graphs, which helps to show that K_4 is KM-forcing.

Theorem 7. *Graph H is KM-forcing if and only if every component of K is KM-forcing.*

Another valuable result for further investigations is that a KM-forcing graph is still KM-forcing if we attach a pending edge to it.

Theorem 8. *Let K be a graph and q be a vertex with a degree of one. If $K - q$ is KM-forcing, then K is KM-forcing as well.*

2.2 Kempe chains and rooted K_7-minors

Definition 11. *A coloring $\mathfrak{C}$ is a* Kempe coloring *if any two vertices from distinct color classes belong to the same Kempe chain.*

Hadwiger [7] asked whether for a given Kempe coloring $\mathfrak{C}$ of a graph G and transversal T, the graph $H := H(G, \mathfrak{C}, T)$ is a complete graph and whether G contains a T-rooted H minor. This would follow if every complete graph were KM-forcing. By Theorem 6, this would imply that *every* graph is KM-forcing. It would prove Hadwiger's conjecture for graphs admitting Kempe colorings if true. However, as we will see, KM-forcing is too restrictive a property—and it already fails for K_7.

Theorem 9. *K_7 is not KM-forcing.*

By Theorem 6, if K_7 were KM-forcing, all its subgraphs would also be KM-forcing. Thus, finding a subgraph of K_7 that is not KM-forcing is enough.

To construct graphs that are not KM-forcing, we need a graph G with:

1. Enough paths between transversal vertices to form a routing graph

2. Not many edges incident to transversal vertices so that the construction of a rooted minor fails.

A good construction with these properties is the $Z(H)$ graph:

Definition 12 ($Z(H)$)**.** *For a graph H, $Z(H)$ is defined as:*

1. ***Vertices:*** $V(Z(H)) := V(H) \times \{1, 2\}$
Example: If $V(H) := \{a, b\}$, then $V(Z(H)) := \{(a, 1), (b, 1), (a, 2), (b, 2)\}$

2. ***Edges:*** *For each edge* (x, y) *from* $E(H)$*, the graph* $Z(H)$ *contains edges* $((x, 2)(y, 2)), ((x, 1)(y, 2)), ((x, 2)(y, 1))$.
   Formally:

$$E(Z(H)) := \{(x, i)(y, j) : xy \in E(H) \text{ and } (i \neq 1 \text{ or } j \neq 1)\}$$

$Z(H)$ has coloring $\mathfrak{C} := \{\{(x, 1), (x, 2)\} : x \in V(H)\}$*, and transversal* $T := V(H) \times \{1\}$.

Kriesell and Mohr found a subgraph H of K_7 (Figure 2.1) where the graph $Z(H)$ (Figure 2.2) with coloring $\mathfrak{C}$ and transversal T defined as above. They showed that H is isomorphic to $H(Z(H), \mathfrak{C}, T)$, but H is not a T-rooted minor of $Z(H)$.

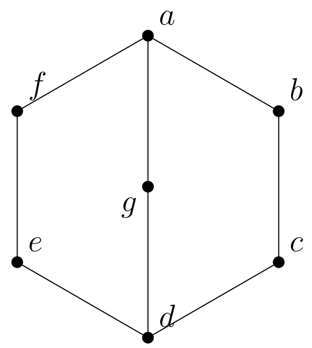

Figure 2.1 The subgraph of K_7 which is not KM-forcing

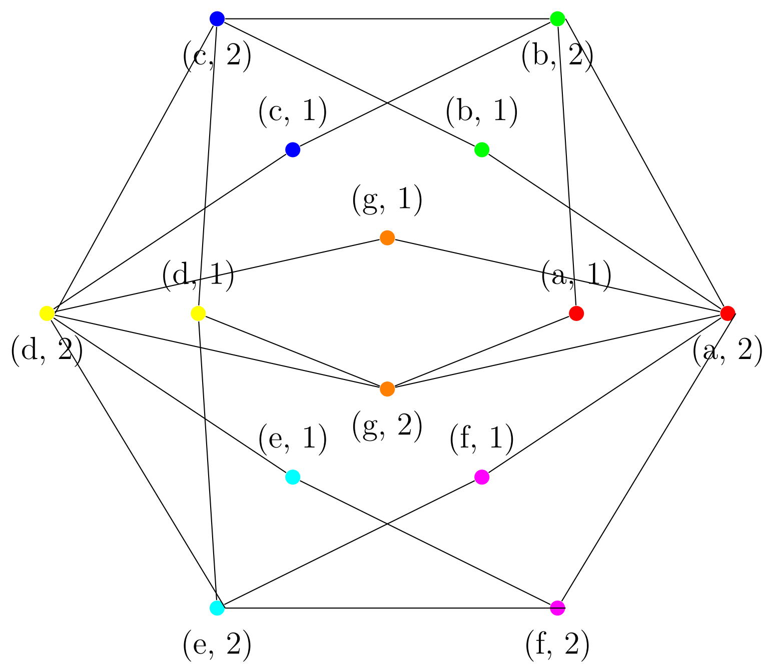

Figure 2.2 The graph $Z(H)$ given H is the graph from 2.1

In Figure 2.2, the graph $Z(H)$ is shown with coloring and transversal described in the definition of Z(H) 12. For clarity, the transversal T is the following

$T := \{(a,1),(b,1),(c,1),(d,1),(e,1),(f,1),(g,1)\}$.

(comment:) I haven't done the computational part yet for the cases with $k = 7$ that we discussed before.

We may be tempted to use the construction of $Z(H)$ to check whether any graph H is KM-forced in $Z(H)$. However, Kriesell and Mohr showed that for any graph H with at most six vertices, $Z(H)$ always contains a T-rooted minor, where T is the transversal of the coloring of $Z(H)$.

Theorem 10. *Let H be any graph with at most six vertices. Consider $Z(H)$ with coloring $\mathfrak{C}$ and the transversal T, as it's defined in 12. Then $Z(H)$ has a T-rooted $H(Z(H), \mathfrak{C}, T)$-minor.*

There are also positive results, one of which is that K_4 is KM-forcing.

Theorem 11. *Every graph on at most four vertices is KM-forcing.*

One might ask whether the class of KM-forcing graphs is bounded. This is not the case, as implied by:

Theorem 12. *Every connected graph with at most one cycle is KM-forcing.*

As we can see, there is a gap between K_4 and K_7. We know that K_4 is KM-forcing and K_7 is not. What about the K_5 and K_6? The question for both of them is open, but there is a partial result on graphs with five vertices, which are the following:

Theorem 13. *Every graph on five vertices with at most six edges is KM-forcing.*

This result naturally leads us to the question of what happens when we consider graphs beyond this bound. In the next chapter, we will look into graphs G, colorings $\mathfrak{C}$ and transversals T where some graph H which has at least five vertices and seven edges appears as the routing graph $H(G, \mathfrak{C}, T)$ but is not a rooted minor of G.

3 Structural Properties of Non-KM-forcing Graphs

We earlier saw in section 1.2 results implying the existence of rooted minors, where some connectivity constraints were assumed. Therefore, it is natural to think that connectivity of a graph G might play a role in determining whether a given graph H is KM-forced in G.

By Theorem 13, all graphs with five vertices and, at most, six edges are KM-forcing. We aim to investigate the connectivity properties of graphs G where some graph H is non-KM-forced. By the following lemma, which proves a stronger statement, we will show that if H has at least seven edges and five vertices, and there exists some graph G in which H is non-KM-forced, then smallest such G is 2-connected.

Lemma 14. *Let H and G be graphs, let $\mathfrak{C}$ be a coloring of G, and let T be the corresponding transversal such that H is isomorphic to $H(G, \mathfrak{C}, T)$.*

Suppose the following hold:

(1) $H(G, \mathfrak{C}, T)$ is not a T-rooted minor in G.

(2) Every proper subgraph H' of H is KM-forcing.

(3) If H is non-KM-forced in any other graph $G' \ncong G$, then

$$|V(G)| + |E(G)| \leq |V(G')| + |E(G')|.$$

Then G is 2-connected.

Proof. Graph H is connected, since if it were not, by condition (2) we would have that every component of H is KM-forcing, and by theorem 7 we would conclude that H is KM-forcing, which is a contradiction.

Every vertex from $V(H)$ has degree at least 2, since if some of the vertices have degree 1, then by theorem 8 and condition (2) we would conclude that H is KM-forcing, which is again a contradiction.

Condition (3) forces G to be a minimal such graph with properties (1) and (2). By minimality of G, for each pair of distinct color classes $A, B \in \mathfrak{C}$, the subgraph of G induced by $A \cup B$ is a set of isolated vertices and a single path connecting the corresponding transversal vertices in T.

Assume, for contradiction, that G is not 2-connected. Then G is either disconnected or 1-connected.

If G is disconnected, since H is connected, then the transversal vertices are in the same component of G, but then condition (3) doesn't hold, since we can have smaller graph G' which is the component where H is non-KM-forced.

If G is 1-connected, then by definition, there exists a cut vertex x splitting G into two subgraphs L and R that intersect only at x (see Figure 3.1).

Case 1: $x \in T$ i.e it is a transversal vertex.

No transversal vertex in $V(L \setminus \{x\})$ can connect to one in $V(R \setminus \{x\})$ via a Kempe chain. Because x is a cut vertex, such a chain must pass through

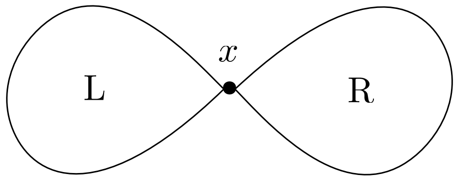

Figure 3.1 A cut vertex x splitting G into subgraphs L and R.

x. However, the Kempe chain between two distinct color classes would contain another color class of x, which is impossible. By condition (3), any Kempe chain between x and another transversal vertex is either fully contained in L or R. Hence, no Kempe chain between the transversal vertices can cross the cut vertex x.

Let T_L be the transversal vertices from subgraph L, and let H_L be the subgraph of $H(G, \mathfrak{C}, T)$ induced by T_L. Let T_R and H_R be defined similarly for R. By condition (2), both H_L and H_R are KM-forcing in the corresponding subgraphs of G, so there is a T_L-rooted minor of H_L in L and a T_R-rooted minor of H_R in R. Since the bags of models of H_L in L and H_R in R contain disjoint vertex sets (except x) and respect the transversal roots, combining them results in a T-rooted minor of $H(G, \mathfrak{C}, T)$ in G, contradicting condition (1).

Case 2: $x \notin T$ i.e x is not a transversal vertex.

If no Kempe chain passes through x between L and R, the same argument as in Case 1 applies. Otherwise, suppose there is a Kempe chain from L passing through x to R. Let a be the end vertex of the Kempe chain in L and b be the end of the Kempe chain in R. Both a and b are transversal vertices by condition (3). WLOG x has the color of b.

Let H_L be the subgraph of $H(G, \mathfrak{C}, T)$ induced by the vertices $(V(L) \cap T) \cup \{b\}$. Let $\mathfrak{C}_L$ be the coloring of L induced by the coloring $\mathfrak{C}$. Let $T_L := (V(L) \cap T) \cup \{x\}$ be the transversal of the coloring $\mathfrak{C}_L$. Since x has color of the transversal vertex b in G, all relevant Kempe chains corresponding to the edges of H_L remain in L, therefore H_L is isomorphic to $H(L, \mathfrak{C}_L, T_L)$. By condition (2), $H(L, \mathfrak{C}_L, T_L)$ is a T_L-rooted minor in L.

Let H_R be the subgraph of $H(G, \mathfrak{C}, T)$ induced by the vertices $(V(R) \cap T) \cup \{a\}$. H_R is a proper subgraph of $H(G, \mathfrak{C}, T)$, otherwise there is only one Kempe chain from L to R and the degree of a is one in H; by theorem 8 and condition (2) we would conclude that H is KM-forcing, which is a contradiction. Let R' be the graph formed from R by adding a single edge from x to a new vertex a as shown in figure 3.2. Let $\mathfrak{C}_R$ be the coloring of R' induced by the coloring $\mathfrak{C}$. Let $T_R := (V(R') \cap T)$ be the transversal of the coloring $\mathfrak{C}_R$ By the construction of H_R, the Kempe chains in R' correspond to edges of H_R, hence H_R is isomorphic to $H(R', \mathfrak{C}_R, T_R)$. By condition (2), $H(R', \mathfrak{C}_R, T_R)$ is a T_R rooted minor in R'.

The models of the rooted minors of H_L in L and H_R in R' have a common color as a root with the color of a. If we can show that there is a model of H_R in R' where the bag B_a only contains a single vertex i.e $B_a = \{a\}$, then we can combine the models of H_L and H_R to get a model of $H(G, \mathfrak{C}, T)$ in G.

We concluded earlier that each vertex of H has a degree of at least 2; hence, all the Kempe chains G passing through x from L to R are connected to b. Hence,

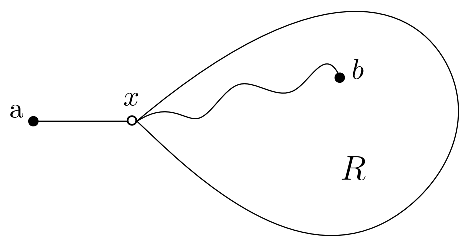

Figure 3.2 R'

a is connected to only b from the transversal vertices from $V(R) \cap T$. So we can extend the bag B_b of the model of H_R in R' to include x. This means we have a valid model of H_R in R' where the bag B_a only contains a single vertex, i.e, $B_a = \{a\}$.

Now we can combine the models of H_L in L and H_R in R' to get a model of $H(G, \mathfrak{C}, T)$ in G.

In all cases, we reach a contradiction. Therefore, G must be 2-connected. □

Corollary 15. *Let H be a graph on five vertices and at least seven edges. Let G be a graph with coloring $\mathfrak{C}$ and transversal T such that H is isomorphic to $H(G, \mathfrak{C}, T)$. If $H(G, \mathfrak{C}, T)$ is not a T-rooted in G, then minimal such G is 2-connected.*

Proof. By Theorem 13, all graphs with five vertices and, at most, six edges are KM-forcing. So every proper subgraph of H is KM-forcing. Hence, we can apply the Lemma 14 and get that G is 2-connected. □

4 Computer Enumeration for Finding Counter-Examples in K_6

Since it remains open whether K_5 and K_6 are KM-forcing, it is natural to attempt to find a counter-example. A single counter-example would show that the graph is non-KM-forcing.

Using computer-assisted enumeration, we tried to find counter-examples for all spanning subgraphs H of K_6 such that every vertex of H has a degree of at least two, and at least one vertex of H has a degree of at least three. More precisely, for each such graph H, we enumerated all (up to isomorphism) triples $(G, \mathfrak{C}, T)$, where G is a graph with at most 13 vertices, $\mathfrak{C}$ is a coloring of G, and T is a transversal such that $H(G, \mathfrak{C}, T)$ is isomorphic to H. More precisely, we restricted the enumeration to the graphs G that contain only the edges of the Kempe paths corresponding to the edges of the routing graph $H(G, \mathfrak{C}, T)$; this is clearly without loss of generality.

We then checked whether $H(G, \mathfrak{C}, T)$ is a T-rooted minor in G.

At a high level, our algorithm for finding possible counter-examples works as follows. First, we fix the vertex set of G and go over all choices of the coloring $\mathfrak{C}$. Since we are generating the graphs up to isomorphism, we can permute the vertices to any way we wish, and thus, it actually suffices to go over all possible choices $n_1, \ldots, n_k$ of the color class sizes and color the first n_1 vertices by color 1, the following n_2 vertices by color 2, etc. Moreover, we can fix the transversal T to contain exactly the first vertex in each color class.

Next, we construct graphs for each coloring where each edge of H corresponds to a Kempe chain between the corresponding transversal vertices. After the Kempe chains are built, we know that H is isomorphic to the routing graph of all those graphs G with their coloring $\mathfrak{C}$ and the transversal T. Then, we check whether those graphs contain $H(G, \mathfrak{C}, T)$ as a T-rooted minor.

Let us now discuss each of the steps in detail.

4.1 Step 1: Generating Colorings

In the first step, we generate the colorings of the n fixed vertices using exactly $k := |V(H)|$ colors. Of these n vertices, k form the transversal T and receive distinct colors. To the rest of $n - k$ vertices, we assign all possible combinations with the replacement of the colors $1, \ldots k$. Thus, there are $\binom{n-1}{k-1}$ choices for the coloring $\mathfrak{C}$.

Example. Let $H = K_3$ (so $|V(H)| = 3$) and let $n = 8$. First, we select three vertices and mark them as the transversal vertices. We give each of them a distinct color from $1, 2, 3$. Then, we have $\binom{7}{2} = 21$ ways to assign colors for the five remaining vertices. Hence, we get 21 different colorings.

Algorithm 1 GenerateColorings(H, n):

Require: Graph H all differently colored vertices, with $k = |V(H)|$, number of vertices n

Ensure: Set of colorings $\mathcal{A}$

Initialize empty coloring $\mathfrak{C} \leftarrow \{\}$
for each vertex $v \in V(H)$ **do**
    Create new vertex v'
    Set $v'.isTransversal \leftarrow True$
    Set $v'.color \leftarrow v.color$
    Add v' to $\mathfrak{C}$
end for
$R \leftarrow$ all combinations with replacement of $|V(H)|$ elements from $n - |V(H)|$
for each combination $r \in R$ **do**
    $\mathfrak{C}' \leftarrow$ copy of $\mathfrak{C}$
    for each $v \in r$ **do**
        Create new vertex v'
        Set $v'.isTransversal \leftarrow False$
        Set $v'.color \leftarrow v.color$
        Add v' to $\mathfrak{C}'$
    end for
    Add $\mathfrak{C}'$ to $\mathcal{A}$
end for
return $\mathcal{A}$

4.2 Step 2: Constructing Kempe Chains

For each coloring $\mathfrak{C}$, we now need to build graphs G such that H is isomorphic to their routing graph. To do so, for each edge of H, we try adding all possible paths joining the corresponding vertices of T and alternating between their colors. To do so, we employ a recursive procedure that receives the part of the graph that we have already finished, consisting of the paths for previously processed edges of H and of an initial segment Q of the path we are constructing for the current edge xy of H, and lists all the ways of extending this path Q to an alternating path ending in the vertex of T corresponding to y and then adding the paths representing the remaining not yet processed edges of H

Let s be the end of Q and let t be the vertex of T corresponding to y. If $s \neq t$, then let c be the color corresponding to y if Q has even length and to x if Q has odd length, i.e., different from the color of s. For each choice of a vertex $v \notin V(Q)$ of color c, we extend the path Q by adding the edge sv, then recursively process the resulting graph. If $s = t$, we have just finished the path representing the edge xy; we move on to the next edge $x'y'$ of H, replacing Q by the trivial single-vertex path consisting of the vertex of T corresponding to x'. Finally, if all edges of H have already been processed, we report the current graph, whose routing graph is necessarily isomorphic to H.

Algorithm 2 BuildKempeChains(G, *chains*, s, t, *visited*, *available*, H)

Input:

- G — adjacency map of the graph
- *chains* — list of pairs (s_i, t_i) to connect
- s — end of current path
- t — the vertex we want to reach by extending the current path
- *visited* — all vertices v such that the current path can potentially be extended by adding the edge sv.
- *available* — list of available vertices to extend chain
- H — minor to test existence against

Procedure:

if $s = t$ **then**
    if $chains = \emptyset$ **then**
        if TestMinor(H, $E(G)$, $V(H)$) **then return**
        else
            report counterexample
        end if
    else ▷ Building new chain
        $(s', t') \leftarrow chains.extract()$
        $c \leftarrow \text{color}(t')$
        $available \leftarrow \{v \in V(G) : \text{color}(v) = c\}$
        BuildKempeChains(G, *chains*, $s', t', \{s'\}$, *available*, H)
    end if
    return
end if
for all $v \in available$ **do**
    add edge (s, v) in G
    $visited' \leftarrow visited \cup \{v\}$
    $available' \leftarrow \{w \in V(G) : w \notin visited' \wedge \text{color}(w) = color(s)\}$
    BuildKempeChains(G, *chains*, v, t, *visited'*, *available'*, H)
    remove edge (s, v) from G
end for

4.3 Step 3: Checking for Rooted Minors

Finally, for each constructed graph G, we check whether it contains $H(G, \mathfrak{C}, T)$ as a T-rooted minor. If not, we have found a counter-example.

For this step, we use a heuristic tool for finding minor embeddings [21]. The primary function we rely on is `find_embedding()`, which implements the heuristic algorithm described in [4]. Since `find_embedding()` is a heuristic, we can be sure it is correct when it returns an embedding of H in G. However, when it returns no embedding, we cannot immediately conclude that one does not exist.

To reduce the false negatives, if no embedding is found, we rerun the function 10 additional times with different random seeds. While the authors of the algorithm do not provide an estimate for the probability of false negatives (returning no embedding when one exists) or any probabilistic quantity that we can rely on, in our experience in most cases, the function returns an embedding on the first try.

The algorithm is the following:

Algorithm 3 TestMinor(H, G, S)

Input:

- H — the target minor graph
- G — the graph to look for the minor in
- S — a set of rooted vertices on which we want to get the rooted minor H

Output: True if H is a rooted minor of G, otherwise False.

```
embedding ← find_embedding(H, G)
if embedding ≠ ∅ then
    return True
end if
for i ← 1 to 10 do
    seed ← random integer
    embedding ← find_embedding(H, G,
    random_seed = seed, suspend_chains = S)
    if embedding ≠ ∅ then
        return True
    end if
end for
return False
```

4.4 Bringing everything together

We combine all the subroutines discussed in the previous paragraphs for the main algorithm. For a given graph H and a number of vertices n that each graph G will have, we first generate all possible colorings for n and H, where the transversal vertices T of each coloring $\mathfrak{C}$ are mapped to the vertices of H. Then, for each such coloring $\mathfrak{C}$, we build graphs G consisting only of Kempe chains, such that there is a Kempe chain between two transversal vertices of the coloring if and only if those two vertices form an edge in H. After constructing all Kempe chains, we test whether $H(G, \mathfrak{C}, T)$ is a T-rooted minor in G. If it is not, then we know that H is non-KM-forcing and that any supergraph of H is also non-KM-forcing.

Algorithm 4 Searches for counter-examples

Input:

- H — rooted minor we look for.
- n — number of vertices in the graph G

Output: Graph G where H is isomorphic to the routing graph of G with its corresponding coloring and transversal but not a rooted minor in G, or **None** if no such pair exists.

```
colorings ← GenerateColorings(H, n)
for all coloring 𝔆 in colorings do
    G ← empty graph on 𝔆
    result ← BuildKempeChains(G, E(H), s, t, ∅, available, H)
    if result ≠ None then
        return result                          ▷ Found a counter-example
    end if
end for
return None                                    ▷ No counter-examples found
```

4.5 Results

First, with the subroutine TestMinor, we verified the result of [16] that K_7 is non-KM-forcing.

We used the main algorithm from 4.4 to primarily test whether K_6 is non-KM-forcing or not. By 6, finding any non-KM-forcing subgraph of K_6 is sufficient to prove that K_6 is non-KM-forcing. Since we also know that K_4 is KM-forcing 9 and that graphs with five vertices and at most six edges are KM-forcing, the subgraphs of K_6 we are looking for are between dense graphs on five vertices and graphs on six vertices. Moreover, we know that all cycles are KM-forcing 12, so we do not need to consider C_6. Adding a pending edge to a KM-forcing graph still keeps this property 8. So, we have a limited amount of subgraphs of K_6 to consider.

Because the algorithm has super-exponential time complexity, we could test for all candidate subgraphs H of K_6 such that H is isomorphic to $H(G, \mathfrak{C}, T)$, where G is a graph with at most 13 vertices, $\mathfrak{C}$ is a coloring of G, and T is a transversal such that $H(G, \mathfrak{C}, T)$ is isomorphic to H.

We found no counter-examples on such graphs.

(comment:) I should add the other results here regarding other subgraphs of K_7, if there are any.

Conclusion

In this thesis, we surveyed the results of Hadwiger's conjecture and Kempe chains, focusing on Kriesell and Mohr's results. We have worked on computer-assisted graph enumeration to search for counterexamples for K_6 to show that it is non-KM-forced in some graph G but found no counterexamples in graphs up to 13 vertices. We believe that there are counterexamples for K_6, but they are larger than 13 vertices, so to find them, we would need to use more computational power or a better algorithm. Furthermore, we showed that any minimal counterexample G such that K_5 is non-KM-forced in G, must be 2-connected, which might be a step towards proving the case $k = 5$.

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List of Figures

A Attachments

A.1 First Attachment