+++ title = ‘What is a Matroid?’ date = 2024-05-15T21:24:35+02:00 author = “Tigran Arsenyan” draft = false tags = [“Math”, “Combinatorial Optimization”] comments = true +++

Matroid is a structure (E, F) where E and F are sets and elements of E are ‘building’ the set F, which has the following properties:

- **(A1):** $\in F$
- **(A2):** If $X \subseteq Y$ and $Y \in F$ then $X \in F$
- **(A3):** Let $X, Y \in F$ and $|X| < |Y|$ then $\exists y \in Y \backslash X$ such that $X \cup y \in F$

If M is a matroid, then E is called the `ground set`, elements of F `independent sets`

Example

- **Ground set** E: Finite set of n elements
- **Independent sets** F All subsets of E that contain no more than k $(k <= n)$ elements.

So let: - $E :=$
$1, 2, 3, 4, 5, 6, 7.$ - $k = 3$ - Then: $F =$
$, 1, 2, 3, 1, 2, 1, 3, 2, 3, 1, 2, 3$

Independent System

If you have (E, F) such that only **(A1)** and **(A2)** hold, then (E, F) is called Independent system.

Now, you may rightfully ask, what is an example of Independent system which is not a Matriod? > {{< collapse summary=“**Answer:**” >}} Let graph $G = (V, E)$ where V is the set of vertices and E is the set of edges, Let’s take (E, F) where E(Ground set) is $E(G)$ and each element of F is a matching in G. (E, F) is a Independent system but not Matroid, because: - **(A1)** $\in F$ - **(A2)** Take matching $Y \in F$, then $\forall X \subseteq Y : X \in F$, because all subsets of Y are also matchings in G - **(A3)** Doesn’t hold: Take G := C_4 and perfect matching M_1, and any non empty other matching M_2 of C_4 such that $M_1 \cap M_2 \neq$, then take any edge edge from M_1 and add to M_2 it won’t be a matching. Hence (A3) doesn’t hold, and (E, F) is an Independent set (A1 and A2 holds), but not a matroid. {{</ collapse >}} —

Some practice with matroids

Problem: Let (E, F) be a matroid, $E' \subseteq E$ and $F' =$
$I \cap E' | I \in F$, prove that (E', F') is also a matroid.

Think of a soltuion for 5 minutes.

{{< collapse summary="**Answer:**" >}} To prove (E', F') is a matroid, we should prove that the 3 properties of the matroid hold: (A1, A2, A3): - **A1:** Since $\in F$, for $I :=$, $I \cap E' =$ hence $\in F'$., so **A1** holds - **A2:** We should show that $\forall X' \in F'$, all subsets of X are also in F'. Take $X' \in F'$, for some $I \in F$, $X' = I \cap E'$ and $I \cap E' \in F$, hence $X' \in F$ as well, this implies that all subsets of X' are also elements F(Because (E, F) is a matroid), hence all subsets of X' are also elements of F' - **A3:** let's prove it by contradiction: Which means, for an $X, Y \in F'$ s.t $|X| > |Y|$, $\forall x \in X \backslash Y |(Y \cup x) \notin F'$, from the proof of **A2** we know that for some $I_x \in F$ such that $X = I_x \cap E'$, $(I_x \cap E') \in F$, similar for some I_y and Y. This means that $|I_x \cap E'| > |I_y \cap E'|$, and property **A3** holds for them, but this is contradiction, because if there is $x \in ((I_x \cap E') \backslash (I_y \cap E'))$ such that $((I_y \cap E') \cup x \in F)$, then $(I_y \cap E') \cup x \in F'$ by definition of F'. Hence proving **A3** {{</ collapse >}} — ### Why should you care? Matroids give framework to prove the correctness of greedy algorithms. If a structure you want to maximize/minimize is a matroid, then you can use a greedy algorithm on it. Conversely if you can solve a problem using greedy algorithm, then the underlying structure must be a matroid.

This was a quick intro to matroids, structures in combinatorial optimization which can be seen in our daily life (if you are wondering where, then you just should look deeper)