LOCAL AND GLOBAL TOMOGRAPHIC IMAGE RECONSTRUCTION WITH DISCRETE RADON TRANSFORM

Nirmal Yadav[1], Tanuja Srivastava[2]
Department of Mathematics, Indian Institute of Technology, Roorkee
Roorkee, Uttrakhand (India) - 247667
[1]Email-nirmaliitr25@gmail.com

***Abstract*— Image reconstructed from its projections or Computerized Tomography is very practical area of research, which is applicable in many other fields such as material testing imaging etc. In this work we give an inversion formula using discrete convolution back projection algorithm on discrete Radon transform, which makes it faster as interpolation is not required and results are comparable.**

***Index Terms*—Radon transform, Convolution Backprojection, Fourier Transform.**

I. INTRODUCTION

In tomography the image reconstruction is referred to recovering the image function form its projections obtained as radiographs. These radiographs are best approximated by Radon transform [2]. The subject image reconstruction from projections or popularly known as computerized tomography (CT) is relatively recent area of research. But in last four decades since the CT Machine is invented, much work has been done on developing the algorithms for reconstruction and still the emphasis for getting better and better reconstructions is on. Since the projections are approximated by Radon transform for actual implementation and development of algorithm we require to either digitize the mathematical transforms for algorithms or the Radon transform and then develop a method to reconstruct the image. The first kind of methods is called transform methods and they are digitized before implementation, one such widely used reconstruction method is Convolution Back Projection (CBP) method [5, 6]. The digitization of Radon transform is quite recent and is known as discrete Radon transform (DRT) or finite Radon transform (FRAT) [1]. In present paper the reconstruction from DRT is proposed and this is compared with direct back projection of DRT.The organisation of paper is as follows; in section 2, the brief introduction of Radon transform and convolution back projection (CBP) algorithm and discrete convolution back projection (CBP) is given. In section 3, discrete Radon transform (DRT) its discrete back projection operator is given then in this section the main result of the paper the discrete convolution back projection on DRT is given, in last section the implementation of these algorithms and comparison with CBP with conclusion is given.

2. *RADON TRANSFORM AND ITS INVERSE*

In this section the classical Radon transform and finite Radon transform is explained and the inversion of Radon transform by Fourier method known as convolution back projection is given. This section is introductory and the notations and definitions taken here are used for main result.

2.1 RADON TRANSFORM

The Radon transform maps a function on N-dimensional space $\mathbb{R}^n$ into the set of its integrals over the hyper plane S^{n-1} of $\mathbb{R}^n$ [2]

$$Rf(\theta,s)=\int_{x.\theta=s} f(x)\,dx=\int_{\theta^{\perp}} f(s\theta+t)\,dt \qquad (1)$$

where $\theta\in S^{n-1}, s\in R$ *and* $\theta^{\perp}$ is hyper plane perpendicular to θ with radial distance s. In two dimension if the density function of an image $f(x,y)$, is defined in unit circle then the projections denoted by $p(s,\theta)$ is given as the line integral of density function $f(x,y)$ along the line specified by direction $\theta\in[0,2\pi]$ and distance s from the origin, with the Dirac delta function δ,

$$p(s,\theta)=\iint_{R^2} f(x,y)\delta(x\cos\theta+y\sin\theta-s)dxdy. \qquad (2)$$

2.2 CONVOLUTION BACKPROJECTION (CBP) ALGORITHM

The projection slice theorem says that the n dimensional Fourier transform of the function is equal to the one dimensional Fourier transform (along the radial direction) of the Radon transform [2]. Thus in notations

$$\hat{f}(\omega)=\hat{R}_{\theta}f(\omega).$$

This gives the inversion formula, in two dimensions this relates the projection data at enough angle to the image function $f(x,y)$ as follows

978-1-4673-4862-1/13/$31.00 ©2013 IEEE

$$\tilde{f}(x,y)=\frac{1}{(2\pi)^2}\int_0^{\pi}\int_{-\infty}^{\infty}\hat{R}_\theta f(\omega)e^{i2\pi\omega(x\cos\theta+y\sin\theta)}|\omega|W(\omega)d\omega d\theta$$

$$=\int_0^{\pi}\int_{-\infty}^{\infty}p(s,\theta)q(x\cos\theta+y\sin\theta-s)dsd\theta,$$

with

$$q(s)=\int_{-\infty}^{\infty}|\omega|W(\omega)e^{-i2\pi\omega s}d\omega$$

this inversion formula is known as convolution backprojection (CBP) algorithm [5] and can be evaluted by performing one dimensional convolution

$$\tilde{p}(s',\theta)=\int_{-\infty}^{\infty}p(s,\theta)q(s'-s)ds \qquad (3)$$

followed by the backprojection

$$\tilde{f}(x,y)=\int_0^{\pi}\tilde{p}(x\cos\theta+y\sin\theta,\theta)d\theta. \qquad (4)$$

2.3 DISCRETE CBP ALGORITHM

The discrete implementation of the CBP algorithm is similar as in [6].

Suppose that the image is discretized on $M\times N$ pixels, and the projection data $p(s,\theta)$ is available for K rays and L views. Let Δs and $\Delta\theta$ be the ray and the angular spacing respectively. Then

$$\tilde{f}(m\Delta x,n\Delta y)=(\Delta\theta)(\Delta s)(\Delta s)$$
$$\sum_{l=1}^{L}\sum_{k'=-K}^{K}\sum_{k=-K}^{K}p(k'\Delta s-k\Delta s,\theta_l)q(k\Delta s)I(m\Delta x\cos\theta_l+n\Delta y\sin\theta_l-k'\Delta s) \qquad (5)$$

3. *DISCRETE RADON TRANSFORM AND ITS INVERSE*

In CBP algorithm the reconstruction formula is obtained for general case and then for actual implementation that obtained reconstruction formula is digitized. In 1987, Beylkin [1] gave the concept of discrete Radon transform (DRT), thus an inversion formula for discrete Radon transform is required. Also for this concept to apply in practice the relationship between actual projection data and DRT is to be established and then only the reconstruction by new obtained reconstruction formula can be achieved. In this section the DRT as defined in [3] is given and using the concept of CBP i.e. projection slice theorem a reconstruction formula is obtained.

3.1 DISCRETE RADON TRANSFORM

The discrete Radon transform (DRT) or finite Radon transform (FRAT) introduced by Beylkin [1], is defined as the sum of discrete image points over a certain set of lines. For a prime number n, consider the finite field Z_n with modulo n operation, so $Z_n=\{0,1,\ldots,n-1\}$, let $L_{k,l}$ denote the set of points that define the line on the lattice Z_n^2 as:

$$L_{l,k}=\{(i,j): j=li+k(\mathrm{mod}\,n), i\in Z_n\}, 0\le k<n,$$
$$L_{n,k}=\{(i,j): j\in Z_n\}.$$

The FRAT for an image function $f[i,j]$ on the finite plane is defined as

$$p(k,l)=DRT_f(k,l)=\sum_{(i,j)\in L_{k,l}}f[i,j]. \qquad (6)$$

3.2 INVERSION FORMULA FOR DISCRETE RADON TRANSFORM

In order to apply the convolution backprojection algorithm for finite Radon transform, the lines in DRT will be considered in normal form in the finite plane Z_n^2 [3]

$$L_{a,b,t}=\{(i,j)\in Z_n^2: ai+bj-t=0(\mathrm{mod}\,n)\},$$

for $a,b,t\in Z_n$.

This set of equation is similar to the set of p lines $\{L_{k,l}: l\in Z_n\}$ with the slope $k=-\frac{a}{b}\,for\ b\neq 0\ and\ k=0\ for\ b=0.$

This line set defines a new DRT as

$$R_f^{a,b}(t)=DRT_f(a,b,t)=\sum_{(i,j)\in L_{a,b,t}}f[i,j]. \qquad (7)$$

Two methods of defining the discrete lines give same line set, Let

$$u_k=\begin{cases}(-k,1), & k=0,1,\ldots\ldots,n-1\\(1,0), & k=n\end{cases}$$

then the complete projection set for the FRAT is

$$\Theta_n=\left\{\begin{matrix}(-k,1), & k=0,1,\ldots..,n-1\\(0,1), & k=n\end{matrix}\right\}.$$

Thus the normal vectors $\{(a_k,b_k): k\in Z_n^*\}$ $where\ Z_n^*=\{0,1,\ldots..,n\}$ give all the n+1 DRT projections.

3.2.1 BACK PROJECTION OF DRT

Since the DRT also satisfy the properties of Radon transform hence the discrete back projection operator on it can be defined as sum of projections (DRT) of all those lines that passes through the point. Hence, Let

$$F_{i,j}=\{(k,l): l=(j-ki)(\mathrm{mod}\,n), k\in Z_n\}\bigcup\{(n,i)\}$$

$$\tilde{f}(i,j)=\frac{1}{n}\sum_{(k,l)\in F_{i,j}}p(k,l)$$

Using finite geometry on Z_n, it can be shown that this gives exact reconstruction [3], i.e.

$$\tilde{f}(i,j)=f(i,j)\quad\forall(i,j)\in Z_n^2$$

3.2.2 DISCRETE CBP ALGORITHM ON DRT

The reconstruction method from DRT is obtained as applying the discrete CBP algorithm on DRT. The discrete CBP algorithm as given in (2.1) is used on $n\times n$ digitization of image and as the DRT is available for n rays and (n+1) views, moreover in applying CBP on DRT the interpolation is not required as the projections are calculated on discrete lines through and reconstruction. Thus discrete CBP algorithm for DRT is derived as,

$$\tilde{f}(m\Delta x, n\Delta y) = (\Delta\theta)(\Delta s)(\Delta s) \sum_{k=1}^{K}\sum_{l'=-L}^{L}\sum_{l=-L}^{L} p(l'\Delta s - l\Delta s, \theta_l) q(l\Delta s) I(m\Delta x \cos\theta_k + n\Delta y \sin\theta_k - l'\Delta s) \tag{8}$$

In the above Eq.8 change $(m\Delta x, n\Delta y)$ with $(i, j), (\Delta s)$ $with \frac{1}{n} and$ $(\Delta\theta)$ $with \frac{1}{n}$ then,

$$\tilde{f}(i, j) = \frac{1}{n^2}\sum_{k=0}^{n}\sum_{l'=0}^{n-1}\sum_{l=0}^{n-1} p(l'-l, k) q(l) I(i\cos k + j \sin k - l') \tag{9}$$

Since (i, j) lies exactly on the line $L_{a,b,k}$ (from the definition of $L_{a,b,k}$), the interpolation operator is not required, thus Eq.9 can be written as

$$\tilde{f}(i, j) = \frac{1}{n^2}\sum_{k=0}^{n}\sum_{l=0}^{n-1} p(ai + bj - l, k) q(l)$$

The function $q(l)$ is same as $q(l\Delta s)$ in discrete CBP for Radon transform. Note that for applying discrete CBP algorithm on DRT the condition that p is prime, is not required. Also for implementation we do "unfold" the DRT for projection data, thus the number of views are increased from n+1 any other number as CBP requires equispaced views.

3.3 CBP FILTERS

In CBP algorithm defined above we use a suitably chosen window function $W(\omega)$ in the convolving function (Filters) $q(s)$. Most used window functions reported in literature [2, 5, 6, 7, 8] with compact support $[-A, A]$ are:

Band Limited

For $0 \le \in \le 1$,

$$W(\omega) = \begin{cases} 1 - \frac{\in \omega}{A}, & |\omega| \le A \\ 0, & |\omega| > A \end{cases}$$

and

Sinc Function (Shepp-Logan)

$$W(\omega) = \begin{cases} \frac{\sin\left(\frac{\pi\omega}{2A}\right)}{\frac{\pi\omega}{2A}}, & |\omega| \le A \\ 0, & |\omega| > A \end{cases}$$

These window functions give the Filters as Bandlimited

$$q(l\Delta s) = \begin{cases} \frac{A^2}{3}(3 - 2\in), & l = 0 \\ \frac{-4A^2}{\pi^2 l^2}(1-\in), & l = odd \\ \frac{-4A^2 \in}{\pi^2 l^2}, & l = even \end{cases}$$

with $0 \le \in \le 1$, in this paper three values $\in = 0, 1, \frac{1}{2}$ are used for experimentation.

Sinc Function

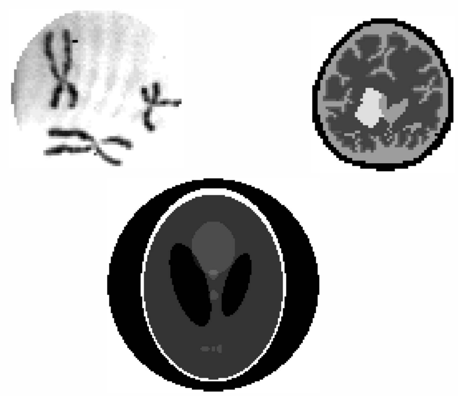
$$q(l\Delta s) = \frac{4A^2}{\pi^2}\frac{2}{1 - 4l^2}$$

With these filters there is one more filter named as optimal filter designed for discrete CBP algorithm [6] is also used for reconstruction in results.

4. IMPLEMENTATION OF ALGORITHMS

4.1 Test Images:

For implementation following three images are considered.



Pic.1: Chromosomes Pic.2: Cross Section of Brain
Pic.3: Shepp Phantom

Fig. 4.1: Digitized test images first four are 64X64 digitized images and last is 128X128 digitized image.

For implementation of discrete CBP algorithm on DRT and discrete CBP algorithm the projection data is simulated for 100 views and 64 rays for 64X64 digitized images and 200 views and 128 rays for 128X128 digitized image, while for implementation of discrete back projection algorithm on DRT, the images are increased to next prime number i.e. 3 rows and columns are extended in 64X64 digitized images and then projection data is simulated for 68 views and 67 rays of extended in 67X67 digitized images, after the reconstruction again these extra 3 rows and columns were deleted and got back the64X64 reconstruction, in last case it is 131X131 and projection data is of 132 views and 131 rays. These reconstructions are also compared with discrete CBP algorithm as well. The reconstruction results are shown in figures 4.2 to 4.4. Since radon transform is not local we require all the projections for any fixed point of an image [4].

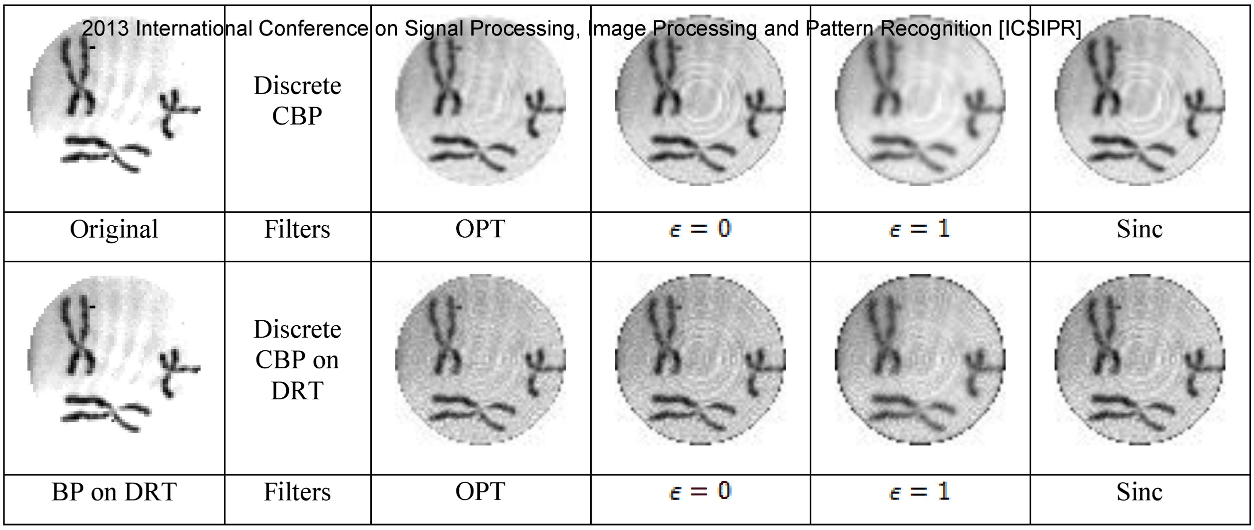


Fig. 4.2: The original image and reconstructions with Discrete Back Projection on DRT, and discrete CBP algorithm on DRT using band limited, sinc and optimal filters for Pic.1 (Chromosomes 64X64).

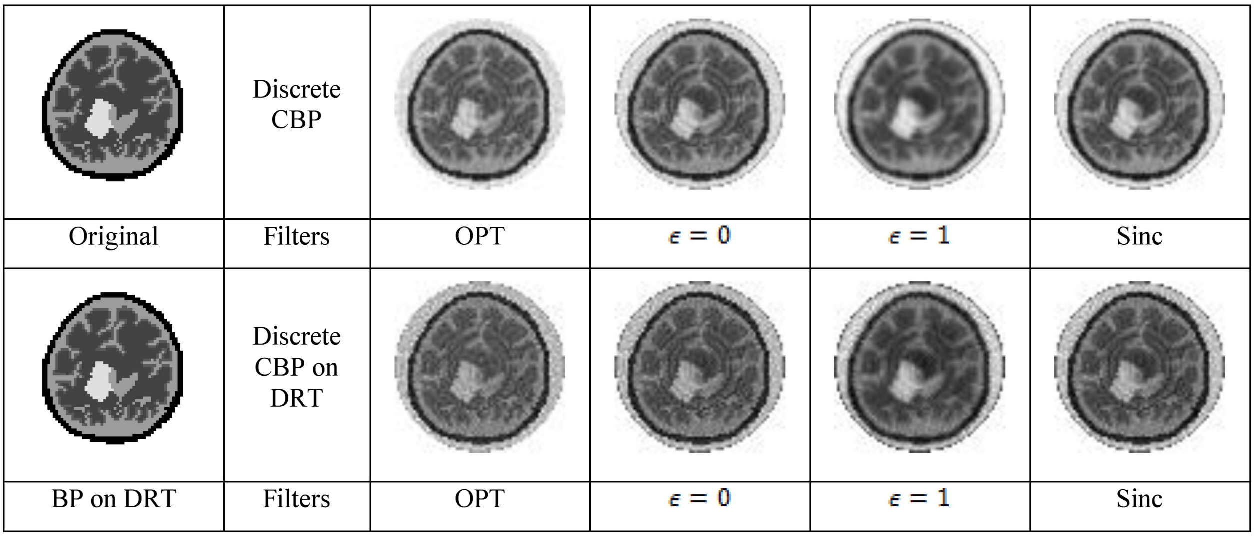


Fig. 4.3: The original image and reconstructions with Discrete Back Projection on DRT, and discrete CBP algorithm on DRT using band limited, sinc and optimal filters for Pic.2 (Cross section of Brain 64X64).

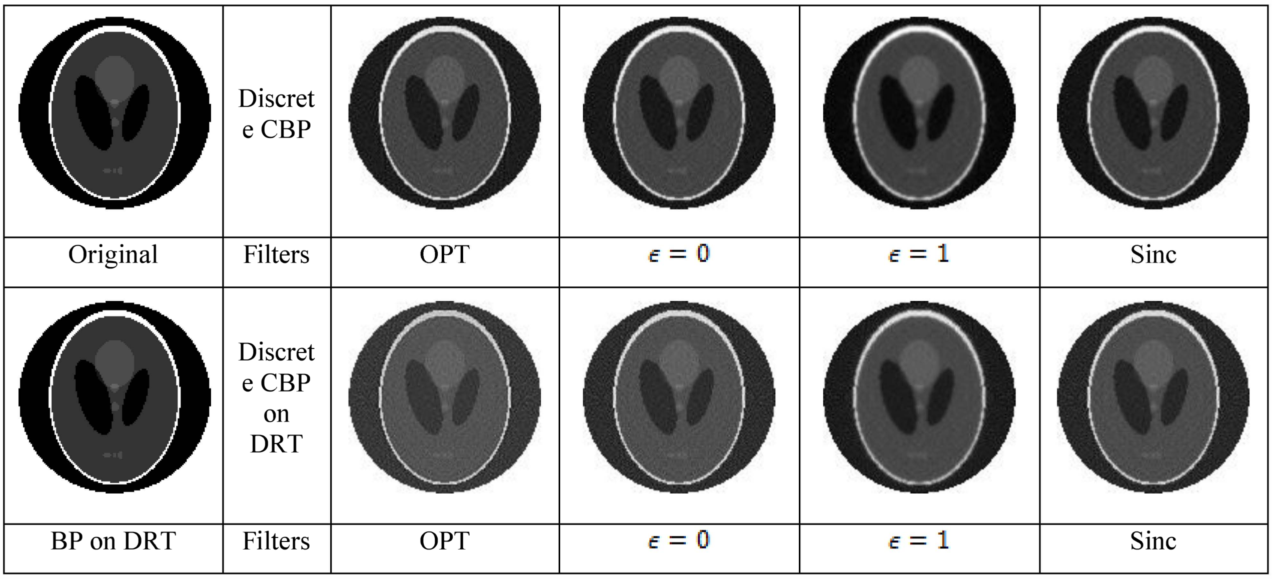


Fig. 4.4: The original image and reconstructions with Discrete Back Projection on DRT, and discrete CBP algorithm on DRT using band limited, sinc and optimal filters for Pic.3 (Shepp Phantom 128X128).

ANALYSIS OF RESULTS AND CONCLUSION

We have obtained the reconstruction images of chromosomes and cross section of brain for 64X64 pixels and the shepp phantom image is reconstructed with 128X128 pixels. Table 5 illustrates the accuracy of the reconstructed images with discrete convolution Backprojection algorithm on discrete radon transform using different filters, defined in section 3.3. From Table 5, we observe that due to bigger data size, the results obtained for shepp phantom are more accurate.

In the above results, the error energy in the reconstruction image with standard convolution Backprojection results are computed by comparing the results of both the algorithms. This paper presents an algorithm discrete convolution Backprojection using discrete radon transform and The CBP on DRT gives an approximation of original image, while when we are directly taking back projection of DRT, due to "folding" of lines in DRT, it gives exact reconstruction.

Table 5 Accuracy of reconstruction using different filters

	OPTIMAL	RAMO	RAM	SHEPP-LOGAN
CHROMOSOME	29%	21%	26%	15%
BRAIN	53%	41%	26%	39%
SHEPP PHANTOM	7%	6%	-8%	-9%

REFERENCES

[1] G. Beylkin, "Discrete Radon Transform", *IEEE Trans. Acoustics, Speech Signal Process. ASSP*, vol.35, pp. 162-172, 1987.

[2] S.R.Deans, "Radon Transform and some of its Applications", John *Wiley and Sons*, *New York*, 1983.

[3] M. N. Do and M. Vetterli, "The finite Ridgelet Transform for Image Representation", *IEEE Transactions on Image Processing*, pp.16-28, 2003.

[4] A. Faridani, E. L Ritman. and K.T. Smith, "Local tomography", *SIAM J. Appl. Math.*, Vol. 52, pp. 459-484, 1992.

[5] Tanuja Srivastava, RKS Rathore and I.D. Dhariyal, "Direct Estimates of Statistical Error for Convolution Back-projection Algorithm in Computerized tomography", *Journal of Combinatorics Information and system sciences*, vol.17, pp. 271-287, 1992.

[6] Tanuja Srivastava, RKS Rathore, I.D. Dhariyal, P.Munshi and R. Rastogi , "Design of Optimal Statistical Filter for Discrete Convolution Back- projection Method", *American Journal of Mathematical and Management Sciences,* vol. 14, pp 229-265, 1994.

[7] Tanuja Srivastava, "Statistical Method of Estimation of ESE in CBP", *American Journal of Mathematical and Management Sciences*, vol 24, pp 291-320, 2004.

[8] Tanuja Srivastava, "Error Analysis in Convolution Backprojection Algorithm: An Empirical Validation of ω_c-Estimate", *Int. J. Tomogr. Stat.*, vol. 4, No. S-06, pp 42-62,2006.