Maximizing volume given a surface area constraint

Math 8

Department of Mathematics

Dartmouth College

Maximizing wih a constraint

We wish to solve the following problem:
Find the maximum volume of a rectangular box that has a surface area of $70cm^2$.

Maximizing wih a constraint

We wish to solve the following problem:
Find the maximum volume of a rectangular box that has a surface area of $70cm^2$.

- We begin by translating this into a set of equations:

$$V = lwh$$

$$SA = 2lw + 2wh + 2lh$$

Maximizing wih a constraint

We wish to solve the following problem:
Find the maximum volume of a rectangular box that has a surface area of $70cm^2$.

- We begin by translating this into a set of equations:

$$V = lwh$$

$$SA = 2lw + 2wh + 2lh$$

- Solving the surface area equation for h yields

$$h = \frac{35 - lw}{l + w}$$

Defining V in terms of l and w

So far, we have

$$h = \frac{35 - lw}{l + w}$$

- Plugging this into the volume equation yields

$$V = lw\left(\frac{35 - lw}{l + w}\right) = \frac{35lw - l^2w^2}{l + w}$$

Defining V in terms of l and w

So far, we have

$$h = \frac{35 - lw}{l + w}$$

- Plugging this into the volume equation yields

$$V = lw\left(\frac{35 - lw}{l + w}\right) = \frac{35lw - l^2w^2}{l + w}$$

- We will now maximize this function for positive values of l and w - negative values do not make sense physically.

Defining V in terms of l and w

So far, we have

$$h = \frac{35 - lw}{l + w}$$

- Plugging this into the volume equation yields

$$V = lw\left(\frac{35 - lw}{l + w}\right) = \frac{35lw - l^2w^2}{l + w}$$

- We will now maximize this function for positive values of l and w - negative values do not make sense physically.

To do this, we follow our maximization/minimization procedure.

1. Find the critical points of V with $l \geq 0$ and $w \geq 0$
2. Test critical points and boundary points to find maximum

Calculating V_l

First, calculate the partial derivative of V with respect to l:

$$\frac{\partial}{\partial l}V = \frac{\partial}{\partial l}\frac{35lw - l^2w^2}{l+w}$$

Calculating V_l

First, calculate the partial derivative of V with respect to l:

$$\frac{\partial}{\partial l} V = \frac{\partial}{\partial l} \frac{35lw - l^2w^2}{l + w}$$

$$= \frac{(35w - 2lw^2)(l + w) - 35lw + l^2w^2}{(l + w)^2} = \frac{35lw - 2l^2w^2 + 35w^2 - 2lw^3 - 35lw + l^2w^2}{(l + w)^2}$$

Calculating V_l

First, calculate the partial derivative of V with respect to l:

$$\frac{\partial}{\partial l} V = \frac{\partial}{\partial l} \frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35w - 2lw^2)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{\color{red}{35lw} - \color{blue}{2l^2w^2} + 35w^2 - 2lw^3 - \color{red}{35lw} + \color{blue}{l^2w^2}}{(l+w)^2}$$

Calculating V_l

First, calculate the partial derivative of V with respect to l:

$$\frac{\partial}{\partial l} V = \frac{\partial}{\partial l} \frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35w - 2lw^2)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{35lw - 2l^2w^2 + 35w^2 - 2lw^3 - 35lw + l^2w^2}{(l+w)^2}$$

$$= \frac{-l^2w^2 + 35w^2 - 2lw^3}{(l+w)^2}$$

Calculating V_l

First, calculate the partial derivative of V with respect to l:

$$\frac{\partial}{\partial l}V = \frac{\partial}{\partial l}\frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35w - 2lw^2)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{35lw - 2l^2w^2 + 35w^2 - 2lw^3 - 35lw + l^2w^2}{(l+w)^2}$$

$$= \frac{-l^2w^2 + 35w^2 - 2lw^3}{(l+w)^2}$$

$$= \frac{w^2}{(l+w)^2}(35 - 2lw - l^2)$$

Calculating V_w

Second, calculate the partial derivative of V with respect to w:

$$\frac{\partial}{\partial w} V = \frac{\partial}{\partial w} \frac{35lw - l^2w^2}{l + w}$$

Calculating V_w

Second, calculate the partial derivative of V with respect to w:

$$\frac{\partial}{\partial w}V = \frac{\partial}{\partial w}\frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35l - 2l^2w)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{35l^2 - 2l^3w + 35lw - 2l^2w^2 - 35lw + l^2w^2}{(l+w)^2}$$

Calculating V_w

Second, calculate the partial derivative of V with respect to w:

$$\frac{\partial}{\partial w} V = \frac{\partial}{\partial w} \frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35l - 2l^2w)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{35l^2 - 2l^3w + 35lw - 2l^2w^2 - 35lw + l^2w^2}{(l+w)^2}$$

Calculating V_w

Second, calculate the partial derivative of V with respect to w:

$$\frac{\partial}{\partial w} V = \frac{\partial}{\partial w} \frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35l - 2l^2w)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{35l^2 - 2l^3w + 35lw - 2l^2w^2 - 35lw + l^2w^2}{(l+w)^2}$$

$$= \frac{35l^2 - l^2w^2 - 2l^3w}{(l+w)^2}$$

Calculating V_w

Second, calculate the partial derivative of V with respect to w:

$$\frac{\partial}{\partial w}V = \frac{\partial}{\partial w}\frac{35lw - l^2w^2}{l+w}$$

$$= \frac{(35l - 2l^2w)(l+w) - 35lw + l^2w^2}{(l+w)^2} = \frac{35l^2 - 2l^3w + 35lw - 2l^2w^2 - 35lw + l^2w^2}{(l+w)^2}$$

$$= \frac{35l^2 - l^2w^2 - 2l^3w}{(l+w)^2}$$

$$= \frac{l^2}{(l+w)^2}(35 - 2lw - w^2)$$

Find critical points

From the previous computations, we have

$$\nabla V(l,w) = \left(\frac{w^2}{(l+w)^2}(35 - 2lw - l^2), \frac{l^2}{(l+w)^2}(35 - 2lw - w^2) \right)$$

Find critical points

From the previous computations, we have

$$\nabla V(l,w) = \left(\frac{w^2}{(l+w)^2}(35 - 2lw - l^2), \frac{l^2}{(l+w)^2}(35 - 2lw - w^2) \right)$$

Assuming, for physical reasons, that l and w are nonzero, we need to solve the system

$$35 - 2lw - l^2 = 0$$

$$35 - 2lw - w^2 = 0$$

Find critical points

From the previous computations, we have

$$\nabla V(l,w) = \left(\frac{w^2}{(l+w)^2}(35-2lw-l^2), \frac{l^2}{(l+w)^2}(35-2lw-w^2)\right)$$

Assuming, for physical reasons, that l and w are nonzero, we need to solve the system

$$35 - 2lw - l^2 = 0$$

$$35 - 2lw - w^2 = 0$$

- Subtracting the second equation from the first yields:

$$(35-2lw-l^2) - (35-2lw-w^2) = 0$$

$$-l^2 + w^2 = 0$$

$$w = \pm l$$

Find critical points

From the previous computations, we have

$$\nabla V(l,w) = \left(\frac{w^2}{(l+w)^2}(35 - 2lw - l^2), \frac{l^2}{(l+w)^2}(35 - 2lw - w^2) \right)$$

Assuming, for physical reasons, that l and w are nonzero, we need to solve the system

$$\begin{aligned} 35 - 2lw - l^2 &= 0 \\ 35 - 2lw - w^2 &= 0 \end{aligned}$$

- Subtracting the second equation from the first yields:

$$\begin{aligned} (35 - 2lw - l^2) - (35 - 2lw - w^2) &= 0 \\ -l^2 + w^2 &= 0 \\ w &= \pm l \end{aligned}$$

- Again, to be physically reasonable, both l and w are positive so, $l = w$

Solving for critical points

So far, we have that $l = w$.
Plugging this into the first equation $35 - 2lw - l^2 = 0$, we get

$$35 - 2l^2 - l^2 = 0$$

Solving for critical points

So far, we have that $l = w$.
Plugging this into the first equation $35 - 2lw - l^2 = 0$, we get

$$35 - 2l^2 - l^2 = 0$$

Or,

$$l^2 = \frac{35}{3}$$

Solving for critical points

So far, we have that $l = w$.
Plugging this into the first equation $35 - 2lw - l^2 = 0$, we get

$$35 - 2l^2 - l^2 = 0$$

Or,

$$l^2 = \frac{35}{3}$$

Or, again noting the l must be positive:

$$l = \left(\frac{35}{3}\right)^{\frac{1}{2}}$$

Solving for critical points

So far, we have that $l = w$.
Plugging this into the first equation $35 - 2lw - l^2 = 0$, we get

$$35 - 2l^2 - l^2 = 0$$

Or,

$$l^2 = \frac{35}{3}$$

Or, again noting the l must be positive:

$$l = \left(\frac{35}{3}\right)^{\frac{1}{2}}$$

Putting this together with the fact that $w = l$ and $h = \frac{35-lw}{l+w}$, we have that the only critical point in our domain is

$$\left(\left(\frac{35}{3}\right)^{\frac{1}{2}}, \left(\frac{35}{3}\right)^{\frac{1}{2}}, \frac{35 - \left(\frac{35}{3}\right)}{2\left(\frac{35}{3}\right)^{\frac{1}{2}}}\right) = \left(\left(\frac{35}{3}\right)^{\frac{1}{2}}, \left(\frac{35}{3}\right)^{\frac{1}{2}}, \left(\frac{35}{3}\right)^{\frac{1}{2}}\right)$$

Finding the absolute maximum

- The boundary of our domain are the lines $l = 0$ and $w = 0$ for $l, w \geq 0$. Noticing that on these lines, the volume and the surface area must be zero, we can ignore these as physically unreasonable solutions.

Finding the absolute maximum

- The boundary of our domain are the lines $l = 0$ and $w = 0$ for $l, w \geq 0$. Noticing that on these lines, the volume and the surface area must be zero, we can ignore these as physically unreasonable solutions.
- Thus our critical point is the only test point. The Volume at this point is

$$V = \left(\left(\frac{35}{3}\right)^{\frac{1}{2}}\right)^{3}$$

Finding the absolute maximum

- The boundary of our domain are the lines $l = 0$ and $w = 0$ for $l, w \geq 0$. Noticing that on these lines, the volume and the surface area must be zero, we can ignore these as physically unreasonable solutions.
- Thus our critical point is the only test point. The Volume at this point is

$$V = \left(\left(\frac{35}{3}\right)^{\frac{1}{2}}\right)^3$$

- The only remaining question is, is this a maximum or not?

Finding the absolute maximum

- The boundary of our domain are the lines $l = 0$ and $w = 0$ for $l, w \geq 0$. Noticing that on these lines, the volume and the surface area must be zero, we can ignore these as physically unreasonable solutions.
- Thus our critical point is the only test point. The Volume at this point is

$$V = \left(\left(\frac{35}{3}\right)^{\frac{1}{2}}\right)^3$$

- The only remaining question is, is this a maximum or not?
- To answer this we make two observations:

Finding the absolute maximum

- The boundary of our domain are the lines $l = 0$ and $w = 0$ for $l, w \geq 0$. Noticing that on these lines, the volume and the surface area must be zero, we can ignore these as physically unreasonable solutions.
- Thus our critical point is the only test point. The Volume at this point is

$$V = \left(\left(\frac{35}{3} \right)^{\frac{1}{2}} \right)^3$$

- The only remaining question is, is this a maximum or not?
- To answer this we make two observations:
- For a fixed l (say $l = 1$), we have $h = \frac{35-w}{1+w}$. As $w \to 0$, we see that $h \to 35$ and the volume tends to zero.

Finding the absolute maximum

- The boundary of our domain are the lines $l = 0$ and $w = 0$ for $l, w \geq 0$. Noticing that on these lines, the volume and the surface area must be zero, we can ignore these as physically unreasonable solutions.
- Thus our critical point is the only test point. The Volume at this point is

$$V = \left(\left(\frac{35}{3}\right)^{\frac{1}{2}}\right)^3$$

- The only remaining question is, is this a maximum or not?
- To answer this we make two observations:
- For a fixed l (say $l = 1$), we have $h = \frac{35-w}{1+w}$. As $w \to 0$, we see that $h \to 35$ and the volume tends to zero.
- As l and w tend towards ∞, we see that h eventually becomes negative, an unreasonable physical assumption.

Conclusion

We conclude that the critical point is a maximum. Thus the absolute maximum volume occurs at the point

$$\left(\left(\frac{35}{3}\right)^{\frac{1}{2}}, \left(\frac{35}{3}\right)^{\frac{1}{2}}, \left(\frac{35}{3}\right)^{\frac{1}{2}}\right)$$

where the volume is

$$\left(\frac{35}{3}\right)^{\frac{3}{2}}$$