Wiskunde 13/14

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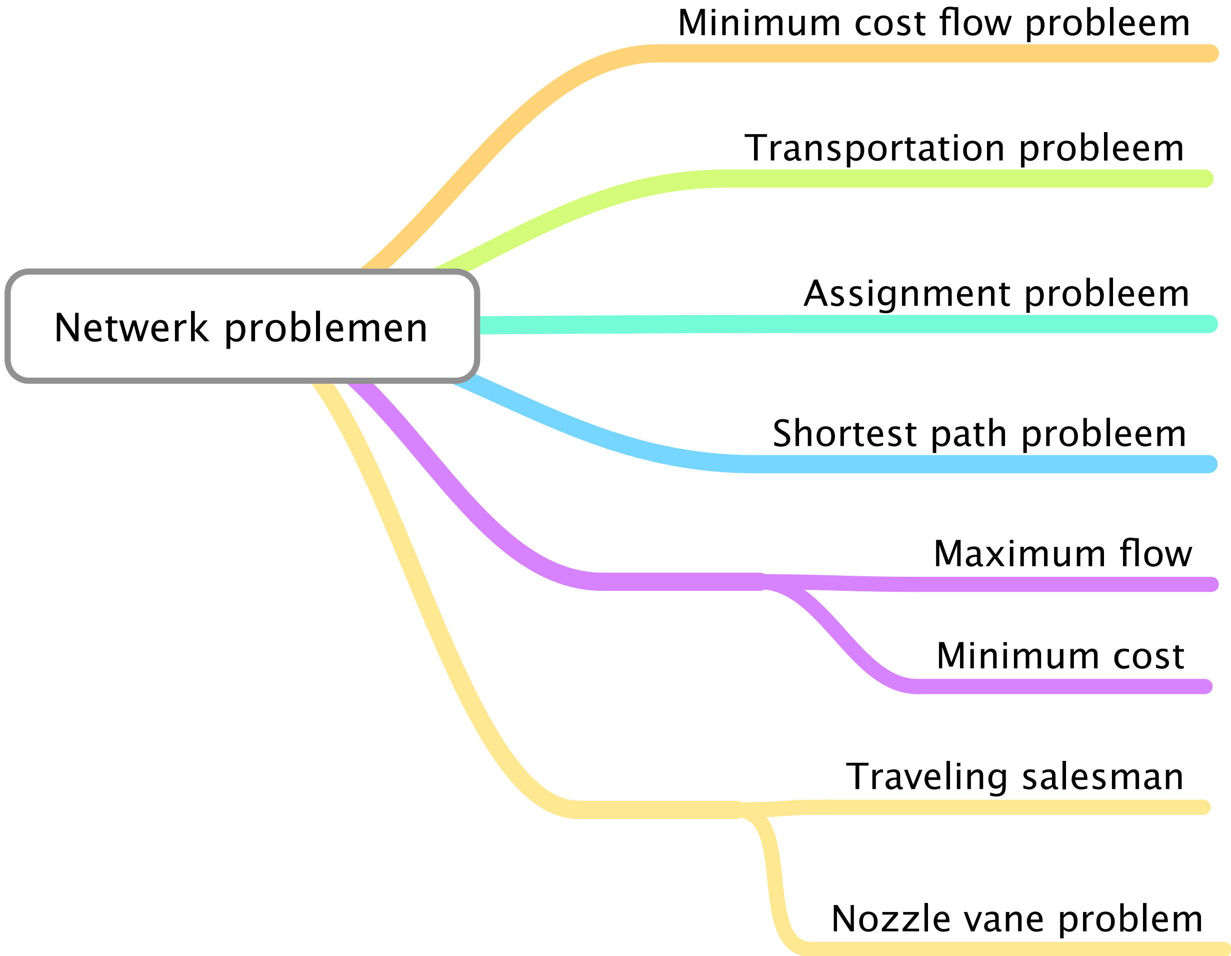
Optimization : Les 3

Organisatie van de cursus

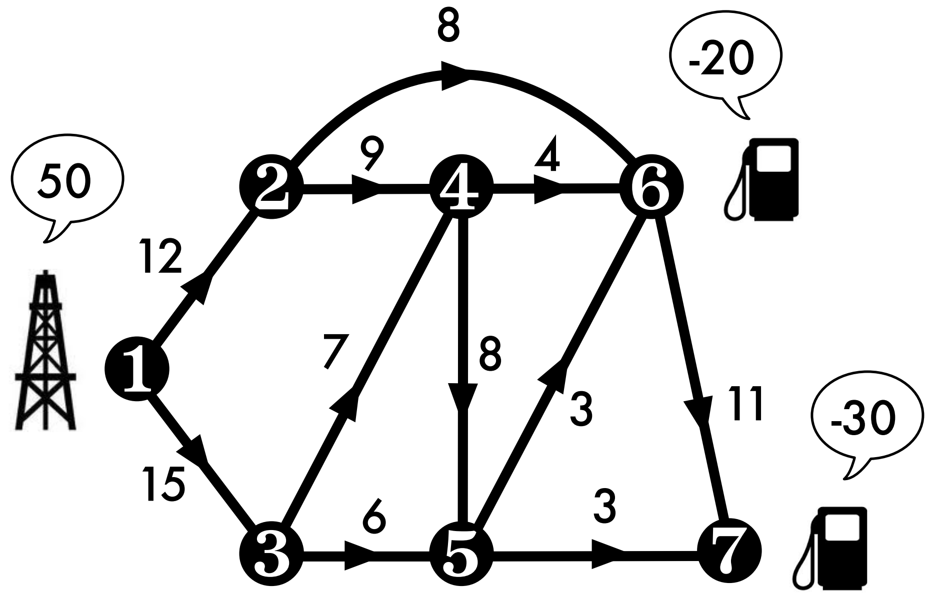
Week	50 min	100 min	100 min	
1	HC	PR	ZS	Opdracht 1
2	HC	PR	ZS	
3	HC	PR	ZS	Opdracht 2
4	HC	PR	ZS	
5		PR	ZS	Opdracht 3
6		PR	ZS	
7		PR	ZS	

Overzicht


Netwerk problemen
Minimum cost flow probleem
Transportation probleem
Assignment probleem
Shortest path probleem
Maximum flow
Minimum cost
Traveling salesman
Nozzle vane problem

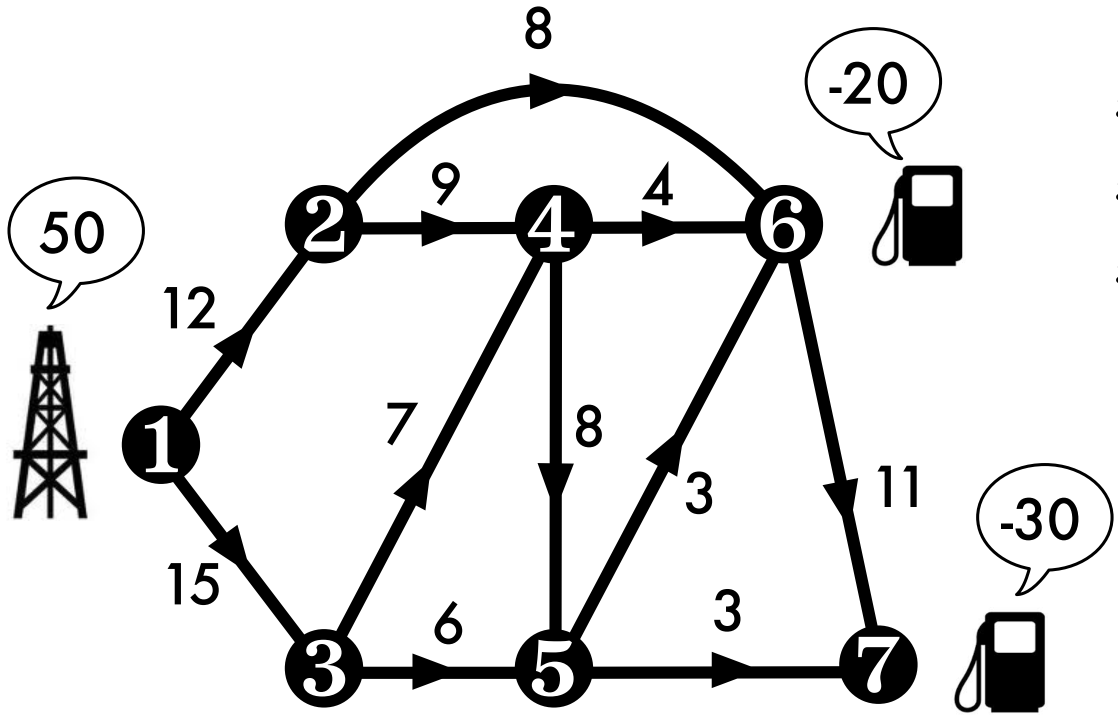
Network problems : Minimum cost flow netwerk

- Nodes : **❶❷,….,❼**
- Arcs : **(1,2)**, **(1,3)**, **(2,4)**,...,**(6,7)**
- Cost coefficients : $c_{1,2},\ c_{2,3},\ \cdots\ ,c_{6,7}$
- Flow : $x_{1,2},\ x_{2,3},\ \cdots\ ,x_{6,7}$
- Sources : Node 1, Sinks : Nodes 6,7
- Constraints :
  - Flow $\sum_j x_{i,j} - \sum_k x_{k,i} = b_i\,,$
  - Source $S \equiv \sum_{\{i:b_i>0\}} b_i$
  - Sink $D \equiv -\sum_{\{i:b_i<0\}} b_i\,.$



Network problems : Minimum cost flow netwerk

$$
\begin{aligned}
\text{minimize} \quad & z = 12x_{1,2} + 15x_{1,3} + 9x_{2,4} + 8x_{1,2} + 7x_{3,4} + \\
& \quad x_{3,5} + 8x_{4,5} + 4x_{4,6} + 3x_{5,7} + 11x_{6,7} \\
\text{subject to} \quad & x_{1,2} + x_{1,3} = 50 \\
& x_{2,4} + x_{2,6} - x_{1,2} = 0 \\
& x_{3,4} + x_{3,5} - x_{1,3} = 0 \\
& x_{4,5} + x_{4,6} - x_{2,4} - x_{3,4} = 0 \\
& x_{5,6} + x_{5,7} - x_{3,5} - x_{4,5} = 0 \\
& x_{6,7} - x_{2,6} - x_{4,6} - x_{5,6} = -20 \\
& -x_{5,7} - x_{6,7} = -30 \\
& 0 \leq x \leq 30
\end{aligned}
$$



Network problems : Minimum cost flow netwerk

Network flow problem example HC 3 Equations

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```
In[59]:= ClearAll["Global`*"]

In[60]:= z = 12 x12 + 15 x13 + 9 x24 + 8 x26 + 7 x34 + 6 x35 + 8 x45 + 4 x46 + 5 x56 + 3 x57 + 11 x67;
         c1 = x12 + x13 == 50;
         c2 = x24 + x26 - x12 == 0 ;
         c3 = x34 + x35 - x13 == 0 ;
         c4 =  x45 + x46 - x24 - x34 == 0 ;
         c5 = x56 + x57 - x35 - x45 == 0 ;
         c6 = x67 - x26 - x46 - x56 == -20 ;
         c7 = -x57 - x67 == -30;
         c8 = x12 ≥ 0 && x13 ≥ 0 && x24 ≥ 0 && x26 ≥ 0 && x34 ≥ 0 && x35 ≥ 0 &&
            x45 ≥ 0 && x46 ≥ 0 && x56 ≥ 0 && x57 ≥ 0 && x67 ≥ 0 ;
         c9 = x12 ≤ 30 && x13 ≤ 30 && x24 ≤ 30 && x26 ≤ 30 && x34 ≤ 30 && x35 ≤ 30 &&
            x45 ≤ 30 && x46 ≤ 30 && x56 ≤ 30 && x57 ≤ 30 && x67 ≤ 30;
         var = {x12, x13, x24 , x26 , x34 , x35 , x45 , x46 , x56, x57, x67};
         Minimize[{z, c1 && c2 && c3 && c4 && c5 && c6 && c7 && c8 && c9}, var, Reals]

Out[71]= {1120, {x12 → 20, x13 → 30, x24 → 0, x26 → 20,
           x34 → 0, x35 → 30, x45 → 0, x46 → 0, x56 → 0, x57 → 30, x67 → 0}}
```

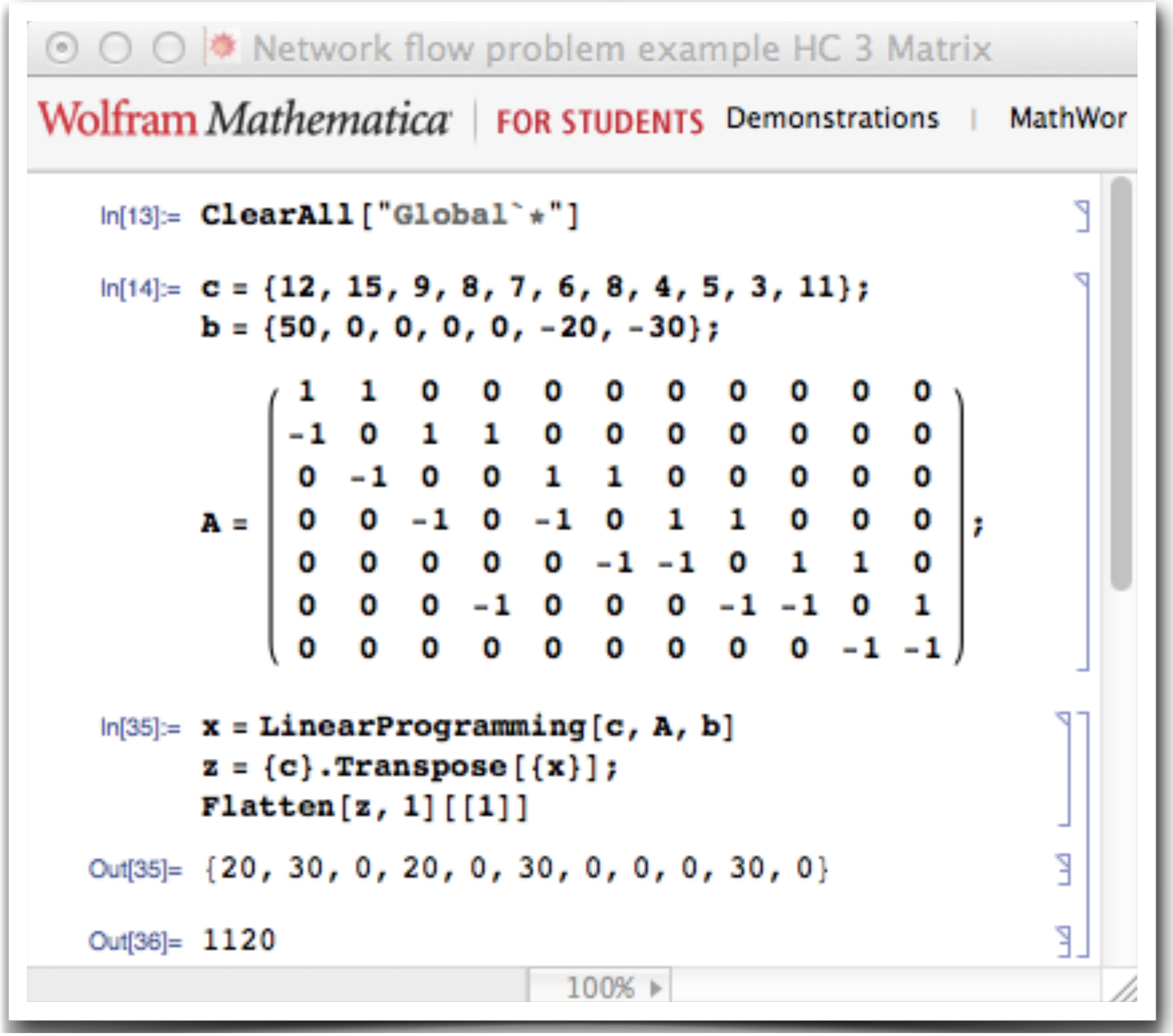
100%

Network problems : Minimum cost flow netwerk

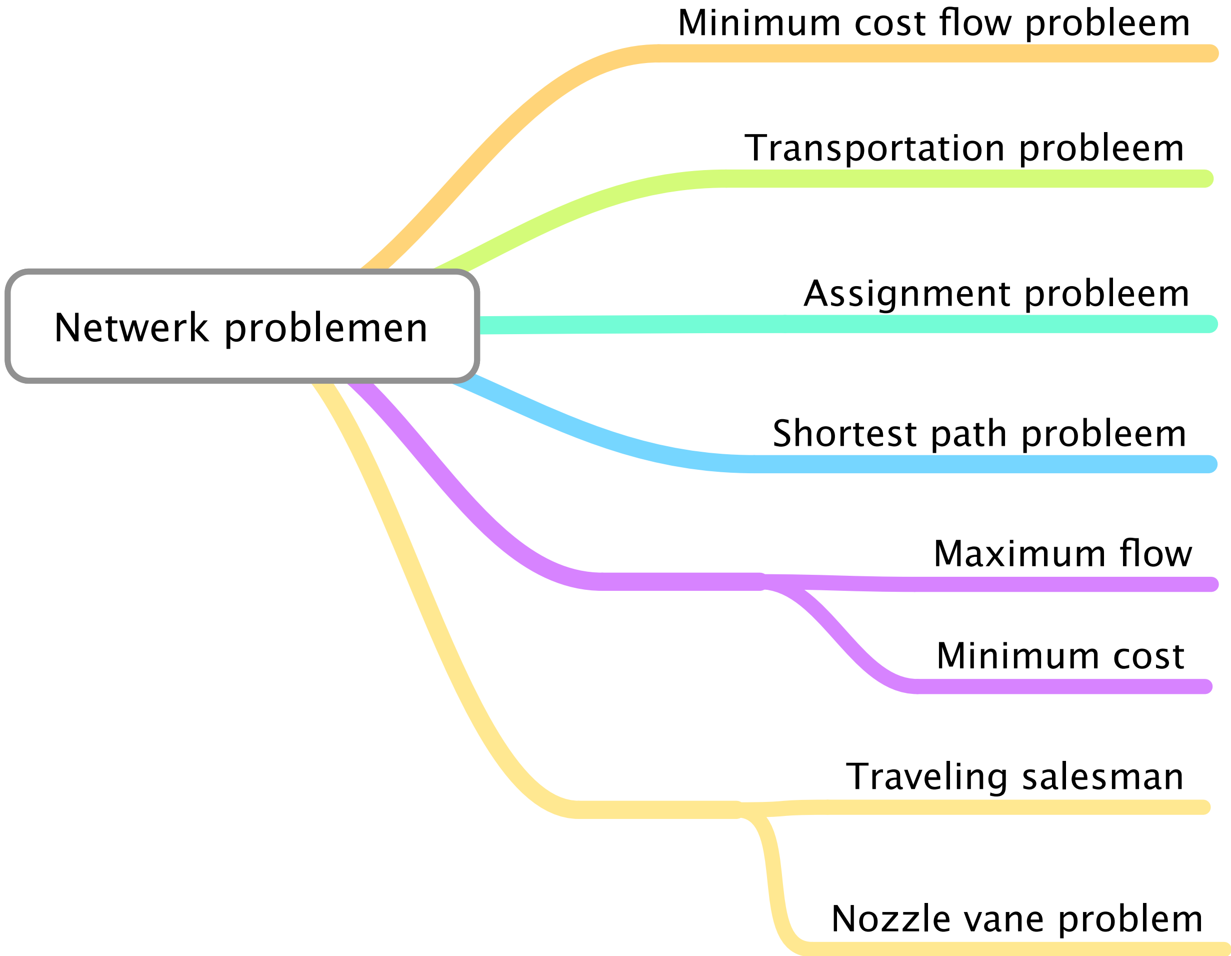
- Formuleer als matrix vergelijking :

$$\begin{aligned} \text{minimize} \quad & z = c^T x \\ \text{subject to} \quad & Ax = b \\ & \ell \le x \le u, \end{aligned}$$

- Los op met Mathematica :



Overzicht : Transportation probleem



Network problems : Transportation probleem

- Een transportation problem modelleert de verplaatsing van goederen van leveranciers naar klanten. Aan verplaatsingen zijn kosten verbonden

- Elke node is of een *source* of een *sink*

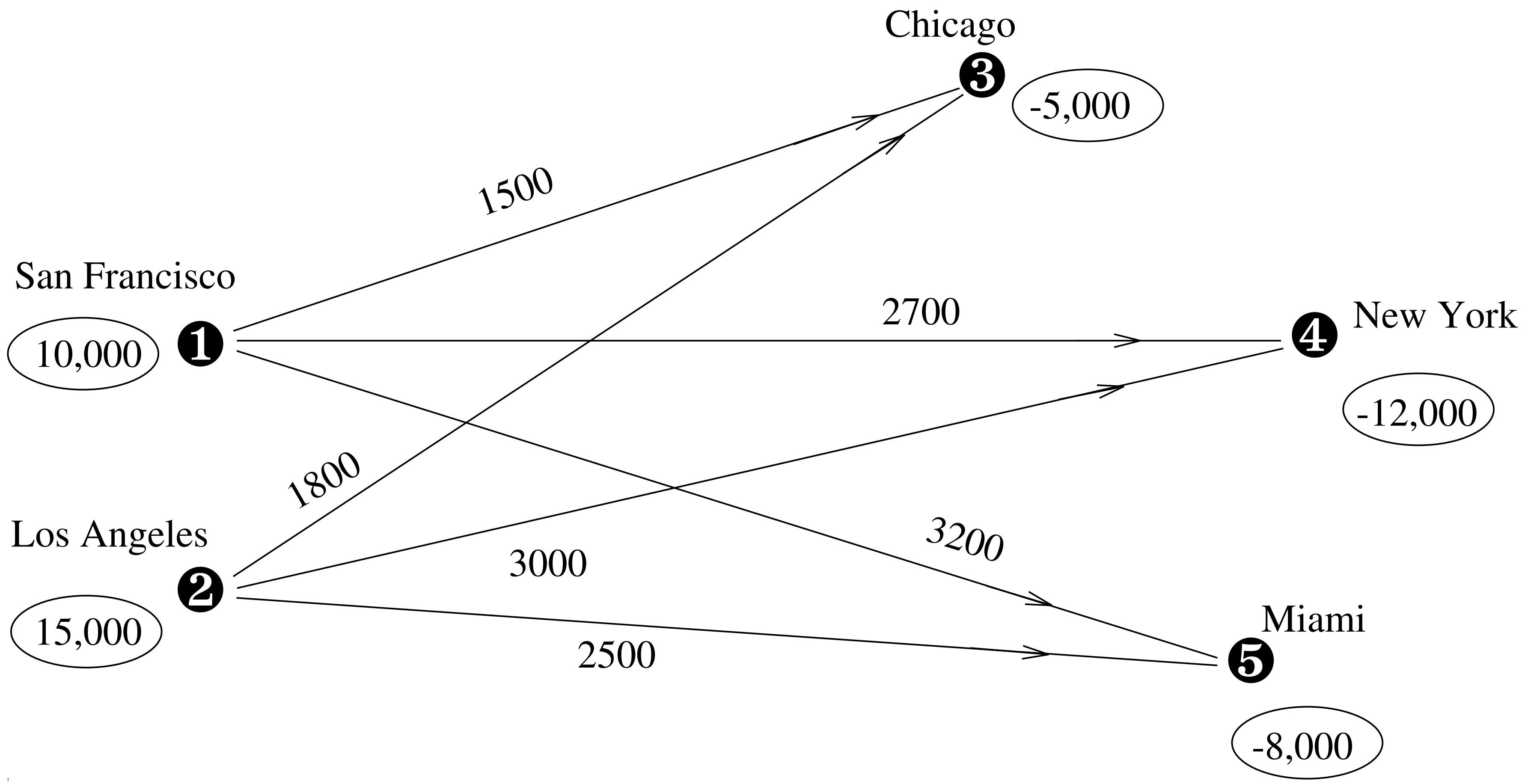
- Elke *arc* gaat van een *source node* naar een *sink node*

- *De flow conservation constraints* zijn :

  - voor een source met $b_i > 0$, $$\sum_j x_{i,j} = b_i$$

  - voor een sink met $b_i < 0$. $$-\sum_k x_{k,i} = b_i$$

Network problems : Transportation probleem



Network problems : Transportation probleem

Transportation problem example HC 3 Equations

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```
In[95]:= ClearAll["Global`*"]

In[105]:= z = 1500 x13 + 1800 x23 + 2700 x14 + 3000 x24 + 3200 x15 + 2500 x25;
c1 = x13 + x14 + x15 == 10000;
c2 = x23 + x24 + x25 == 15000;
c3 = -x13 - x23 == -5000 ;
c4 = -x14 - x24 == -12000 ;
c5 = -x15 - x25 == -8000 ;
c6 = x13 ≥ 0 && x14 ≥ 0 && x15 ≥ 0 && x23 ≥ 0 && x24 ≥ 0 && x25 ≥ 0;
var = { x13, x14 , x15 , x23 , x24 , x25};
Minimize[{z, c1 && c2 && c3 && c4 && c5 && c6}, var, Reals]

Out[113]= {62000000, {x13 → 0, x14 → 10000, x15 → 0, x23 → 5000, x24 → 2000, x25 → 8000}}
```

100%

Network problems : Transportation probleem

Transportation problem example HC 3 Matrix

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```
In[114]:= ClearAll["Global`*"]

In[115]:= c = {1500, 2700, 3200, 1800, 3000, 2500};
          b = {10 000, 15 000, -5000, -12 000, -8000};
```

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix};$$

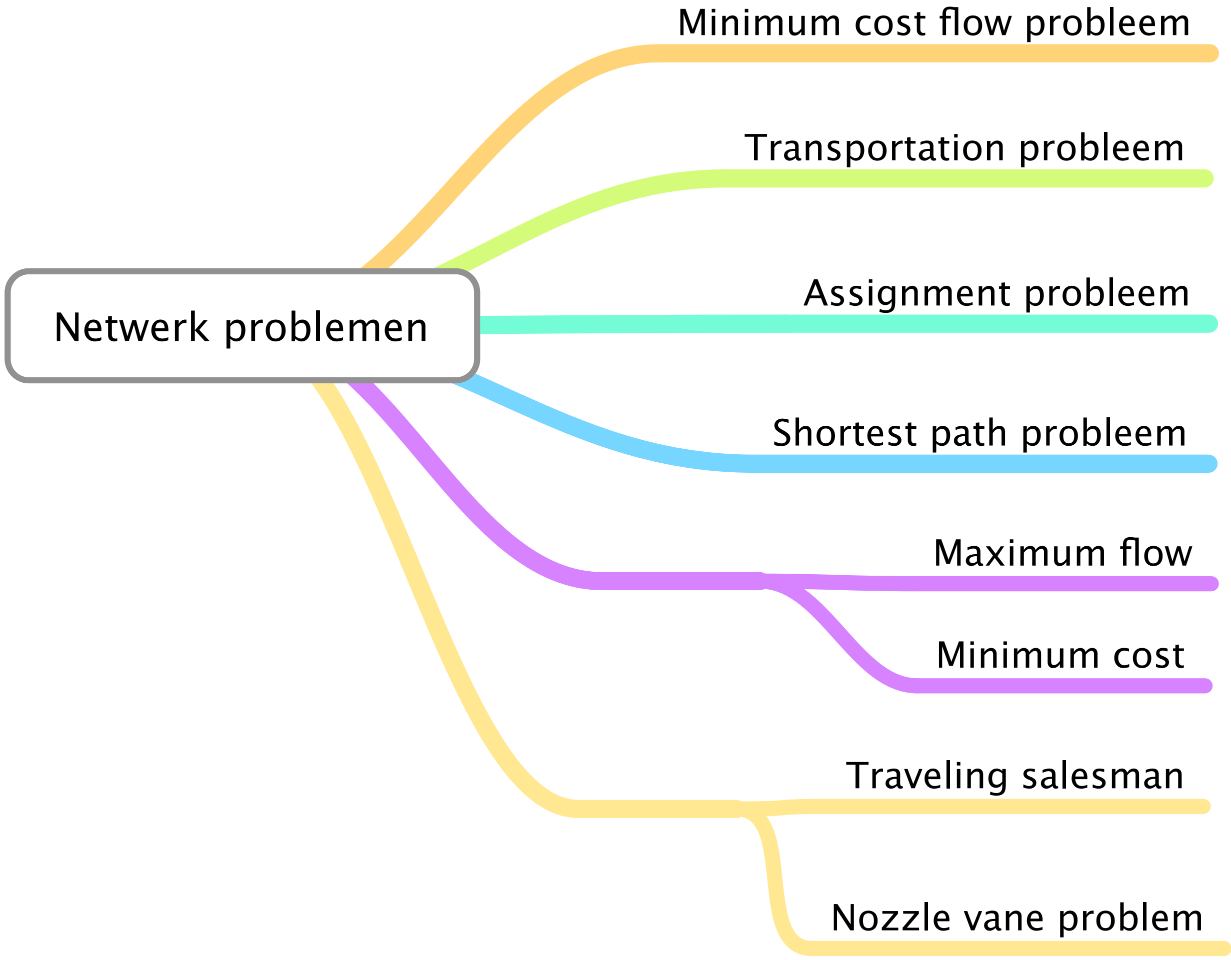
```
In[118]:= x = LinearProgramming[c, A, b]
          z = {c}.Transpose[{x}];
          Flatten[z, 1][[1]]

Out[118]= {0, 10 000, 0, 5000, 2000, 8000}

Out[119]= 62 000 000
```

100%

Overzicht : Assignment probleem

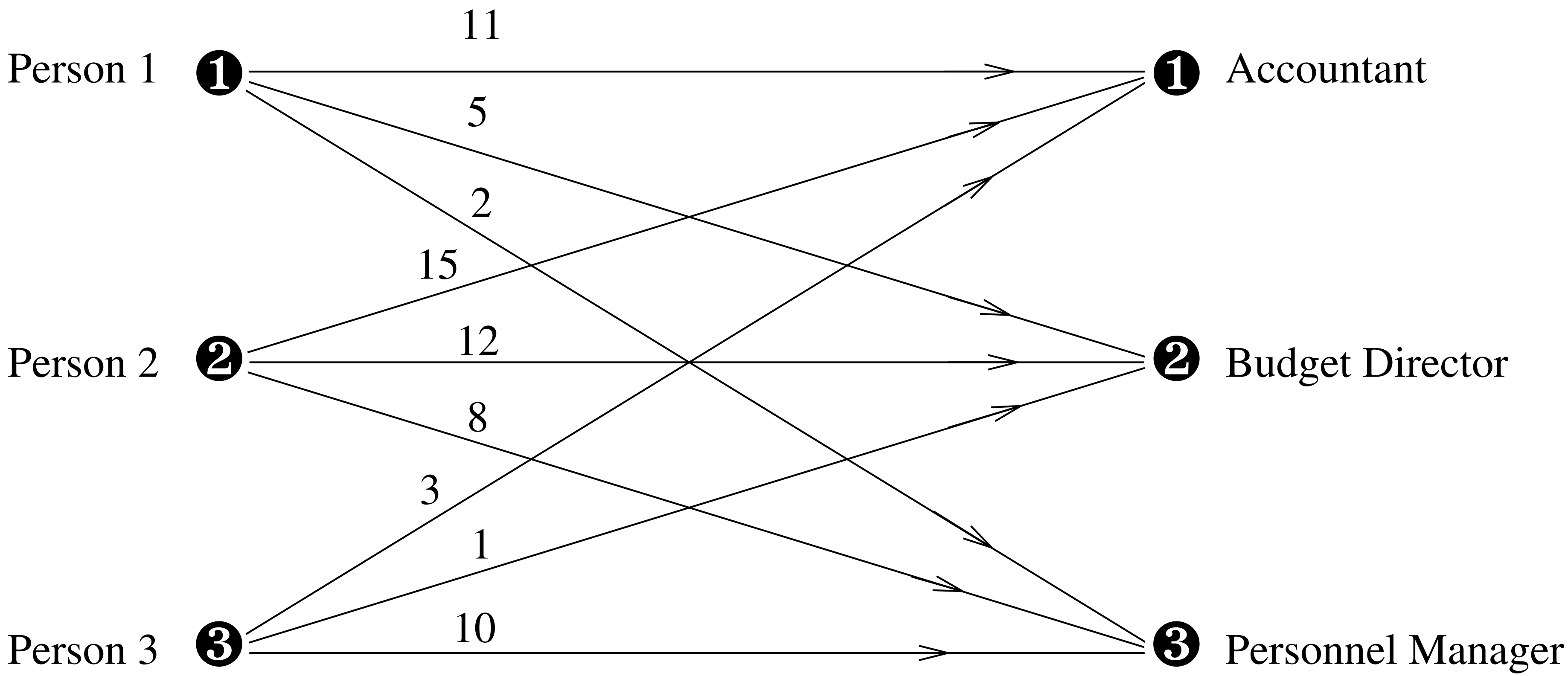


Network problems : Assignment probleem

- Persoon 1 :
  - Business degree
  - Just out of school
- Persoon 2 :
  - Business degree
  - 10 years of corporate experience
  - Management experience
- Persoon 3:
  - Management experience
  - Anthropology

- Functie 1: Accountant
- Functie 2: Budget Director
- Functie 3 : Personnel Manager

Network problems : Assignment probleem



Network problems : Assignment probleem

$$\text{maximize} \quad z = 11x_{1,1} + 5x_{1,2} + 2x_{1,3} + 15x_{2,1} + 12x_{2,2} + 8x_{2,3} + 3x_{3,1} + 1x_{3,2} + 10x_{3,3}$$

subject to the constraints

$$\begin{aligned}
x_{1,1} + x_{1,2} + x_{1,3} &= 1 \\
x_{2,1} + x_{2,2} + x_{2,3} &= 1 \\
x_{3,1} + x_{3,2} + x_{3,3} &= 1 \\
-x_{1,1} - x_{2,1} - x_{3,1} &= -1 \\
-x_{1,2} - x_{2,2} - x_{3,2} &= -1 \\
-x_{1,3} - x_{2,3} - x_{3,3} &= -1 \\
0 \le x \le 1.
\end{aligned}$$

Network problems : Assignment probleem

Assignment problem example HC 3 Equations

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```
In[154]:= ClearAll["Global`*"]

In[155]:= z = 11 x11 + 5 x12 + 2 x13 + 15 x21 + 12 x22 + 8 x23 + 3 x31 + 1 x32 + 10 x33;
          c1 = x11 + x12 + x13 == 1;
          c2 = x21 + x22 + x23 == 1;
          c3 = x31 + x32 + x33 == 1;
          c4 = -x11 - x21 - x31 == -1;
          c5 = -x12 - x22 - x32 == -1;
          c6 = -x13 - x23 - x33 == -1;
          c6 =  x11 ≥ 0 && x12  ≥ 0 && x13  ≥ 0 &&
                x21  ≥ 0 && x22  ≥ 0 && x23  ≥ 0 &&
                x31  ≥ 0 && x32  ≥ 0 && x33  ≥ 0;
          var = { x11, x12 , x13 , x21 , x22 , x23 , x31 , x32 , x33 };
          Minimize[{z, c1 && c2 && c3 && c4 && c5 && c6}, var, Reals]

Out[164]= {16, {x11 → 0, x12 → 1, x13 → 0, x21 → 0, x22 → 0, x23 → 1, x31 → 1, x32 → 0, x33 → 0}}
```

100%

Network problems : Assignment probleem

Assignment problem example HC 3 Matrix

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```
In[137]:= ClearAll["Global`*"]

In[138]:= c = {11, 5, 2, 15, 12, 8, 3, 1, 10};
          b = {1, 1, 1, -1, -1, -1};
```

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix};$$

```
In[141]:= x = LinearProgramming[c, A, b]
          z = {c}.Transpose[{x}];
          Flatten[z, 1][[1]]

Out[141]= {0, 1, 0, 0, 0, 1, 1, 0, 0}

Out[142]= 16
```

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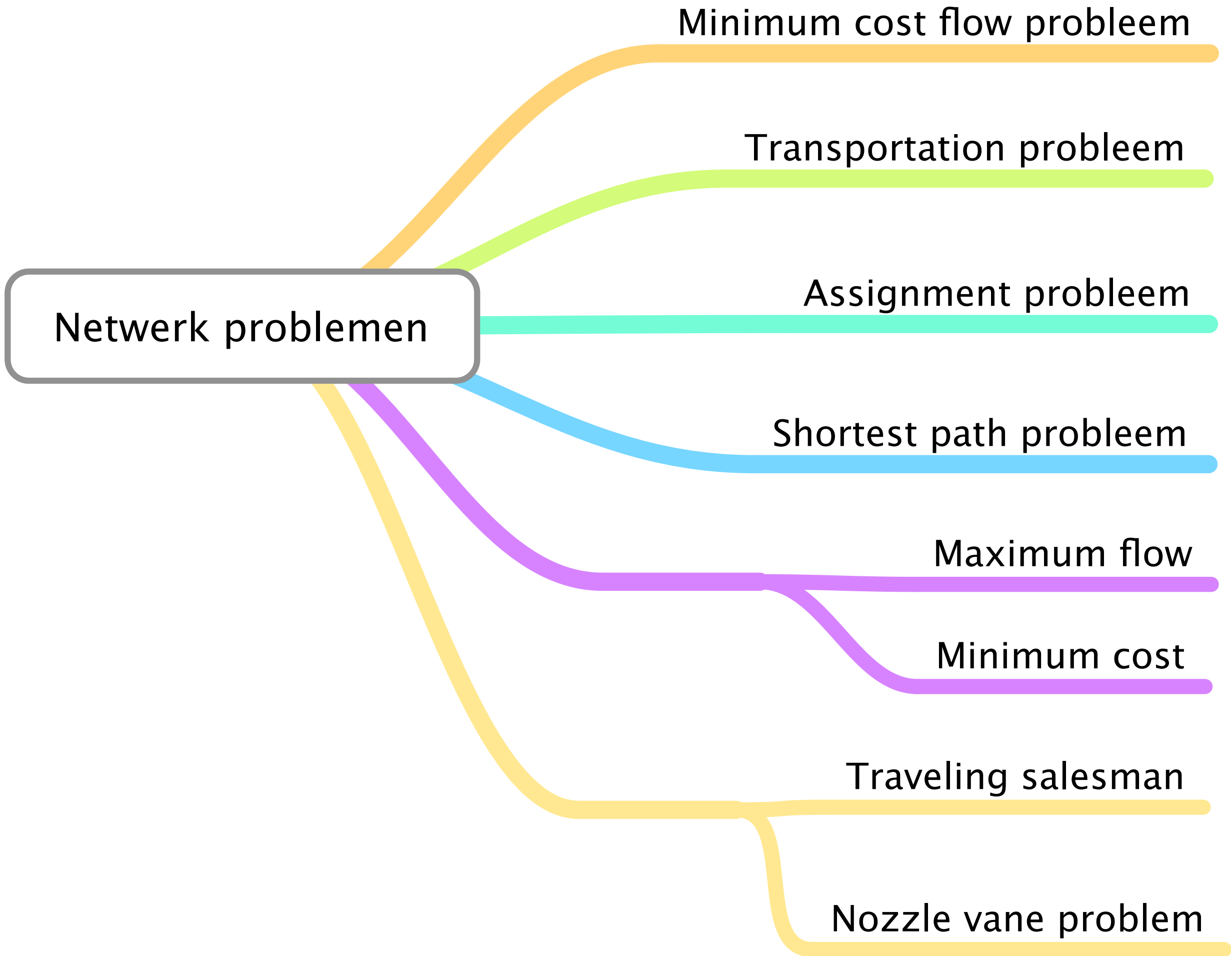
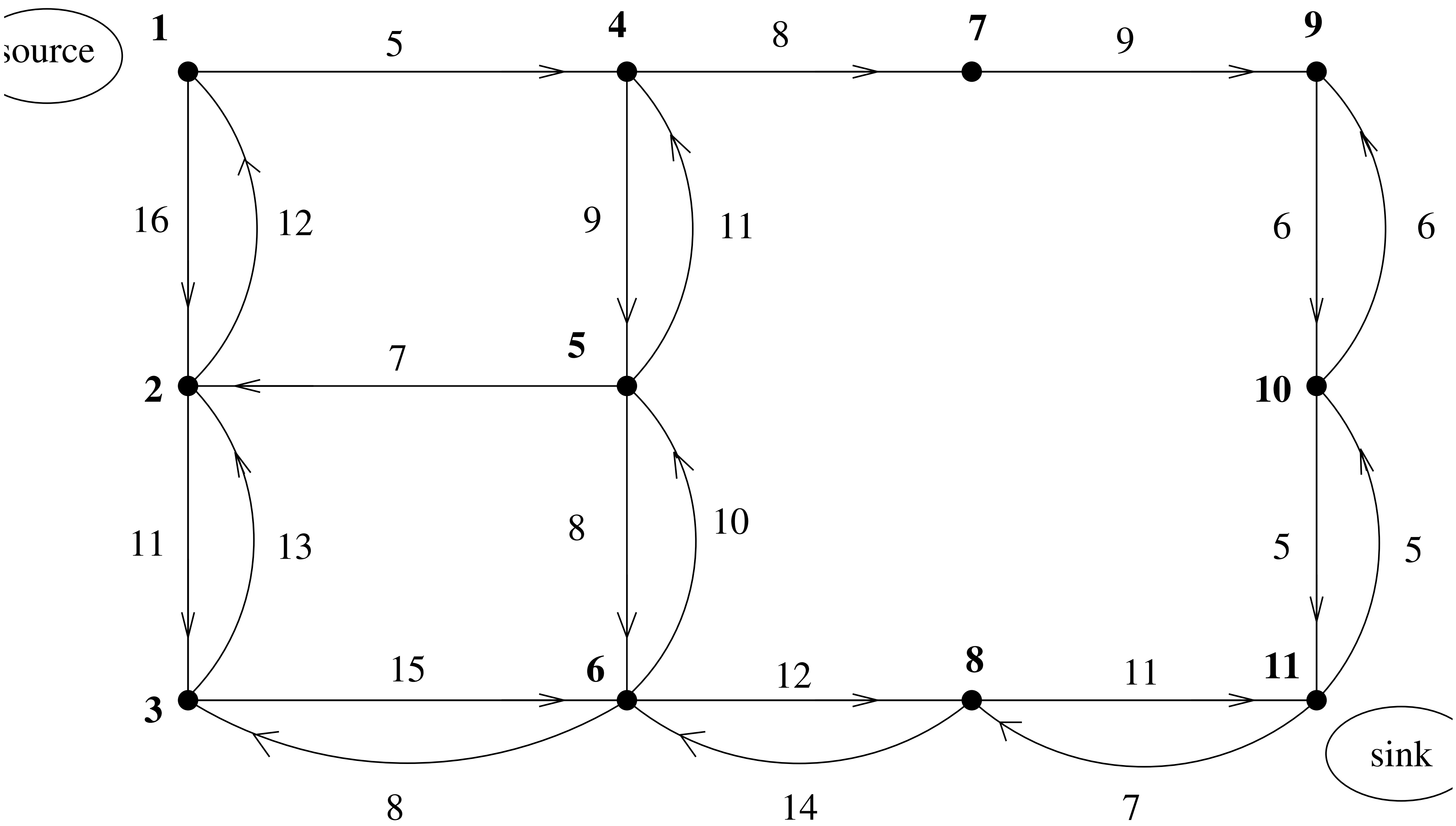
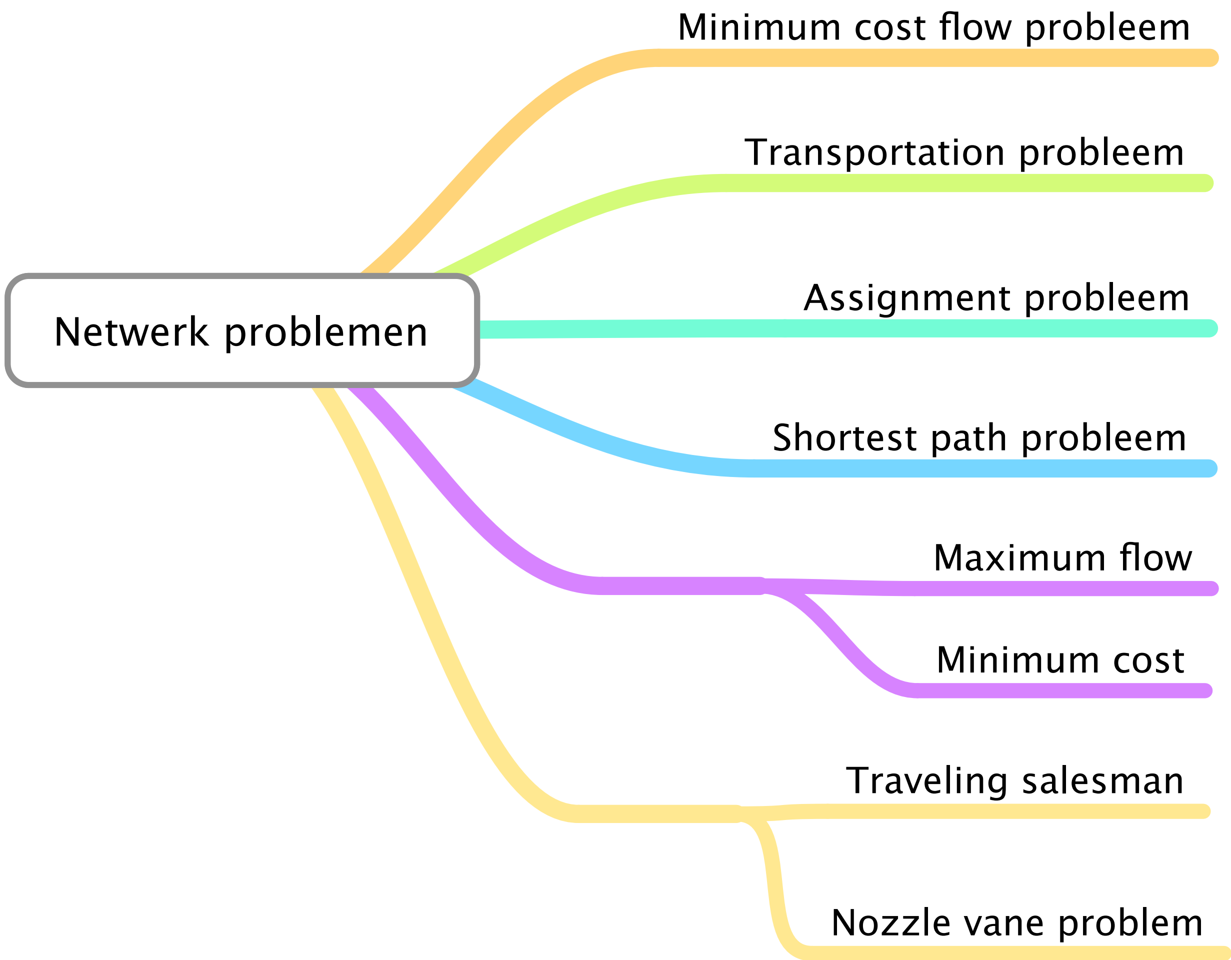
Overzicht : Shortest path probleem



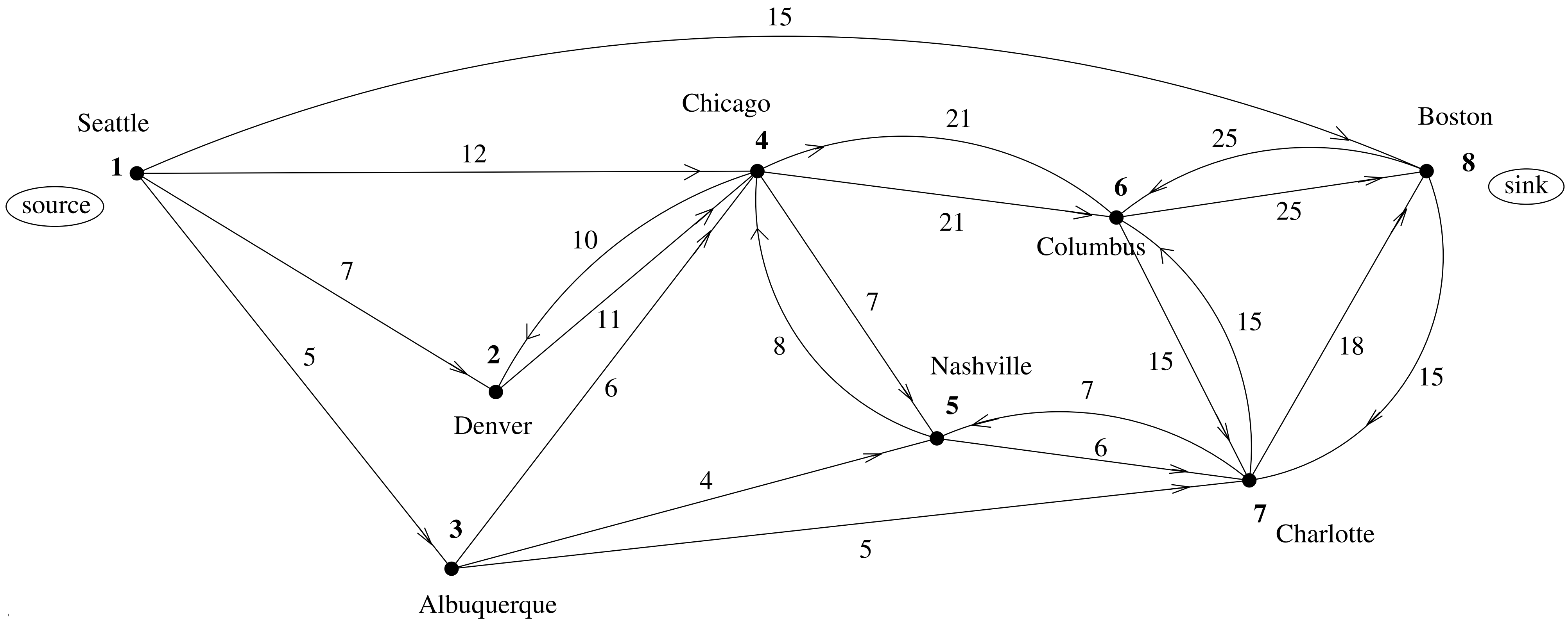
Network problems : Shortest path probleem



Overzicht : Maximum flow/Minimum cost probleem



Network problems : Maximum flow



Network problems : Maximum flow probleem

$$
\begin{aligned}
\underset{x,f}{\text{maximize}} \quad & z = f \\
\text{subject to} \quad & \sum_j x_{1,j} - \sum_k x_{k,1} = f \\
& \sum_j x_{i,j} - \sum_k x_{k,i} = 0, \quad i = 2, \ldots, m-1 \\
& \sum_j x_{m,j} - \sum_k x_{k,m} = -f \\
& 0 \le x_{i,j} \le u_{i,j} .
\end{aligned}
$$

Network problems : Maximum flow / Minimum cost

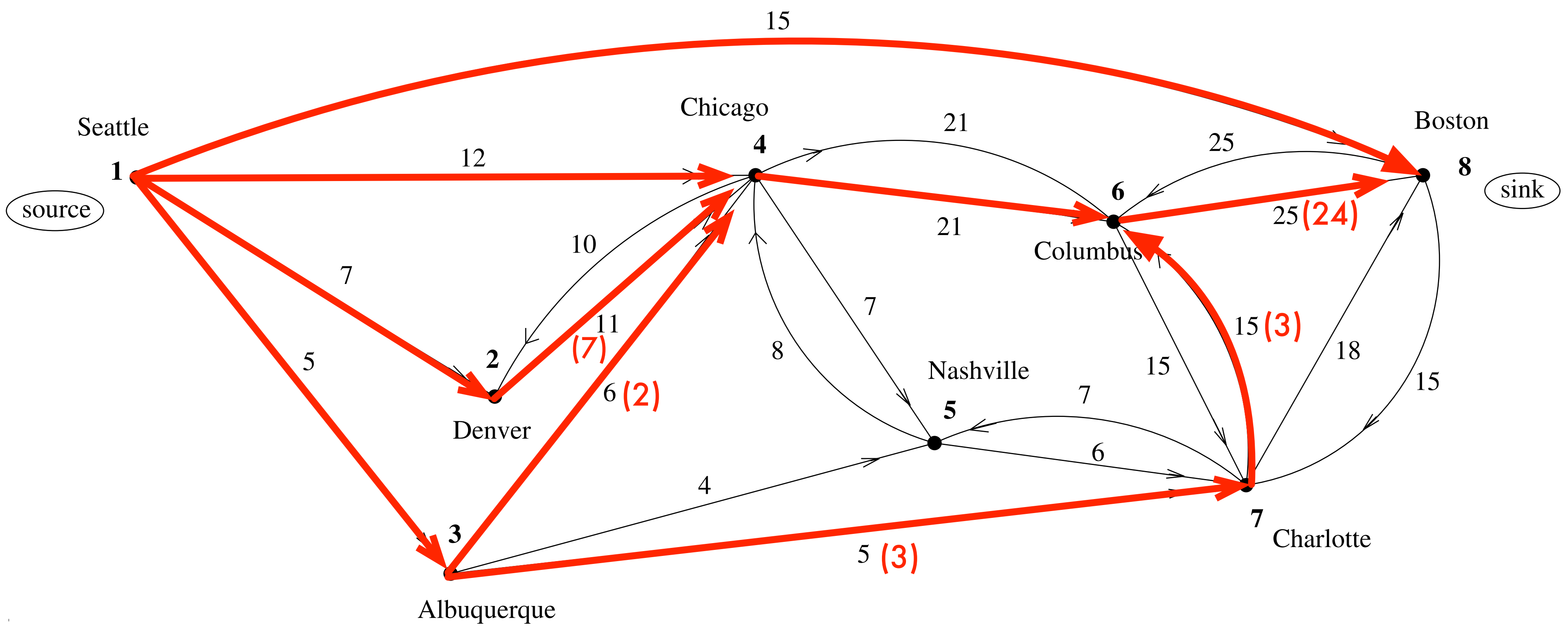
$$
\begin{aligned}
\underset{x,f}{\text{maximize}} \quad & z = f \\
\text{subject to} \quad & \sum_j x_{1,j} - \sum_k x_{k,1} = f \\
& \sum_j x_{i,j} - \sum_k x_{k,i} = 0, \quad i = 2, \ldots, m-1 \\
& \sum_j x_{m,j} - \sum_k x_{k,m} = -f \\
& 0 \leq x_{i,j} \leq u_{i,j}.
\end{aligned}
$$

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & z = -x_{m,1} \\
\text{subject to} \quad & \sum_j x_{i,j} - \sum_k x_{k,i} = 0, \quad i = 1, \ldots, m \\
& 0 \leq x_{i,j} \leq u_{i,j}.
\end{aligned}
$$

Network problems : Minimum cost flow probleem

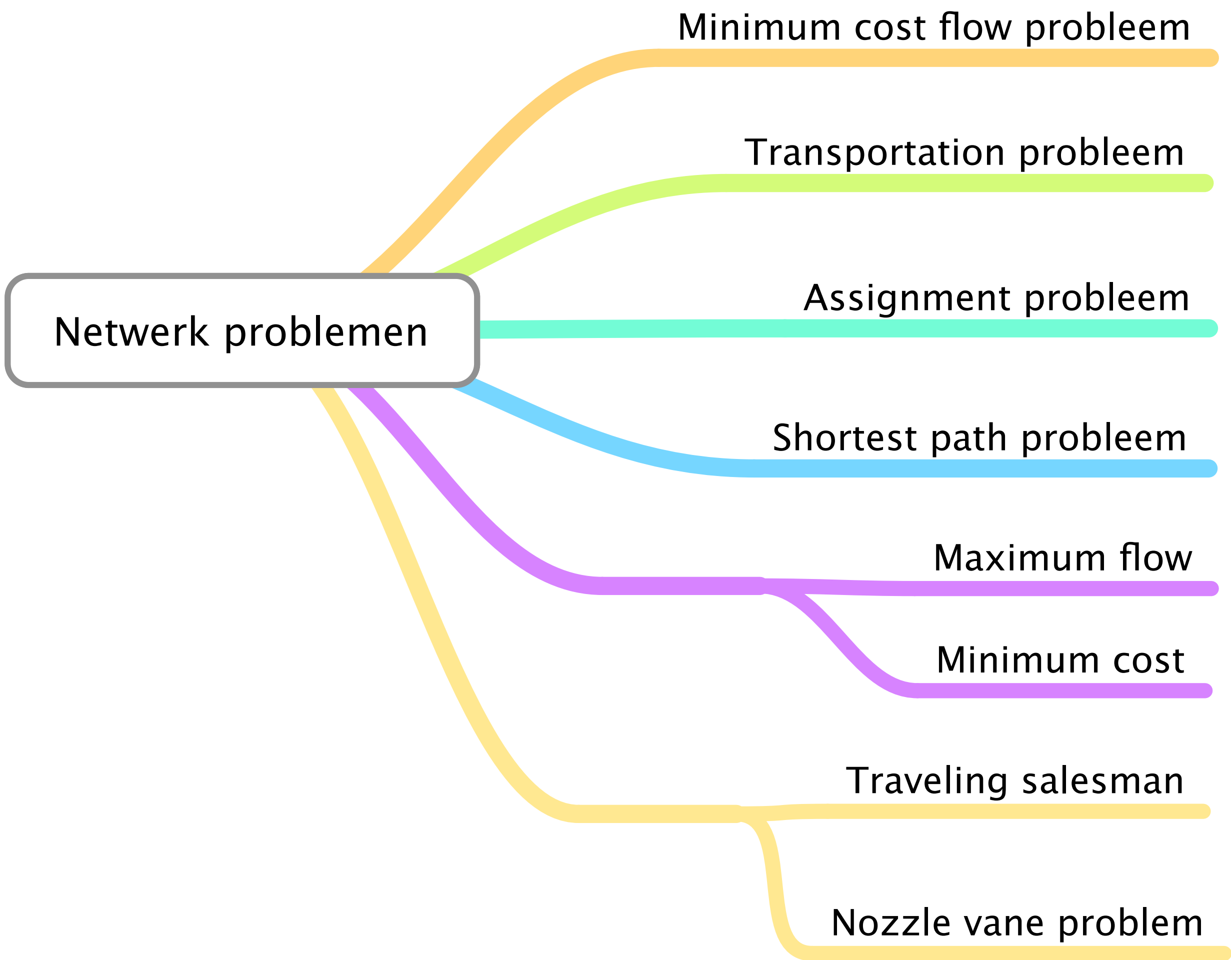
```
In[1]:= ClearAll["Global`*"]

z = -x81;

c1 = x12 + x13 + x14 + x18           - x81 == 0;
c2 = x24                             - x12 - x42 == 0;
c3 = x34 + x35 + x37                 - x13 == 0;
c4 = x45 + x46 + x42                 - x14 - x24 - x34 - x54 - x64 == 0;
c5 = x54 + x57                       - x45 - x35 - x75 == 0;
c6 = x64 + x67 + x68                 - x46 - x76 - x86 == 0;
c7 = x78 + x76 + x75                 - x37 - x57 - x67 - x87 == 0;
c8 = x86 + x87 + x81                 - x18 - x68 - x78 == 0;

c9 = x12 ≥ 0 && x13 ≥ 0 && x14 ≥ 0 && x18 ≥ 0 && x81 ≥ 0 &&
     x24 ≥ 0 && x12 ≥ 0 && x42 ≥ 0 &&
     x34 ≥ 0 && x35 ≥ 0 && x37 ≥ 0 && x13 ≥ 0 &&
     x45 ≥ 0 && x46 ≥ 0 && x42 ≥ 0 && x14 ≥ 0 && x24 ≥ 0 && x34 ≥ 0 && x54 ≥ 0 && x64 ≥ 0 &&
     x54 ≥ 0 && x57 ≥ 0 && x45 ≥ 0 && x35 ≥ 0 && x75 ≥ 0 &&
     x64 ≥ 0 && x67 ≥ 0 && x68 ≥ 0 && x46 ≥ 0 && x76 ≥ 0 && x86 ≥ 0 &&
     x78 ≥ 0 && x76 ≥ 0 && x75 ≥ 0 && x37 ≥ 0 && x57 ≥ 0 && x67 ≥ 0 && x87 ≥ 0 &&
     x86 ≥ 0 && x87 ≥ 0 && x81 ≥ 0 && x18 ≥ 0 && x68 ≥ 0 && x78 ≥ 0 ;

c10 = x12 ≤ 7  && x13 ≤ 5  && x14 ≤ 12 && x18 ≤ 15 &&
      x24 ≤ 11 && x12 ≤ 7  && x42 ≤ 10 &&
      x34 ≤ 6  && x35 ≤ 4  && x37 ≤ 5  && x13 ≤ 5  &&
      x45 ≤ 7  && x46 ≤ 21 && x42 ≤ 10 && x14 ≤ 12 && x24 ≤ 11 && x34 ≤ 6 && x54 ≤ 8 && x64 ≤ 21 &&
      x54 ≤ 8  && x57 ≤ 6  && x45 ≤ 7  && x35 ≤ 4  && x75 ≤ 7  &&
      x64 ≤ 21 && x67 ≤ 15 && x68 ≤ 25 && x46 ≤ 21 && x76 ≤ 15 && x86 ≤ 25 &&
      x78 ≤ 18 && x76 ≤ 15 && x75 ≤ 7  && x37 ≤ 5  && x57 ≤ 6  && x67 ≤ 15 && x87 ≤ 18 &&
      x86 ≤ 25 && x87 ≤ 15 && x18 ≤ 15 && x68 ≤ 25 && x78 ≤ 18 ;

var = {x12, x13, x14, x18, x42, x24, x34, x35, x37, x54, x64, x45, x46, x75, x57, x76, x86, x67, x68, x78, x87, x81};

Minimize[{z, c1 && c2 && c3 && c4 && c5 && c6 && c7 && c8 && c9 && c10}, var, Reals]

Out[14]= {-39, {x12 → 7, x13 → 5, x14 → 12, x18 → 15, x42 → 0, x24 → 7, x34 → 2, x35 → 0, x37 → 3, x54 → 0,
  x64 → 0, x45 → 0, x46 → 21, x75 → 0, x57 → 0, x76 → 3, x86 → 0, x67 → 0, x68 → 24, x78 → 0, x87 → 0, x81 → 39}}
```

Network problems : Maximum flow / minimum cost



$\{-39, \{x12 \to 7, x13 \to 5, x14 \to 12, x18 \to 15, x42 \to 0, x24 \to 7, x34 \to 2, x35 \to 0, x37 \to 3, x54 \to 0, x64 \to 0, x45 \to 0, x46 \to 21, x75 \to 0, x57 \to 0, x76 \to 3, x86 \to 0, x67 \to 0, x68 \to 24, x78 \to 0, x87 \to 0, x81 \to 39\}\}$

Overzicht : Traveling salesman



Network problems : Traveling salesman

- Een verkoper bezoekt n steden op een ronde, de steden zijn genummerd als

$$0, 1, \ldots, n-1.$$

- Elke stad wordt één keer bezocht en de reis eindigt bij het begin
- De afstand tussen de steden is gegeven als c_{ij}
- De reis wordt beschreven door $s_0 = 0, s_1, s_2, \ldots, s_{n-1}$
- De variabele x_{ij} wordt 1 als stad j bezocht wordt na stad i
- Er zijn veel combinatie mogelijk, voor $n-1$ steden zijn dit $(n-1)!$routes

$$50! = 3.041 \times 10^{64}$$

Network problems : Traveling salesman

- De objective functie is

$$\text{minimize} \sum_i \sum_j c_{ij} x_{ij}$$

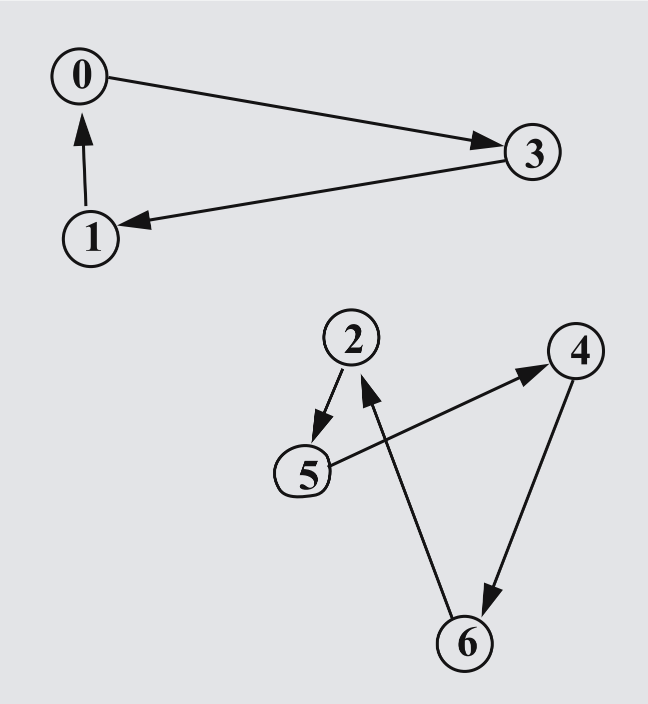
- met de constraints

$$\sum_j x_{ij} = 1, \qquad i = 0, 1, \ldots, n-1$$

$$\sum_i x_{ij} = 1, \qquad j = 0, 1, \ldots, n-1$$

- De constraints dwingen af dat de verkoper bij het bezoeken van elke stad afkomstig is uit één voorgaande stad en op reis gaat naar één volgende stad

- Er zijn echter meer constraints nodig...



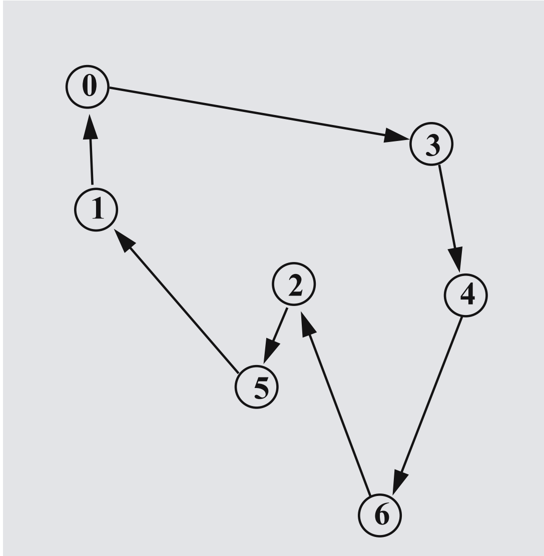
Network problems : Traveling salesman

- Er zijn constraints nodig die afdwingen dat de reis continu verloopt
  - Gegeven een reis $s_0 = 0, s_1, s_2, \ldots, s_{n-1}$
  - dan is t_i met $i = 0, 1, \ldots, n$ het volgorde nummer van stad i
  - dus $t_0 = 0,\ t_{s_1} = 1,\ t_{s_2} = 2,$ etc dus : $t_{s_i} = i, \quad i = 0, 1, \ldots, n-1$
  - het volgt dat $t_j = t_i + 1$ als $x_{ij} = 1$
  - en dus omdat t_i een geheel getal is volgt dat $t_j \geq \begin{cases} t_i + 1 - n & \text{if } x_{ij} = 0, \\ t_i + 1 & \text{if } x_{ij} = 1. \end{cases}$
  - of $t_j \geq t_i + 1 - n(1 - x_{ij}), \qquad i \geq 0, j \geq 1, i \neq j$

Network problems : Traveling salesman

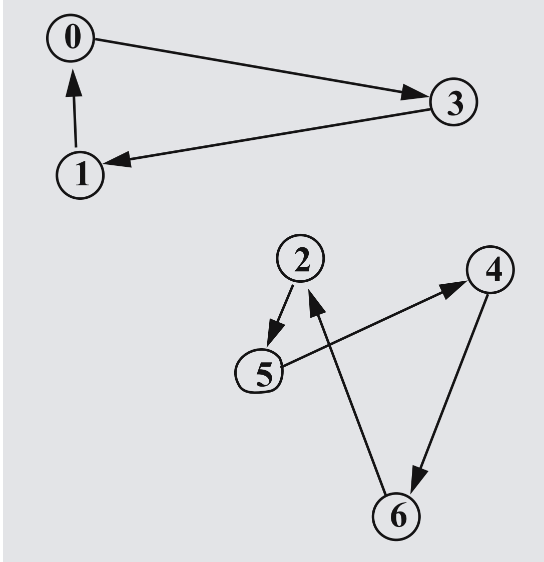
- Case 1

S	$t_S = 0$	(i,j)	$x_{i,j} = 1$	$t_j \geq t_i + 1$	$t_j \geq t_i + 1$
$S_0 = 0$	$t_0 = 0$	$(0,3)$	$x_{0,3} = 1$	$t_3 \geq t_0 + 1$	$1 \geq 0 + 1$
$S_1 = 3$	$t_3 = 1$	$(3,4)$	$x_{3,4} = 1$	$t_4 \geq t_3 + 1$	$2 \geq 1 + 1$
$S_2 = 4$	$t_4 = 2$	$(4,6)$	$x_{4,6} = 1$	$t_6 \geq t_4 + 1$	$3 \geq 2 + 1$
$S_3 = 6$	$t_6 = 3$	$(6,2)$	$x_{6,2} = 1$	$t_2 \geq t_6 + 1$	$4 \geq 3 + 1$
$S_4 = 2$	$t_2 = 4$	$(2,5)$	$x_{2,5} = 1$	$t_5 \geq t_2 + 1$	$5 \geq 4 + 1$
$S_5 = 5$	$t_5 = 5$	$(5,1)$	$x_{5,1} = 1$	$t_1 \geq t_5 + 1$	$6 \geq 5 + 1$

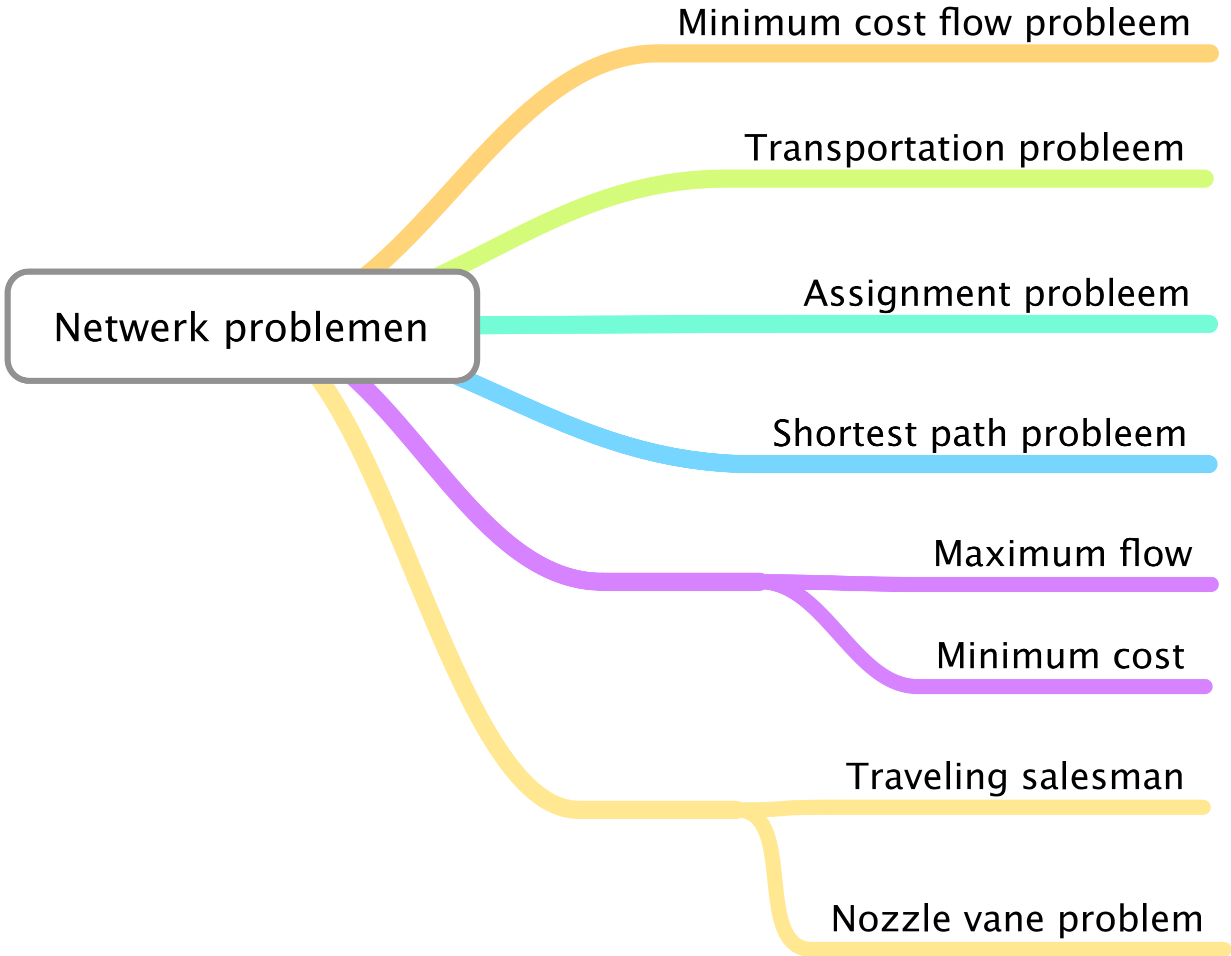


- Case 2

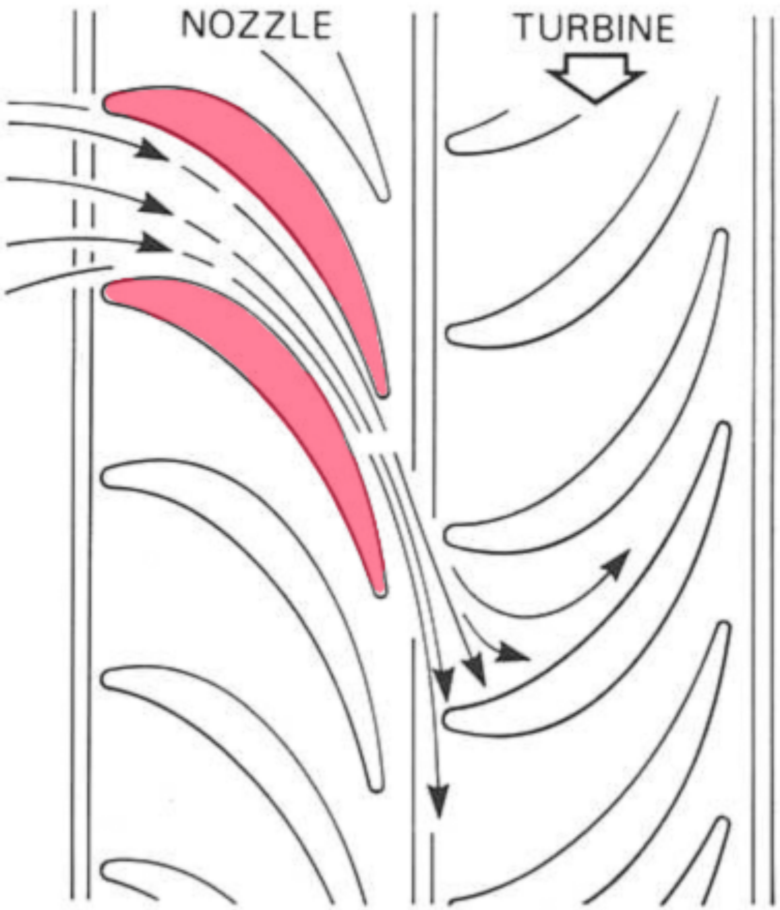
S	$t_S = 0$	(i,j)	$x_{i,j} = 1$	$t_j \geq t_i + 1$	$t_j \geq t_i + 1$
$S_0 = 2$	$t_2 = 0$	$(2,5)$	$x_{2,5} = 1$	$t_5 \geq t_2 + 1$	$1 \geq 0 + 1$
$S_1 = 5$	$t_5 = 1$	$(5,4)$	$x_{5,4} = 1$	$t_4 \geq t_5 + 1$	$2 \geq 1 + 1$
$S_2 = 4$	$t_4 = 2$	$(4,6)$	$x_{4,6} = 1$	$t_6 \geq t_4 + 1$	$3 \geq 2 + 1$
$S_3 = 6$	$t_6 = 3$	$(6,2)$	$x_{6,2} = 1$	$t_2 \geq t_6 + 1$	$0 \leq 3 + 1$

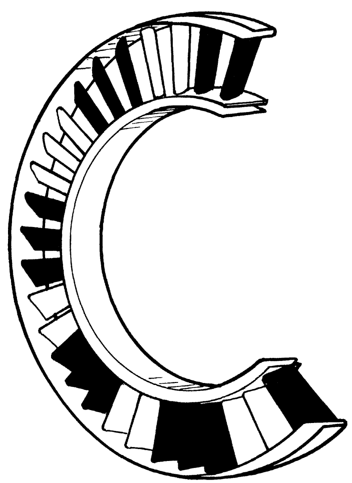


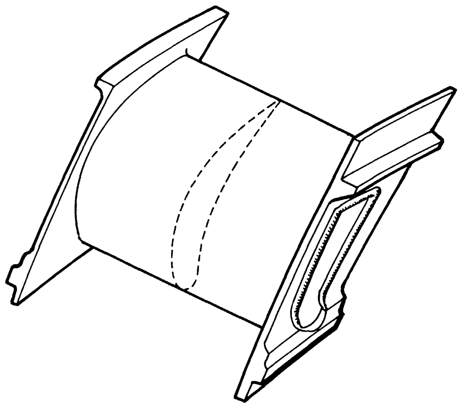
Overzicht : Nozzle vane probleem

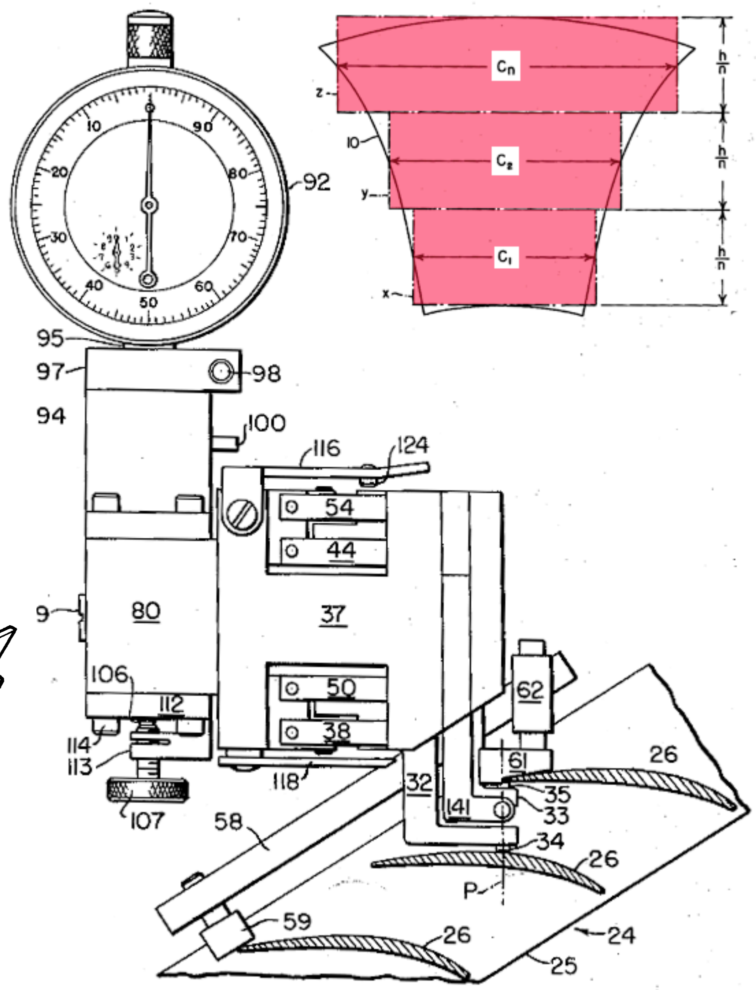


Nozzle vane probleem





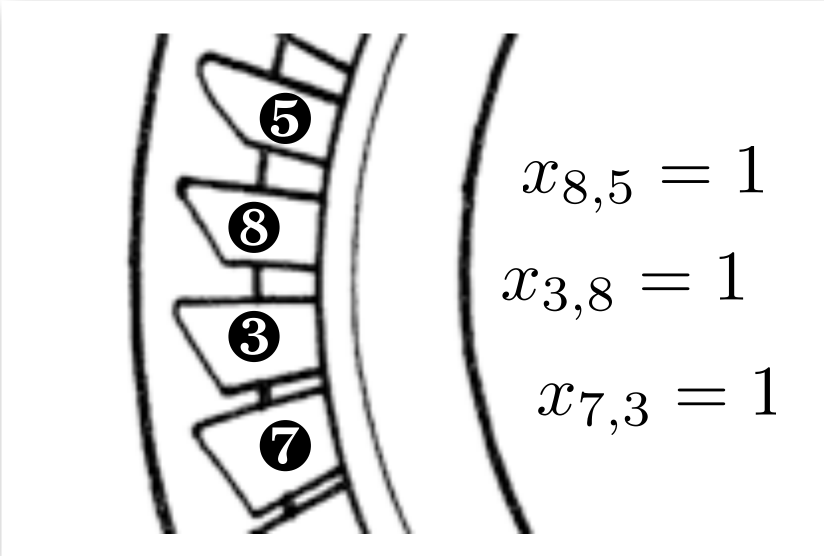




Nozzle vane problem : Oppervlakte

- Een nozzle vane *j* die direct rechts (met de klok mee) van nozzle vane *i* geplaatst wordt vormt een doorgang met een oppervlakte van c_{ij}

- Het totale oppervlak van het doorstroom opening is :



$$T = \sum_i \sum_j c_{ij} x_{ij},$$

where

$$x_{ij} = \begin{cases} 1 & \text{if vane } j \text{ is placed} \\ & \quad \text{clockwise-adjacent to vane } i \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Nozzle vane problem : Oppervlakte

- Met een meetinstrument worden twee waarden gemeten

$$c_{ij} = A_i + B_j \quad \text{for all } i \text{ and } j.$$

- De ideale nozzle vane guide plaatsing wordt bereikt als alle c_{ij} gelijk zijn
  - Er geldt dan dat $c_{ij} = c_{kl}$ voor alle $i,\ j$ en $k,\ l$ als $x_{ij} = 1$ en $x_{kl} = 1$

- De het totale doorstroom oppervlak wordt gegeven door

$$T = \sum_{i=1}^{n} (A_i + B_i)$$

- Het gemiddelde oppervlak per nozzle kanaal is

$$\bar{d} = \frac{\sum_i (A_i + B_i)}{n}$$

- Waarmee het ideale kanaal gegeven wordt door

$$c_{ij} - \bar{d} = 0 \quad \text{for all } i \text{ and } j \text{ with } x_{ij} = 1.$$

Nozzle vane problem : Probleemstelling

Problem P1

Minimize

$$U(X) = \sum_i \sum_j (\bar{d} - (A_i + B_j))^2 x_{ij}$$

subject to

$$\sum_j x_{ij} = 1 \quad \text{for all } i,$$

$$\sum_i x_{ij} = 1 \quad \text{for all } j,$$

$$x_{ij} = \{0, 1\} \quad \text{for all } i \text{ and } j,$$

no subtours.

Problem P2

Minimize

$$h(X) = \sum_i \sum_j (A_i B_j) x_{ij}$$

subject to

$$\sum_j x_{ij} = 1 \quad \text{for all } i,$$

$$\sum_i x_{ij} = 1 \quad \text{for all } j,$$

$$x_{ij} = \{0, 1\} \quad \text{for all } i \text{ and } j,$$

no subtours.

Vragen?

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