1 Introduction

The goal of this project is to solve the lid driven cavity problem with using a stream function vorticity method.

2 Methods

The lid driven cavity problem can be formulated as:

$$\bar{x} = \frac{x}{L_{ref}}, \bar{y} = \frac{y}{L_{ref}}, \quad \bar{t} = \frac{tu_{ref}}{L_{ref}}, \bar{u} = \frac{u}{u_{ref}}, \quad \bar{v} = \frac{v}{v_{ref}}, \bar{\zeta} = \frac{\zeta L_{ref}}{u_{ref}}, \bar{\psi} = \frac{\psi}{u_{ref} L_{ref}}$$

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = -\bar{\zeta} \tag{1}$$

and

$$\frac{\partial \bar{\zeta}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{\zeta}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{\zeta}}{\partial \bar{y}} = \frac{1}{Re}\left(\frac{\partial^2 \bar{\zeta}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\zeta}}{\partial \bar{y}^2}\right) \tag{2}$$

where the velocities are given as:

$$\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}} \tag{3}$$

and

$$\bar{v} = -\frac{\partial \bar{\psi}}{\partial \bar{x}} \tag{4}$$

The boundary conditions are

$$\bar{u} = \bar{v} = 0 \tag{5}$$

for all the walls except the top wall where:

$$\bar{u} = 1 \tag{6}$$

$$\bar{v} = 0 \tag{7}$$

This implies $u_{lid} = u_{ref}$. The equations is discretized on a regular grid with grid spacing Δx and Δy which results in $nx = \Delta x/L$ and $ny = \Delta y/L$. We indicate a discrete point $x_i = i\Delta x$ and $y_j = j\Delta y$ and use the notation

$$\phi_{i,j} = \phi(x_i, y_i) \tag{8}$$

The second derivatives in the vorticity equation are discretized with a centered second order scheme, which leads to the discretized streamfunction equation.

$$\left.\frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2}\right|_{x_i, y_j} = \frac{\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}}{\Delta x^2} \tag{9}$$

$$\left.\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right|_{x_i,y_j} = \frac{\psi_{i,j-1} - 2\psi_{i,j} + \psi_{i,j+1}}{\Delta y^2} \tag{10}$$

This leads to a system of $(nx-2)(ny-2)$ unknowns for the internal grid points. We use a homogeneous Dirchlet boundary condition for the stream function since

$$\bar{v} = \frac{\partial\bar{\psi}}{\partial\bar{x}} = 0 \tag{11}$$

on the top and bottom boundary and

$$\bar{u} = -\frac{\partial\bar{\psi}}{\partial\bar{y}} = 0 \tag{12}$$

on the left and right boundary. Therfore the stream function is a constant on the boundary which we choose to set to zero. The terms in the vorticity equation are discretized as follows:

$$\bar{u}_{i,j} = -\frac{\bar{\psi}_{i,j+1} - \bar{\psi}_{i,j-1}}{2\Delta y} \tag{13}$$

$$\bar{v}_{i,j} = \frac{\bar{\psi}_{i+1,j} - \bar{\psi}_{i-1,j}}{2\Delta x} \tag{14}$$

$$\left.\frac{\partial\bar{\zeta}}{\partial\bar{x}}\right|_{i,j} = \frac{\bar{\zeta}_{i+1,j} - \bar{\zeta}_{i-1,j}}{2\Delta x} \tag{15}$$

$$\left.\frac{\partial\bar{\zeta}}{\partial\bar{y}}\right|_{i,j} = \frac{\bar{\zeta}_{i,j+1} - \bar{\zeta}_{i,j-1}}{2\Delta y} \tag{16}$$

The diffusive term in the vorticity equation is discretized using second order finite differences

$$\left.\frac{\partial^2\bar{\zeta}}{\partial\bar{x}^2}\right|_{x_i,y_j} = \frac{\zeta_{i-1,j} - 2\zeta_{i,j} + \zeta_{i+1,j}}{\Delta x^2} \tag{17}$$

$$\left.\frac{\partial^2\bar{\zeta}}{\partial\bar{y}^2}\right|_{x_i,y_j} = \frac{\zeta_{i,j-1} - 2\zeta_{i,j} + \zeta_{i,j+1}}{\Delta y^2} \tag{18}$$