The Analysis of Various Basic Pendulum Systems

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Abstract

This paper is an analysis of different basic pendulum systems and their governing ordinary differential equations. The simple pendulum problem is characterized by a mass anchored through a pivot by a mass-less rod, with no forces working on it other than gravity, however it can be developed into alternate scenarios through the addition of drag for and other types of pendulums. We develop ODEs that characterize 4 different pendulum scenarios, and take on analytical and numerical methods to find their solutions.

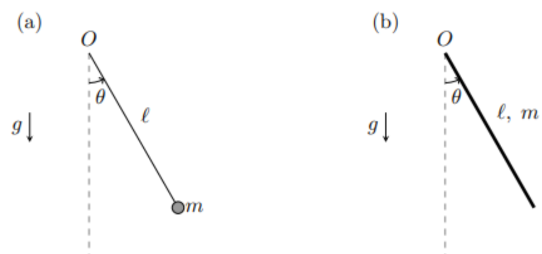

Figure 1: (a) A simple pendulum of mass m, attached to an anchor O via a mass-less rod with length ℓ (b) A circular rod of radius a (thickness $2a$), mass m, and length ℓ connected to an anchor O. Source: projectIMM Assignment

Contents

1 Introduction

The pendulum system is a long studied problem, and there are many different motivations for addressing it, such as for keeping track of time, finding longitude, and for solving problems of perpetual motion. The first known instance of someone studying it was in 1583, when Galileo found that regardless of the amplitude, the period of a pendulum is consistent, and only depends on the gravitational acceleration and on the length of the pendulum, specifically for small angles. Further properties of the pendulum systems came about in 1656, when the first clock with a pendulum was built by Christian Huygens, and it was confirmed that even though the motion of pendulum on a circle is not truly independent of amplitude, there does exist other curves that do have that property, specifically the tautochrone. The objective of this paper is to address some of the core discoveries made about the simple pendulum through dimensional analysis and the development of governing ordinary differential equations through the fundamentals of linear and angular motion. In particular, we look at 4 different systems: the simple pendulum system, the simple pendulum with drag force, the rod pendulum without and with drag force, and the double pendulum system. Each pendulum system is further analyzed through linear approximations and Euler's method, to get an idea of the solution of the developed ODEs through analytical and numerical techniques respectively. All code for this project can be found via Appendix A.

2 Simple Pendulum System

2.1 Conducting a Dimensional Analysis

In this section, I will break down the variables and dimensions of the simple pendulum problem. We deal with 5 variables: mass m, gravitational acceleration g, angle θ, length of rod l, and period of oscillation τ.

$$m : M$$

$$g : LT^{-2}$$

$$\theta : \text{dimensionless}$$

$$l : L$$

$$\tau : T$$

There are 3 dimensions: M, L, T and there are 5 variables, so there ultimately will be 2 dimensionless parameters. If we choose m, l, τ to be the repeating parameters, then:

$$\pi_1 = gm^a l^b \tau^c$$

$$\doteq LT^{-2}M^aL^bT^c$$

$$a = 0,\ b = -1,\ c = 2$$

$$\pi_1 = \frac{\tau^2 g}{l}$$

and $\pi_2 = \theta$ since it is already dimensionless. We have

$$\pi_1 = \Phi_1(\pi_2)$$

$$\frac{\tau^2 g}{l} = \Phi_1(\theta)$$

$$\tau = \sqrt{\frac{l}{g}}\Phi_2(\theta)$$

Then the time scale ψ of this system is $\sqrt{\frac{l}{g}}$.

2.2 Development of General ODE for Simple Pendulum System

Utilizing Newton's Second Law, which states $ma = \sum_i \vec{F}_i$, we can evaluate the different forces acting on the mass of the pendulum. Generally, we have the force of gravity acting on the mass, but in particular we want to deal with the force of gravity in the direction tangent to the movement of the mass. Normally, the force of gravity is mg, however in the tangential direction, it is $-mg\sin\theta$. So we have

$$ma_s = -mg\sin\theta \tag{1}$$

where a_s is the acceleration in the direction of movement and s is the position of the mass on the curve path. Note that $s = l\theta$, so $a_s = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$, and

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{2}$$

2.3 Solution of General ODE, for sufficiently small θ

This ODE is not easy to solve analytically for normal scenarios, but if θ is close to 0, we can consider the Taylor Expansion of $\sin\theta$ about $\theta = 0$.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \tag{3}$$

In particular, we can just look at the linear term, so replacing $\sin\theta$ with θ, we have:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

This is an ODE whose characteristic equation is

$$r^2 + \frac{g}{l} = 0. \tag{4}$$

Since $\frac{g}{l}$ is a strictly positive value, this equation has complex roots

$$r_{1,2} = \frac{\sqrt{-4\frac{g}{l}}}{2} = 0 \pm i\sqrt{\frac{g}{l}}. \tag{5}$$

The general solution to an ODE with complex roots $r_{1,2} = \lambda \pm \mu i$ is

$$\theta(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \tag{6}$$

so our resulting equation is

$$\theta(t) = c_1 \cos(\sqrt{\frac{g}{l}}t) + c_2 \sin(\sqrt{\frac{g}{l}}t). \tag{7}$$

To solve for the constants c_1 and c_2, we can use the initial conditions $\theta(0) = \theta_0$ and $\frac{d\theta(0)}{dt} = \dot{\theta}(0) = 0$.

$$\theta(0) = \theta_0 = c_1\cos(\sqrt{\frac{g}{l}}\cdot 0) + c_2\sin(\sqrt{\frac{g}{l}}\cdot 0)$$

$$\theta(0) = \theta_0 = c_1 \cdot 1 + 0 \to c_1 = \theta_0$$

$$\dot{\theta}(t) = -\sqrt{\frac{g}{l}}c_1 \sin(\sqrt{\frac{g}{l}}t) + \sqrt{\frac{g}{l}}c_2 \cos(\sqrt{\frac{g}{l}}t)$$

$$\dot{\theta}(0) = 0 = -\sqrt{\frac{g}{l}}c_1 \sin(\sqrt{\frac{g}{l}} \cdot 0) + \sqrt{\frac{g}{l}}c_2 \cos(\sqrt{\frac{g}{l}} \cdot 0)$$

$$\dot{\theta}(0) = 0 = 0 + \sqrt{\frac{g}{l}}c_2 \cdot 1 \to c_2 = 0$$

Therefore, our solution to the ODE is

$$\theta(t) = \theta_0 \cos(\sqrt{\frac{g}{l}}t). \tag{8}$$

To find a time scale based on (2), we consider the dimensions of the left and right sides of the equation. The left side $\theta(t)$ is dimensionless because the function returns an angle, so the right hand side must be dimensionless. Since θ_0 is dimensionless, the main concern is the input to the cosine function, $\sqrt{\frac{g}{l}}t$ should be dimensionless. Clearly, t has the dimension of T, that means $\sqrt{\frac{g}{l}}$ has the dimension $\frac{1}{T}$. So, the timescale is the reciprocal $\sqrt{\frac{l}{g}}$, which is the same result from the previous dimensional analysis. Furthermore, the period of oscillation of $\theta(t)$ is $2\pi\sqrt{\frac{l}{g}}$ when we assume θ to be sufficiently small.

2.4 Euler's Method to Solve General ODE

Now, recall the original equation (1). We can utilize Euler's Method to numerically solve this. Because this is a second-order differential equation, we need to go through Euler's method twice, once to estimate $\frac{d\theta}{dt}$, and then based on that, solve for θ.

Let $z = \frac{d\theta}{dt}$. Then

$$\frac{dz}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

The iterations for Euler's method are:

$$\theta_{k+1} = \theta_k + z_k\Delta t$$
$$z_{k+1} = z_k - \frac{g}{l}\sin\theta\Delta t$$

2.5 Results of Euler's Method

I ran Euler's Method multiple times to get an idea of its performance using different time steps. To do this, I set the variables $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $\frac{d\theta(0)}{dt} = 0$, and $\theta_0 = \frac{\pi}{4}$. I defined the timescale ψ to be $2\pi\sqrt{\frac{l}{g}}$, and decided to let the final time point to be 5ψ. The time steps that I tested across were $[0.0001\psi, 0.005\psi, 0.01\psi]$.

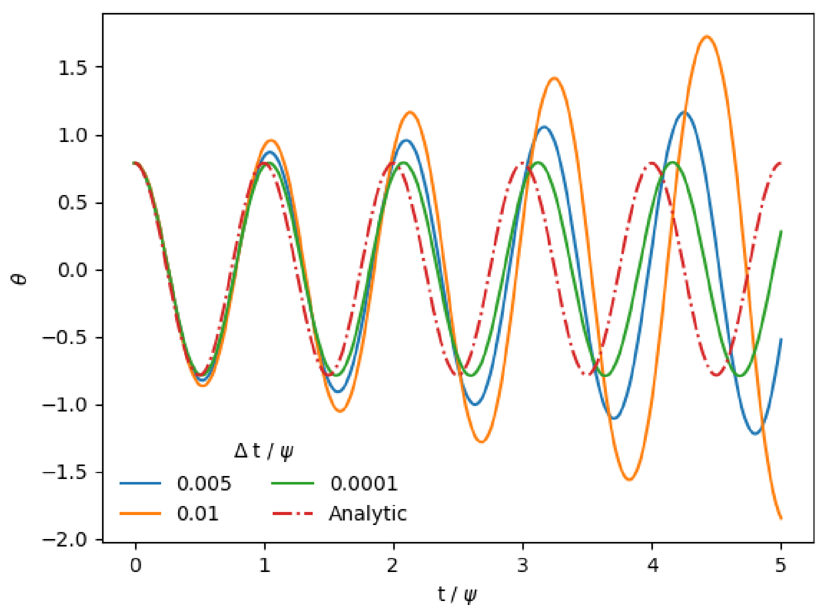

Figure 2: Results of numerical solutions $\theta(t)$, which describes the motion of a simple pendulum with respect to the pivot, from using Euler's method on equation (2) with varying timesteps, under the following conditions: $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $\frac{d\theta(0)}{dt} = 0$, $\theta_0 = \frac{\pi}{4}$, and timescale $\psi = 2\pi\sqrt{\frac{l}{g}}$.

The smaller the time step, the more accurate the function created by Euler's iterations is. In particular, the time step 0.01ψ generated a function that started widening out of the frame very quickly (an indication of it "blowing up"), and we can see how the function generated by a time step of 0.005ψ has an increasing amplitude as well. The 0.0001ψ time step seem to be holding out well on this range of times. Ultimately, for the Euler's iterations going forward, I will use the time step of 0.0001ψ as by nature it has the best approximation out of the ones tested, but it still doesn't take very long to run.

I also ran Euler's Method to see how the oscillation periods change as a result of changing the initial angle that the pendulum is dropped from. So, the values I gave were $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $\frac{d\theta(0)}{dt} = 0$, and $\Delta t = 0.0001\psi$.

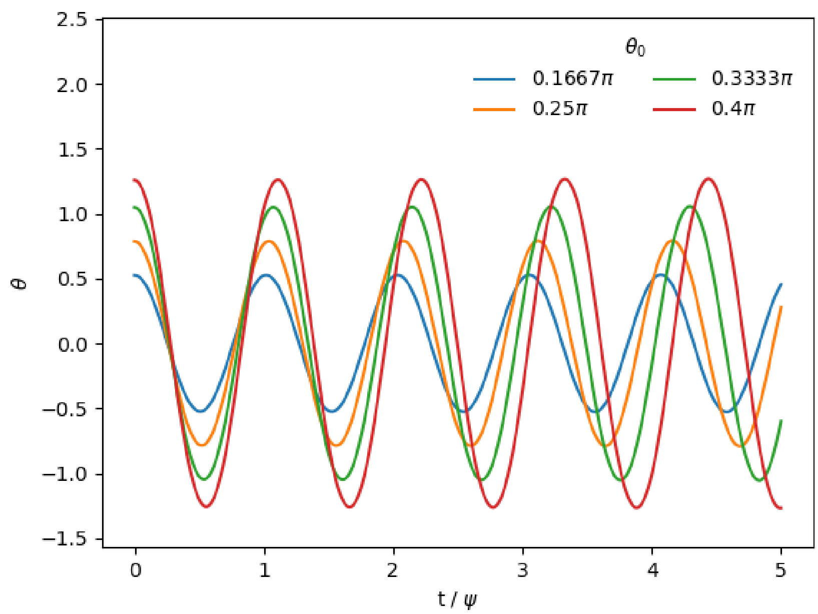

Figure 3: Results of numerical solutions $\theta(t)$, which describes the motion of a simple pendulum with respect to the pivot, from using Euler's method on equation (2) with varying initial θ, under the following conditions: $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $\frac{d\theta(0)}{dt} = 0$, timescale $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.0001\psi$.

Let's compare each of these individually with their corresponding analytical solutions, from when we used the linearization of $\sin\theta$.

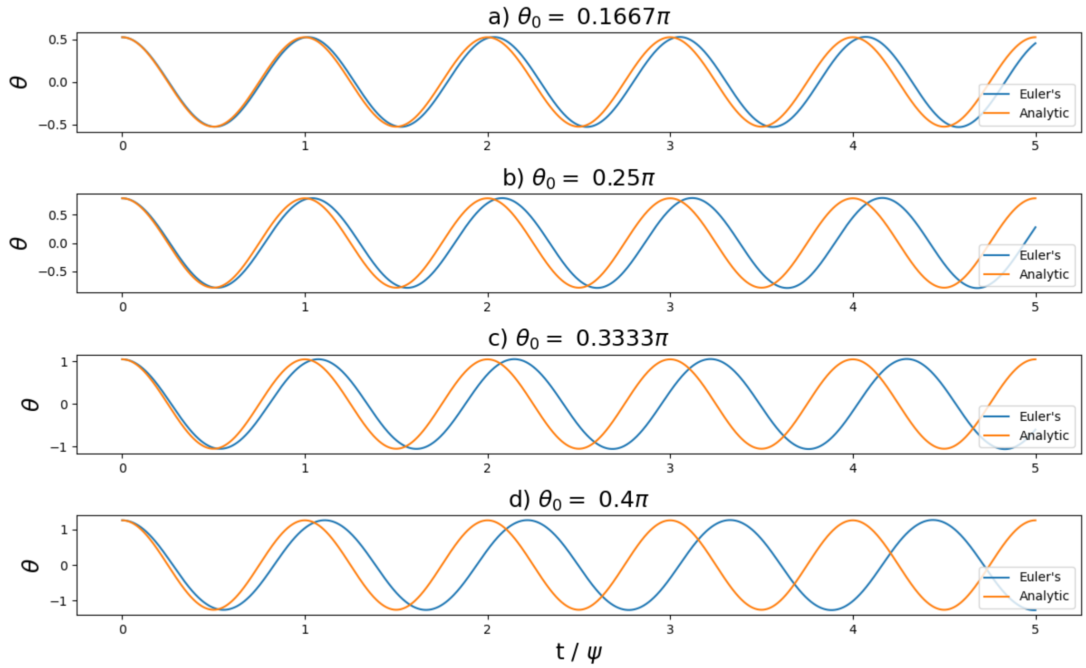

Figure 4: Each subplot shows the results, for a specific θ_0, of Euler's method to find the solution $\theta(t)$ of equation (2), which describes the motion of a simple pendulum with respect to the pivot, in comparison to their corresponding analytic solutions from equation (8), when $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $\frac{d\theta(0)}{dt} = 0$, timescale $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.0001\psi$. Figure a, b, c, d are based on θ_0: $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{2\pi}{5}$ respectively.

This graphic shows how if the pendulum begins at a small initial θ, the analytical solution and the solution from Euler's Method are very similar to each other. As θ_0 grows larger, the differences between the two solutions become more and more obvious. This makes sense, because the analytical solution was based on the linear Taylor series about $\theta = 0$, so it is known that the approximation of $\sin\theta$ will is better with small θ. That's why we made the analytical solution with the assumption that θ is small in the first place.

3 Simple Pendulum System with Drag Force

3.1 Development of ODE

We follow through with the same steps in section 2.2, except include an additional term in the sum of all forces. If we assume drag force to be $F_D = -kAv$, where A is the projected area, k is some constant, and v is the velocity and we state a to be the radius of the spherical mass, we have

$$ma_s = -mg\sin\theta - k(\pi a^2)v_s. \tag{9}$$

Note that there is no need to adjust for the direction of F_D since drag force is perpendicular to the direction of movement.

Considering that $a_s = l\frac{d^2\theta}{dt^2}$ and $v_s = l\frac{d\theta}{dt}$

$$\begin{aligned} ml\frac{d^2\theta}{dt^2} &= -mg\sin\theta - kl(\pi a^2)\frac{d\theta}{dt} \\ \frac{d^2\theta}{dt^2} &= -\frac{g}{l}\sin\theta - \frac{k}{m}(\pi a^2)\frac{d\theta}{dt}. \end{aligned} \tag{10}$$

3.2 Analytical Solution for sufficiently small θ

If we assume θ to be close to 0, we can utilize Taylor Series to find an analytical solution to (10).

$$\frac{d^2\theta}{dt^2} + \frac{k}{m}(\pi a^2)\frac{d\theta}{dt} + \frac{g}{l}\theta = 0$$

We can solve for the roots of the corresponding characteristic equation.

$$r_{1,2} = \frac{1}{2}(-\frac{k}{m}(\pi a^2) \pm \sqrt{(\frac{k}{m}(\pi a^2))^2 - 4(\frac{g}{l})}) \tag{11}$$

However, we don't know whether these roots are complex valued, real valued, or distinct, as it is dependent on the specific values of g, l, k, m, a.

If they are real and distinct roots ($r_1 \neq r_2 \in R$), the general solution is:

$$\theta(t) = c_1e^{r_1t} + c_2e^{r_2t}. \tag{12}$$

If they are complex roots ($r_{1,2} = \lambda \pm \mu i$), the general solution is:

$$\theta(t) = c_1e^{\lambda t}\cos(\mu t) + c_2e^{\lambda t}\sin(\mu t). \tag{13}$$

If they are repeated roots ($r = r_1 = r_2$), the general solution is:

$$\theta(t) = c_1e^{rt} + c_2te^{rt}. \tag{14}$$

Therefore, once the roots of the specific ODE are determined and the initial values are given, it is easy to determine the solution!

3.3 Euler's Method to Solve ODE

Let $z = \frac{d\theta}{dt}$. Then

$$\frac{dz}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - \frac{k}{m}(\pi a^2)\frac{d\theta}{dt}.$$

$$\theta_{k+1} = \theta_k + z_k \Delta t$$

$$z_{k+1} = z_k - \frac{g}{l}\sin\theta\Delta t - \frac{k}{m}(\pi a^2) z_k \Delta t.$$

3.4 Results of Euler's Method

I ran Euler's method again, choosing to vary k, the drag force constant. To do this, I set $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$ and $\theta_0 = \frac{\pi}{4}$. I defined the timescale to be $\psi = 2\pi\sqrt{\frac{l}{g}}$, and decided to let the final time point to be 5ψ. The k values that I tested across were $[0\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}, 0.1\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}, 0.5\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}, 1\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}]$.

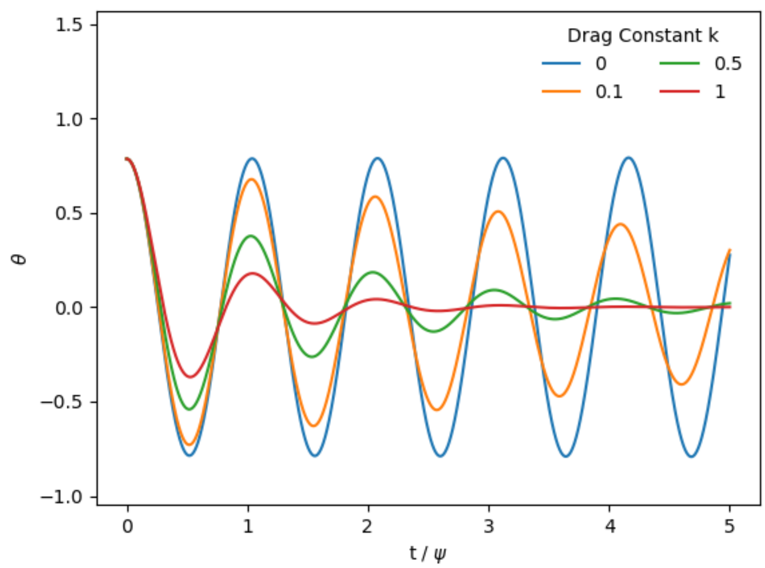

Figure 5: Results of numerical solution $\theta(t)$ from using Euler's method on equation (10), which describes the motion of a simple pendulum with drag force with respect to the pivot, varying the drag constant k (in units $\frac{\mathrm{kg}}{\mathrm{m^2 s}}$), under the following conditions: $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$, $\theta_0 = \frac{\pi}{4}$, $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.0001\psi$.

Just as we would expect, the larger the drag force constant is, the lesser amount of time it takes for the pendulum to lose momentum (the angle of the pendulum starts decreasing at a quicker rate). Of course, when the constant is 0, we are dealing with a system that doesn't experience any drag force so the pendulum stays in motion.

Let us compare Euler's Method to an analytical solution for when $k = 0.5\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}$. The analytical solution and development can be found in Appendix B.

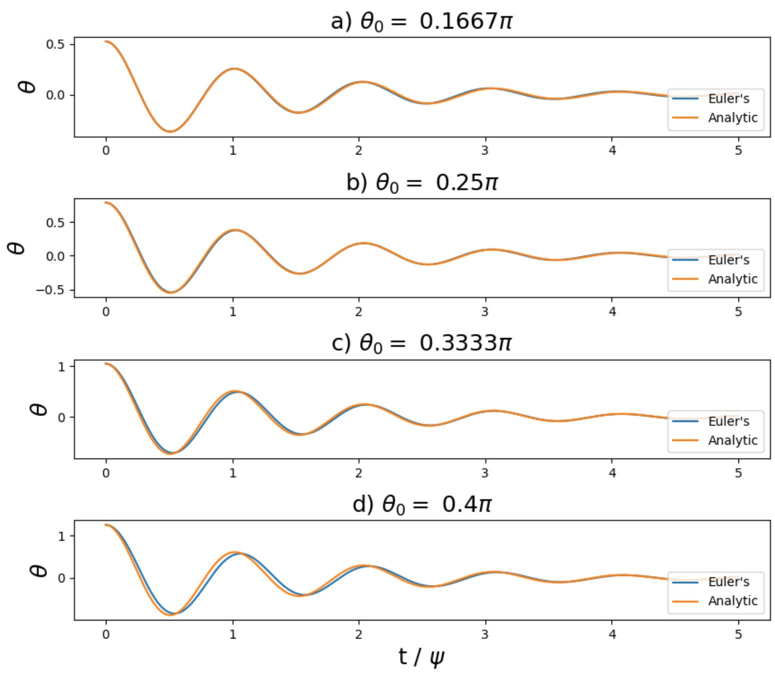

Figure 6: Each subplot shows the results, for a specific θ_0, of Euler's method to find the solution $\theta(t)$ of equation (10), which describes the motion of a simple pendulum with drag force with respect to the pivot, in comparison to their corresponding analytic solutions (found in Appendix B), when $g = 9.81\ \mathrm{m\,s^{-2}}$, $k = 0.5\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$, $\theta_0 = \frac{\pi}{4}$, $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.0001\psi$.

It is quite clear that the offset between the analytical solution and the solution from Euler's method when there is drag force acting on the pendulum is not nearly as large as the offset when there is no drag force affecting the pendulum. It can be argued that this is because if the drag force is large enough, the damping it causes dominates how the system behaves.

We can also look at the graphs of $\frac{d\theta}{dt}$ against θ.

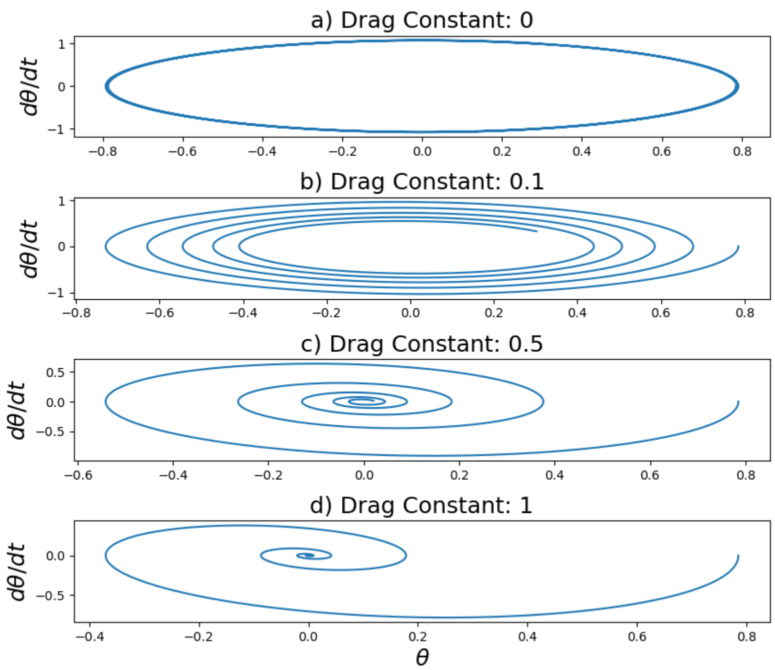

Figure 7: Graphs of $\frac{d\theta}{dt}$ against θ generated through Euler's method on equation (10), which describes the motion of a simple pendulum with drag force with respect to the pivot, for different values of drag constant k, where $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$, $\theta_0 = \frac{\pi}{4}$, $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.0001\psi$. Figure a, b, c, d are based on k (in units $\frac{\mathrm{kg}}{\mathrm{m^2 s}}$): 0, 0.1, 0.5, and 1 respectively.

To interpret this, we need to recognize how one loop around the center represents one period of oscillation occurring. In the case where the drag constant is 0, we have a perfect ellipse, because nothing is slowing the pendulum down and forcing the amplitude of its period to decrease. However, as we increase k, we see how there are less and less full rounds, because increasing drag force decreases the number of oscillations the pendulum can go through. Furthermore, when drag force is not negligible, we see that each round is closer to the origin than the last, which represents the fact that θ is changing at slower and slower rates as time goes on. The interpretation for this is that each oscillation is associated with a lesser amplitude (maximum θ value) than the last.

4 Rod Pendulum System

4.1 Development of ODE

The change from the massless string in the simple pendulum scenario to a circular rod that has mass requires the approach to obtaining the governing differential equation to be reframed from linear motion to angular motion. This is because each point within the rod is moving at a different velocity. When dealing with the simple pendulum, all of the mass was concentrated at the end of the string, so treating it as a point mass was acceptable, but that is not a valid argument for a Rod Pendulum System.

For bookkeeping, the circular rod has radius a (thickness $2a$), mass m, and length l. The acceleration of gravity is g and k is the drag force constant.

To make this change, we use Newton's Second Law for Rotation, which states $I\frac{d^2\theta}{dt^2} = \sum_i \vec{\tau_i}$. I is the moment of inertia of the object in question, and τ is the torque of the object, both with respect to the same axis.

The moment of inertia of a circular rod about its end can be calculated as follows:

$$I = \int_0^l x^2 \frac{m}{l} dx = \frac{1}{3} m l^2. \tag{15}$$

As for the torque, just as how we considered the forces of gravity and of drag as two components of the net force, we do the same, but with torque from gravity and from drag force.

We utilize this formula to calculate torque: $\tau = \vec{r} \times \vec{F}$ where $\vec{r}$ is the position vector and $\vec{F}$ is the force acting on the object.

Starting with gravity, we keep in mind that the center of mass for the rod is not located at the end, like how it was for the string, but is right in the middle. The distance between the pivot and the center of mass is then $\frac{l}{2}$, so the torque due to gravity is

$$\vec{\tau_g} = \vec{r} \times \vec{F_g} = -mg\frac{l}{2}\sin\theta\hat{k}. \tag{16}$$

It should be kept in mind that if we wanted to consider the scenario where there is no drag force acting on the rod, all that would be needed is this previous analysis. Since torque is only pointing in the $\hat{k}$ direction, we could consider just its scalar component.

$$\frac{1}{3} m l^2 \frac{d^2\theta}{dt^2} = -mg\frac{l}{2}\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2l}\sin\theta \tag{17}$$

For the the torque due to drag force, we have to recognize that drag force applies throughout the rod, and the velocity at each point of the rod vary depending on their position vector. Since drag force depends on velocity, we will have to use an integral to solve for τ_D.

We start by identifying what the torque is at a infinitesimally small slice of the rod dr, and denote that $d\tau_D$

$$d\vec{\tau}_D = \vec{r} \times d\vec{F_D} \tag{18}$$

where $d\vec{F_D}$ is the drag force on that slice of the rod and $\vec{r}$ is the position of the slice with respect to the pivot. If we allow $v(\vec{r})$ to be the velocity of the slice at located at $\vec{r}$, and denote $r = |\vec{r}|$, then

$$d\vec{F_D} = -k(2a)(dr)v(\vec{r}). \tag{19}$$

Similar to the derivations of (1) and (2), $v(r) = r\frac{d\theta}{dt}$ so we have

$$d\vec{F_D} = -k(2a)(dr)r\frac{d\theta}{dt}.$$

So,

$$d\vec{\tau}_D = d\vec{F_D} \times \vec{r} = k(2a)(dr)r^2\frac{d\theta}{dt}\hat{k} = (-k(2a)r^2\frac{d\theta}{dt}dr)\hat{k}. \tag{20}$$

We integrate:

$$\vec{\tau_D} = (\int_0^l -k(2a)r^2\frac{d\theta}{dt}dr)\hat{k} = -\frac{2}{3}kal^3\frac{d\theta}{dt}\hat{k}. \tag{21}$$

Since both $\vec{\tau_D}$ and $\vec{\tau_g}$ are in the direction of $\hat{k}$, we can consider their scalars alone, rather than their vectors.

$$\frac{1}{3}ml^2\frac{d^2\theta}{dt^2} = -mg\frac{l}{2}\sin\theta - \frac{2}{3}kal^3\frac{d\theta}{dt}$$
$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2l}\sin\theta - \frac{2kal}{m}\frac{d\theta}{dt}. \tag{22}$$

4.2 Analytical Solution for sufficiently small θ

With equation (4), if we replace $\sin\theta$ with its linear approximation θ, we have:

$$\frac{d^2\theta}{dt^2} + \frac{3g}{2l}\theta = 0.$$

Then, the characteristic equation has complex roots

$$r_{1,2} = \frac{\sqrt{-4\frac{3g}{2l}}}{2} = 0 \pm i\sqrt{\frac{3g}{2l}}. \tag{23}$$

Through the same process in section 2.3, we find that the solution is

$$\theta(t) = \theta_0 \cos(\sqrt{\frac{3g}{2l}} t). \tag{24}$$

As for equation (5),

$$\frac{d^2\theta}{dt^2} + \frac{2kal}{m}\frac{d\theta}{dt} + \frac{3g}{2l}\theta = 0$$

the roots of the corresponding characteristic equation are

$$r_{1,2} = \frac{1}{2}(-\frac{2kal}{m} \pm \sqrt{(\frac{2kal}{m})^2 - 4(\frac{3g}{2l})}). \tag{25}$$

Similar to the scenario of a simple pendulum with drag force, the types of roots depend on the exact values chosen for the parameters of the system, but once those are determined, an analytic solution can be very easily found with the same general solutions described in section 3.2.

4.3 Euler's Method to Solve ODE

Let $z = \frac{d\theta}{dt}$. Then

$$\frac{dz}{dt} = \frac{d^2\theta}{dt^2} = -\frac{3g}{2l}\sin\theta - \frac{2kal}{m}\frac{d\theta}{dt}$$

So the iterations are:

$$\theta_{k+1} = \theta_k + z_k \Delta t$$
$$z_{k+1} = z_k - \frac{3g}{2l}\sin\theta\Delta t - \frac{2kal}{m}\frac{d\theta}{dt}\Delta t.$$

4.4 Results of Euler's Method

For this, I repeated the same process as in Section 2.4: I chose to vary k and I set $g = 9.81\ \mathrm{m\,s^{-2}}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$ and $\theta_0 = \frac{\pi}{4}$. I defined the timescale to be $\psi = 2\pi\sqrt{\frac{l}{g}}$, and decided to let the final time point to be 5ψ. I also made each timestep $\Delta t = 0.000001\psi$, which is much smaller than the previous cases because this scenario is more complicated than the previous. The k values that I tested across were $[0\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}, 0.1\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}, 0.5\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}, 1\ \frac{\mathrm{kg}}{\mathrm{m^2 s}}]$.

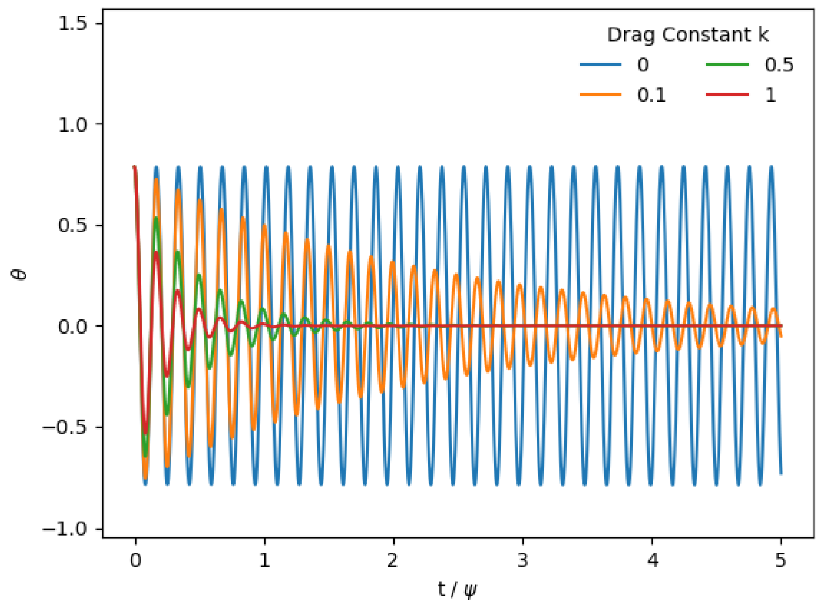

Figure 8: Results of numerical solution $\theta(t)$ from using Euler's method on equation (22), which describes the motion of a rod pendulum with drag force with respect to the pivot, varying the drag constant k (in units $\frac{\text{kg}}{\text{m}^2\text{s}}$), under the following conditions: $g = 9.81 \text{ m s}^{-2}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$, $\theta_0 = \frac{\pi}{4}$, $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.000001\psi$.

There are a few things that stand out in this rod pendulum with drag, compared to the simple pendulum with drag. Despite both having similar initial conditions, we see that the rod pendulum makes many more oscillations in a same period of time, but it slows down to a stable state of equilibrium much faster. The first is probably due to the change to a pendulum that has a different center of mass that is closer to the axis of rotation, leading to a smaller period of oscillation. The second difference can be attributed to the fact that the rod has a higher surface area exposed to drag force, which causes it to slow down at a faster rate.

One might wonder how the analytical solution compares to the solution from Euler's method. It is clear based on the simple pendulum with drag force scenario that the difference between the analytical solution and the numerical solution would be minimal for the rod scenario, as drag force plays an even bigger role in slowing down the pendulum, therefore dominating the behavior of the rod.

5 Extra Credit: Double Pendulum System (No Drag Force)

5.1 Development of ODE

In this scenario, we have a simple pendulum attached to another. We denote the top mass m_1, attached to the main pivot by a string of length l_1 with no mass, and its angle with respect to the pivot θ_1. The second mass is denoted m_2, attached to the above mass by a string of length of l_2, and its angle with respect to the pivot θ_2.

First, looking at m_2, there are two forces working on it: the force of tension from the string, and gravitational force. This is the exact same as the forces working on the mass in the simple pendulum system. So if we assume that the force of tension of the string **is not** dependent on the movement of the mass above,

$$\frac{d^2\theta_2}{dt^2} = -\frac{g}{l_2}\sin\theta_2$$

As for m_1, not only do we have to consider the the gravitational force of its own mass, we also have to consider the tension from the string below.

We set it up similarly to how we did in the simple pendulum system. The gravitational force of the object is still $-m_1 g \sin\theta_1$. The tension of the string below the mass is due to m_2 it is attached to, that is, it is dependent on the gravitational force acting on m_2. We need to identify how much of that force is tangential to the movement of m_1, by incorporating both θ_1 and θ_2. Particularly, we care about the difference between θ_1 and θ_2: when l_2 is exactly perpendicular to l_1, the difference between their angles would be $\frac{\pi}{2}$, and we would expect that the effect of m_2 of the motion of m_1 along its arc is maximized. Then, the tension of the the string can then be defined as $-m_2 g \sin(\theta_1 - \theta_2)$.

$$m_1 a_s = -m_1 g \sin\theta_1 - m_2 g \sin(\theta_1 - \theta_2)$$

Again, $a_{s_1} = l_1 \frac{d^2\theta_1}{dt^2}$, so

$$\frac{d^2\theta_1}{dt^2} = -\frac{g}{l_1}\sin\theta_1 - \frac{m_2 g}{m_1 l_1}\sin(\theta_1 - \theta_2) \tag{26}$$

In reality, though, it is unrealistic to assume that the second mass's behavior is independent of the behavior of the first, particularly when the initial θ for either of the masses are large. While I go forward with the analysis of this scenario using equation (26), it should be noted that a useful resource depicting the double pendulum scenario and the development of its equation of motion have been provided in Appendix C.

5.2 Euler's Method to Solve ODE

Let $z_1 = \frac{d\theta_1}{dt}$ and $z_2 = \frac{d\theta_2}{dt}$. Then

$$\frac{dz_1}{dt} = \frac{d^2\theta_1}{dt^2} = -\frac{g}{l}\sin\theta_2 - \frac{m_2 g}{m_1 l}\sin(\theta_1 - \theta_2)$$

$$\frac{dz_2}{dt} = \frac{d^2\theta_2}{dt^2} = -\frac{g}{l_2}\sin\theta_2$$

.

So the iterations are:

$$\theta_{1,k+1} = \theta_{1,k} + z_{1,k}\Delta t$$

$$z_{1,k+1} = z_{1,k} - \frac{g}{l}\sin\theta_1\Delta t - \frac{m_2 g}{m_1 l}\sin(\theta_1 - \theta_2)\Delta t$$

$$\theta_{2,k+1} = \theta_{2,k} + z_{2,k}\Delta t$$

$$z_{2,k+1} = z_{2,k} - \frac{g}{l}\sin\theta_2\Delta t$$

.

5.3 Results of Euler's Method

For this problem, I set $g = 9.81\ \mathrm{m\,s^{-2}}$, $l_1 = l_2 = 10$ m, $m_1 = m_2 = 10$ kg, $\frac{d\theta_1(0)}{dt} = \frac{d\theta_2(0)}{dt} = 0$ and $\theta_{1_0} = \frac{\pi}{4}$. I defined the timescale to be $\psi = 2\pi\sqrt{\frac{l}{g}}$, and decided to let the final time point to be 5ψ, making each timestep $\Delta t = 0.000001\psi$. Under these conditions, I varied θ_{2_0}.

The following graphic is interesting to see for values of initial θ_2 that are small, the motion of m_1 seems to be somewhat reasonable, as we can see that it doesn't have perfect oscillations, but the amplitude is less than or equal to it's initial θ_1. When the initial θ_2 is high, at $\frac{\pi}{2}$, the motion generated through Euler's method is much more unpredictable, seemingly chaotic. This can be for a few reasons: as I said before, the differential equation (26) that I developed is not representative of how the double pendulum truly behaves, as the assumption that the motion of m_2 doesn't affect m_1 is not true in most cases. Another reason is that the system itself is chaotic by nature. Finally, it could also be that Euler's method doesn't do well enough to numerically approximate the differential equation. All of these issues are addressed in the resource in Appendix C.

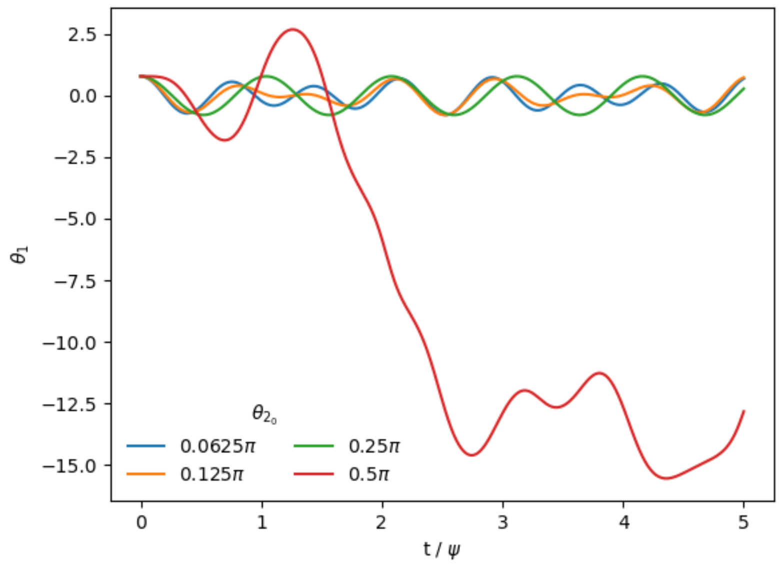

Figure 9: Results of numerical solution $\theta(t)$ from using Euler's method on equation (7), varying θ_{2_0}, under the following conditions: $g = 9.81 \text{ m s}^{-2}$, $l_1 = l_2 = 10$ m, $m_1 = m_2 = 10$ kg, $\frac{d\theta_1(0)}{dt} = \frac{d\theta_2(0)}{dt} = 0$ and $\theta_{1_0} = \frac{\pi}{4}$, $\psi = 2\pi\sqrt{\frac{l}{g}}$, and $\Delta t = 0.000001\psi$.

6 Conclusions

Overall, we developed 4 different ODEs that characterize 4 common pendulum scenarios. If it is safe to assume that the angles related to the systems are small, generalized closed form solutions to the ODEs can be found and used in real world scenarios where we know finer details of the pendulum, such as mass, length, etc. If not, Euler's method can provide a good estimate of what a solution might look like, given that the time step used is small enough, and assuming that the true solution to the ODE is well conditioned. We conclude with the statement that the derivations of these ODEs can be generalized to many other pendulum systems through adjustments in the parameters defining the system (such as using the same methods to derive ODEs for pendulums of different shapes, centers of mass, and areas exposed to drag force).

References

[1] Alexandre Eremenko, *Brief history of clock making and of the longitude problem*, https://www.math.purdue.edu/~eremenko/clockmaking.

[2] LibreTexts Physics, *The Simple Pendulum*, https://phys.libretexts.org/Bookshelves/College_Physics/Book%3A_College_Physics_1e_(OpenStax)/16%3A_Oscillatory_Motion_and_Waves/16.04%3A_The_Simple_Pendulum

[3] Martin Braun, *Differential Equations and Their Applications*, Springer, New York, NY, 4th edition, 1993

[4] Moebs et al., *University Physics Volume 1, 10.6-7*, OpenStax, 2016

Appendix A. Code

https://github.com/JanyaMirpuri-NYU/IMM_Project

Appendix B. Development of Equation

Under the conditions $g = 9.81 \text{ m s}^{-2}$, $l = 5$ m, $a = 1$ m, $m = 5$ kg, $\frac{d\theta(0)}{dt} = 0$ and $\theta_0 = \frac{\pi}{4}$, the roots for the characteristic equation are

$$r_{1,2} = \frac{1}{2}(-\frac{\pi}{10} \pm \sqrt{(\frac{\pi^2}{100}) - 4(\frac{9.81}{5})}) \approx -0.157 \pm i\sqrt{1.937}.$$

Clearly, these are complex roots, so the general solution is

$$\theta(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$$

To solve for the constants c_1 and c_2, we can use the initial conditions $\theta(0) = \theta_0$ and $\frac{d\theta(0)}{dt} = \dot{\theta}(0) = 0$.

$$\theta(0) = \theta_0 = c_1 e^{\lambda\cdot 0}\cos(\mu\cdot 0) + c_2 e^{\lambda\cdot 0}\sin(\mu\cdot 0)$$

$$\theta(0) = \theta_0 = c_1 \cdot 1 + 0 \rightarrow c_1 = \theta_0$$

Additionally

$$\begin{gathered}\dot{\theta}(t) = -\mu c_1 e^{\lambda t}\sin(\mu t) + \mu c_2 e^{\lambda t}\cos(\mu t) \\ +\lambda c_1 e^{\lambda t}\cos(\mu t) + \lambda c_2 e^{\lambda t}\sin(\mu t)\end{gathered}$$

$$\begin{gathered}\dot{\theta}(0) = -\mu c_1 e^{\lambda\cdot 0}\sin(\mu\cdot 0) + \mu c_2 e^{\lambda\cdot 0}\cos(\mu\cdot 0) \\ \lambda c_1 e^{\lambda\cdot 0}\cos(\mu\cdot 0) + \lambda c_2 e^{\lambda\cdot 0}\sin(\mu\cdot 0)\end{gathered}$$

$$\dot{\theta}(0) = 0 = \mu c_2 + \lambda\theta_0$$

$$c_2 = \frac{-\lambda\theta_0}{\mu}$$

Therefore, the analytic solution with these conditions are

$$\theta(t) = \theta_0 e^{\lambda t}\cos(\mu t) - \frac{\lambda\theta_0}{\mu} e^{\lambda t}\sin(\mu t)$$

Let's plug in the values from the roots:

$$\theta(t) = \theta_0 e^{-0.157t}\cos(\sqrt{1.937}t) - \frac{-0.157\cdot\theta_0}{\sqrt{1.937}} e^{-0.157t}\sin(\sqrt{1.937}t)$$

Appendix C. Double Pendulum

https://www.myphysicslab.com/pendulum/double-pendulum-en.html