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# A general approach to minimal mass tensegrity

Muhao Chen*, Robert E. Skelton

*Department of Aerospace Engineering, Texas A&M University, College Station, TX, USA*

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ABSTRACT

This work is motivated by the fundamental questions in engineering mechanics: Does continuum guarantee a minimal mass structure? What is a more mass efficient way to design structures? Moreover, what are the fundamental laws of the new approach? This paper provides a general unified approach for the minimal mass design of any solid or hollow bar tensegrity structures with any given external forces subject to the structure equilibrium conditions and the maximum stress constraints of structure members (strings yield, bars yield or buckle) in a compact matrix form. The methodology yields several nonlinear programming problems. Local stability is assured by checking the modes of failure of all the structure members, and global stability is guaranteed by solving a linear matrix inequality with the derived stiffness matrix. To further reduce mass, the choice of the cross-section of bars are also discussed. For practical problems, joint mass is considered as a penalty to the total structure mass. The principles developed in this paper demonstrates a fundamental insight into both materials and structures.

## 1. Introduction

There are five fundamental problems in engineering mechanics: compression, tension, torsion, cantilever, and simply supported. Each subject has been well studied in continuum mechanics for hundreds of years. However, does continuum guarantee a minimal mass structure? The answer is not always. For example, Michell [25] showed that a discrete truss is more mass efficient in taking cantilever load. Skelton et al. [32] proved T-Bar and D-Bar systems cost much less mass than a single continuum bar in taking compression. In fact, we found all these five fundamental questions in engineering mechanics have tensegrity solutions [15,28,30,32,39]. Biological systems provide perhaps the greatest evidence that tensegrity concepts yield the most efficient structures [17,20,35,36,38]. This leads us to believe tensegrity is a way to design structures. And of course not all of them, but a lot.

A tensegrity structure is a stable equilibrium of axially-loaded elements, of which compressive members are bars/struts, and tensile members are strings/cables [32]. There are many benefits of tensegrity structures: 1. Mass efficiency: the material is only needed in the essential load path, not the orthogonal one [3,7,19]. Tensegrity networks can be arranged to achieve strength with little mass. 2. Tension stabilising: One can change shape/stiffness without changing stiffness/shape [6,16,21]. 3. Accurate model: there is no material bending, the deformation of each structure member is only one dimensional, so the uncertainty is only in one direction, and we can get better stability margins with dedicated materials [40]. Moreover, more accurate structure models can give more precise control [1]. 4. Less control energy: the flexible structure has an infinite number of equilibrium states, by sliding along the equilibrium, less control energy is required to morph and maintain the structure shape [9,33]. 5. Energy absorber/harvester: the soft structure and strings can be used to absorb impact energy [23,27,42,43]. 6. Promote integration of structure and control design [5,12,13,29,37]: A misconception is that "The best system is made from the best components". That is certainly not true. Often, we gain more in integrating two disciplines than making exceptional improvements in one discipline [31,8]. Thus, tensegrity structures provide a promising paradigm for integrating these disciplines.

To design a tensegrity structure, there are two critical issues: 1. specify the connection patterns of bars and strings. 2. determine the thickness of all the structure members. The first one is called a form-finding problem, which has been widely studied. For example, Tibert and Pellegrino [34] classified and discussed the properties of kinematical and statical form-finding approaches for tensegrity structures. Milenko et al. [24] presented a general method for symmetric tensegrity structures with shape constraints. Zhang et al. [41] derived an efficient form-finding method based on the structural stiffness matrix and the total potential energy. Koohestani [18] proposed a computationally efficient algorithm for the analytical form-finding of tensegrity

* Corresponding author.
*E-mail addresses:* muhaochen@tamu.edu (M. Chen), bobskelton@tamu.edu (R.E. Skelton).

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structures. Lee et al. [19] implemented a genetic algorithm by using a force density method to get structure topology. The second one is called a minimal mass problem, few studies have been conducted. For example, Skelton et al. [32] introduced a minimal mass method by introducing a self-similar fractal considering yielding and buckling of structures. Nagase et al. [26] presented major aspects of minimal mass tensegrity problems, but the linear programming algorithms they presented are for yielding case calculations. Carpentieri et al. [4] gave a minimal mass design of tensile reinforcements of masonry structures with arbitrary shapes. Fraternali et al. [10] showed a topology optimization and minimal mass design strategy for masonry domes and vaults. Ma et al. [22] presented a minimal mass tensegrity cantilever structure subject to yielding and buckling constraints of bars. However, few of them start from a unified point of view and give a general form of minimal mass for any tensegrity structures. This paper presents a systematic minimal mass tensegrity framework in a unified compact matrix form. The principles developed in this paper can be used to compute the minimal mass and stiffness of solid and hollow bar tensegrity structures considering yielding and buckling constraints of each structural member with or without gravity. As a matter of fact, many composite structure studies have been conducted using the tensegrity paradigm. For example, softening–stiffening response of tensegrity prisms [1], minimal mass and self-stress analysis for v-expander tensegrity cells [8], domes and vaults [18]. This research also paves a road for minimal mass design, self-stress and stiffness analysis for any tensegrity based composite structures.

This paper is structured as follows: Section 2 formulates the tensegrity equilibrium equation in terms of force densities in the strings and bars. Section 3 gives the minimal mass formulation for both solid and hollow bar tensegrity structures without gravity. Section 4 presents the minimal mass formulation for both solid and hollow bar tensegrity structures in the presence of gravity. Section 5 derives the general form of structure stiffness. Section 6 shows another way of mass saving by changing the shape of bar cross-sections. Section 7 discusses the joint mass penalty for practical considerations. Section 8 demonstrates several numerical examples for verification. Section 9 presents the conclusions.

## 2. Tensegrity statics

For a given tensegrity structure and external force at an equilibrium state we have:

$$NK = W, \tag{1}$$

where $K = C_s^T\hat{\gamma}C_s - C_b^T\hat{\lambda}C_b, N \in R^{3\times n}$ is the nodal matrix (n is number of nodes), $C_s \in R^{\alpha\times n}$ ($\alpha$ is number of strings) and $C_b \in R^{\beta\times n}$ ($\beta$ is number of bars) are string and bar connectivity matrices, $W \in R^{3\times n}$ is external force matrix (each column is the force vector on each corresponding node), $\gamma \in R^{\alpha\times 1}$ and $\lambda \in R^{\beta\times 1}$ are force densities (force over structure member length) in the strings and bars ($\hat{\bullet}$ is a diagonal operator that converts a vector into a diagonal matrix). The bar and string matrices are $S = [\mathbf{s_1} \quad \mathbf{s_2} \quad \cdots \quad \mathbf{s}_\alpha] \in R^{3\times\alpha}$ and $B = [\mathbf{b_1} \quad \mathbf{b_2} \quad \cdots \quad \mathbf{b}_\beta] \in R^{3\times\beta}$, which satisfy $S = NC_s^T$ and $B = NC_b^T$. Then we get:

$$S\hat{\gamma}C_s - B\hat{\lambda}C_b = W. \tag{2}$$

Take the $i^{th}$ $(i = 1, 2, \cdots, n)$ column of Eq. (2):

$$S\hat{\gamma}C_se_i - B\hat{\lambda}C_be_i = We_i, \tag{3}$$

where $e_i = [0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0]^T$ is a column operator with 1 in the $i^{th}$ element and zeros elsewhere.

Using the identity $\hat{x}y = \hat{y}x$ for x and y being column vectors:

$$S(\widehat{C_se_i})\gamma - B(\widehat{C_be_i})\lambda = We_i. \tag{4}$$

This equation can be written as:

$$\begin{bmatrix} S(\widehat{C_se_i}) & -B(\widehat{C_be_i}) \end{bmatrix} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = We_i \tag{5}$$

Stacking all the columns from $i = 1$ to $i = n^{th}$ column, we get:

$$\underbrace{\begin{bmatrix} S(\widehat{C_se_1}) & -B(\widehat{C_be_1}) \\ S(\widehat{C_se_2}) & -B(\widehat{C_be_2}) \\ \vdots & \vdots \\ S(\widehat{C_se_n}) & -B(\widehat{C_be_n}) \end{bmatrix}}_{A} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} We_1 \\ We_2 \\ \vdots \\ We_n \end{bmatrix}}_{W_{vec}}, \tag{6}$$

which can be simply written as:

$$Ax = W_{vec}, x = \begin{bmatrix} \gamma^T & \lambda^T \end{bmatrix}^T. \tag{7}$$

## 3. Minimal mass tensegrity without gravity

Let us consider two general structure cases, bars are solid rods or hollow pipes. They are discussed separately.

### 3.1. Mass formulation for solid bar tensegrity

Assume we use same material for all the strings and same material for all the bars. Mass of a solid bar is:

$$m = \rho_b||b_j||\pi r^2, \quad j = 1, \cdots, \beta. \tag{8}$$

Euler buckling force for a solid bar is [32]:

$$f = \frac{\pi^3 E_b}{4||b_j||^2}r^4. \tag{9}$$

Using $f = \lambda_j||b_j||$, the mass of a solid bar subject to buckling can be written in terms of its force density:

$$m_B^S = 2\rho_b\lambda_j^{\frac{1}{2}}\left(\frac{||b_j||^5}{\pi E_b}\right)^{\frac{1}{2}}. \tag{10}$$

Mass of a solid bar subject to yielding is:

$$m_Y^S = \frac{\rho_b}{\sigma_b}\lambda_j||b_j||^2. \tag{11}$$

Since by definition of tensegrity structures, strings only take tension and bars only take compression, yielding is the mode of failure for strings, yielding and buckling are modes of failure for bars. The structure minimal mass $M$ can be achieved when every string yields and every bar yields or buckles at the same time. We cannot know the mode of failure for each bar in advance, but the minimal bar mass is equivalent to take the maximum mass value of the two failure modes. Then, total mass can be written as:

$$M^S = \frac{\rho_s}{\sigma_s}\sum_{i=1}^{\alpha}\gamma_i||s_i||^2 + \sum_{j=1}^{\beta}max\left\{\frac{\rho_b}{\sigma_b}\lambda_j||b_j||^2, 2\rho_b\lambda_j^{\frac{1}{2}}\left(\frac{||b_j||^5}{\pi E_b}\right)^{\frac{1}{2}}\right\},$$

where $\rho_s, \rho_b, \sigma_s, \sigma_b$ are the density and yield strength of strings and solid bars, $\gamma_i, \lambda_j, ||s_i||$, and $||b_j||$ $(i = 1, 2, \cdots, \alpha, j = 1, 2, \cdots, \beta)$ are the force densities and length of each string and each bar, and $E_b$ is Young's modulus of bars.

The mass ratio of a solid bar subject to the same given load can be written as:

$$\frac{m_Y^S}{m_B^S} = \frac{1}{2\sigma_b}\sqrt{\frac{\lambda_j\pi E_b}{||b_j||}}. \tag{12}$$

For a given force $f$ of a solid bar, when yielding is mode of failure, we have $m_Y^S > m_B^S$, when buckling is mode of failure, we have $m_B^S > m_Y^S$. And when yielding and buckling happens at the same time,

$m_Y^S = m_B^S$. To simplify the notations, we introduce a label matrix $Q \in R^{\beta \times \beta}$ to identify and record the mode of failure for each bar:

$$Q_{jj} = \begin{cases} 0 & \lambda_j \geqslant \frac{4\sigma_b^2||b_j||}{\pi E_b}, \text{ bar yields} \\ 1 & \lambda_j < \frac{4\sigma_b^2||b_j||}{\pi E_b}, \text{ bar buckles} \end{cases}, \tag{13}$$

and the off diagonal elements of $Q$ are zeros.

The diagonal elements of matrix $Q$ with ones denote buckling is the mode of failure of those bars, and the diagonal elements of the matrix $(I - Q)$ with ones represent yielding is the mode of failure of those bars. The bars are now separated into two parts, and the minimal mass formula can be well defined in a matrix form:

$$M^S = \frac{\rho_s}{\sigma_s}\left(vec(\lfloor S^T S \rfloor)\right)^T \gamma + \frac{\rho_b}{\sigma_b}\left(vec(\lfloor B^T B \rfloor (I - Q))\right)^T \lambda + \frac{2\rho_b}{\sqrt{\pi E_b}}\left(vec\left(\lfloor B^T B \rfloor^{\frac{5}{4}} Q\right)\right)^T \lambda^{\frac{1}{2}}, \tag{14}$$

where $\lfloor \bullet \rfloor$ is an operator taking the diagonal elements of a matrix, $vec(\bullet)$ is an operator taking the diagonal elements of the matrix and form a vector.

### 3.2. Mass formulation for hollow bar tensegrity

Mass of a hollow bar is:

$$m = \rho_b ||b_j|| \pi \left(r_{j_{out}}^2 - r_{j_{in}}^2\right) = \rho_b \pi ||b_j|| r_{j_{in}}^2 \left(\left(\frac{r_{j_{out}}}{r_{j_{in}}}\right)^2 - 1\right), \tag{15}$$

where $r_{j_{out}}$ and $r_{j_{in}}$ are outer and inner radius of the $j^{th}$ hollow bar.

Euler buckling force for a hollow bar is [32]:

$$f = \frac{\pi^3 E_b}{4||b_j||^2}\left(r_{j_{out}}^4 - r_{j_{in}}^4\right) = \frac{\pi^3 E_b}{4||b_j||^2} r_{j_{in}}^4 \left(\left(\frac{r_{j_{out}}}{r_{j_{in}}}\right)^4 - 1\right). \tag{16}$$

Using $f = \lambda_j ||b_j||$, the mass of a hollow bar can be written in terms of force densities:

$$m_B^H = \rho_b \pi ||b_j|| r_{j_{in}}^2 \left(\sqrt{1 + \frac{4\lambda_j ||b_j||^3}{\pi^3 E_b r_{j_{in}}^4}} - 1\right). \tag{17}$$

The mass of a hollow bar subject to yielding is:

$$m_Y^H = \frac{\rho_b}{\sigma_b} \lambda_j ||b_j||^2. \tag{18}$$

Similarly to solid bars, the structure mass $M$ for hollow bars can be written as:

$$M^H = \frac{\rho_s}{\sigma_s}\sum_{i=1}^{\alpha} \gamma_i ||s_i||^2 + \sum_{j=1}^{\beta} \max\left\{\frac{\rho_b}{\sigma_b}\lambda_j ||b_j||^2, \frac{\rho_b ||b_j||}{\sqrt{\pi E_b}}\left(\sqrt{\pi^3 E_b r_{j_{in}}^4 + 4\lambda_j ||b_j||^3} - \pi r_{j_{in}}^2 \sqrt{\pi E_b}\right)\right\},$$

where $\rho_s, \rho_b, \sigma_s, \sigma_b$ are the density and yield strength of strings and hollow bars, $\gamma_i, \lambda_j, ||s_i||$, and $||b_j||$ $(i = 1, 2, \cdots, \alpha, j = 1, 2, \cdots, \beta)$ are the force densities and length of each string and each bar, and $E_b$ is Young's modulus of bars.

The mass ratio of a hollow bar subject to buckling and yielding can be written as:

$$\frac{m_B^H}{m_Y^H} = \frac{\sigma_b \pi r_{j_{in}}^2}{\lambda_j ||b_j||}\left(\sqrt{1 + \frac{4\lambda_j ||b_j||^3}{\pi^3 E_b r_{j_{in}}^4}} - 1\right). \tag{19}$$

For a given force $f$ and inner radius $r_{j_{in}}$ of a hollow bar, when yielding is the mode of failure, $m_Y^H > m_B^H$, when buckling is the mode of failure, $m_Y^H < m_B^H$, and when buckling and yielding happen at the same time, $m_Y^H = m_B^H$. Similarly, we define a label matrix $Q \in R^{\beta \times \beta}$ to identify whether one bar is yielding or buckling:

$$Q_{jj} = \begin{cases} 0 & \lambda_j \geqslant \frac{4\sigma_b^2||b_j||}{\pi E_b} - \frac{2\sigma_b \pi r_{j_{in}}^2}{||b_j||}, \text{ bar yields} \\ 1 & 0 < \lambda_j < \frac{4\sigma_b^2||b_j||}{\pi E_b} - \frac{2\sigma_b \pi r_{j_{in}}^2}{||b_j||}, \text{ bar buckles} \end{cases}, \tag{20}$$

the off diagonal elements of $Q$ are zeros.

The minimal mass formula for hollow bar tensegrity is given below in a matrix form:

$$M^H = \frac{\rho_s}{\sigma_s}\left(vec(\lfloor S^T S \rfloor)\right)^T \gamma + \frac{\rho_b}{\sigma_b}\left(vec(\lfloor B^T B \rfloor (I - Q))\right)^T \lambda + \frac{\rho_b\left(vec\left(\lfloor B^T B \rfloor^{\frac{1}{2}} Q\right)\right)^T}{\sqrt{\pi E_b}}\left(\sqrt{\pi^3 E_b r_{in}^4 + 4\left(\lfloor B^T B \rfloor^{\frac{3}{2}} Q\right)\lambda} - \pi r_{in}^2 \sqrt{\pi E_b}\right), \tag{21}$$

where $r_{in} = \begin{bmatrix} r_{1_{in}} & \cdots & r_{j_{in}} & \cdots & r_{\beta_{in}} \end{bmatrix}^T$ is a vector of the inner radius of all the hollow bars.

### 3.3. Minimal mass tensegrity without gravity

The two minimal mass problems can be formed as:

$$\begin{cases} \underset{x}{\text{minimize}}\, M \\ \text{subject to } Ax = W_{vec}, \quad x \geqslant \varepsilon(\varepsilon \geqslant 0) \end{cases}, \tag{22}$$

where $M$ is $M^S$ or $M^H$ depends on using solid or hollow bars for the structure, $\varepsilon$ is the prestress assigned to the strings, and $\varepsilon \geqslant 0$ guarantees that all strings are in tension and all bars are in compression.

Notice that to solve Eq. (22), one needs to specify the label matrix $Q$. However, since $Q$ is determined by structure typology and external forces, one cannot exactly specify $Q$ for any structure in advance. To obtain a global solution, the nonlinear optimization problem can be solved in an iterative manner, see Algorithm 1.

**Algorithm 1:** Minimal Mass No Gravity

**1)** Assumes all bar buckles, $Q = I^{\beta \times \beta}$.
**2)** Compute force densities $x$:
**while** $Q_{i+1} \neq Q_i$ **do**

$$\begin{cases} \underset{x}{\text{minimize}}\ M \\ \text{subject to } Ax = W_{vec},\ x \geq \epsilon\ (\epsilon \geq 0). \end{cases}$$

Take $\lambda$ out of $x$, check Eqn. (13) or (20), update $Q$
$i \leftarrow i + 1$.

**end while**

## 4. Minimal mass tensegrity with gravity

Since gravity is unlike a given set of specific external forces that apply to the structure, it is determined by the mass of the structure itself. In other words, the statics mass optimization process is coupled with the gravity force. We separate total force $W$ into two parts $W = W_e + W_g$, where $W_g$ is the gravity force, $W_e$ is other applied external force, and $\mathbf{g}$ is the gravitational acceleration, i.e. $\mathbf{g} = \begin{bmatrix} 0 & 0 & -9.8 \end{bmatrix}^T$ is the gravity on the Earth. The gravity force can be modeled by lumped forces equally distributed on the member nodes [26].

### 4.1. Gravity formulation for solid bar tensegrity

Thus, the gravity force of solid bars and strings can be expressed as:

$$W_g^S = \frac{1}{2}\mathbf{g}\frac{\rho_s}{\sigma_s}\left(vec(\lfloor S^T S \rfloor)\right)^T \hat{\gamma} |C_s| + \frac{1}{2}\mathbf{g}\frac{\rho_b}{\sigma_b}\left(vec(\lfloor B^T B \rfloor (I - Q))\right)^T \hat{\lambda} |C_b| + \frac{1}{2}\mathbf{g}\frac{2\rho_b}{\sqrt{\pi E_b}}\left(vec\left(\lfloor B^T B \rfloor^{\frac{5}{4}} Q\right)\right)^T \hat{\lambda}^{\frac{1}{2}} |C_b|, \tag{23}$$

where $|\bullet|$ is an operator getting the absolute value of each element for a given matrix. Take the $i^{th}$ column of the above equation, we get:

$$\begin{aligned}W^S_{gi} &= \tfrac{1}{2}\mathbf{g}\tfrac{\rho_s}{\sigma_s}\left(vec\left(\lfloor S^TS\rfloor\right)\right)^T\hat{\gamma}|C_s|e_i \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{\rho_b}{\sigma_b}\left(vec\left(\lfloor B^TB\rfloor(I-Q)\right)\right)^T\hat{\lambda}|C_b|e_i \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{2\rho_b}{\sqrt{\pi E_b}}\left(vec\left(\lfloor B^TB\rfloor^{\frac{5}{4}}Q\right)\right)^T\hat{\lambda}^{\frac{1}{2}}|C_b|e_i \\ &= \tfrac{1}{2}\mathbf{g}\tfrac{\rho_s}{\sigma_s}\left(vec\left(\lfloor S^TS\rfloor\right)\right)^T|\widehat{C_se_i}|\gamma \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{\rho_b}{\sigma_b}\left(vec\left(\lfloor B^TB\rfloor(I-Q)\right)\right)^T|\widehat{C_be_i}|\lambda \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{2\rho_b}{\sqrt{\pi E_b}}\left(vec\left(\lfloor B^TB\rfloor^{\frac{5}{4}}Q\right)\right)^T|\widehat{C_be_i}|\lambda^{\frac{1}{2}}.\end{aligned} \tag{24}$$

Arrange all the columns, we get:

$$W^S_{gvec} = \begin{bmatrix} W^{S\;T}_{g1} & \cdots & W^{S\;T}_{gi} & \cdots & W^{S\;T}_{gn}\end{bmatrix}^T. \tag{25}$$

The steps for solving statics in the presence of gravity are as the following.

*4.2. Gravity formulation for hollow bar tensegrity*

For hollow bar tensegrity, the gravity force of bars and strings can be expressed as:

$$\begin{aligned}W^H_{g} &= \tfrac{1}{2}\mathbf{g}\tfrac{\rho_s}{\sigma_s}\left(vec\left(\lfloor S^TS\rfloor\right)\right)^T\hat{\gamma}|C_s| \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{\rho_b}{\sigma_b}\left(vec\left(\lfloor B^TB\rfloor(I-Q)\right)\right)^T\hat{\lambda}|C_b| \\ &+\tfrac{1}{2}\mathbf{g}\frac{\rho_b\left(vec\left(\lfloor B^TB\rfloor^{\frac{1}{2}}Q\right)\right)^T}{\sqrt{\pi E_b}} \\ &\left(\sqrt{\pi^3E_b\hat{r}^4_{in}+4\left(\lfloor B^TB\rfloor^{\frac{3}{2}}Q\right)\hat{\lambda}}-\pi\hat{r}^2_{in}\sqrt{\pi E_b}\right)|C_b|,\end{aligned}$$

where $|\bullet|$ is an operator getting the absolute value of each element for a given matrix.

Take the $i^{th}$ column of the above equation, we get:

$$\begin{aligned}W^H_{gi} &= \tfrac{1}{2}\mathbf{g}\tfrac{\rho_s}{\sigma_s}\left(vec\left(\lfloor S^TS\rfloor\right)\right)^T\hat{\gamma}|C_s|e_i \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{\rho_b}{\sigma_b}\left(vec\left(\lfloor B^TB\rfloor(I-Q)\right)\right)^T\hat{\lambda}|C_b|e_i \\ &+\tfrac{1}{2}\mathbf{g}\frac{\rho_b\left(vec\left(\lfloor B^TB\rfloor^{\frac{1}{2}}Q\right)\right)^T}{\sqrt{\pi E_b}} \\ &\left(\sqrt{\pi^3E_b\hat{r}^4_{in}+4\left(\lfloor B^TB\rfloor^{\frac{3}{2}}Q\right)\hat{\lambda}}-\pi\hat{r}^2_{in}\sqrt{\pi E_b}\right)|C_b|e_i \\ &= \tfrac{1}{2}\mathbf{g}\tfrac{\rho_s}{\sigma_s}\left(vec\left(\lfloor S^TS\rfloor\right)\right)^T|\widehat{C_se_i}|\gamma \\ &+\tfrac{1}{2}\mathbf{g}\tfrac{\rho_b}{\sigma_b}\left(vec\left(\lfloor B^TB\rfloor(I-Q)\right)\right)^T|\widehat{C_be_i}|\lambda \\ &+\tfrac{1}{2}\mathbf{g}\frac{\rho_b\left(vec\left(\lfloor B^TB\rfloor^{\frac{1}{2}}Q\right)\right)^T}{\sqrt{\pi E_b}}|\widehat{C_be_i}| \\ &\left(\sqrt{\pi^3E_br^4_{in}+4\left(\lfloor B^TB\rfloor^{\frac{3}{2}}Q\right)\lambda}-\pi r^2_{in}\sqrt{\pi E_b}\right).\end{aligned} \tag{26}$$

Arrange all the columns, we get:

$$W^H_{gvec} = \begin{bmatrix} W^{H\;T}_{g1} & \cdots & W^{H\;T}_{gi} & \cdots & W^{H\;T}_{gn}\end{bmatrix}^T. \tag{27}$$

*4.3. Minimal mass tensegrity with gravity*

The two minimal mass problems can be formulated as:

$$\begin{cases}\underset{x}{\text{minimize}}\, M \\ \text{subject to } Ax = W_{evec} + W_{gvec}, \quad x \geqslant \varepsilon(\varepsilon \geqslant 0)\end{cases}, \tag{28}$$

where $M, W_{gvec}$ are $M^S, W^S_{gvec}$ or $M^H, W^H_{gvec}$ which depend on using solid or hollow bars for the structure, $\varepsilon$ is the prestress assigned to the strings, and $\varepsilon \geqslant 0$ guarantees that all strings are in tension and all bars in compression.

Similarly to Algorithm 1 for the solid bar tensegrity, to solve Eq. (28), one needs to specify the label matrix $Q$. To obtain a global solution for hollow bar tensegrity considering gravity, the nonlinear optimization problem can be solved in an iterative manner, see Algorithm 2.

**Algorithm 2:** Minimal Mass with Gravity

**1)** Assumes all bar buckles and gravity forces are zero, $Q = I^{\beta\times\beta}$ and $W_{g\ vec} = 0$.
**2)** Compute force densities $x$:
**while** $Q_{i+1} \neq Q_i$ **do**

$$\begin{cases}\underset{x}{\text{minimize}}\ M \\ \text{subject to } Ax = W_{e\ vec} + W_{g\ vec},\ x \geq \epsilon\ (\epsilon \geq 0).\end{cases}$$

Take $\lambda$ out of $x$, check Eqn. (13) or (20), update $Q$.
Update $W_{g\ vec}$ from Eqn. (25) or (27).
$i \leftarrow i + 1$.

**end while**

## 5. Tensegrity structure stiffness matrix

*5.1. Stiffness matrix formulation*

Consider a small variation around the equilibrium, $N + dN = [\cdots \quad (n_k + dn_k) \quad \cdots]$, $W + dW = [\cdots \quad (w_k + dw_k) \quad \cdots]$, $\gamma + d\gamma = [\cdots \quad (\gamma_i + d\gamma_i) \quad \cdots]^T$, $\lambda + d\lambda = [\cdots \quad (\lambda_j + d\lambda_j) \quad \cdots]^T$, the statics equation can be written as:

$$(N+dN)C_s^T\left(\widehat{\gamma+d\gamma}\right)C_s - (N+dN)C_b^T\left(\widehat{\lambda+d\lambda}\right)C_b = W + dW. \tag{29}$$

Notice $\gamma = [\cdots \quad \gamma_i \quad \cdots]^T$ and $\lambda = [\cdots \quad \lambda_j \quad \cdots]^T$, we have $d\gamma = [\cdots \quad d\gamma_i \quad \cdots]^T$ and $d\lambda = [\cdots \quad d\lambda_i \quad \cdots]^T$, then we get:

$$(N+dN)C_s^T(\hat{\gamma}+d\hat{\gamma})C_s - (N+dN)C_b^T(\hat{\lambda}+d\hat{\lambda})C_b = W + dW. \tag{30}$$

Then, we have:

$$\begin{aligned}&NC_s^T\hat{\gamma}C_s + dNC_s^T\hat{\gamma}C_s + NC_s^Td\hat{\gamma}C_s + dNC_s^Td\hat{\gamma}C_s - \\ &NC_b^T\hat{\lambda}C_b - dNC_b^T\hat{\lambda}C_b - NC_b^Td\hat{\lambda}C_b - dNC_b^Td\hat{\lambda}C_b \\ &= W + dW.\end{aligned} \tag{31}$$

Using the fact $N\left(C_s^T\hat{\gamma}C_s - C_b^T\hat{\lambda}C_b\right) = W$ and neglect the higher order terms:

$$\left(dNC_s^T\hat{\gamma} + NC_s^Td\hat{\gamma}\right)C_s - \left(dNC_b^T\hat{\lambda} + NC_b^Td\hat{\lambda}\right)C_b = dW.$$

Since $S + dS = [\cdots \quad (s_i + ds_i) \quad \cdots]$, $B + dB = [\cdots \quad (b_j + db_j) \quad \cdots]$, we have $dS = dNC_s^T$, $dB = dNC_b^T$,

$$\left(dS\hat{\gamma} + Sd\hat{\gamma}\right)C_s - \left(dB\hat{\lambda} + Bd\hat{\lambda}\right)C_b = dW.$$

Let us take a close look at each string and each bar:

$$\begin{aligned}&\Big[\cdots\underbrace{(ds_i\gamma_i + s_id\gamma_i)}_{K_{si}ds_i}\cdots\Big]C_s - \\ &\Big[\cdots\underbrace{(db_j\lambda_j + b_jd\lambda_j)}_{K_{bj}db_j}\cdots\Big]C_b = dW.\end{aligned} \tag{32}$$

Eq. (32) can be written as:

$$\begin{aligned}&[K_{s1}ds_1 \quad \cdots \quad K_{s\alpha}ds_\alpha]C_s - \\ &[K_{b1}db_1 \quad \cdots \quad K_{b\beta}db_\beta]C_b = dW\end{aligned} \tag{33}$$

Vectorize the equation on both sides, we have:

$$\begin{aligned}&vec([K_{s1}ds_1 \quad \cdots \quad K_{s\alpha}ds_\alpha]C_s)-\\&vec([K_{b1}db_1 \quad \cdots \quad K_{b\beta}db_\beta]C_b) = vec(dW).\end{aligned} \tag{34}$$

Use the properties of the vec operator $vec(AXB) = (B^T \otimes A)vec(X)$:

$$\begin{aligned}&(C_s^T \otimes I_3)vec([K_{s1}ds_1 \quad \cdots \quad K_{s\alpha}ds_\alpha]) - (C_b^T \otimes I_3)\\&vec([K_{b1}db_1 \quad \cdots \quad K_{b\beta}db_\beta]) = vec(dW).\end{aligned} \tag{35}$$

This can be written as:

$$\begin{aligned}&(C_s^T \otimes I_3)\begin{bmatrix}K_{s1} & & \\ & \ddots & \\ & & K_{s\alpha}\end{bmatrix}vec([ds_1 \quad \cdots \quad ds_\alpha])-\\&(C_b^T \otimes I_3)\begin{bmatrix}K_{b1} & & \\ & \cdots & \\ & & K_{b\beta}\end{bmatrix}vec([db_1 \quad \cdots \quad db_\beta])\\&= vec(dW).\end{aligned} \tag{36}$$

As $dS = [ds_1 \quad \cdots \quad ds_\alpha]$ and $dB = [db_1 \quad \cdots \quad db_\beta]$,

$$\begin{aligned}&(C_s^T \otimes I_3)\mathbf{b.d.}(K_{s1}, \cdots, K_{s\alpha})vec(dS)\\&-(C_b^T \otimes I_3)\mathbf{b.d.}(K_{b1}, \cdots, K_{b\beta})vec(dB) = vec(dW),\end{aligned} \tag{37}$$

where $\mathbf{b.d.}(\bullet)$ is an operator that generates block diagonal matrices in the parentheses.

Use $dS = dNC_s^T$ and $dB = dNC_b^T$, we get:

$$\begin{aligned}&(C_s^T \otimes I_3)\mathbf{b.d.}(K_{s1}, \cdots, K_{s\alpha})vec\{dNC_s^T\}\\&-(C_b^T \otimes I_3)\mathbf{b.d.}(K_{b1}, \cdots, K_{b\beta})vec\{dNC_b^T\} = vec(dW).\end{aligned} \tag{38}$$

Use the properties of the vec operator $vec(AXB) = (B^T \otimes A)vec(X)$:

$$\begin{aligned}&(C_s^T \otimes I_3)\mathbf{b.d.}(K_{s1}, \cdots, K_{s\alpha})(C_s \otimes I_3)vec(dN)-\\&(C_b^T \otimes I_3)\mathbf{b.d.}(K_{b1}, \cdots, K_{b\beta})(C_b \otimes I_3)vec(dN)\\&= vec(dW).\end{aligned} \tag{39}$$

Then, we get:

$$\underbrace{\begin{Bmatrix}(C_s^T \otimes I_3)\mathbf{b.d.}(K_{s1}, \cdots, K_{s\alpha})(C_s \otimes I_3)\\-(C_b^T \otimes I_3)\mathbf{b.d.}(K_{b1}, \cdots, K_{b\beta})(C_b \otimes I_3)\end{Bmatrix}}_{K_n}vec(dN) = vec(dW), \tag{40}$$

where $K_n$ is the stiffness matrix.

Let us take a look at $K_{si}$ and $K_{bj}$:

$$\begin{aligned}K_{si} &= \tfrac{ds_i\gamma_i + s_i d\gamma_i}{ds_i} = \gamma_i I_3 + s_i\tfrac{d\gamma_i}{ds_i},\\K_{bj} &= \tfrac{db_j\lambda_j + b_j d\lambda_j}{db_j} = \lambda_j I_3 + b_j\tfrac{d\lambda_j}{db_j}.\end{aligned} \tag{41}$$

We assume the materials are Hookean, the force densities (string in tension, bar in compression) can be expressed as:

$$\gamma_i = k_{si}\left(1 - \frac{||s_{i0}||}{||s_i||}\right),\ \lambda_j = -k_{bj}\left(1 - \frac{||b_{j0}||}{||b_j||}\right), \tag{42}$$

where $||s_{i0}||$ and $||b_{j0}||$ are the rest length of the $i^{th}$ string and $j^{th}$ bar. Take the derivative of Eq. (42), we get:

$$\frac{d\gamma_i}{ds_i} = k_{si}||s_{i0}||\frac{s_i^T}{||s_i||^3},\quad \frac{d\lambda_j}{db_j} = -k_{bj}||b_{j0}||\frac{b_j^T}{||b_j||^3}, \tag{43}$$

where $k_{si}$ and $k_{bj}$ are spring constants. They satisfy:

$$k_{si} = \frac{E_{si}A_{si}}{||s_{i0}||},\quad k_{bj} = \frac{E_{bj}A_{bj}}{||b_{j0}||}, \tag{44}$$

where $A_{si}$ and $A_{bj}$ are cross section areas, $E_{si}$ and $E_{bj}$ are Young's modules of the strings and bars. Substitute Eq. (43) and Eqs. (44)–(41), we get:

$$\begin{aligned}K_{si} &= \gamma_i I_3 + s_i\left(k_{si}||s_{i0}||\tfrac{s_i^T}{||s_i||^3}\right) = \gamma_i I_3 + \tfrac{E_{si}A_{si}}{||s_i||^3}s_i s_i^T,\\K_{bj} &= \lambda_j I_3 + b_j\left(-k_{bj}||b_{j0}||\tfrac{b_j^T}{||b_j||^3}\right) = \lambda_j I_3 - \tfrac{E_{bj}A_{bj}}{||b_j||^3}b_j b_j^T.\end{aligned} \tag{45}$$

### 5.2. Stiffness matrix for solid bar tensegrity

For solid bar tensegrity, use the information of label matrix $Q$, the mass of a string and mass of a bar are:

$$\begin{aligned}m_{si} &= \tfrac{\rho_s}{\sigma_s}||s_i||^2\gamma_i,\\m_{bj} &= (1 - Q_{jj})\tfrac{\rho_b}{\sigma_b}||b_j||^2\lambda_j + Q_{jj}\tfrac{2\rho_b}{\sqrt{\pi E_{bj}}}||b_j||^{\frac{5}{2}}\lambda_j^{\frac{1}{2}}.\end{aligned} \tag{46}$$

Then, the cross section area of a bar and string are given by:

$$\begin{aligned}A_{si} &= \tfrac{m_{si}}{\rho_s||s_i||} = \tfrac{||s_i||\gamma_i}{\sigma_s},\\A_{bj} &= \tfrac{m_{bj}}{\rho_b||b_j||} = (1 - Q_{jj})\tfrac{||b_i||\lambda_j}{\sigma_b} + Q_{jj}\tfrac{2||b_j||^{\frac{3}{2}}\lambda_j^{\frac{1}{2}}}{\sqrt{\pi E_{bj}}}.\end{aligned} \tag{47}$$

Substitute Eq. (47) into Eq. (45):

$$\begin{aligned}K_{si} &= \gamma_i\left(I_3 + \tfrac{E_{si}}{\sigma_s}\tfrac{s_i s_i^T}{||s_i||^2}\right),\\K_{bj} &= \lambda_j\left(I_3 - (1 - Q_{jj})\tfrac{E_{bj}}{\sigma_b}\tfrac{b_j b_j^T}{||b_j||^2}\right) - 2Q_{jj}\sqrt{\tfrac{E_{bj}}{\pi}}\tfrac{b_j b_j^T}{||b_j||^{\frac{3}{2}}}\lambda_j^{\frac{1}{2}}.\end{aligned} \tag{48}$$

Summarize Eqs. (40) and (48) we get stiffness matrix $K_n$ for tensegrity structure subject to yielding and buckling constraints is given by:

$$K_n vec(dN) = vec(dW), \tag{49}$$

where

$$\begin{aligned}K_n = &(C_s^T \otimes I_3)\mathbf{b.d.}(K_{s1}, \cdots, K_{s\alpha})(C_s \otimes I_3)-\\&(C_b^T \otimes I_3)\mathbf{b.d.}(K_{b1}, \cdots, K_{b\beta})(C_b \otimes I_3),\end{aligned} \tag{50}$$

and

$$\begin{aligned}K_{si} &= \gamma_i\left(I_3 + \tfrac{E_{si}}{\sigma_s}\tfrac{s_i s_i^T}{||s_i||^2}\right),\\K_{bj} &= \lambda_j\left(I_3 - (1 - Q_{jj})\tfrac{E_{bj}}{\sigma_b}\tfrac{b_j b_j^T}{||b_j||^2}\right) - 2Q_{jj}\sqrt{\tfrac{E_{bj}}{\pi}}\tfrac{b_j b_j^T}{||b_j||^{\frac{3}{2}}}\lambda_j^{\frac{1}{2}}.\end{aligned} \tag{51}$$

### 5.3. Stiffness matrix for hollow bar tensegrity

For hollow bar tensegrity, use the information of label matrix $Q$, the mass of a string and a bar are:

$$\begin{aligned}m_{si} &= \tfrac{\rho_s}{\sigma_s}||s_i||^2\gamma_i,\\m_{bj} &= (1 - Q_{jj})\tfrac{\rho_b}{\sigma_b}||b_j||^2\lambda_j + Q_{jj}\tfrac{\rho_b||b_j||}{\sqrt{\pi E_b}}\\&\left(\sqrt{\pi^3 E_{bj} r_{j_{in}}^4 + 4\lambda_j||b_j||^3} - \pi r_{j_{in}}^2\sqrt{\pi E_b}\right).\end{aligned} \tag{52}$$

Then, the cross section area of a bar and string are given by:

$$A_{si} = \frac{m_{si}}{\rho_s||s_i||} = \frac{||s_i||\gamma_i}{\sigma_s}, \tag{53}$$

$$\begin{aligned}A_{bj} &= \tfrac{m_{bj}}{\rho_b||b_j||} = (1 - Q_{jj})\tfrac{||b_i||\lambda_j}{\sigma_b} + Q_{jj}\tfrac{1}{\sqrt{\pi E_{bj}}}\\&\left(\sqrt{\pi^3 E_b r_{j_{in}}^4 + 4\lambda_j||b_j||^3} - \pi r_{j_{in}}^2\sqrt{\pi E_b}\right).\end{aligned} \tag{54}$$

Substitute Eq. (53) into Eq. (45):

$$\begin{aligned}K_{si} &= \gamma_i\left(I_3 + \tfrac{E_{si}}{\sigma_s}\tfrac{s_i s_i^T}{||s_i||^2}\right),\\K_{bj} &= \lambda_j\left(I_3 - (1 - Q_{jj})\tfrac{E_{bj}}{\sigma_b}\tfrac{b_j b_j^T}{||b_j||^2}\right) - Q_{jj}\tfrac{E_{bj}}{\sqrt{\pi E_{bj}}}\\&\quad\left(\sqrt{\pi^3 E_{bj} r_{j_{in}}^4 + 4\lambda_j||b_j||^3} - \pi r_{j_{in}}^2\sqrt{\pi E_{bj}}\right)\tfrac{b_j b_j^T}{||b_j||^3}.\end{aligned} \tag{55}$$

Summarize Eqs. (40) and (55) we get stiffness matrix $K_n$ for tensegrity structure subject to yielding and buckling constraints is given by:

$$K_n vec(dN) = vec(dW), \tag{56}$$

where

$$K_n = (C_s^T \otimes I_3)\mathbf{b.d.}(K_{s1}, \cdots, K_{s\alpha})(C_s \otimes I_3) - (C_b^T \otimes I_3)\mathbf{b.d.}(K_{b1}, \cdots, K_{b\beta})(C_b \otimes I_3), \tag{57}$$

and

$$K_{si} = \gamma_i \left( I_3 + \frac{E_{si}}{\sigma_s} \frac{s_i s_i^T}{||s_i||^2} \right),$$
$$K_{bj} = \lambda_j \left( I_3 - (1 - Q_{jj}) \frac{E_{bj}}{\sigma_b} \frac{b_j b_j^T}{||b_j||^2} \right) - Q_{jj} \frac{E_{bj}}{\sqrt{\pi E_{bj}}} \left( \sqrt{\pi^3 E_{bj} r_{j_{in}}^4 + 4\lambda_j ||b_j||^3} - \pi r_{j_{in}}^2 \sqrt{\pi E_{bj}} \right) \frac{b_j b_j^T}{||b_j||^3}. \tag{58}$$

## 6. Variable bar cross-sections

Let a solid or hollow bar has length $l_0$, density $\rho_b$, cross-section area $A$, moment of inertia $I(A)$, which is a function of bar cross-section shape), Young's modulus $E_b$, and yield strength $\sigma_b$, and the bar is under compressive load $f(l_0)$. If a bar yields, the bar mass is:

$$m_Y(l_0) = \rho_b A l_0, \; f(l_0) = \sigma_b A, \;\; m_Y(l_0) = \frac{\rho_b}{\sigma_b} f(l_0) l_0. \tag{59}$$

If a bar buckles, the bar mass is:

$$m_B(l_0) = \rho_b A l_0, f(l_0) = \frac{\pi^2 E_b}{l_0^2} I(A). \tag{60}$$

From the above equation, we know that the mass of a bar subject to yielding is independent of the bar cross-section shape. In other words, given compressive load $f(l_0)$, bar length $l_0$, and bar material, bar mass cannot be reduced by changing the solid bar cross-section shape if the bar yields. However, if a bar buckles, bar mass is related to the bar cross-section shape, which is discussed as follows.

### 6.1. Variable bar cross-sections for solid bar tensegrity

Let a solid bar with regular polygon with $p$ sides, as shown in Fig. 1, C is centroid (at the center of the polygon), $p$ is number of sides ($p \geqslant 3$), b is length of the side, $\theta = \frac{2\pi}{p}$ is central angle for one side, moment of inertia of the regular polygon $I_{ppoly}$ with $p$ sides is given as [11]:

$$I_{ppoly} = \frac{A_{ppoly}^2}{12p} \left( 3cot\frac{\pi}{p} + tan\frac{\pi}{p} \right), \;\; A_{ppoly} = \frac{pb^2}{4} cot\frac{\pi}{p}, \tag{61}$$

where $A_{ppoly}$ is the area of the polygon cross-section. For a constant cross-section area $A_{ppoly}$, Take the first derivative of Eq. (61) with respect to p, we get:

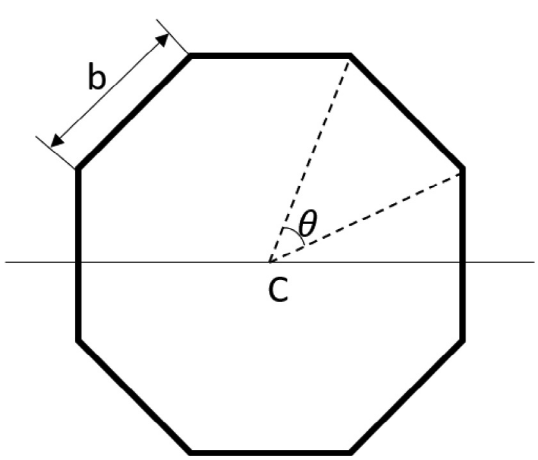


**Fig. 1.** Solid regular polygon bar cross-section with $p$ sides.

$$\frac{\partial I_{ppoly}}{\partial p} = \frac{2A_{ppoly}}{12p} \left( 3cot\frac{\pi}{p} + tan\frac{\pi}{p} \right) \left( \frac{b^2}{4} cot\frac{\pi}{p} + \frac{\pi b^2}{4p} csc^2\frac{\pi}{p} \right) - \frac{A_{ppoly}^2}{12p^2} \left( 3cot\frac{\pi}{p} + tan\frac{\pi}{p} \right) - \frac{\pi A_{ppoly}^2}{12p^3} \left( -3csc^2\frac{\pi}{p} + sec^2\frac{\pi}{p} \right) < 0. \tag{62}$$

From Eq. (62), we know as $p$ increases, the moment of inertia $I_{ppoly}$ decreases. Mass of a solid bar subject to buckling with a polygon cross-section is given by:

$$m_{ppoly} = \frac{\rho_b l_0^2}{\pi} \sqrt{\frac{12pf(l_0)}{\left(3cot\frac{\pi}{p} + tan\frac{\pi}{p}\right) E_b}}. \tag{63}$$

Moment of inertia of the circle is:

$$I_{cir} = \frac{\pi r^4}{4}, A_{cir} = \pi r^2. \tag{64}$$

Mass of a solid bar subject to buckling with a circular cross-section is given by:

$$m_{cir} = 2\rho_b l_0^2 \sqrt{\frac{f(l_0)}{\pi E_b}}. \tag{65}$$

Then to take the same compressive load $f(l_0)$, we can compare the mass of a polygon and a circular cross-section bar:

$$\frac{m_{ppoly}}{m_{cir}} = \frac{1}{\pi} \sqrt{\frac{3p\pi}{3cot\frac{\pi}{p} + tan\frac{\pi}{p}}}, p = 3, 4, 5, \cdots. \tag{66}$$

From Fig. 2, we know that to take the same compressive load, the mass of a polygon cross-section bar is smaller than a circular one. As the polygon sides increase, the bar mass increases. A triangle cross-section gives the minimal mass, which is 90.94% of a circular one. In other words, a triangular cross-section bar can save 9.06% mass compared to a circular one.

Here, we introduce a mass reduction coefficient $\zeta_{sp}$, which satisfies $m_{ppoly} = \zeta_{sp} m_{cir}, (p = 3, 4, 5, \cdots)$. For different cross section shape, for example, triangle $\zeta_3 = 0.9094$, square $\zeta_4 = 0.9772$, pentagon $\zeta_5 = 0.9916$, hexagon $\zeta_6 = 0.9962$.

Eq. (14) are for circular solid bars, when buckling is the mode of failure of one bar, we can replace it with a regular polygon bar to save mass. Then, Eq. (14) can be written as:

$$M^S = \frac{\rho_s}{\sigma_s} \left( vec\left( \lfloor S^T S \rfloor \right) \right)^T \gamma + \frac{\rho_b}{\sigma_b} \left( vec\left( \lfloor B^T B \rfloor (I - Q) \right) \right)^T \lambda + \zeta_p \frac{2\rho_b}{\sqrt{\pi E_b}} \left( vec\left( \lfloor B^T B \rfloor^{\frac{5}{4}} Q \right) \right)^T \lambda^{\frac{1}{2}}. \tag{67}$$

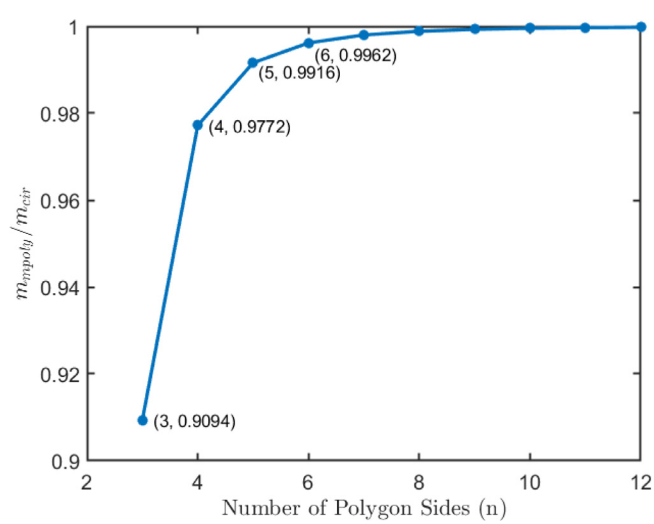


**Fig. 2.** Mass ratio of a solid bar with polygon and circular cross section v.s. polygon sides ($p$).

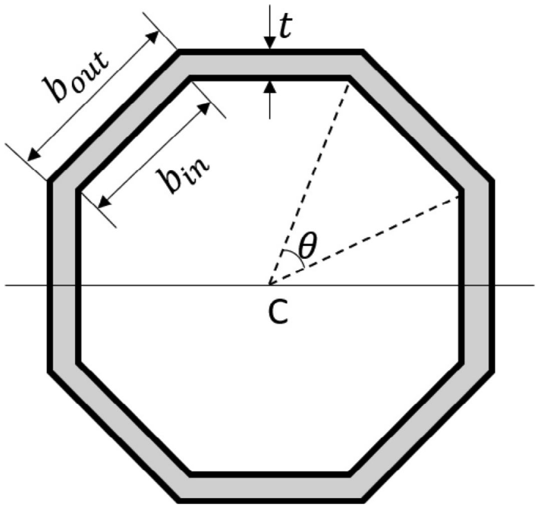


**Fig. 3.** Hollow regular polygon bar cross-section with $p$ sides.

### *6.2. Variable bar cross-sections for hollow bar tensegrity*

Let a hollow bar with regular polygon with $p$ sides, as shown in Fig. 3, C is centroid (at center of polygon), $p$ is number of sides ($p \geqslant 3$), $b_{out}$ and $b_{in}$ are length of the outer and inner sides, $A_{out}$ and $A_{in}$ are area of the outer and inner polygon, $\theta = \frac{2\pi}{p}$ is central angle for one side, moment of inertia of the regular polygon $I_{ppoly}$ with $p$ sides is given as [11]:

$$\begin{aligned} I_{ppoly} &= \frac{(A_{out}^2 - A_{in}^2)}{12p}\left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right), \\ A_{out} &= \frac{pb_{out}^2}{4}\cot\frac{\pi}{p}, \quad A_{in} = \frac{pb_{in}^2}{4}\cot\frac{\pi}{p}. \end{aligned} \tag{68}$$

Mass of a hollow bar can be written as:

$$m = \rho_b l_0 (A_{out} - A_{in}) = \rho_b l_0 \frac{p}{4}\cot\frac{\pi}{p} b_{in}^2 \left(\left(\frac{b_{out}}{b_{in}}\right)^2 - 1\right).$$

Moment of inertia for a hollow bar can be written as:

$$\begin{aligned} I_{ppoly} &= \frac{1}{12p}\left(\frac{p^2}{16} b_{out}^4 \cot^2\frac{\pi}{p} - \frac{p^2}{16} b_{in}^4 \cot^2\frac{\pi}{p}\right)\left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right) \\ &= \frac{p}{192}\cot^2\frac{\pi}{p} b_{in}^4 \left(\left(\frac{b_{out}}{b_{in}}\right)^4 - 1\right)\left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right) \\ &= \frac{p}{192}\left(b_{out}^4 - b_{in}^4\right)\cot^2\frac{\pi}{p}\left(1 + 3\cot^2\frac{\pi}{p}\right). \end{aligned}$$

Since we have $f(l_0) = \frac{\pi^2 E_b}{l_0^2} I$, one can get:

$$\left(\frac{b_{out}}{b_{in}}\right)^2 = \sqrt{\frac{192 f(l_0) l_0^2}{\pi^2 p E_b \cot^2\frac{\pi}{p} b_{in}^4 \left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right)} + 1}. \tag{69}$$

Mass of a hollow bar with a polygon cross-section is given by:

$$m_{ppoly} = \frac{\rho_b l_0 p}{4}\cot\frac{\pi}{p} b_{in}^2 \left(\sqrt{\frac{192 f(l_0) l_0^2}{\pi^2 p E_b \cot^2\frac{\pi}{p} b_{in}^4 \left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right)} + 1} - 1\right). \tag{70}$$

Moment of inertia of a bar with circular cross-section is:

$$I_{cir} = \frac{\pi(r_{out}^4 - r_{in}^4)}{4}, \quad A_{cir} = \pi(r_{out}^2 - r_{in}^2). \tag{71}$$

Euler buckling force for a hollow bar with a circular cross-section is:

$$f(l_0) = \frac{\pi^3 E_b}{4 l_0^2}(r_{out}^4 - r_{in}^4) = \frac{\pi^3 E_b}{4 l_0^2} r_{in}^4 \left(\left(\frac{r_{out}}{r_{in}}\right)^4 - 1\right).$$

Then, we get:

$$\left(\frac{r_{out}}{r_{in}}\right)^2 = \sqrt{1 + \frac{4 f l_0^2}{\pi^3 E_b r_{in}^4}}.$$

Mass of a hollow bar with a circular cross-section is given by:

$$m_{cir} = \rho_b \pi l_0 r_{in}^2 \left(\sqrt{1 + \frac{4 f(l_0) l_0^2}{\pi^3 E_b r_{in}^4}} - 1\right).$$

Then to take the same compressive load $f(l_0)$, we can compare the mass of a polygon and a circular cross section bar:

$$\begin{aligned} \frac{m_{ppoly}}{m_{cir}} = \frac{p\cot\frac{\pi}{p}}{4\pi}\left(\frac{b_{in}}{r_{in}}\right)^2 \left(\sqrt{\frac{192 f(l_0) l_0^2}{\pi^2 p E_b \cot^2\frac{\pi}{p} b_{in}^4 \left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right)} + 1} - 1\right) \Big/ \\ \left(\sqrt{1 + \frac{4 f(l_0) l_0^2}{\pi^3 E_b r_{in}^4}} - 1\right), \quad p = 3, 4, 5, \cdots. \end{aligned} \tag{72}$$

In order to make a fair comparison of polygon cross-sections, we assume the inner cross section area of the polygon and circular hollow bar are the same, which satisfies $\pi r_{in}^2 = \frac{pb_{in}^2}{4}\cot\frac{\pi}{p}$. The physical meaning is that the polygon and circular hollow bars have the same space to fit in the strings, instruments, electronics etc. Substitute $b_{in}^2 = \frac{4\pi}{p\cot\frac{\pi}{p}} r_{in}^2$ into Eq. (72):

$$\frac{m_{ppoly}}{m_{cir}} = \left(\sqrt{\frac{12 f(l_0) l_0^2 p}{\pi^4 E_b r_{in}^4 \left(3\cot\frac{\pi}{p} + \tan\frac{\pi}{p}\right)} + 1} - 1\right) \Big/ \left(\sqrt{1 + \frac{4 f(l_0) l_0^2}{\pi^3 E_b r_{in}^4}} - 1\right), \quad p = 3, 4, 5, \cdots. \tag{73}$$

Let us take a close look at this equation by an example, for a steel pipe, $E_b = 200$ GPa, $r_{in} = 0.01$ m, $l_0 = 1$ m, and $f(l_0) = 1, 10, 100, 1000$ N, Fig. 4 gives the results. The results show that to take the same compressive load, the mass of a polygon cross-section bar is smaller than a circular one. As the polygon sides increase, the bar mass increases, a triangle cross-section gives the minimal mass, which is 82.7% of a circular one. In other words, a triangular cross-section bar can save 17.3% mass compared to a circular one. And the compressive load $f(l_0)$ has little influence on the mass reduction ratio as long as the buckling is the mode of failure of the bar.

Similarly, we use the same mass reduction coefficient $\zeta_{hp}$, which satisfies $m_{ppoly} = \zeta_{hp} m_{cir}$, $(p = 3, 4, 5, \cdots)$. For different cross section shape, for example, triangle $\zeta_3 = 0.8270$, square $\zeta_4 = 0.9549$, pentagon $\zeta_5 = 0.9833$, hexagon $\zeta_6 = 0.9924$.

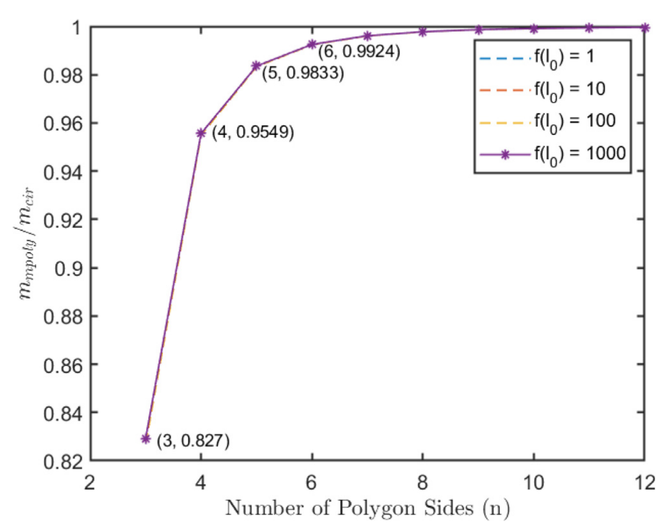


**Fig. 4.** Mass ratio of a hollow bar with polygon and circular cross-section v.s. polygon sides ($p$).

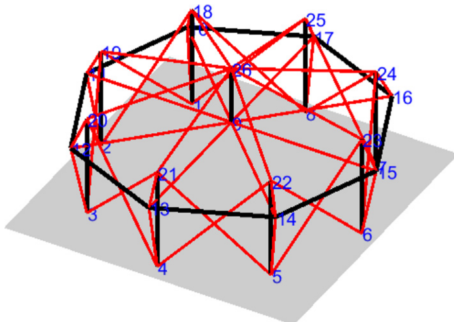


**Fig. 5.** Tensegrity Shelter, structure complexity is 8 (gives an octagon in the middle), bars are in black and strings are in red.

Eq. (21) are for circular hollow bars, when buckling is the mode of failure of one bar, one can replace the bar with a regular hollow polygon bar to save mass. Then, Eq. (21) can be written as:

$$\begin{aligned} M &= \tfrac{\rho_s}{\sigma_s}\left(vec\left(\lfloor S^T S\rfloor\right)\right)^T\gamma + \tfrac{\rho_b}{\sigma_b}\left(vec\left(\lfloor B^T B\rfloor(I-Q)\right)\right)^T\lambda \\ &+\zeta_p \frac{\rho_b\left(vec\left(\lfloor B^T B\rfloor^{\frac{1}{2}}Q\right)\right)^T}{\sqrt{\pi E_b}}\left(\sqrt{\pi^3 E_b r_{in}^4 + 4\left(\lfloor B^T B\rfloor^{\frac{3}{2}}Q\right)\lambda} - \pi r_{in}^2\sqrt{\pi E_b}\right). \end{aligned} \tag{74}$$

## 7. Bar joint mass penalty

It is also important to know the joint information of a structure. Because as structure complexity increases, numbers of nodes also increase, in a practical problem, joint mass can take a bigger portion of the total structure mass. Here, the joint mass penalty is assigned to the total structure mass according to the class number of each node (number of bars touching each other). Joint information vector $J_{info}$ can be obtained from the absolute values of bar connectivity matrix $|C_b|$:

$$J_{info}^T = \mathbf{1}^{1\times\beta}|C_b|. \tag{75}$$

Each column of $J_{info}^T$ gives the class number of each node. For example, a tensegrity shelter is shown in Fig. 5. We can obtain $J_{info}^T = [1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,1,1,1,1,1,1,1,1]$.

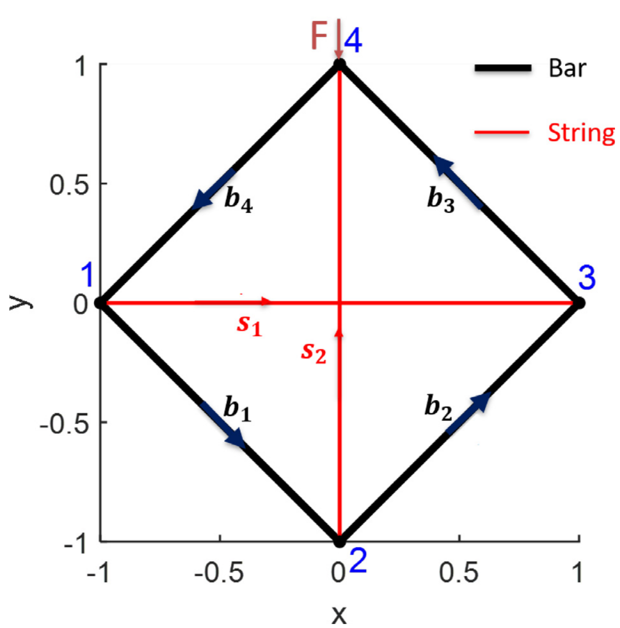


**Fig. 6.** 2D tensegrity D-Bar structure, downward vertical force F (marked in maroon) is applied at node 4 (marked in blue), node 2 is fixed to the ground.

**Table 1**
Material Property (Aluminum).

| Properties | Value | Units |
|---|---|---|
| Yield Stress | $\sigma = 1.1\times10^8$ | Pa |
| Young's Modulus | $E = 6\times10^{10}$ | Pa |
| Mass Density | $\rho = 2.7\times10^3$ | kg/m$^3$ |

It is also easily accessible by programming to abstract the information of all classes of the structure as well as a class of each joint below. This is a Class 2 structure. Class 1 joints are 1, 2, 3, 4, 5, 6, 7, 8, 9, 18, 19, 20, 21, 22, 23, 24, 25, 26. Class 2 joints are 10, 11, 12, 13, 14, 15, 16, 17.

The penalty mass for joints can be usually treated as a small ratio of a bar mass [32]. For example, if a bar mass is 1 kg, one can assign Class 1 joint mass is 0.01 kg, Class 2 joint mass is 0.02 kg, and Class K joint mass is $0.01\times K$ kg as a linear fashion.

## 8. Numerical example

D-Bar structure has been studied in many composite structure studies. For example, Boz et al. [2] showed the D-Bar structure can be used as both actuators and sensors. Skelton and de Oliveira presented that the microstructure of a spider fiber is like a D-Bar structure, and they implemented a D-Bar structure to build a wave-powered station-keeping buoy (WPSB) System. Goyal et al. [14] studied mass efficient landers based on D-Bar structures. Zhao and Hernandez [43] analyzed tunable energy dissipation by D-Bar structures. Thus, we choose a 2D D-Bar structure as an example, shown in Fig. 6 is chosen to verify the minimal mass algorithms.

Four kinds of combination of solid and hollow bars with and without gravity are studied. Using the same material (aluminum) for all the bars and strings, material information is given in Table 1. For convenience in subsequent comparison, the inner radii for all the hollow bars are set to be 0.02 m.

### 8.1. D-Bar without gravity

Firstly, a vertical downward external load $F = 1.0\times10^4$ N is applied at the top, and the bottom node is fixed in both x and y direc-

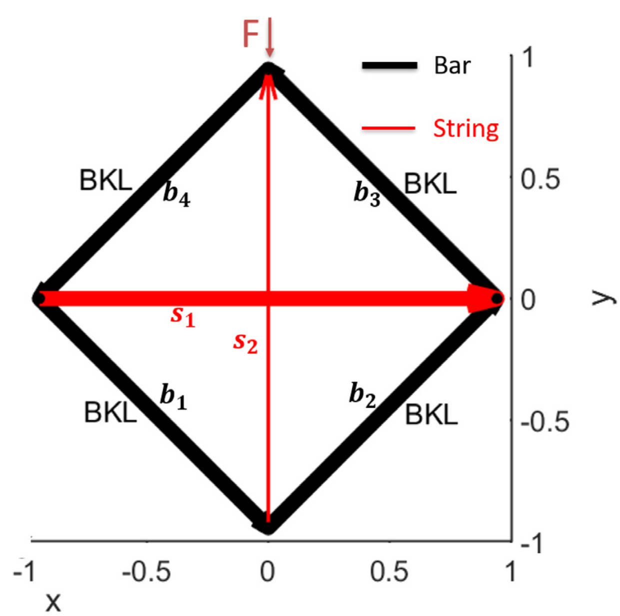


**Fig. 7.** Tensegrity D-Bar structure, $F = 1.0\times10^4$ N, all bar buckle.

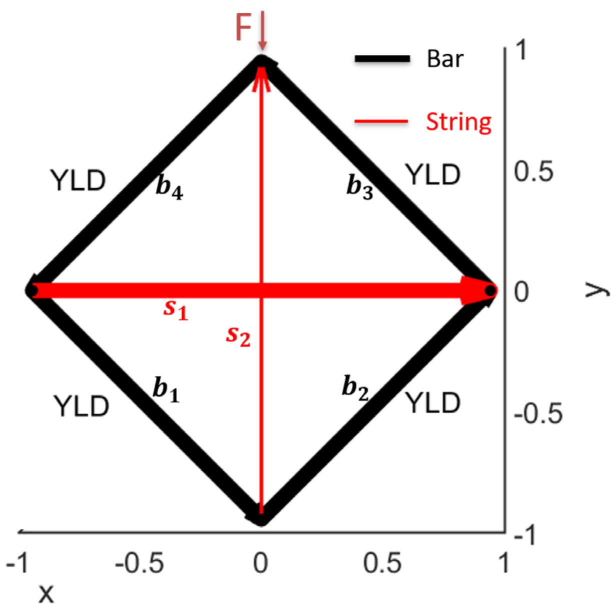


**Fig. 8.** Tensegrity D-Bar structure, $F = 1.0 \times 10^6$ N, all bar yield.

tions to the ground. The results show that all bars buckle under this load, shown in Fig. 7. The load is then increased to $F = 1.0 \times 10^6$ N where all bars yield, shown in Fig. 8. These results agree with the analytical solutions.

Fig. 7 gives the failure information of bars of the D-Bar structures with solid bars and hollow bars, respectively. The results are listed in Table 2. It is shown that for solid bar structure, buckling is the mode of failure for all the bars. The bar mass distributes evenly, and only string 1 has mass. For hollow bars, the mass distribution is the same. The only difference is that the hollow bar structure has less mass than the solid bar one.

Table 3 indicates that all bars yield when the external load is $F = 1 \times 10^6$ N. This supplemental case also proves that the hollow bar design does not save mass when yielding in the mode of failure of bars.

### 8.2. D-Bar with gravity

Table 4 gives similar information as Table 2: hollow bar tensegrity cost less mass than a solid bar one. And Table 4 together with Table 2 indicates that for both solid bar and hollow bar design, minimal structure mass with gravity is slightly larger than without gravity, which agrees well with the fact.

**Table 2**
Member information of D-Bar structure, $F = 1.0 \times 10^4$ N, without gravity. BKL and YLD represent buckling and yielding.

| Solid/Hollow<br>Load[N] | | No Gravity<br>$1.0 \times 10^4$ | | |
|---|---|---|---|---|
| Solid Bars | Failure | $\lambda$ [N/m] | Mass [kg] | r [m] |
| $b_1$ | BKL | 5000 | 2.092 | 0.0132 |
| $b_2$ | BKL | 5000 | 2.092 | 0.0132 |
| $b_3$ | BKL | 5000 | 2.092 | 0.0132 |
| $b_4$ | BKL | 5000 | 2.092 | 0.0132 |
| Strings | Failure | $\gamma$ [N/m] | Mass [kg] | R [m] |
| $s_1$ | YLD | 5000 | 0.491 | 0.005 |
| $s_2$ | YLD | 0 | 0 | 0 |
| Mass Sum | | | 8.858 | |
| Hollow Bars | Failure | $\lambda$ [N/m] | Mass [kg] | $r_{out}$ [m] |
| $b_1$ | BKL | 5000 | 2.080 | 0.021 |
| $b_2$ | BKL | 5000 | 2.080 | 0.021 |
| $b_3$ | BKL | 5000 | 2.080 | 0.021 |
| $b_4$ | BKL | 5000 | 2.080 | 0.021 |
| Strings | Failure | $\gamma$ [N/m] | Mass [kg] | r [m] |
| $s_1$ | YLD | 5000 | 0.491 | 0.011 |
| $s_2$ | YLD | 0 | 0 | 0 |
| Mass Sum | | | 8.810 | |

**Table 3**
Structure mass of D-Bar structure, $F = 1.0 \times 10^6$ N, without gravity.

| Solid/Hollow | Value | Units |
|---|---|---|
| External Load | $F = 1.0 \times 10^6$ | N |
| Minimal Mass (Solid) | $M = 147.273$ | kg |
| Minimal Mass (Hollow) | $M = 147.273$ | kg |

**Table 4**
Member information of D-Bar structure, $F = 1.0 \times 10^4$ N, with 1-g gravity. BKL and YLD represent buckling and yielding.

| Solid/Hollow<br>Load [N] | | With Gravity<br>$1.0 \times 10^4$ | | |
|---|---|---|---|---|
| Solid Bars | Failure | $\lambda$ [N/m] | Mass [kg] | r [m] |
| $b_1$ | BKL | 5033.220 | 2.099 | 0.0132 |
| $b_2$ | BKL | 5033.220 | 2.099 | 0.0132 |
| $b_3$ | BKL | 5033.220 | 2.099 | 0.0132 |
| $b_4$ | BKL | 5033.220 | 2.099 | 0.0132 |
| Strings | Failure | $\gamma$ [N/m] | Mass [kg] | r [m] |
| $s_1$ | YLD | 5021.740 | 0.493 | 0.005 |
| $s_2$ | YLD | 0 | 0 | 0 |
| Mass Sum | | | 8.878 | |
| Hollow Bars | Failure | $\lambda$ [N/m] | Mass [kg] | $r_{out}$ [m] |
| $b_1$ | BKL | 5033.044 | 2.087 | 0.021 |
| $b_2$ | BKL | 5033.044 | 2.087 | 0.021 |
| $b_3$ | BKL | 5010.202 | 2.082 | 0.021 |
| $b_4$ | BKL | 5010.202 | 2.082 | 0.021 |
| Strings | Failure | $\gamma$ [N/m] | Mass [kg] | r [m] |
| $s_1$ | YLD | 5021.623 | 0.493 | 0.011 |
| $s_2$ | YLD | 0 | 0 | 0 |
| Mass Sum | | | 8.830 | |

## 9. Conclusions

This paper presents a unified framework of the minimal mass design for any solid or hollow bar tensegrity structures with and without gravity. The methodology yields several nonlinear programming problems. The choice of the cross-section of bars is discussed. Results show that for a solid and hollow bar (with the same inner cross-section area), if yielding is the mode of failure, one cannot save mass. If buckling is the mode of failure, one can use a solid or hollow triangular cross-section shape to save 9.06% or 17.3% mass compared with a circular one. We also parameterized the class number of a joint versus joints mass. Structure mass is slightly larger when considering gravity than one without gravity. The principles developed in this paper facilitates the understanding of both materials and structures.

## CRediT authorship contribution statement

**Muhao Chen:** Conceptualization, Methodology, Software, Validation, Writing - original draft. **Robert E. Skelton:** Supervision, Writing - review & editing.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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