Álgebra Linear 8/11/24

Cálculo

Teste Segunda Derivada

$f: \mathbb{R} \to \mathbb{R} \to f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \dots + \frac{d f^n}{d x^n}(x)\frac{(x-x_0)^n}{n!}$

- Olhamos os pontos críticos:

$f'(x_0) = 0 \Rightarrow f(x) - f(x_0) = \frac{f''(x_0)\boxed{(x-x_0)^2}}{2} + \dots$ >0

$\to f''(x_0) > 0 \Rightarrow f(x) - f(x_0) > 0$ para x próximo de x_0

se $f(x) > f(x_0) \Rightarrow x_0$ é mínimo local

se $f(x) < f(x_0) \Rightarrow x_0$ é máximo local

- Agora vamos olhar $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = f(x_0, y_0)$

$\to f(x,y) = f(x_0,y) + \frac{\partial f}{\partial x}(x_0,y)(x-x_0) + \frac{\partial^2 f}{\partial x^2}(x_0,y)\frac{(x-x_0)^2}{2} + \dots$

$\to f(x_0,y) = f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)\frac{(y-y_0)^2}{2} + \dots$

$\frac{\partial f}{\partial x}(x_0,y) = \frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(y-y_0) + \frac{\partial^3 f}{\partial x \partial y^2}(x_0,y_0)\frac{(y-y_0)^2}{2} + \dots$

$\frac{\partial^2 f}{\partial x^2}(x_0,y) = \frac{\partial^2 f}{\partial x^2}(x_0,y_0) + \frac{\partial^3 f}{\partial^2 x \partial y}(x_0,y_0)(y-y_0) + \dots$

LOGO:

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + 2\cdot\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(y-y_0)^2\right) + \dots$$

- ELABORANDO O TESTE DA SEGUNDA DERIVADA

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{bmatrix} \qquad z = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$Hf(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{bmatrix}$$

$$\Downarrow$$

$$f(x,y) = (z-z_0)^T Hf(x,y)(z-z_0)$$

Ponto crítico $\Rightarrow \nabla f = 0$

$$(z-z_0)^T \underbrace{Hf(x,y)}_{\text{Positiva Definida}} (z-z_0) > 0 \Rightarrow f(x,y) - f(x_0,y_0) > 0$$

(x_0,y_0) mínimo local

$$(z-z_0)^T \underbrace{Hf(x,y)}_{\text{Negativa Definida}} (z-z_0) < 0 \Rightarrow f(x,y) - f(x_0,y_0) < 0$$

(x_0,y_0) máximo local

$$\det(Hf(x,y)) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{vmatrix}$$

$> 0 \Rightarrow \lambda_1 > 0$ e $\lambda_2 > 0$ ou $\lambda_1 < 0$ e $\lambda_2 < 0$

$< 0 \Rightarrow \lambda_1 > 0$ e $\lambda_2 < 0$

Derivadas $\mathbb{R}^n \to \mathbb{R}^m$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$x \longmapsto (f_1(x), \dots, f_m(x))$ VETOR LINHA

$$\frac{\partial f}{\partial x} = \text{Jacobiana de } f = \begin{bmatrix} - & \frac{\partial f_1}{\partial x} & - \\ & \vdots & \\ - & \frac{\partial f_m}{\partial x} & - \end{bmatrix}_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Regra da cadeia

$f: \mathbb{R}^n \to \mathbb{R}^m$

$g: \mathbb{R}^m \to \mathbb{R}^k$ $\longrightarrow$ $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^k, \; h(x) = g(f(x))$

$$\to \underset{k \times n}{\frac{\partial h}{\partial x}} = \underset{k \times m}{\frac{\partial g}{\partial x}(f(x))} \; \underset{m \times n}{\frac{\partial f}{\partial x}(x)} \to h_i(x) = g_i(f(x)) \quad i = 1, \dots, k$$

$$\frac{\partial h_i}{\partial x_j}(x) = \sum_{k=1}^{m} \frac{\partial g_i}{\partial x_k}(f(x)) \cdot \frac{\partial f_k}{\partial x_j}(x)$$