ÁLGEBRA LINEAR NUMÉRICA

23/03/25

LISTA 3

CAP. 6

6.1. If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

$$(I-2P)^T(I-2P) \to I - 2P - 2\bar{P}^T + 4\bar{P}^TP$$

$$(I-2P^T)(I-2P) \qquad I - 4P + 4P^2 = I - 4P = I$$

6.4. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Answer the following questions by hand calculation.

(a) What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1,2,3)^*$?

(b) Same questions for B.

a) $P = A(A^*A)^{-1}A^*$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}\Bigg)^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\to \det(\bar{A}^TA) = 2$

$$Px = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$x^T = [1\ 2\ 3]$

b) $P = B(B^*B)^{-1}B^*$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}}\Bigg)^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \\ 2 & 2 & -2 \end{bmatrix}$$

$\det = 10 - 4 = 6$

$$= \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \Rightarrow Py = \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$y^T = [1, 2, 3]$

6.5. Let $P \in \mathbb{C}^{m \times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

Ⓘ $\rightarrow P^2 = P \Rightarrow \|P\|_2 \geq 1$

Ⓘⓘ $\rightarrow P^2 = P, \ \|P\|_2 = 1 \Leftrightarrow P^* = P$

I) $\|P\|_2 = \sup\left\{ \frac{\|Px\|_2}{\|x\|_2} \right\}$, Vamos supor que $\nexists x \,/\, \|Px\|_2 > \|x\|_2$ isso é impossível pela própria definição de P. Seja $y = Px$, então $Py = y$, logo $\|Py\|_2 = \|y\|_2$, o que faz nossa suposição ser ABSURDA. Logo, $\|P\|_2 \geq 1$

II) $P^* = P \Leftrightarrow \|P\|_2 = 1$

Seja $P = U\Sigma\bar{V}^T$ a S.V.D de P:

$$P = U\Sigma\bar{U}^T \Leftrightarrow U = V \Leftrightarrow P = Q\Sigma Q^*$$

$$\Leftrightarrow P^2 = Q\Sigma Q^* Q\Sigma Q^* \Leftrightarrow P = Q\Sigma^2 Q^* \Leftrightarrow \Sigma = \Sigma^2$$

Se $\Sigma = \Sigma^2$, isso significa que todos os valores singulares de P são iguais a 1 ou 0.

CAP. 7

7.2. Let A be a matrix with the property that columns $1, 3, 5, 7, \ldots$ are orthogonal to columns $2, 4, 6, 8, \ldots$. In a reduced QR factorization $A = \hat{Q}\hat{R}$, what special structure does $\hat{R}$ possess? You may assume that A has full rank.

$$A = \hat{Q}\hat{R} \Rightarrow \hat{Q}^* A = \hat{R}, \quad \begin{bmatrix} - & q_1 & - \\ & & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$$

Ideia:

$a_i^* v_j = 0 \rightarrow i \in \{1, 3, \ldots\}, j \in \{2, 4, \ldots\}$ $(a_i^* v_j = 0 \Leftrightarrow a_i^* q_j = 0)$

Base

$q_1 = \frac{a_1}{\|a_1\|} \Rightarrow q_1^* a_j = 0 \rightarrow j \in \{2, 4, \ldots\}$

$v_2 = a_2 - \cancel{q_1^* a_2}^{\,0} q_1 \rightarrow \quad a_3^* v_2 = 0$

$$a_4^* v_3 = \underbrace{\cancel{a_4^* a_3}}_{0} - \underbrace{\cancel{a_4^* q_1^*}}_{0} a_3 q_1 - a_4^* \underbrace{\cancel{q_2^* a_3}}_{0} q_2^*$$

Passo indutivo

Vamos supor que vale para a_i e v_j, vale para a_i e v_{j+2}?

$$a_{i+2}^* v_j = a_{i+2}^* \left(a_j - \sum_{i=1}^{j-1} q_i^* a_j q_i\right) = \overset{0}{\cancel{a_{i+2}^* a_j}} - a_{i+2}^* \sum_{i=1}^{j-1} q_i^* a_j q_i$$

$$= -(\underbrace{\cancel{a_{i+2}^* q_2^* a_j}}_{0} q_2 + \underbrace{\cancel{a_{i+2}^* q_4^*}}_{0} a_j q_4 + \underbrace{\cancel{a_{i+2}^* q_6^* a_j q_6}}_{0} + \dots) = 0$$

Logo, vemos que a matriz $\hat{R}$ alterna de forma que, a partir da diagonal principal, as diagonais, de dois em dois, são iguais a 0.

7.3. Let A be an $m \times m$ matrix, and let a_j be its jth column. Give an algebraic proof of *Hadamard's inequality*:

$$|\det A| \le \prod_{j=1}^{m} \|a_j\|_2.$$

Also give a geometric interpretation of this result, making use of the fact that the determinant equals the volume of a parallelepiped.

$$|\det A| \lesseqgtr \prod_{j=1}^{m} \|a_j\|_2 \Leftrightarrow |\det(QR)| \lesseqgtr \prod_{j=1}^{m} \|Q r_j\|_2 \Leftrightarrow |\det R| \lesseqgtr \prod_{j=1}^{m} \|r_j\|_2$$

$$\left|\prod_{i=1}^{m} r_{ii}\right| \lesseqgtr \sqrt{\prod_{j=1}^{m} r_j^* r_j}$$

m termos

↳ $\prod_{i=1}^{m} r_{ii}^2$ é um fator desse produtório

$$\Rightarrow |\det A| \le \prod_{j=1}^{m} \|a_j\|_2$$

Isso significa que a transformação linear A nunca aumenta o volume de uma região em $\mathbb{C}^m$ mais do que o volume da região formada pelos vetores em suas colunas

7.5. Let A be an $m \times n$ matrix ($m \geq n$), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

(a) Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are nonzero.

(b) Suppose $\hat{R}$ has k nonzero diagonal entries for some k with $0 \leq k < n$. What does this imply about the rank of A? Exactly k? At least k? At most k? Give a precise answer, and prove it.

a) $A = \hat{Q}\hat{R}$, $\text{posto}(A) = n \Leftrightarrow \nexists x \neq 0 \mid Ax = 0$

Pela definição, $r_{ii} = \left\| a_i - \sum_{k=1}^{i-1} q_k^* a_i q_k \right\|_2$, logo, $r_{ii} \geqslant 0$ e $r_{ii} = 0$ $\Leftrightarrow a_i - \sum_{k=1}^{i-1} q_k^* a_i q_k = 0$. ou seja, $\{a_i, q_1, \dots, q_{i-1}\}$ não são L.I (a_i tem coeficiente 1), ou seja, as colunas de A não são L.I, portanto $\text{posto}(A) < n$

b) $A = \hat{Q}\hat{R} \Rightarrow \text{posto}(\hat{R}^*) = \text{posto}(\hat{R})$

$\rightarrow \underbrace{\cancel{\text{posto}(\hat{R}^*)}}_{k} \geq \text{posto}(\underbrace{\hat{R}^*\hat{Q}^*}_{A^*}) \Leftrightarrow k \geq \text{posto}(A^*)$

$(\text{posto}(A^*) = \text{posto}(A)) \Rightarrow k \geq \text{posto}(A)$

Logo, A tem posto, no máximo, igual a k

CAP. 8

8.1. Let A be an $m \times n$ matrix. Determine the exact numbers of floating point additions, subtractions, multiplications, and divisions involved in computing the factorization $A = \hat{Q}\hat{R}$ by Algorithm 8.1.

for $i = 1$ to n $\rightarrow O(mn)$
  $v_i = a_i$

for $i = 1$ to n
  $r_{ii} = \|v_i\| \rightarrow O(m)$
  $q_i = v_i / r_{ii} \rightarrow O(m)$
  for $j = i+1$ to n
    $r_{ij} = q_i^* v_j$
    $v_j = v_j - r_{ij} q_i \rightarrow O(m(n-i)) \Rightarrow O(mn^2)$

$\underline{O(mn^2)}$

8.3. Each upper-triangular matrix R_j of p. 61 can be interpreted as the product of a diagonal matrix and a unit upper-triangular matrix (i.e., an upper-triangular matrix with 1 on the diagonal). Explain exactly what these factors are, and which line of Algorithm 8.1 corresponds to each.

$$R_j = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \frac{1}{r_{jj}} & \frac{-r_{jj+1}}{r_{jj}} & \cdots \\ & & & & \ddots & \\ & & & & & \ddots \end{bmatrix}$$

É fácil ver que, ao fatorar $R_j = DU$ com D diagonal e U triangular superior, temos

$$D = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & \frac{1}{r_{jj}} & \\ & & & \ddots \end{bmatrix} \qquad U = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & -r_{jj+1} & r_{jj+2} \cdots \\ & & & & \ddots & \\ & 0 & & & & \ddots \end{bmatrix}$$

A linha 5 é a que simula essa multiplicação