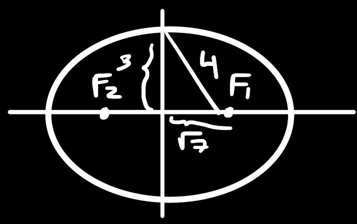
GA LISTA 5

ELIPSE

1) Faça um esboço do gráfico da curva $9x^2 + 16y^2 = 144$.

$$\frac{9x^2+16y^2=144}{144} \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$



$16 = 9 + 7$

$c = \sqrt{7}$

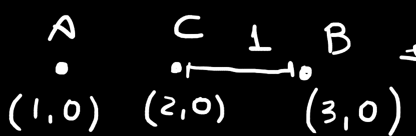
2) Encontre os focos da elipse $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$.

$a^2 = 9 \quad b^2 = 5 \Rightarrow 9-5=4 \Rightarrow c=2$

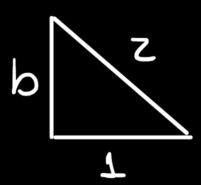
$F_1 = (-2,0) \quad F_2 = (2,0)$

3) Dados $A=(1,\ 0)$, $B=(3,\ 0)$ e $P=(x,\ y)$ determine a equação da curva descrita pelo ponto P de forma que $d(P,A)+d(P,B)=4$.

$A = F_1 \quad B = F_2 \Rightarrow a = 2,$



$\Rightarrow$

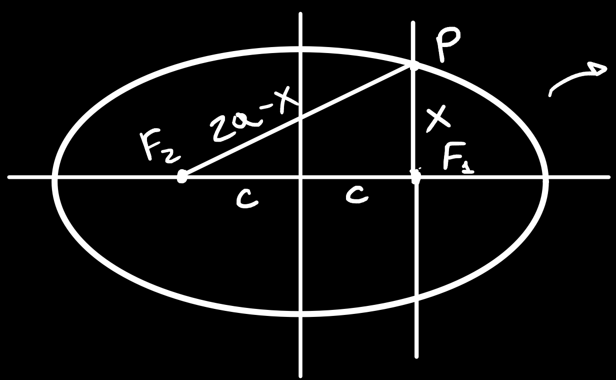


$4 = 1 + b^2$

$b^2 = 3$

$$\frac{(x-2)^2}{4}+\frac{y^2}{3}=1$$

4) Se $a > b$ determine, na elipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, o comprimento da corda focal perpendicular ao eixo maior.



$4c^2 + \cancel{x^2} = 4a^2 - 2ax + \cancel{x^2}$

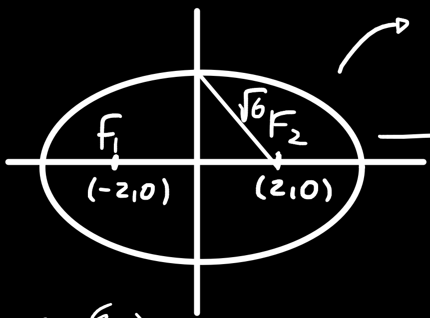
$\cancel{4c^2} = \cancel{4c^2} + 4b^2 - 2ax$

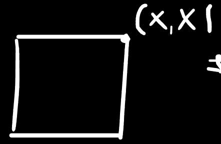
$x = \dfrac{4b^2}{2a} \Rightarrow x = \dfrac{2b^2}{a}$

$2x = \dfrac{4b^2}{a}$

5) Determine a área do quadrado inscrito na elipse $\frac{x^2}{6}+\frac{y^2}{2}=1.$

$6=2+4 \Rightarrow c=2$



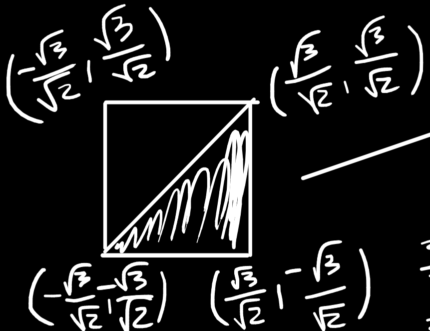


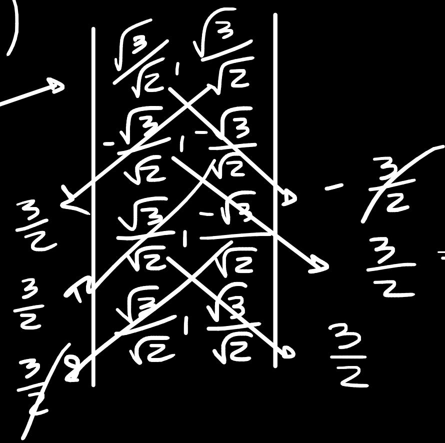
$\Rightarrow \frac{x^2}{6}+\frac{x^2}{2}=1$

$2x^2+6x^2=12$

$x^2=\frac{12}{8} \Rightarrow x=\frac{2\sqrt{3}}{2\sqrt{2}}$

$x=\frac{\sqrt{3}}{\sqrt{2}}$





$\Rightarrow \frac{12}{2} \Rightarrow \boxed{6}$

6) Determine k para que a reta $y=\frac{x}{2}+k$ seja tangente à elipse $\frac{x^2}{4}+y^2=1.$

$\frac{x^2}{4}+\left(\frac{x}{2}+k\right)^2=1 \Rightarrow \frac{x^2}{4}+\frac{x^2}{4}+xk+k^2-1=0$

$\frac{x^2}{2}+xk+k^2-1=0 \rightarrow k^2-\cancel{4}^2\cdot\left(\frac{1}{\cancel{2}}\right)(k^2-1)=0 \rightarrow \Delta=0$

$k^2-2k^2+2=0 \Rightarrow -k^2+2=0$

$\underline{k=\pm\sqrt{2}}$

8) Determine as tangentes à elipse $\frac{x^2}{10}+\frac{2y^2}{5}=1$ que são paralelas à reta $3x+2y+7=0$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \frac{x_0}{a^2}x+\frac{y_0}{b^2}y=1 \text{ TANGENTE} \qquad \frac{x^2}{10}+\frac{4y^2}{10}=1$$

$$3x+2y+K=0$$
$$2y=-K-3x \rightarrow \frac{x^2+K^2+6xK+9x^2}{10}=1$$
$$10x^2-6xK+K^2-10=0$$

$$\rightarrow \Delta=0: \ (-6K)^2-4\cdot 10\cdot(K^2-10)=0$$
$$36K^2-40K^2+400=0 \rightarrow 4K^2=400$$
$$2K=20$$
$$K=\pm 10$$

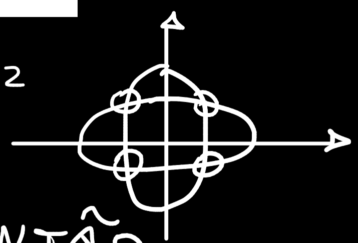
9) Mostre que $P=(2,\ 1)$ pertence à elipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ e encontre a equação da reta tangente em P a essa elipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \frac{x_0}{a^2}x+\frac{y_0}{b^2}y=1$ TANGE

$$\frac{\cancel{4}^{\,2}}{\cancel{6}_{\,3}}+\frac{1}{3}=1 \Rightarrow \frac{3}{3}=1 \ ✓ \qquad \frac{2}{6}\cdot x+\frac{1}{3}y=1$$

$$\boxed{x+y=3}$$

10) Sendo $a\neq b$, quantos pontos possuem em comum as curvas $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ e $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$?

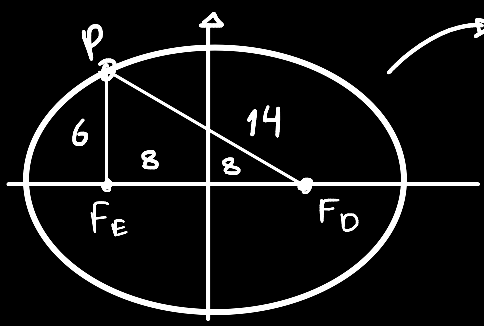
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2}{b^2}+\frac{y^2}{a^2} \Rightarrow x^2b^2+y^2a^2=x^2a^2+y^2b^2$$



→ O POLINÔMIO EM X E Y É DO 4º GRAU, ENTÃO EXISTEM 4 PONTOS

11) Determine um ponto da elipse $\frac{x^2}{100}+\frac{y^2}{36}=1$ cuja distância ao foco da direita é igual a 14. $a=10,\ b=6\ c=8 \quad a^2=b^2+c^2$

$$\begin{array}{r} 196 \\ -\ 64 \\ \hline 132 \end{array}$$



$$F_D=(8,0),\ P=(x,y) \qquad |\overrightarrow{F_DP}|=14$$
$$14=\sqrt{(x-8)^2+y^2} \qquad \frac{x^2}{100}+\frac{132+16x-x^2}{36}=1$$
$$196=x^2-16x+64+y^2$$
$$y^2=-x^2+16x+132$$

$$36x^2 + 13.200 + 1600x - 100x^2 = 3600$$

$$\begin{array}{r} 13200 \\ -\ 3600 \\ \hline 9600 \end{array}$$

$$\frac{-64x^2 + 1600x + 9600 = 0}{8}$$

$$-8x^2 + 200x + 1200 = 0 \Rightarrow -x^2 + 25x + 150 = 0$$

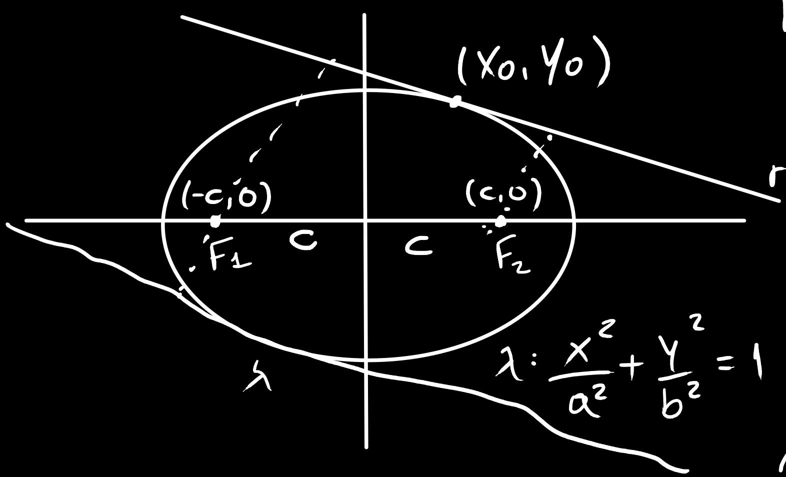
$$\Delta = 625 + 600 \rightarrow x = \frac{-25 \pm 35}{-2} \rightarrow \boxed{x' = -5}, \quad x'' = 30 \rightarrow$$

$$\Delta = 1225$$

X=30 NÃO SATISFAZ A ELIPSE

(I) $$\frac{25}{100} + \frac{y^2}{36} = 1 \qquad \frac{1}{4} + \frac{y^2}{36} = 1 \rightarrow \frac{y^2}{\cancel{36}_{9}} = \frac{3}{\cancel{4}} \qquad \boxed{y = \pm 3\sqrt{3}} \rightarrow P = (-5, -3\sqrt{3}), (-5, 3\sqrt{3})$$

12) Em uma elipse mostre que o produto das distâncias dos focos a uma tangente qualquer é constante.



$$r: \overset{a}{\frac{x_0}{a^2}} x + \overset{b}{\frac{y_0}{b^2}} y \overset{c}{-1} = 0$$

$$x_0^2 = \underbrace{\left(1 - \frac{y_0^2}{b^2}\right) a^2}$$

$$\rightarrow d_{F_1 r} \cdot d_{F_2 r}$$

$$\rightarrow \left|\frac{x_0}{a^2} c + 1\right| \qquad \rightarrow a^2 - \frac{y_0^2 a^2}{b^2} + \frac{y_0^2 a^4}{b^4}$$

$$\rightarrow \frac{\left|\frac{x_0}{a^2} \cdot c - 1\right|}{\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4}}} \cdot \frac{\left|-\frac{x_0}{a^2} \cdot c - 1\right|}{\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4}}} = \frac{\left|\frac{x_0^2}{a^4} c^2 - 1\right|}{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4}}$$

$$\frac{\frac{|x_0^2 c^2 - a^4|}{\cancel{a^4}}}{\frac{1}{\cancel{a^4}}\left(x_0^2 + \frac{y_0^2 a^4}{b^4}\right)} \Rightarrow \frac{\left|a^2 c^2 - \frac{y_0^2 a^2 c^2}{b^2} - a^4\right|}{a^2 - \frac{y_0^2 a^2}{b^2} + \frac{y_0^2 a^4}{b^4}} \rightarrow \frac{\cancel{a^2}\left|\cancel{c^2} - \frac{y_0^2 c^2}{b^2} \overset{-b^2}{\cancel{- a^2}}\right|}{\cancel{a^2}\left(1 - \frac{y_0^2}{b^2} + \frac{y_0^2 a^2}{b^4}\right)}$$

$$\cancel{\frac{b^4 + y_0^2 c^2}{b^2}} \cdot \cancel{\frac{b^4}{b^4 + y_0^2 (a^2 - b^2)}} = \boxed{b^2}$$

HIPÉRBOLE

13. $a = 3 \quad b = 2$

(I) FOCOS

$c^2 = 9 + 4$

$c^2 = 13$

$c = \sqrt{13}$

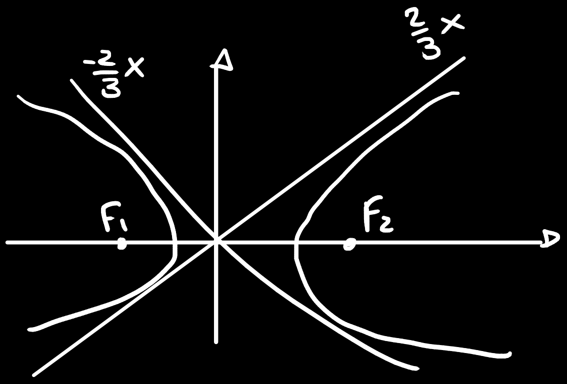
$F_1 = (-\sqrt{13}, 0)$

$F_2 = (\sqrt{13}, 0)$

(II) ASSÍNTOTAS

$y = \frac{2}{3}x \quad y = \frac{-2}{3}x$

(III)

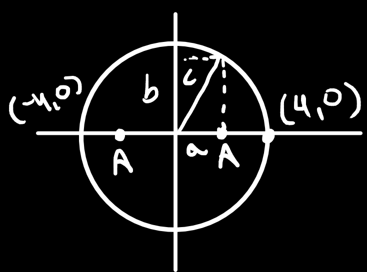


14. $a = \sqrt{6} \longrightarrow m_1 = -\frac{\sqrt{2}}{\sqrt{6}} \Rightarrow -\frac{\cancel{2}\sqrt{3}}{\cancel{6}} \Rightarrow -\frac{\sqrt{3}}{3} \text{ e } \frac{\sqrt{3}}{3}$

$b = \sqrt{2}$

$m > -\frac{\sqrt{3}}{3} \wedge m < \frac{\sqrt{3}}{3}$

15.



$c^2 = 2a^2$

$16 = 2a^2$

$8 = a^2$

$\frac{x^2}{8} - \frac{y^2}{8} = 1$

16. $y = 3x + k \qquad x^2 - \frac{y^2}{4} = 1$

$4x^2 - 9x^2 - 6xk - k^2 = 4$

$-5x^2 - 6xk - k^2 - 4 = 0$

$\Delta = 36k^2 - 4(-5)(-k^2 - 4)$

$0 = 36k^2 - 20k^2 - 80$

$16k^2 = 80$

$k^2 = 5$

$k = \sqrt{5}$

17. $y = k \cdot x + m$ $\quad P: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $\quad x = \dfrac{y-m}{k}$

$$\frac{x^2}{a^2} - \frac{(kx+m)^2}{b^2} = 1 \rightarrow b^2x^2 - a^2k^2x^2 - 2kxma^2 + m^2a^2 = a^2b^2$$

$$\rightarrow -a^2k^2x^2 - 2kxma^2 + m^2a^2 + b^2x^2 - a^2b^2 = 0$$

$$\rightarrow x^2(-a^2k^2+b^2) - 2kxma^2 - a^2(m^2+b^2) = 0$$

$$\Delta = 4k^2m^2a^4 + 4(-a^2k^2+b^2)a^2(m^2-b^2)$$

$$\Delta = 4k^2m^2a^4 + (-4a^4k^2 + 4b^2a^2)(m^2-b^2)$$

$$\Delta: \frac{\cancel{4k^2m^2a^4} - \cancel{4a^4k^2m^2} + 4a^4k^2b^2 + 4b^2a^2m^2 - 4b^4a^2 = 0}{4a^2b^2}$$

$$\rightarrow a^2k^2 + m^2 - b^2 = 0 \rightarrow \boxed{a^2k^2 + m^2 = b^2}$$

18) Se $P = (x_0,\ y_0)$ é um ponto da hipérbole $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ mostre que a reta $\dfrac{x_0}{a^2}x - \dfrac{y_0}{b^2}y = 1$ é tangente a essa elipse no ponto P.

$b^2x^2 - a^2y^2 = a^2b^2 \rightarrow$ DERIVANDO

$$2xb^2 - 2yy'a^2 = 0$$

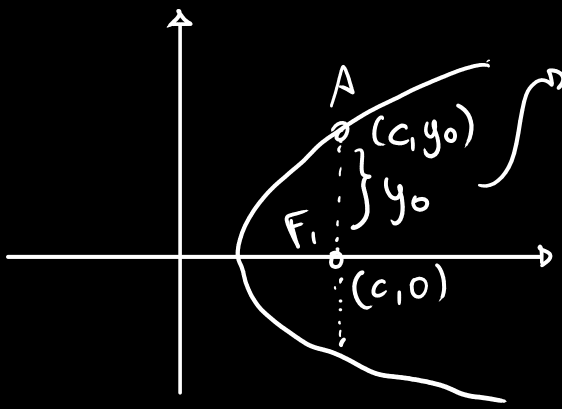
$$\rightarrow y' = \frac{2xb^2}{2ya^2} \Rightarrow y' = \frac{xb^2}{ya^2} \Rightarrow y - y_0 = y'(x - x_0)$$

$$\rightarrow y - y_0 = \frac{xb^2}{ya^2}(x - x_0) \Rightarrow y^2a^2 - y_0ya^2 = x^2b^2 - x_0xb^2$$

$$\frac{y^2a^2 - x^2b^2}{a^2b^2} = \frac{y_0ya^2 - x_0xb^2}{a^2b^2} \Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{y_0}{b^2}y - \frac{x_0}{a^2}x$$

$$\Rightarrow \boxed{\frac{x_0}{a^2}x - \frac{y_0}{b^2}y = 1}$$

19) Na hipérbole $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ determine o comprimento da corda focal perpendicular ao eixo transverso.

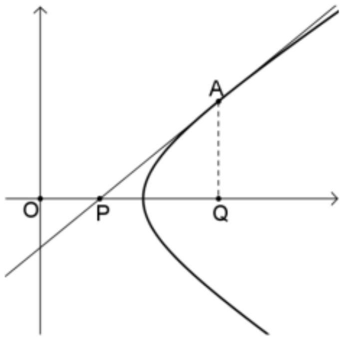


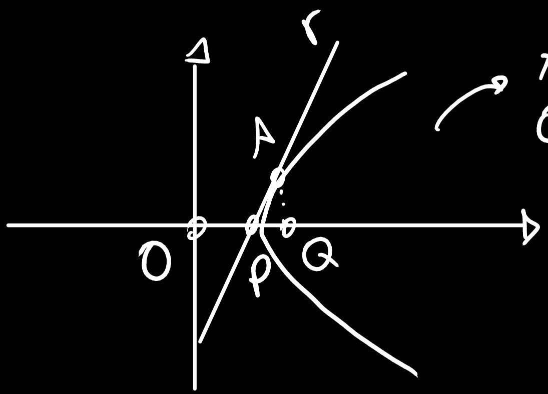
$$y_0 = b\sqrt{\frac{c^2}{a^2} - 1}$$

$$2y_0 = 2b\sqrt{\frac{b^2}{a^2} \cancel{+1} - \cancel{1}}$$

$$\boxed{\frac{2b^2}{a} = 2y_0}$$

20) A figura ao lado mostra uma tangente em um ponto A de uma hipérbole e o segmento AQ perpendicular ao eixo X. Mostre que $OP \cdot OQ = a^2$ onde a e o semieixo transverso.





$A = (x_0, y_0) \Rightarrow P = (?, 0)$

$Q = (x_0, 0)$

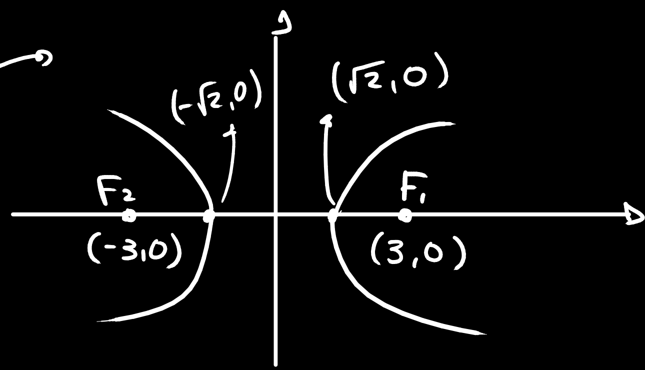
$$r: \frac{x_0}{a^2}x - \frac{y_0}{b^2}y = 1$$

$$r: \frac{x_0 x}{a^2} = 1 \Rightarrow x = \frac{a^2}{x_0}$$

$$|\vec{P}| \cdot |\vec{Q}| \Rightarrow \boxed{\frac{a^2}{x_0} \cdot x_0 = a^2}$$

21) Faça um esboço do gráfico das curvas $x^2 - 2y^2 = 2$ e $x^2 - 2y^2 = -2$.

(i) $x^2 = 2 + 2y^2 \rightarrow \frac{x^2}{2} - y^2 = 1 \rightarrow$



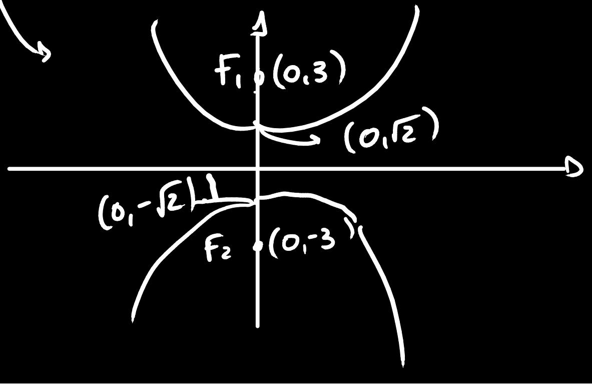
$a = \sqrt{2}$

$b = 1$

$c = 3$

$x^2 = 2y^2 - 2$

$\frac{x^2}{2} - y^2 = -1$



22) Sendo $x > y$ determine a equação que x e y devem satisfazer para que o produto das distâncias de $P = (x,\ y)$ às retas $x + y = 0$ e $x - y = 0$ seja igual a 3.

$$\frac{|x+y|}{\sqrt{2}} \cdot \frac{|x-y|}{\sqrt{2}} = 3$$

$$\frac{x^2 - y^2}{2} = 3 \rightarrow \boxed{x^2 - y^2 = 6}$$

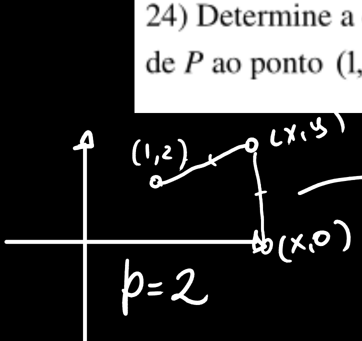
23) Determine o foco e a diretriz da parábola $y = x^2$.

P DISTÂNCIA DA DIRETRIZ AO FOCO

$y = x^2 \Rightarrow 2py = x^2$

$2p = 1 \rightarrow p = \frac{1}{2} \Rightarrow \frac{p}{2} = \frac{1}{4} \rightarrow \boxed{\left(0, \frac{1}{4}\right)}$

24) Determine a equação da curva descrita pelo ponto $P = (x,\ y)$ de forma que a distância de P ao ponto $(1,\ 2)$ seja igual à sua distância ao eixo OX.



$p = 2$

$\rightarrow \sqrt{(x-1)^2 + (y-2)^2} = y$

$x^2 - 2x + 1 + \cancel{y^2} - 4y + 4 = \cancel{y^2} \rightarrow (x^2 - 2x + 5) \cdot \frac{1}{4} = y$

25) Sendo $a>0$ determine os pontos comuns às parábolas $y^2 = ax$ e $x^2 = ay$.

$$x^2 - ay = y^2 - ax$$

$$x^2 - y^2 = a(y - x)$$

QUANDO $x = y$, HÁ PONTO EM COMUM

E QUANDO $x = 0 \wedge y = 0$

26) Determine k para que a reta $y = 4x + k$ seja tangente à curva $x^2 = 3y$.

$$x^2 = 3(4x + K)$$

$$x^2 - 12x - 3K = 0$$

$$\Delta = 144 - 4 \cdot (-3K)$$

$$\Delta = 144 + 12K \rightarrow 12K = -144$$

$$K = -12$$

27) Determine a equação da tangente à parábola $x^2 = 2py$ no ponto $(x_0,\ y_0)$ pertencente à parábola.

$$x^2 = 2py \Rightarrow y = \frac{x^2}{2p} \Rightarrow y' = \frac{x}{p}$$

$$y = \frac{x_0}{p} \cdot x + n$$

$$y_0 = \frac{x_0^2}{p} + n$$

$$n = y_0 - \frac{x_0^2}{p}$$

$$y = \frac{x_0 x}{p} + y_0 - \frac{x_0^2}{p}$$

$$yp = x_0 x + y_0 p - x_0^2$$

$$\boxed{p(y - y_0) = x_0(x - x_0)}$$