Álgebra Linear Numérica

08/03/25

Lista 2

Cap. 4

4.1. Determine SVDs of the following matrices (by hand calculation):

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

4.1 a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \to A^TA = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$

$\Rightarrow \sigma_1 = 3,\ \sigma_2 = 2$

$A^TA - \lambda I = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Av_j = \sigma_j u_j \Rightarrow u_j = \frac{Av_j}{\sigma_j}$

$u_1 = \frac{1}{3} \cdot \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow u_1 = \frac{1}{3} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_2 = \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow u_2 = \frac{1}{2}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\Sigma} \cdot \underbrace{I}_{V^T}$

4.1 b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \to U = I,\ V^T = I,\ \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

4.1 c) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

$\sigma_1 = 2,\ \sigma_2 = 0$ $\boxed{Av_j = \sigma_j u_j}$

$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_1 = \frac{1}{2} \cdot \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow u_1 = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

4.1 d) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\det(A^TA - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda + \cancel{1} \cancel{-1} = 0$

$\lambda(\lambda - 2) = 0$

$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 0 \Rightarrow \sigma_1 = \sqrt{2},\ \sigma_2 = 0$

$A^TA - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Normalizando v_i

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Av_j = \sigma_j u_j$

$u_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \frac{1}{2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

4.1 e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = 0 \Rightarrow \cancel{4} - 4\lambda + \lambda^2 - \cancel{4} = 0$

$\lambda^2 - 4\lambda = 0$

$\sigma_1 = 2,\ \sigma_2 = 0 \leftarrow \lambda(\lambda - 4) = 0$

$\frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\frac{v_2}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$Av_j = \sigma_j u_j \Rightarrow u_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \frac{1}{2} \Rightarrow \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} \cdot \frac{1}{2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix}$

$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

4.2. Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not exactly a standard mathematical transformation!). Do A and B have the same singular values? Prove that the answer is yes or give a counterexample.

$\underset{m\times n}{A} = \underset{m\times m}{U}\,\overset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$

4.2 $A \in \mathbb{C}^{m\times n} \wedge B \in \mathbb{C}^{n\times m}$

$\underset{m\times n}{A} = \begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_m & - \end{bmatrix} \Rightarrow \underset{n\times m}{B} = \begin{bmatrix} | & & | \\ a_m & \cdots & a_1 \\ | & & | \end{bmatrix}$

$Av_j = \sigma_j u_j$ $v_j^T \begin{bmatrix} | & & | \\ a_m & \cdots & a_1 \\ | & & | \end{bmatrix} = [v_j^T a_m \cdots v_j^T a_1]$

$A\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \sigma_1 & & 0 \\ 0 & \ddots & \sigma_n \\ & 0 & \end{bmatrix}\begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}$ $[v_j^T a_m \cdots v_j^T a_1] = [\sigma_j u_m \cdots \sigma_j u_1]$

$\underbrace{\underset{n\times n}{\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}}\underset{n\times m}{\begin{bmatrix} | & & | \\ a_m & \cdots & a_1 \\ | & & | \end{bmatrix}} = \underset{n\times m}{\begin{bmatrix} \sigma_1 & & 0 \\ 0 & \ddots & \sigma_n \\ & 0 & \end{bmatrix}}\underset{m\times m}{\begin{bmatrix} - & u_1 & - \\ & \vdots & \\ - & u_m & - \end{bmatrix}}}_{\text{Mesmos valores singulares}}$

4.4. Two matrices $A, B \in \mathbb{C}^{m\times m}$ are *unitarily equivalent* if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values?

$A = QB\bar{Q}^T \iff$ A e B tem mesmos valores singulares

$\Rightarrow$) $A = QB\bar{Q}^T \iff A = QU\Sigma\bar{V}^T\bar{Q}^T \Rightarrow A = W\Sigma\bar{W}^T$ com W sendo uma matriz unitária

QU é unitária e $\bar{V}^T\bar{Q}^T$ também

ou seja, A tem os mesmos valores singulares que B

$\Leftarrow$) A e B tem mesmos autovalores

→ Eu posso expressar Q (Matriz unitária) como a multiplicação de outras 2 matrizes unitárias? Sim!

$A = Q\bar{V}^T \Rightarrow AB = (Q\bar{V}^T)V = Q$
$B = V$

Ou seja, se $A = U_A\Sigma\bar{V}_A^T$ e $B = U_B\Sigma\bar{V}_B^T$, eu posso decompor A em $A = WU_B\Sigma\bar{V}_B^T\bar{W}^T$

porém, eu preciso que $WU_B\bar{V}_B^T\bar{W}^T = I$, logo, $U_B\bar{V}_B^T = I$

$\Rightarrow U_B = V_B$, logo, a volta NÃO VALE

CAP. 5

5.3. Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

(a) Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

(c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(d) Find A^{-1} not directly, but via the SVD.

(e) Find the eigenvalues λ_1, λ_2 of A.

(f) Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.

(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

a) $AA^T = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$

$\det(AA^T - \lambda I) = 0 \iff \begin{vmatrix} 125-\lambda & 75 \\ 75 & 125-\lambda \end{vmatrix} = 0$

$(125-\lambda)^2 - 5525 = 0$

$\lambda^2 - 250\lambda + 15.625 - 5625 = 0$

$\rightarrow \lambda^2 - 250\lambda + 10000 = 0 \rightarrow \lambda_1 = 50 \quad \lambda_2 = 200$

$\Rightarrow \sigma_1 = 5\sqrt{2} \wedge \sigma_2 = 10\sqrt{2}$

$$u_1 \rightarrow \underbrace{\begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix}}^{AA^T - 50I} \Rightarrow u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \qquad u_2 \rightarrow \overbrace{\begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix}}^{AA^T - 200I} \Rightarrow u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$Av_j = \sigma_j u_j \Rightarrow \begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix} v_1 = 5\cdot\cancel{\sqrt{2}}\cdot\left(\frac{1}{\cancel{\sqrt{2}}}\right)\cdot\begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow \begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix} v_1 = \begin{bmatrix}5\\-5\end{bmatrix}$$

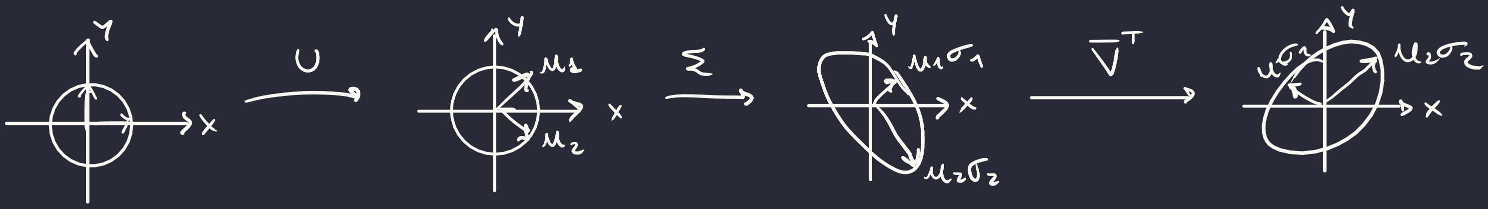
$$\det(A) = -10+110 = 100 \Rightarrow v_1 = \frac{1}{100}\cdot\begin{bmatrix}5 & -11\\ 10 & -2\end{bmatrix}\begin{bmatrix}5\\-5\end{bmatrix} \Rightarrow v_1 = \frac{1}{10\cancel{0}}\cdot\begin{bmatrix}8\cancel{0}\\6\cancel{0}\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}$$

$$v_2 = \frac{1}{100}\begin{bmatrix}5 & -11\\ 10 & -2\end{bmatrix}\begin{bmatrix}10\\10\end{bmatrix} \Rightarrow v_2 = \frac{1}{100}\begin{bmatrix}-60\\80\end{bmatrix} \Rightarrow v_2 = \frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}5\sqrt{2} & 0\\ 0 & 10\sqrt{2}\end{bmatrix}\begin{bmatrix}4/5 & 3/5\\ -3/5 & 4/5\end{bmatrix}$$

b) $\sigma_1 = 5\sqrt{2}$, $\sigma_2 = 10\sqrt{2}$

$$v_1 = \begin{bmatrix}1\\-1\end{bmatrix}\cdot\frac{1}{\sqrt{2}} \qquad v_2 = \begin{bmatrix}1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}} \qquad u_1 = \begin{bmatrix}4\\3\end{bmatrix}\cdot\frac{1}{5} \qquad u_2 = \begin{bmatrix}3\\4\end{bmatrix}\cdot\frac{1}{5}$$



c) $\|A\|_1 = \sup\{\|Ax\|_1 \,/\, \|x\|_1 = 1\}$ $\qquad$ $\|A\|_2 = \sup\{\|Ax\|_2 \,/\, \|x\|_2 = 1\}$

$\|A\|_1 = 16$ $\qquad$ $\|A\|_2 = \sqrt{\lambda_{max}} = \sigma_{max} = 10\sqrt{2}$

$$\|A\|_F = (\cancel{4+121+100}+\cancel{25})^{1/2} = (250)^{1/2} = \sqrt{125\cdot 2} = \sqrt{25\cdot 5\cdot 2} = 5\sqrt{10}$$

221 $\quad$ 29

250

$\|A\|_\infty = 11$

d) $A^{-1} = V\begin{bmatrix}\frac{1}{\sigma_1} & \\ & \frac{1}{\sigma_2}\end{bmatrix}\bar{U}^T \Rightarrow A^{-1} = \begin{bmatrix}4/5 & -3/5\\ 3/5 & 4/5\end{bmatrix}\begin{bmatrix}\frac{1}{5\sqrt{2}} & \\ & \frac{1}{10\sqrt{2}}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

$$A^{-1} = \begin{bmatrix}\frac{4}{25\sqrt{2}} & \frac{-3}{50\sqrt{2}}\\ \frac{3}{25\sqrt{2}} & \frac{4}{50\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix}\frac{5}{100} & \frac{-11}{100}\\ \frac{10}{100} & \frac{-2}{100}\end{bmatrix}$$

e) $\det(A-\lambda I) = 0 \Leftrightarrow \begin{vmatrix}-2-\lambda & 11\\ -10 & 5-\lambda\end{vmatrix} = 0 \Leftrightarrow (2+\lambda)(\lambda-5)+110 = 0$

$2\lambda - 10 + \lambda^2 - 5\lambda + 110 = 0$

$\Delta = 9 - 4\cdot 100 = -391$ $\qquad$ $\lambda^2 - 3\lambda + 100 = 0$

$$\lambda = \frac{3 \pm i\sqrt{391}}{2} \Rightarrow \lambda_1 = \frac{3+i\sqrt{391}}{2} \wedge \lambda_2 = \frac{3-i\sqrt{391}}{2}$$

f) $\det A = \left(\frac{3+i\sqrt{391}}{2}\right)\left(\frac{3-i\sqrt{391}}{2}\right) = \frac{9+391}{4} = \frac{400}{4} = 100$

$|\det A| = 5\sqrt{2} \cdot 10\sqrt{2} = 50 \cdot 2 = 100$

g) Como o círculo tem área π, o elipsoide terá área 100π $(|\det A| \cdot \pi)$

5.4. Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition (5.1) of the $2m \times 2m$ hermitian matrix

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

Se $A = U\Sigma\bar{V}^T$, $A = \sum_{i=1}^{r} \sigma_i u_i \bar{v}_i^T$

$$A = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \ddots \; 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

$$\begin{bmatrix} 0 & \bar{A}^T \\ A & 0 \end{bmatrix} \cdot \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \begin{bmatrix} Av_i \\ \bar{A}^T u_i \end{bmatrix} = \begin{bmatrix} \sigma_i u_i \\ \sigma_i v_i \end{bmatrix} = \sigma_i \begin{bmatrix} u_i \\ v_i \end{bmatrix}$$

$N(A) = \{v_{r+1} \cdots v_n\}$

$C(A) = \{u_1 \cdots u_r\}$

$C(A^T) = \{v_1 \cdots v_r\}$

$N(A^T) = \{u_{r+1} \cdots u_m\}$

$$\begin{bmatrix} 0 & \bar{A}^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v_i \\ -u_i \end{bmatrix} = \begin{bmatrix} Av_i \\ -Au_i \end{bmatrix} = \sigma_i \begin{bmatrix} u_i \\ -v_i \end{bmatrix}$$

logo $\begin{bmatrix} 0 & \bar{A}^T \\ A & 0 \end{bmatrix} = \left[\begin{array}{c|c} U & U \\ \hline V & -V \end{array}\right] \left[\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & \Sigma \end{array}\right] \left[\begin{array}{c|c} \bar{U}^T & \bar{V}^T \\ \hline \bar{U}^T & -\bar{V}^T \end{array}\right]$