Álgebra Linear Numérica

27/02/25

Lista Revisão

Capítulo 1

1.1. Let B be a 4×4 matrix to which we apply the following operations:

1. double column 1,
2. halve row 3,
3. add row 3 to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1 (so that the column dimension is reduced by 1).

(a) Write the result as a product of eight matrices.

(b) Write it again as a product ABC (same B) of three matrices.

1.1 a)

I. Dobrar a coluna 1: $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

II. Metade da linha 3: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

III. Adicionar a linha 3 na 1: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

IV. Trocar colunas 1 e 4: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

V. Subtrair linha 2 de todas as outras linhas $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$

VI. Substituir a linha 4 pela 3: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

VII. Diminuir uma dim (col): $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$(VI)(V)(III)(II \cdot B \cdot I)(IV)(VII)$$

1.3. Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices.)

1.3 $R \Rightarrow$ Triangular sup.

$R^{-1} = E_{kw} \cdots E_{pq} \rightarrow$ Várias matrizes de eliminação

Porém, como R é Δ sup., não usamos um pivô para alterar as linhas abaixo dele, pois as entradas são 0, logo, nenhuma matriz de eliminação terá uma entrada diferente de 0 abaixo da diagonal, logo R^{-1} também é uma diagonal superior.

1.4. Let $f_1, \ldots, f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1, \ldots, 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.

(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the i, j entry of A^{-1}?

Se $\forall d_i$, $d_i = \sum_{j=1}^{8} c_j f_j(i)$, isso quer dizer que, independente do d_i que eu escolher, vai existir um conjunto $\{c_1, \ldots, c_8\}$ que satisfaz a propriedade, isso implica que o sistema $Ac = d$ com $A = \begin{bmatrix} f_1(1) & \cdots & f_8(1) \\ \vdots & \ddots & \vdots \\ f_1(8) & \cdots & f_8(8) \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ \vdots \\ c_8 \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ \vdots \\ d_8 \end{bmatrix}$ é sempre solúvel! Logo, isso significa que A é uma transformação linear bijetiva, $\Rightarrow \exists A^{-1}$, logo, $c = A^{-1}d$. Como A é bijetiva, todo vetor d é fruto da transformação de um único vetor c.

CAPÍTULO 2

2.1. Show that if a matrix A is both triangular and unitary, then it is diagonal.

2.1 Provar que se Q é Δ sup. e ortogonal então ela é diagonal.

Q é triang. sup., então Q^{-1} é triang. sup também, mas $Q^{-1} = Q^T$, e Q^T deveria ser triang. inf. logo, se Q^T é Δ sup. e inf. ao mesmo tempo, então Q^T é diagonal, tal qual Q.

2.2. The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

(a) Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$.

(b) Show that this computation also establishes the general case, by induction.

2.2 a) $n = 2$

$\|x_1 + x_2\|^2$

$(x_1 + x_2)^H (x_1 + x_2)$

$x_1^H x_1 + x_2^H x_1 + x_1^H x_2 + x_2^H x_2$

$\|x_1\|^2 + \|x_2\|^2 + \underbrace{\cancel{2x_1^H x_2}}_{0} = \|x_1\|^2 + \|x_2\|^2$

b) Vale para k

b) $(x_1 + \cdots + x_n)^H (x_1 + \cdots + x_n)$

$x_1^H x_1 + x_1^H x_2 + \cdots + x_n^2$

Todos os produtos, com exceção dos $x_i^H x_i$ dão 0 pela ortogonalidade, logo

$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$ com $x_i^H x_j = 0$

2.4. What can be said about the eigenvalues of a unitary matrix?

2.4

$$Qx = \lambda x$$

$$x^T \underbrace{Q^T Q}_{I} x = \lambda^2 x^T x$$

$$\|\cancel{x}\|^2 = \lambda^2 \|\cancel{x}\|^2$$

$$1 = \lambda^2$$

$$\boxed{|\lambda| = 1}$$

2.5. Let $S \in \mathbb{C}^{m\times m}$ be *skew-hermitian*, i.e., $S^* = -S$.

(a) Show by using Exercise 2.3 that the eigenvalues of S are pure imaginary.

(b) Show that $I - S$ is nonsingular.

(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the *Cayley transform* of S, is unitary. (This is a matrix analogue of a linear fractional transformation $(1+s)/(1-s)$, which maps the left half of the complex s-plane conformally onto the unit disk.)

Provar que $\bar{\lambda} = -\lambda$

a) $S^H = -S$

$$Sx = \lambda x \rightarrow \bar{x}^T \bar{S}^T = \bar{x}^T \bar{\lambda}$$

$$\bar{x}^T \bar{S}^T S x = \bar{\lambda} \lambda \underbrace{\cancel{\bar{x}^T x}}_{\|x\|^2}$$

$$\bar{\lambda}\lambda = \frac{-\bar{x}^T S^2 x}{\|x\|^2} \Rightarrow \frac{-\bar{x}^T S \overset{\lambda x}{(\cancel{-\bar{S}^T})} \cancel{x}}{\|x\|^2} \Rightarrow \bar{\lambda} = \frac{-\bar{x}^T \overset{-\bar{S}^T}{S} x}{\|x\|^2}$$

$$\bar{S}^T = -S \Rightarrow \bar{S}^T x = -Sx$$

$$\Rightarrow -\bar{S}^T x = \lambda x$$

$$\bar{\lambda} = -\frac{\cancel{\bar{x}^T \lambda x}}{\cancel{\|x\|^2}}$$

$$\boxed{\bar{\lambda} = -\lambda}$$

b) Mostrar que $I - S$ é inversível

Dado $Sx = \lambda x$ sabemos que $\bar{\lambda} = -\lambda$

$$(I - S)x = x - \lambda x = (1-\lambda)x$$

ou seja, se λ é autovalor de S, $I-S$ tem $1-\lambda$ como autovalor. Como λ é puramente imaginário, $1-\lambda$ não pode ser 0, logo, como $\det(I-S) = \prod_{i=1}^{n} \lambda_i \Rightarrow \det(I-S) \neq 0$

c) $Q = (I-S)^{-1}(I+S)$

$$\bar{Q}^T = \overline{(I+S)}^T \overline{(I-S)}^{T^{-1}} = (I+\bar{S}^T)(I-\bar{S}^T)^{-1}$$

$$= (I-S)(I+S)^{-1}$$

$$(I-S)(I+S)^{-1}(I-S)^{-1}(I+S)$$

$$(I-S)\big(\underbrace{(I-S)(I+S)}_{=}\big)^{-1}(I+S)$$

$$\underbrace{(\cancel{I-S})\big((\cancel{I+S})}_{I}\underbrace{(\cancel{I-S})\big)^{-1}(\cancel{I+S})}_{I}$$

$$= I$$

2.6. If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is null(A)?

2.6 $A = I + uv^T$

$$(I + uv^T)(I + \alpha uv^T) = I$$

$$I + \alpha uv^T + uv^T + \alpha uv^T uv^T = I$$

$$\Rightarrow \alpha uv^T + uv^T + \alpha (v^T u) uv^T = 0$$

$$\alpha (uv^T + [v^T u] uv^T) = -uv^T$$

$$\alpha uv^T (1 + v^T u) = -uv^T$$

$$\Rightarrow \alpha (1 + v^T u) = -1 \Rightarrow \boxed{\alpha = \frac{-1}{v^T u + 1}}$$

If $v^T u = -1$, then A is singular

If $N(A) \neq 0 \Rightarrow v^T u = -1$

$\hookrightarrow (I + uv^T)(u) = u + u\overset{-1}{v^T u} \Rightarrow u \in N(A)$

CAPÍTULO 3

3.2. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the *spectral radius* of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

3.6 $\rho(A) \leq \|A\|_n$ (I)

Vamos supor que $\boxed{\rho(A) > \|A\|_n}$. Por definição:

$\|A\|_n = \sup\left\{\frac{\|Ax\|_n}{\|x\|_n}\right\}$. Logo, se $\rho(A)$ é o autovalor associado ao autovetor w $(Aw = \rho(A)w)$, a proporção $\frac{\|Aw\|_n}{\|w\|_n}$ está no conjunto.

$$\frac{\left(\sum_{k=1}^{m} |Aw_k|^n\right)^{1/n}}{\left(\sum_{k=1}^{m} |w_k|^n\right)^{1/n}} = \frac{\left(\sum_{k=1}^{m} |\rho(A) w_k|^n\right)^{1/n}}{\left(\sum_{k=1}^{m} |w_k|^n\right)^{1/n}} = \frac{\left(|\rho(A)|^n\right)^{1/n} \cdot \cancel{\left(\sum_{k=1}^{m} |w_k|^n\right)^{1/n}}}{\cancel{\left(\sum_{k=1}^{m} |w_k|^n\right)^{1/n}}}$$

$= \boxed{|\rho(A)|}$ ou seja, $|\rho(A)|$ está no conjunto, logo, pela definição de $\|A\|$, $\rho(A) \leq \|A\|$, ou seja, (I) é ABSURDO

3.3. Vector and matrix p-norms are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.

(a) $\|x\|_\infty \leq \|x\|_2$,

(b) $\|x\|_2 \leq \sqrt{m}\,\|x\|_\infty$,

(c) $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$,

(d) $\|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$.

3.3

a) $\|x\|_2 \geq \|x\|_\infty$

$$\sqrt{\sum_{i=1}^{m} x_i^2} \geq \max_{i=1,\dots,m} |x_i|$$

$$\sqrt{\underbrace{\sum_{i=1}^{m-1} x_i^2}_{\geq 0} + \left(\max_{i=1\dots m} |x_i|\right)^2} \geq \max_{i=1,\dots,m} |x_i|$$

$\hookrightarrow$ Por construção eu posso chegar de $\|x\|_\infty$ até $\|x\|_2$

$\rightarrow \|x\|_\infty \geq \|x\|_\infty \Leftrightarrow \sqrt{\|x\|_\infty^2} \geq \|x\|_\infty$

Se eu adiciono termos POSITIVOS AO QUADRADO dentro da raíz $(|x_i|^2)$, o valor direito da desigualdade sempre irá aumentar.

b) $\|x\|_2 \leq \sqrt{m}\,\|x\|_\infty$

$$\sqrt{\sum_{i=1}^{m} |x_i|^2} \leq \sqrt{m}\,\|x\|_\infty$$

$\rightarrow \|x\|_2 \geq 1$

$$\sum_{i=1}^{m} |x_i|^2 \leq m \cdot \|x\|_\infty^2$$

Na esquerda eu tenho uma soma de termos.
Do outro, eu tenho m vezes o maior termo do lado esquerdo da desigualdade.

$\rightarrow \|x\|_2 < 1 \Leftrightarrow \|x\|_\infty < 1 \Rightarrow \sum_{i=1}^{m} |x_i|^2 \geq m \cdot \|x\|_\infty^2$

$\Rightarrow \|x\|_2 \leq \sqrt{m} \cdot \|x\|_\infty$

c)

d)

3.5. Example 3.6 shows that if E is an outer product $E = uv^*$, then $\|E\|_2 = \|u\|_2\|v\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|u\|_F\|v\|_F$? Prove it or give a counterexample.

3.5 Provar ou refutar que

$$\|u\bar{v}^T\|_F = \|u\|_F \cdot \|v\|_F$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow u\bar{v}^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} \bar{v}_1 & \dots & \bar{v}_n \end{bmatrix}$$

$$\Rightarrow u\bar{v}^T = \begin{bmatrix} u_1\bar{v}_1 & \dots & u_1\bar{v}_n \\ \vdots & & \vdots \\ u_n\bar{v}_1 & \dots & u_n\bar{v}_n \end{bmatrix}$$

$$\|u\bar{v}^T\|_F = \left(\sum_{i,j=1}^{n} (u_{ij}\bar{v}_{ij})^2\right)^{1/2}$$

$$\|u\|_F = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2} \quad \|v\|_F = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$$

$$\left((u_1^2 + \dots + u_n^2)(v_1^2 + \dots + v_n^2)\right)^{1/2}$$

$$\left((u_1v_1)^2 + (u_1v_2)^2 + \dots + (u_1v_n)^2 + \dots + (u_nv_n)^2\right)^{1/2}$$

$= \|uv^T\|_F \rightarrow$ Vale para matrizes reais

Matrizes Complexas não vale!

Contra-exemplo: $\left\|\begin{bmatrix} 1 \\ i \end{bmatrix}\right\|_F = 2$ $\quad \left\|\begin{bmatrix} i \\ 2 \end{bmatrix}\right\|_F = 3$

$$\left\|\begin{bmatrix} i & 2 \\ -1 & 2i \end{bmatrix}\right\|_F = 10$$

1
2
25
25
125
50
625

3.6. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$. The corresponding *dual norm* $\|\cdot\|'$ is defined by the formula $\|x\|' = \sup_{\|y\|=1} |y^*x|$.

(a) Prove that $\|\cdot\|'$ is a norm.

(b) Let $x, y \in \mathbb{C}^m$ with $\|x\| = \|y\| = 1$ be given. Show that there exists a rank-one matrix $B = yz^*$ such that $Bx = y$ and $\|B\| = 1$, where $\|B\|$ is the matrix norm of B induced by the vector norm $\|\cdot\|$. You may use the following lemma, without proof: given $x \in \mathbb{C}^m$, there exists a nonzero $z \in \mathbb{C}^m$ such that $|z^*x| = \|z\|'\|x\|$.

3.4 a) Definição norma:

$$\|\alpha x\| = |\alpha| \cdot \|x\|$$
$$\|x\| \geq 0 \wedge \|x\| = 0 \Leftrightarrow x = 0$$
$$\|x+y\| \leq \|x\| + \|y\|$$

$$\|x\|' = \sup_{\|y\|=1} \{|\bar{y}^T x|\}$$

(I) $|\alpha\beta| \leq |\alpha||\beta|$

$$\|\alpha x\| = \sup_{\|y\|=1}\{|\bar{y}^T(\alpha x)|\} = \sup_{\|y\|=1}\{|\alpha \bar{y}^T x|\} \overset{(I)}{=} \sup_{\|y\|=1}\{|\alpha| \cdot |\bar{y}^T x|\} = |\alpha| \cdot \sup_{\|y\|=1}\{|\bar{y}^T x|\}$$

$\sup\{|\bar{y}^T x|\} = 0$, if $x \neq 0$, $\exists y \in \mathbb{C}^n / \bar{y}^T x \neq 0 \Rightarrow \|x\|' \neq 0 \Rightarrow \|x\|' = 0 \Leftrightarrow x = 0$

$$\sup_{\|y\|=1}\{|\bar{y}^T(x+z)|\} = \sup_{\|y\|=1}\{|\bar{y}^T x + \bar{y}^T z|\} \leq \sup_{\|y\|=1}\{|\bar{y}^T x| + |\bar{y}^T z|\}$$
$$= \sup_{\|y\|=1}\{|\bar{y}^T x|\} + \sup_{\|y\|=1}\{|\bar{y}^T z|\}$$

$$\Rightarrow \|x+z\|' \leq \|x\|' + \|z\|'$$

b) $\|x\|' = \sup_{\|y\|=1} \{|\bar{y}^T x|\}$

Dados $\|x\| = \|y\| = 1$ com $x, y \in \mathbb{C}^m$, mostre que $\exists B \in \mathbb{C}^{m \times m}$ de posto 1 ($B = y\bar{z}^T$) tal que $Bx = y$ e $\|B\| = 1$ onde $\|B\|$ é a máxima norma de B induzida pela norma $\|\cdot\|$.